Quantal response equilibria in sequential discrete choice models: a framework for imposing equilibrium conditions on a microscopic travel demand model

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Abstract

This thesis further develops the microscopic travel demand model Octavius by extending it with the ability to impose equilibrium conditions. In order to do this, an extensive literature research into microscopic travel demand modelling and equilibria is conducted. Then a mathematical framework for sequential discrete choice models is analyzed with equilibrium conditions. For this framework, existence conditions are analyzed and a relaxation of an existing uniqueness condition is proven. Moreover, an efficient solution algorithm based on Monte Carlo simulation is developed and implemented. This algorithm is validated and computationally tested. Finally, two case studies are conducted to showcase the relevance of the developed theory and algorithm.

Keywords— Quantal response equilibrium, discrete choice model, random utility model, Logit-QRE, microscopic travel demand model, best-response algorithm

Preface

This thesis marks the end of almost six years as an Applied Mathematics student at the University of Twente. Starting at the first lectures, I immediately knew that I chose the right study. It took me a year or two to develop the correct work ethic, but once I did, I started enjoying the study even more than I already did. After successfully finishing my bachelor, I started my master Applied Mathematics at the Stochastic Operations Research department, specialising in queueing theory and (discrete) optimization. Almost without exception, I enjoyed the courses in my master program, and after a year I finished all the necessary subjects. In March 2021, I started my internship at DAT.Mobility, a company specialised in data and software to support mobility-related decision making, part of the mobility consultancy bureau Goudappel. My internship was both enjoyable and successful, despite working from home. Hence the logical step was to do my master thesis there as well. Finishing my master thesis would not have been possible without the help of several people.

First of all, I want to thank DAT.Mobility for allowing me to do both my internship and master thesis at their company. I want to thank Bernike Rijksen and Luuk Brederode for their excellent supervision. Especially during my internship, Bernike helped me a lot to get used to the company environment. And I want to specifically thank Luuk for coming up with the idea for this master thesis assignment. I also want to thank my colleagues at DAT.Mobility and Goudappel for providing me with an excellent work environment, especially during the non-lockdown months. I am looking forward to continue working with you in the future.

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Contents

Chapter 1

Introduction

Since the 1950s, travel demand models have been used to support governing decisions. Knowing and being able to predict travel demand has since then been an important tool for determining the required infrastructure and public transport services to accommodate travel demand. However, with modern problems, for example regarding sustainability and new forms of mobility, details such as the heterogeneity of individuals and modelling interactions between individuals have become a relevant aspect of travel demand models, resulting in a shift from macroscopic to microscopic travel demand models.

1.1 Octavius

At DAT.Mobility, the microscopic travel demand model, used for long-term strategic forecasting, is Octavius. Octavius contains a population synthesizer, that generates a synthetic population consisting of agents, representing people, with certain characteristics. To estimate the travel demand, these agents move through a system of discrete choice models. The choice probabilities in each discrete choice model are derived as a random utility model, more specifically a multinomial logit model. The modelled probabilities are translated into individual choices using Statistical Noise Elimination Technique (SNET) [24]. The discrete choice models currently included in Octavius can be grouped into three stages:

1. Tourgenerator: in the tourgenerator, all agents decide how many tours they undertake, along with the number, purposes and order of the trips within each tour.
2. Destination choice: in the destination choice model, for each available mode, the agents decide what destination they would choose for each tour (first and second tour and possibly a secondary destination in the first tour) if they would use that mode.
3. Mode choice: in the mode choice model each agent chooses one of the available modes (and with that also the corresponding set of destinations).

An elaborate description of the models currently included within Octavius is presented in [24]. After this sequence of discrete choice models, apart from the timing of the trips and the route, a full activity-travel pattern is determined.

1.2 Equilibria in travel demand models

Since the development of the user equilibrium by Wardrop in 1952 [48], equilibria have played an important role in travel demand modelling, especially in modelling route assignment. The main motivation to impose equilibria is that it has been shown to be a realistic method of modelling route assignment, for example by Dixit and Denant-Boemont [13]. Another important reason is the stability of the model: if the model satisfies a certain equilibrium condition, than it is stable. Moreover, if the equilibrium is unique, the model is comparable to any other model satisfying the same equilibrium condition. Until the development of microscopic and activity-based travel demand models, applying the notion of equilibria in other stages than route assignment had not often been done. In microscopic and activity-based models often feedback loops are used to determine an equilibrium for relations and interactions more detailed than only the route assignment. An important example of this is MATSim [3]. MATSim is an open-source traffic simulation software package, that can be used to simulate travel demand, including route assignment, and that defines an equilibrium which incorporates all aspects of the travel plans of agents. Many other travel demand models use MATSim as a tool for route assignment, such as MITO [35] and CEMDAP [51]. Other models, such as MobiTopp [49] use MATSim to attain an equilibrium between the destination, mode and route choice.

Equilibrium conditions regarding route assignment are usually imposed since route assignment results in congestion, increasing the travel times on which the initial choice of route was based. Other than this obvious feedback relation, there are more possible applications of equilibria in travel demand models. An example is the dependence of a choice on the influence it has on future opportunities, as incorporated by Habib [20]: if a person decides to choose a certain mode, this might make some destinations unavailable. In this way, the choice of mode influences the future opportunities of possible destinations. Another example of an application of equilibria would be in shopping: shops that are busy at a certain time are less attractive to visit, which could result in people changing their departure time to this shop. A final example would be the availability of shared mobility without a reservation system, such as the Dutch "ov-fietsen" (publicly available shared bicycles at most train stations in the Netherlands). The less of these that are available, the higher the risk that once you arrive at the train station there are none available. Hence, when more people choose this shared mobility as their mode, it may become less attractive, or even unavailable.

Other than these realistic relations for which equilibria can be imposed, imposing equilibria can also be used as a modelling technique. When using feedback loops to enforce capacity or availability constraints it can be useful to analyze the stability and uniqueness of the outcome of the model with these feedback loops. Another example is attraction constraints, which are very common in travel demand models. An attraction constraint essentially ensures that the number of trips arriving at a destination matches the expected number of trips arriving at that destination, given the number of jobs, shops and other variables. These constraints could be modelled as a negative (positive) feedback loop if too many (few) agents choose a certain destination. In that way it can be ensured that the attraction constraints are satisfied. Analyzing these feedback loops using the notion of equilibria can then ensure that the model with attraction constraints has a stable and unique solution.

1.3 Research goal

Based on the relevance of equilibria in travel demand modelling, the goal of the research is to extend Octavius with the ability of imposing equilibrium conditions. In order to do this systematically, we undertake four steps:

1. Perform an extensive literature research to investigate the possibility of integrating an existing travel demand model that incorporates equilibrium conditions (such as MATSim) with Octavius.
2. Formulate the problem as a mathematical model.
3. Analyze the model theoretically, focusing on conditions for existence and uniqueness of solutions.
4. Determine and implement solution algorithms for the model and analyze the computational effort required to find a solution.

1.4 Outline of the report

Chapter 2 discusses the results of the literature research. Chapter 3 describes the mathematical model, together with some preliminary knowledge and notation of various mathematical fields. Chapter 4 concerns existence and uniqueness conditions of quantal response equilibria. Chapter 5 presents the necessary theory about systems of discrete choice models and how these relate to the uniqueness conditions. Chapter 6 describes and validates the solution algorithms, and presents a computational analysis of these algorithms. Chapter 7 concerns the case studies that were conducted to show the practical use of the developed theory. Finally, Chapter 8 concludes and discusses the research results and provides ideas for future research. In Appendix A some additional theoretical background and preliminary knowledge is presented.

Chapter 2

Literature research

This chapter describes the results of the literature research conducted in September 2021. The database of Scopus was used for this literature research. The keywords searched were “Agent-based travel demand”, “Activity-based travel demand” and “Travel demand microsimulator”. resulting in a total of 846 papers. The database of MathSciNet was used as well, without resulting in any additional papers. Finally, the Google search engine was used specifically for finding discussions on the current state and problems with microscopic travel demand models. This chapter firstly presents a historical review of travel demand models, describing the different aspects that constitute a travel demand model. Subsequently, this chapter describes the current state of (microscopic) travel demand models. Finally, this chapter presents a list of desired properties of strategic microscopic travel demand models, which are used to analyze the possibility of integrating an existing travel demand model with Octavius to achieve the research goal.

2.1 Historical review

The classic trip-based travel demand model consists of four steps:

1. Trip generation
2. Trip distribution
3. Mode choice
4. Route assignment

Later, also the scheduling of the trip(s) was added to the travel demand model, but it is still mostly referred to as the four-step travel demand model.

Trip generation

In the first step, trip generation, the total number of undertaken trips in a given time frame is determined. The earliest trip generation model, at the Chicago Area Transportation Study (CATS) in the 1950s [29], was build around the idea of the number of trips directly relating to the land use[1] of a certain area. In later years, more and more detail has been added to this phase of travel demand modelling. Nowadays, purposes of trips are considered (e.g. work or shopping) and also distinctions in trip rates based on demographic characteristics (e.g. gender, household size, car ownership) are made. On top of that, many modern models, such as MITO [35] and MobiTopp [27], make clear distinctions between mandatory and discretionary trips.

Trip distribution

In the trip distribution step, origins are matched with destinations for each trip, thus determining the destination choices of travellers. This is traditionally done using a gravity model: the attractiveness of choosing a certain destination decreases with increasing distance, with distance represented by travel time or cost, whilst it increases with higher attractiveness, for example caused by a larger number of available

[1]Land use is a term describing the economic and cultural activities that are practiced at a certain place.

shops. Reilly's law of retail gravitation, presented in 1931 [40], was the first notion of such an analogy between destination choice and gravity. In the analogy with Newton's law of gravitation, the attractiveness of a retail center becomes the mass. Correspondingly, Reilly defined the *point of indifference*, the point on the line between two retail centers A and B where both retail centers are equally attractive. This point is closer to A when B is larger and vice versa. The concept of gravity models is still to this day used in many models in the trip distribution phase of travel demand modelling. Other trip distribution models include utility based models, often logit models or other S-shaped utility models, such as for example Aurora [2]. Another method is the Fratar method, which is an iterative proportional fitting method. In a paper by Heanue et al. [21] from 1966 the Fratar method is explained and compared to a gravity model.

Mode choice

In the early CATS there was already some form of mode choice model. Based on historic data they attempted to predict the influence of increased car ownership and car use on transit use. Later, discrete choice modelling has been the most prominent model for mode choice, mostly logit based models or other S-shaped utility models. An example of a travel demand model using discrete choice modelling for the mode choice is SIMBA MOBi [43].

Route assignment

Route assignment is a topic that has been researched to great extent. It is also often referred to as traffic assignment or route choice modelling. In the route assignment phase each trip is assigned a route. The first models in route assignment were so-called "all or nothing" assignments, where every traveller takes the shortest route. It became apparent that this was not the most realistic method of assigning routes, due to congestion effects. At first, heuristic methods were implemented to deal with this. Later, formal mathematical foundations were laid, initiated by Wardrop's first principle of route choice [48], or user equilibrium (UE). The user equilibrium is the equivalent of a Nash equilibrium in the game of route choice. In later studies, it was found that this definition of equilibrium is not necessarily the most realistic. The main reason for this is that the user equilibrium assumes that all travelers have perfect knowledge of the system, which in practice is often not the case. This led to the development of the stochastic user equilibrium (SUE) by Daganzo and Sheffi in 1977 [12]. The stochastic user equilibrium assumes that travelers do not always choose the optimal route, and it does so by adding noise (randomness) to the utilities of the different options. In this way, there is a probability that a traveler chooses a non-optimal route, but the probability of a traveler choosing a better route is always higher. It has been shown, for example by Dixit and Denant-Boemont [13], that this is a more accurate representation of human behavior in route choice modelling. Other than the introduction of equilibria, most of the improvements in route assignment models in the last decades have been adding capacity constraints, storage constraints and removing the stationary travel demand assumption. Currently, one of the most used route assignment models is Dynamic Traffic Assignment (DTA), which the agent-based traffic assignment in MATSim is based on. An elaborate review of route assignment models is presented by Bliemer et al. [7].

Trip scheduling

The developments of models concerning scheduling of trips dates back to the work of Vickrey [47], whose departure time choice model, known as the Vickrey bottleneck model, describes a dynamic equilibrium between departure times and road congestion. The Vickrey bottleneck model has been extended to allow for multiple destinations, heterogeneous preferences in departure times and heterogeneous valuation of travel time [25]. Most microscopic travel demand models use some form of heuristic or rule-based approach for the timing of activities, for example TASHA [41]. The model MITO [35] uses the concept of a time budget in scheduling the activities. On top of departure time choice and scheduling models, arrival time choice is another method of determining the timing of trips.

2.2 Modern travel demand models

The first real deviation from the traditional four-step modelling paradigm was the tour-based model [8]. A disadvantage of the trip-based four-step model is that a trip chain, or tour, is not in general accurately modelled. It may occur that modes differ between a trip back and forth, and in the case of a tour with multiple destinations, trip-based models are unable to accurately represent these tours (an A-B-C-A tour for example is modelled as an A-B-A and A-C-A tour instead). This is illustrated in [9] and [24]. Tour-based models take into account these complete trip chains and can hence accurately represent them.

In scientific literature, around the 2000s, a shift from trip- and tour-based models towards activity-based models occurred. The main principle behind activity-based models is that travel demand is a result from activity demand, and that hence the activity demand should be determined in order to predict the travel demand. In these models, the demand of activities is determined and the traditional four steps of the travel demand model are seen more as a scheduling problem to fit these activities. An advantage of activity-based models is for example that purposes of trips and other detailed information can be modelled more accurately. It quickly became apparent that activity-based models could provide great advantages for environmental purposes, which was firstly recognized by Shiftan in 2000 [45]. In [39], Int Panis argues the advantages of activity-based models for solving the air pollution epidemiology: due to the higher level of detail in these models, exposure to air pollution can be modelled more precisely. For example, in Beckx et al. [6] the total estimate of air pollution exposure is better, and at the same time they also enable the disaggregation of exposure over different activity types. Moreover, environmental concerns and new forms of mobility, such as autonomous vehicles and shared mobility, require governments to shift the focus from expanding infrastructure to managing travel demand [15]. An elaborate discussion of the advantages of activity-based model is presented by McNally and Rindt [32].

A recent trend is from activity-based models towards agent-based models. In agent-based models, the behaviour of agents, which most of the time represent persons or households, is autonomous, meaning that there is no strict (mathematical) model defining the behaviour of agents. In for example MATSim, agents' plans are sometimes randomly adjusted and subsequently evaluated (scored). In this way, agents learn what better and worse plans are. In non agent-based models, agents make such choices based on deterministic models, most of which fall into the following two categories:

1. Discrete choice modelling: a finite set of possible choices, from which each agent chooses one based on a known probability distribution.
2. Rule-based methods: a set of rules, sometimes in the form of a decision tree, which determine the choice of each agent.

Note that these non agent-based models, similar to agent-based models, still have a stochastic component to them. However, the probability that agents make a certain decision is explicitly modelled by the modeller. In the field of transportation engineering, the term agent-based is often used for any model that incorporates some type of individual behaviour, however we prefer reserving the term for autonomous behaviour since that is more in line with the definition of agent-based models as used in other scientific fields such as computer science. Nguyen et al. [37] give an overview of existing microscopic traffic simulators, categorized as fully agent-based, featuring agent technology and activity-based.

Other than the distinction between trip-, tour-, activity- and agent-based models there is another way that we can categorize travel demand models, based on [37]:

1. Macroscopic: low level of detail; e.g. aggregated traffic flows.
2. Microscopic: high level of detail; entities such as individuals, household and vehicles modeled with a high level of detail (e.g. demographic characteristics).
3. Mesoscopic: a mixture of macroscopic and microscopic; individuals are modeled but not with a high level of detail in behaviour and interaction.
4. Nanoscopic: an extremely high level of detail. You could think of autonomous vehicles where aspects such as gear shifting are modelled.

The nanoscopic level is as of yet not relevant for travel demand models. Since the term mesoscopic is ambiguous we prefer to only use the terms microscopic and macroscopic. In the early stages of travel demand modelling, macroscopic results (e.g. traffic flows, modal split) were enough to meet the requirements for determining governing policies. However, in recent years a need for disaggregate macroscopic results developed: macroscopic results disaggregated per purpose, or per demographic characteristic. In [9] and [24] it is discussed why microscopic travel demand models are computationally advantageous for obtaining disaggregate macroscopic results. On top of the computational disadvantages, macroscopic models are unable to explicitly model interaction between persons and within or between households. Moreover, macroscopic models are unable to take into account dependencies of choice probabilities on earlier decisions (e.g. destination choice depending on trip purpose).

2.3 Desired properties of a microscopic travel demand model

Now that the relevance of microscopic models is established, it is important to set up a list of desired properties of such a model, since microscopic models come with a variety of problems that do not occur in macroscopic models. Based on some discussions and comments in various papers, and mainly based on the paper by Kagho et al. [23], we set up the following list of desired properties of a microscopic travel demand model:

1. Realism: the results of the model should resemble observed mobility patterns such as trip length frequencies, modal splits and traffic flows.
2. Comparability: in order to use the model for strategic decision making, comparability is crucial. Comparability means that any minor change in input should lead to a minor change in output. Moreover, the changes in output must be explainable by the changes in input. Two main checks are required to verify comparability:
   - Sensitivity analysis: evaluating effects of minor changes in input.
   - Forecasting ability: testing the ability of the model to provide an accurate prognosis for a later time, given that the required data for that later time is available.

   In [23], Kagho et al. state that the latter of these has never been validated in microscopic models. However, in [41], the TASHA model has successfully been validated for its forecasting ability.
3. Transparency: a model is transparent if:
   - The effects of stochasticity in the model are accounted for.
   - The model is reproducible: another researcher can, provided the necessary data is publicly available, replicate the model and its outcomes.

The first property is often the focal point of the validations of travel demand models. Advanced calibration techniques are used to make sure that on an aggregated level, such as traffic flows, the outcome of the model is realistic. The second and third properties are what most models lack, which is also stated by Kagho et al. There are a couple of reasons for this. One of them is that most models use some form of calibration in order to replicate aggregate measures. And although this is not necessarily a bad methodology, it is important to analyze the effects of using such a calibration technique on the forecasting ability (do changes in input yield realistic changes in output). Moreover, from a scientific point of view, it is important to account for the stochasticity of the outcome due to the calibration techniques. A final problem is that not all models describe their calibration techniques in great detail, meaning that reproducibility is not satisfied.
In general, agent-based models do not fit these requirements. Due to the lack of a (mathematical) model behind the simulated choices, both comparability and transparency can not be guaranteed. Moreover, the outcomes are highly stochastic, especially on an individual level.
Octavius aims to satisfy these three properties in the following way:

1. Behavioral models are estimated in the form of multinomial logit models. These models (sets of parameters) are estimated using log-likelihood maximization on datasets containing observed (tourfrequency, destination and mode) choices of travellers as the dependent discrete choice variable amended with data on properties of both the chosen and non-chosen discrete alternatives and properties of the traveller and their context as the independent variables. Only models that fit sufficiently to the data (adjusted r squared values aimed to be as high as possible) are being used and only parameters that are statistically significant[2]. The choice allocation technique used, SNET, is proven to return the results closest to the modelled reality based on the probability distributions [24]. Combining these two facts we see that Octavius is able to replicate an average situation with high accuracy.
2. The choice allocation technique SNET is also proven to compare two situations with as little change in output as required to represent the change in input [24].
3. The causes of stochasticity in Octavius are limited, although an extensive research into the precise effects of this has not yet been done. The model is reproducible: given that the same data is used (and thus the same estimated parameters are derived), the results in Octavius will only differ due to the known causes of stochasticity.

[2]With statistically significant here is meant that the parameters is larger than 0 (hence contributing to the utility) with significance level 95%.

The main reason that Octavius satisfies these properties is because Octavius is a framework built upon well-known mathematical formulations and conditions. The population synthesizer satisfies maximum entropy conditions, and the choice probabilities are derived under the assumption of random utility maximization. On top of that the framework is modular; it consists of a sequence of well-defined discrete choice models, which can easily be extended, replaced or removed. These properties make Octavius both comparable (each outcome satisfies these conditions) and reproducible (there is no unpredictable heuristic or complex method used). The use of SNET further increases the comparability of Octavius and ensures the realism of the outcome.

Using agent-based technology or heuristics generally leads to losing the performance guarantees, especially comparability and transparency. In the literature research, no existing travel demand model came up that imposes equilibria whilst not using any of these methodologies. Hence, the conclusion of the literature research is that there is to our knowledge no existing travel demand model which we can integrate with Octavius to be able to impose equilibrium conditions whilst keeping the desired properties satisfied. Therefore, the remainder of the research will focus on extending the sequential discrete choice model based framework of Octavius with the ability to impose equilibrium conditions.

Chapter 3

Mathematical framework

This chapter firstly discusses some preliminaries concerning discrete choice models and random utility models. Moreover, an analysis of different equilibria is presented, including the quantal response equilibrium. Using this theory, the general mathematical model which can be applied to Octavius is setup.

3.1 Preliminaries

3.1.1 Discrete Choice Model

Discrete choice models are models that describe the choices of certain entities, e.g. persons or firms, among some discrete set of alternatives. There are three important characteristics the set of alternatives must exhibit in order to fit within the framework of a discrete choice model:

1. The alternatives are mutually exclusive; choosing one alternative implies not choosing any of the other alternatives.
2. The set of alternatives is exhaustive; all possible choices are included.
3. The set of alternatives is finite.

These necessary characteristics are not restrictive: as long as your choice options are finite, the first two characteristics can be satisfied. Consider the first characteristic: if the choice set would consist of some alternatives $\mathcal{J} = (1, \ldots, J)$ and any possible combination of these alternatives, a new set of alternatives can be constructed consisting of all possible combinations and the first characteristic is satisfied. Similarly, if the set of alternatives is not exhaustive, a "dummy" alternative can be added representing all choices that are not included, yielding an exhaustive alternative set. The final characteristic is restrictive: there are certainly scenarios where the choice set is not finite (think for example of the amount of money to invest in a certain stock). This characteristic distinguishes discrete choice models from regression models, which allow for continuous (and thus infinite) choice options.

3.1.2 Random Utility Model

Usually, discrete choice models are derived under the assumption of utility-maximizing behavior. This is a principle in behavioral sciences that assumes that any person aims to maximize their (expected) utility, or reward, when making a decision. A model that derives the choice probabilities in this way, originating from the work of Marschak [28], is called a Random Utility Model (RUM). It is important to note that random utility models can be used to represent decision making that is not based on utility maximization. The derivation ensures consistency with random utility maximization, rather than exclude the model from being consistent with other forms of behavior. In principle, random utility models can simply be seen as describing the relation of explanatory variables (such as demographic characteristics) to the choices made, without specifying how the choice is made [46]. The basic principle is as follows: consider some decision-maker i, the utility that this decision-maker obtains from choosing some alternative j equals U_{ij}. However, a researcher does not observe this exact utility. Depending on the type of data available, the researcher observes some attributes x_{ij} of the alternatives (possibly different among different decision-makers) and some attributes s_i of the decision-maker. Then, the researcher can set up a function $V_{ij} = V(x_{ij}, s_i)$, which is often called the representative utility. In general, $U_{ij} \neq V_{ij}$, and hence some error terms ϵ_{ij} are introduced to capture the part of utility that the researcher can not observe. Hence the utility U_{ij} is decomposed as $U_{ij} = V_{ij} + \epsilon_{ij}$. There are many different models resulting from this notion of utility, depending on the error distribution that is assumed. If a type I extreme value distribution[1] is assumed, the probability that the utility U_{ij} for agent i and alternative j is larger than the utilities for all other alternatives j' follows the Logit formula:

$$P_{ij} = \frac{e^{V_{ij}/\lambda_i}}{\sum_{j' \in \mathcal{J}} e^{V_{ij'}/\lambda_i}} \tag{3.1}$$

Another realistic error distribution that is often assumed is a Normal distribution, which results in the Probit model. However, for this model no closed form solution exists.

3.1.3 Game Theory

As discussed in Chapter 1, equilibria have played a large role in traffic modelling. The focus initially was on the problem of route choice and the effects congestion has on the route choice of agents. This led to the introduction of the User Equilibrium by Wardrop in 1952 [48]. This is essentially the Nash Equilibrium applied to the strategic interaction in the route choice problem. To formally define the different notions of equilibria, consider a game $\Gamma = (\mathcal{A}, \{\Delta_i\}_{i=1}^n, \{\boldsymbol{V}_i\}_{i=1}^n)$, where $\mathcal{A}$ is the set of agents $(1, \dots, n)$, Δ_i represents the set of strategies $\boldsymbol{\pi}_i = (\pi_{ij})_{j \in \mathcal{J}}$ for agent i and alternatives $j \in \mathcal{J}$. Such a strategy is essentially a probability distribution over the set of alternatives. Clearly, such a strategy $\boldsymbol{\pi}_i$ must satisfy $\sum_{j \in \mathcal{J}} \pi_{ij} = 1$. $\boldsymbol{\pi}_i$ is called a pure strategy if $\pi_{ij} = 1$ for some j and a mixed strategy otherwise. $\boldsymbol{V}_i = (V_{ij})_{j \in \mathcal{J}}$ is the set of representative utilities for agent i. By $\boldsymbol{\pi}_{-i}$ and $\boldsymbol{V}_{-i}$ the strategies and utilities of all agents except agent i are denoted.

Definition 3.1 (Nash Equilibrium)**.** Let $\Gamma = (\mathcal{A}, \{\Delta_i\}_{i=1}^n, \{V_i\}_{i=1}^n)$. A Nash Equilibrium (NE) is a set of strategies $\{\boldsymbol{\pi}_i\}_{i=1}^n$ such that no agent can unilaterally improve their utility. Hence, a Nash Equilibrium (NE) is a set of strategies $\boldsymbol{\pi}^*$ such that

$$V_i(\boldsymbol{\pi}_i^*, \boldsymbol{\pi}_{-i}^*) \geq V_i(\boldsymbol{\pi}_i, \boldsymbol{\pi}_{-i}^*) \quad \forall \boldsymbol{\pi}_i \in \Delta_i \quad \forall i \in \mathcal{A}$$

Definition 3.2 (User Equilibrium)**.** A User Equilibrium (UE) is a solution to a route choice problem such that no agent can unilaterally improve (decrease) their travel time.

The equivalence should be clear: in both the Nash and the User equilibrium, an agent can not improve their utility by changing their strategy. Note that in modern models where the user equilibrium is applied, most of the time some general cost is used to describe the utility of a route, which is only partially based on travel time.

In 1977, Daganzo et al. [12] introduced the concept of the Stochastic User Equilibrium (SUE). This concept of equilibrium relaxes the assumption that all agents always choose the best action (i.e. the shortest route). Two main reasons why this is behaviorally applicable is because not all agents have perfect information of all travel times, and moreover not all agents perceive the utility of a certain route in the same way, leading to what seems random behavior. The concept of agents perceiving alternatives in different ways is captured by the *perceived utility*.

Definition 3.3 (Perceived utility)**.** Given a game $\Gamma = (\mathcal{A}, \{\Delta_i\}_{i=1}^n, \{\boldsymbol{V}_i\}_{i=1}^n)$ and some error terms ϵ_{ij} with possibly different distributions Q_{ij} for every combination of agent i and alternative j. We define the game $\tilde{\Gamma} = (\mathcal{A}, \{\Delta_i\}_{i=1}^n, \{\boldsymbol{U}_i\}_{i=1}^n)$ as the game with utilities:

$$U_{ij} = V_{ij} + \epsilon_{ij} \quad \forall i \in \mathcal{A} \quad \forall j \in \mathcal{J} \tag{3.2}$$

U_{ij} is referred to as the perceived utility of agent i for alternative j. Note that this definition of perceived utility is exactly the same as the utility definition in random utility models (utility consisting of a random and a representative part).

With this notion of perceived utility, the Stochastic User Equilibrium can be formally defined:

[1] The type I extreme value distribution, also referred to as Gumbel distribution is the probability distribution with cumulative distribution function $F(x; \mu, \lambda) = e^{-e^{-(x-\mu)/\lambda}}$.

Definition 3.4 (Stochastic User Equilibrium). A Stochastic User Equilibrium is a solution to a route choice problem such that no agent can unilaterally improve their perceived travel time.

It has been shown, for example by Dixit and Denant-Boemont [13] that the SUE is behaviorally more consistent in traffic equilibrium models, meaning that it can predict average traffic behavior and variability in traffic behavior more accurately than the UE. In 1995, McKelvey and Palfrey [31] introduced the Quantal Response Equilibrium (QRE), which is also based on the concept of perceived utility. In the QRE model, similar to the SUE model, agents choose actions based on perceived utility. The QRE is formally defined as:

Definition 3.5 (Quantal Response Equilibrium). Let $\Gamma = (\mathcal{A}, \{\Delta_i\}_{i=1}^n, \{U_i\}_{i=1}^n)$ with utilities $\boldsymbol{U}$ as in (3.2) where the error terms ϵ_{ij} follow some distribution Q_{ij}. The Quantal Response Equilibrium (QRE) is a set of mixed strategies $\{\boldsymbol{\pi}_i\}_{i=1}^n$ such that no agent can unilaterally improve their perceived utility. Hence, a Quantal Response Equilibrium is a set of strategies $\boldsymbol{\pi}^*$ such that

$$\boldsymbol{\pi}_{ij}^* = \mathbb{P}\big(U_{ij}(\boldsymbol{\pi}^*) > U_{ij'}(\boldsymbol{\pi}^*)\ \forall j' \neq j \in \mathcal{J}\big)$$

i.e. in equilibrium, the (mixed) strategy of an agent is exactly equal to the probability distribution of an alternative having the maximal perceived utility.

Note that the QRE is a generalization of the SUE. Where the SUE was specifically designed to analyze route choice problems, the QRE can be used in many different applications. The QRE essentially combines the strategic context of Nash equilibria with the theory of random utility modelling, which makes the QRE a perfect fit for Octavius.

3.2 Mathematical formulation

As discussed in Section 1.1, Octavius is a microscopic travel demand model, consisting of discrete choice models, derived as random utility models. As such, Octavius can be represented as a system of discrete choice models with agents, representing people, moving through this system and making choices according to these choice models, using SNET for allocating these choices.

3.2.1 Notation

Define a system of discrete choice models as a tuple $(\mathcal{A}, \mathcal{M})$ where $\mathcal{A}$ is the set of agents ($|\mathcal{A}| = n$) and $\mathcal{M}$ the set of discrete choice models ($|\mathcal{M}| = M$). For any model $m \in \mathcal{M}$ the (representative) utilities and the set of alternatives are denoted with the superscript (m): $\boldsymbol{V}^{(m)}$ and $\mathcal{J}^{(m)}$. The corresponding game is defined as $\Gamma = (\mathcal{A}, \{\Delta_i\}_{i=1}^n, \{\{\boldsymbol{V}_i^{(m)}\}_{i=1}^n\}_{m\in\mathcal{M}})$ where the strategies in Δ_i define a complete set of probability distributions for all discrete choice models in the system. Note that the representative utilities of a certain agent may be dependent on the strategies of other agents.

3.2.2 Graphical representation

To visualize these systems we define the following graphical representation for a system of discrete choice models:

- The nodes of the network are the models $m \in \mathcal{M}$.
- Directed solid edges: these indicate that an agent making a choice in a certain model implies the next choice the agent makes is in the model at the other end of the edge. In the case where the next model depends on which alternative is chosen, the subset of alternatives for which the arrow holds is written on the arrow.
- Directed dashed edges: a directed dashed edge (m_i, m_k) is included if the utilities in model m_k are dependent on the outcome of model m_i.

An example is provided in Figure 3.1

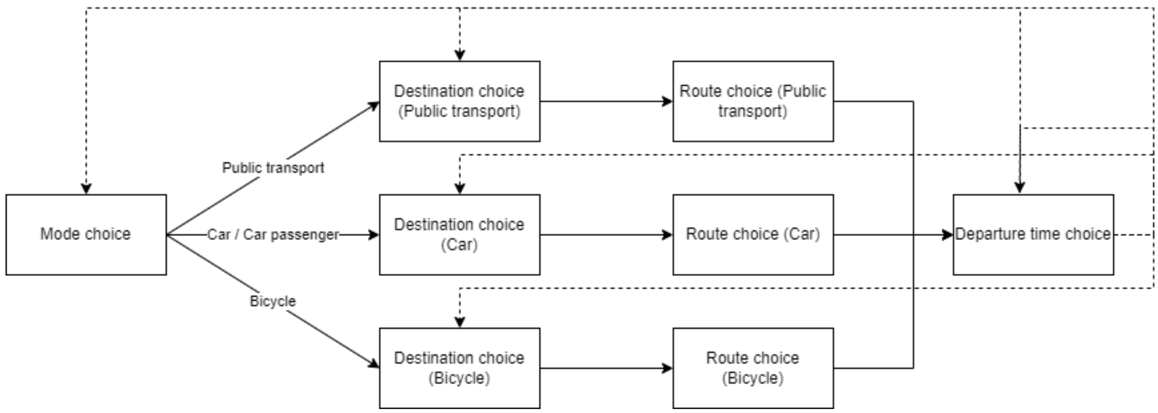


Figure 3.1: Example of a graphical representation of a system of discrete choice models.

Figure 3.1 describes the choice of mode, destination, route and departure time. Both the destination choice and route choice are dependent on which mode was chosen. After the departure time choice all decisions are made and congestion effects can be determined. These effects are given as input to the mode, destination, route and departure time choice using a feedback loop.

The only part of Octavius that does not necessarily fit this structure is the population synthesizer, since this is not a random utility based model. However, this is also not the part of the model for which we are interested in imposing equilibrium conditions. For the relevant parts for imposing equilibrium conditions, the mathematical framework as presented above applies to Octavius. Hence, the remainder of the research can be split up into three parts:

1. Analysis of existence and uniqueness conditions of quantal response equilibria.
2. Relating these existence and uniqueness conditions to systems of discrete choice models.
3. Determining, implementing and testing solution algorithms.

Chapter 4

Existence and uniqueness of QRE

This section concerns existence and uniqueness conditions of the quantal response equilibrium. The analysis largely follows from and builds upon the work by Melo [34]. The notation differs slightly due to the explicit connection that is made with random utility models and the use of notation in accordance with Train [46], as introduced in Chapter 3. Firstly, the general existence and uniqueness conditions of a QRE are presented. Then, the special case of QRE in logit models is addressed. Subsequently, relaxations of the uniqueness condition are presented. Some definitions and theorems regarding convex functions, variational inequalities and matrices are included in Appendix A.

4.1 Existence

From the definition of the quantal response equilibrium (Definition 3.5), one can see that it is equivalent to a fixed-point of the strategies and the utilities given these strategies. This notion of a fixed-point problem provides a large set of tools to prove existence and uniqueness of the equilibrium. The following sufficient conditions for a QRE to exist were presented by McKelvey and Palfrey [31]. Let $\boldsymbol{P}_i : \mathbb{R}^{\mathcal{J}} \to \Delta_i$ with $\mathcal{J}$ the set of alternatives and Δ_i the set of mixed strategies be defined as

$$P_{ij} = P\Big(U_{ij} \geq U_{ij'} \quad \forall j' \in \mathcal{J}\Big)$$

Hence $\boldsymbol{P}_i$ is a function that maps the representative utilities $V_{ij}(\boldsymbol{\pi})$ to a probability distribution, given the error distributions Q_{ij} (recall that $\boldsymbol{V}_i$ represent the set of representative utilities of agent i for alternatives $\mathcal{J}$).

Theorem 4.1. *There exists a QRE if the following conditions hold:*

(a) $\boldsymbol{P}_i$ *is continuous on* $\mathbb{R}^J$*.*

(b) P_{ij} *is monotonically increasing in* V_{ij}*.*

(c) If ϵ_{ij} *are i.i.d., it holds that*

$$V_{ij} > V_{ik} \implies P_{ij}(\boldsymbol{V}_i) > P_{ik}(\boldsymbol{V}_i) \quad \forall i \in \mathcal{A} \quad \forall j, k \in \mathcal{J}$$

(d) The error terms ϵ_{ij} *have a density* Q_{ij}*, for which the marginal distribution exists and* $\mathbb{E}[\epsilon_{ij}] = 0$ *for all* i, j*.*

Proof. Following [31]: A QRE is a fixed point of $(\boldsymbol{P} \circ \boldsymbol{V})(\boldsymbol{\pi})$, which is continuous on $\boldsymbol{\pi} \in \Delta$ since $\boldsymbol{\epsilon}$ has a density. Hence, by Brouwer's fixed point theorem [10], $\boldsymbol{P} \circ \boldsymbol{V}$ has a fixed point. □

The conditions for existence of a QRE are relatively straightforward: the error terms must have some density such that the probability distribution $\boldsymbol{P}_i$ that results from the error terms is continuous. And moreover, the probability of an alternative being chosen must be an increasing function of the representative utility. Finally, if the error terms are i.i.d. it must be the case that a larger representative utility implies a larger probability of being the largest perceived utility.

4.2 Uniqueness

In 1980, Fisk [17] presented a proof of uniqueness of the Stochastic User Equilibrium, using equivalence to and uniqueness of a convex optimization problem. Although his work was on the Stochastic User Equilibrium in route assignment problems, this can be extended to quantal response equilibria in general discrete choice models. This is due to the fact that although the Stochastic User Equilibrium is a specific application of the QRE, mathematically, it is not less general. In 2021, Melo [34] proved uniqueness of quantal response equilibria under well-defined conditions that are economically interpretable. Moreover, the conditions presented are more general than the conditions found by using convex optimization and the results, by for example Zhang [50], using contraction mapping theorems. The uniqueness proof by Melo requires two general assumptions. Using these assumptions, an equivalence between a QRE in a game Γ and a NE in a perturbed game Γ_R can be established by exploiting theory on convex conjugates. Then, using theory of Variational Inequalities the uniqueness condition of the NE and hence the QRE can be obtained. Throughout this section the assumptions and the most important aspects of the proof by Melo are described.

Assumptions

Throughout this section, we consider a game $\Gamma = (A, \{\Delta_i\}_{i=1}^n, \{\boldsymbol{U}_i\}_{i=1}^n)$ with a set of alternatives $\mathcal{J} = (1, \dots, J)$.

Assumption 4.1. *The error terms ϵ_{ij} for agent i and alternative j are drawn from an absolutely continuous distribution with full support and zero mean and these distributions are independent among the agents.*

In general, requiring error terms to have an absolutely continuous distribution with full support is not restrictive. For example, two popular random utility models, logit and probit, satisfy this requirement. Requiring the distributions to have zero mean might appear to be restrictive, however as illustrated by Train [46] (Section 2.5.1), only differences in utility matter, so a random utility model can always be adjusted to having error terms with mean zero[1]. Requiring the error terms to be independent among agents can definitely be restrictive. Consider for example people in the same household being modelled with correlated error terms because it is expected that they will perceive utilities in a similar manner.

The second assumption uses the social surplus function:

Definition 4.1. The *social surplus function*, as introduced by McFadden [30], is a function ϕ of the representative utilities V, given some error distribution Q_{ij} for error terms ϵ_{ij}.

$$\phi_i(\boldsymbol{V}_i) = \mathbb{E}\Big[\max_{j\in\mathcal{J}}\{V_{ij} + \epsilon_{ij}\}\Big] \quad \text{for each } i \in A$$

Given some QRE $\boldsymbol{\pi}^*$, $\phi_i(\boldsymbol{V}_i(\boldsymbol{\pi}^*_{-i}))$ defines the agent i's expected utility in equilibrium: each alternative is chosen with the probability that it is maximal, with the resulting utility being the expected utility of the alternative, given that it is maximal. This exactly yields $\phi_i(\boldsymbol{V}_i)$ as the expected utility.

Assuming that the error distribution satisfies Assumption 4.1, the social surplus function is a convex function of $\boldsymbol{V}$ that is continuous and differentiable everywhere [42].

Theorem 4.2. *Let ϕ_i be the social surplus function, defined in Definition 4.1. Then the QRE $\boldsymbol{\pi}_i^*$ for each agent $i \in \mathcal{A}$ can be expressed as follows:*

$$\boldsymbol{\pi}_i^*(\boldsymbol{V}_i) = \nabla\phi_i(\boldsymbol{V}_i(\boldsymbol{\pi}^*_{-i})) \quad \forall i \in A$$

Proof. This follows directly from the Williams-Daly-Zachery Theorem (Rust [42], Theorem 3.1). □

The intuition behind Theorem 4.2 is that

$$\nabla_{V_{ij}}(\max_{j\in\mathcal{J}} V_{ij} + \epsilon_{ij}) = \begin{cases} 1 & \text{if } j = \arg\max_{j\in\mathcal{J}} V_{ij} + \epsilon_{ij} \\ 0 & \text{else} \end{cases}$$

Hence $\nabla_{V_{ij}}\phi_i(\boldsymbol{V}_i)$ is exactly the probability that $V_{ij} + \epsilon_{ij} > V_{ij'} + \epsilon_{ij'}$ for all $j' \neq j$. Theorem 4.2 hence describes the same fixed-point as the fixed-point in Definition 3.5 that defines the QRE.

[1] If a researcher would model an error term ϵ_{ij} with mean nonzero, the researcher would implicitly increase or decrease the representative utility of agent i for alternative j.

Assumption 4.2. *The Hessian of the social surplus function satisfies the following condition:*

$$||\nabla^2 \phi_i(\boldsymbol{V}_i)||_{\infty,1} \leq L \quad \forall i \in A \qquad \text{for some } L > 0$$

where the derivative is taken with respect to $\boldsymbol{V}_i$, and $||A||_{\infty,1} = \max_{||v||_1 \leq 1} ||Av||_1$ with $||\cdot||_1$ the standard l_1 norm.

Assumption 4.2 implies that agents' choice probabilities change in a smooth way. $||\nabla^2 \phi_i(\boldsymbol{V}_i)||_{\infty,1}$ essentially represents the variability in the expected payoff: if $||\nabla^2 \phi_i(\boldsymbol{V}_i)||_{\infty,1}$ is small, the variability caused by the error terms is relatively high and hence the derivative of the social surplus becomes constant (changes in representative utility barely affect the expected maximum utility). If, on the other hand, the error terms vanish, $||\nabla^2 \phi_i(\boldsymbol{V}_i)||_{\infty,1} \to \infty$ since the probability of choosing the alternative with highest representative utility V_{ij} converges to 1, which causes the mixed second derivatives to go to infinity. Hence, the choice probabilities of an agent do not change smoothly anymore.

Equivalence with a NE

Define $R(\boldsymbol{\pi})$ as the convex conjugate function of ϕ:

$$R_i(\boldsymbol{\pi}_i) := \sup_{\boldsymbol{V}_i} \{\langle \boldsymbol{\pi}_i, \boldsymbol{V}_i \rangle - \phi_i(\boldsymbol{V}_i)\}$$

where $\langle x, y \rangle := \sum_i x_i y_i$. This convex conjugate function R can be used to define a perturbed game Γ_R.

Definition 4.2. Corresponding to the game $\Gamma = (A, \{\Delta_i\}_{i=1}^n, \{\boldsymbol{U}_i\}_{i=1}^n)$ define the perturbed game $\Gamma_R := (A, \{\Delta_i\}_{i=1}^n, \{\hat{\boldsymbol{U}}_i\}_{i=1}^n)$ with payoff functions $\hat{\boldsymbol{U}}_i(\boldsymbol{\pi}_i, \boldsymbol{\pi}_{-i}) := \langle \boldsymbol{\pi}_i, \boldsymbol{V}_i(\boldsymbol{\pi}_{-i}) \rangle - R_i(\boldsymbol{\pi}_i)$.

This payoff function can be interpreted as each agent maximizing their expected payoff minus $R_i(\boldsymbol{\pi}_i)$ which can be interpreted as a cost function quantifying the role of the error terms in agent i's decision. This is illustrated in Chiong et al [11]. Using Fenchel's equality they obtain the following expression for the convex conjugate.

$$R_i(\boldsymbol{\pi}_i) = -\sum_j \pi_{ij} \mathbb{E}[\epsilon_{ij} | V_{ij} + \epsilon_{ij} > V_{ij'} + \epsilon_{ij'} \quad \forall j' \neq j \in \mathcal{J}]$$

which illustrates the interpretation of $R_i(\boldsymbol{\pi}_i)$ quantifying the role of the error terms in the decision of agent i. In [33], Melo proved differentiability of R_i. Using this result combined with Fenchel's inequality and the Baillon-Haddad Theorem, Melo established some important properties of the perturbed game Γ_R.

Proposition 4.1. *Suppose that Assumptions 4.1 and 4.2 hold. Then, for each agent $i \in A$*

(a) $\phi_i(\boldsymbol{V}_i)$ has a gradient-mapping that is Lipschitz continuous with constant $L > 0$.

(b) $R_i(\boldsymbol{\pi}_i)$ is $\frac{1}{L}$-strongly convex and differentiable on $int(\Delta_i)$.

(c) $\hat{\boldsymbol{U}}_i(\boldsymbol{\pi}_i, \boldsymbol{\pi}_{-i})$ is strictly $\frac{1}{L}$-strongly concave with respect to $\boldsymbol{\pi}_i$ for all $\boldsymbol{\pi}_{-i} \in \Delta_{-i}$.

(d) $\nabla \phi_i(\boldsymbol{V}_i(\boldsymbol{\pi}_{-i})) = \arg\max_{\boldsymbol{\pi}_i \in \Delta_i} \hat{\boldsymbol{U}}_i(\boldsymbol{\pi}_i, \boldsymbol{\pi}_{-i})$ for every $\boldsymbol{\pi}_{-i} \in \Delta_{-i}$.

Proof. See Melo [34], Appendix A.4. □

Using these properties the equivalence relation between the QRE in Γ and NE in Γ_R can be formalized.

Proposition 4.2. *Suppose Assumptions 4.1 and 4.2 hold. Then $\boldsymbol{\pi}^*$ is a QRE of Γ if and only if $\boldsymbol{\pi}^*$ is a Nash Equilibrium of Γ_R.*

Proof. Following Melo [34], Appendix A.5: By Theorem 4.2 if $\boldsymbol{\pi}^*$ is a QRE of Γ then

$$\boldsymbol{\pi}_i^* = \nabla \phi_i(\boldsymbol{V}_i(\boldsymbol{\pi}_{-i}^*)) \qquad \forall i \in \mathcal{A}$$

Moreover, if $\tilde{\boldsymbol{\pi}}$ is a NE of Γ_R, by definition (Definition 3.1) for any player i $\tilde{\pi}_i$ maximizes $\hat{\boldsymbol{U}}_i(\boldsymbol{\pi}_i, \tilde{\boldsymbol{\pi}}_{-i})$ and hence

$$\tilde{\boldsymbol{\pi}}_i = \arg\max_{\boldsymbol{\pi}_i \in \Delta_i} \hat{\boldsymbol{U}}_i(\boldsymbol{\pi}_i, \boldsymbol{\pi}_{-i}) \qquad \forall i \in \mathcal{A}$$

Therefore we see that, by Proposition 4.1(d)

$$\boldsymbol{\pi}_i^* = \nabla \phi_i(\boldsymbol{V}_i(\boldsymbol{\pi}_{-i}^*)) = \arg\max_{\boldsymbol{\pi}_i \in \Delta_i} \hat{\boldsymbol{U}}_i(\boldsymbol{\pi}_i, \boldsymbol{\pi}_{-i}^*) = \tilde{\boldsymbol{\pi}}_i \qquad \forall i \in \mathcal{A}$$

and we conclude that if some set of strategies $\boldsymbol{\pi}$ is a QRE of Γ, then it must be a NE of Γ_R and vice versa. □

This proposition allows for using the knowledge on uniqueness of Nash equilibria to prove uniqueness of the QRE.

Uniqueness of NE

Definition 4.3. The payoff gradient of the game Γ_R is the operator $\nabla \hat{\boldsymbol{U}}_i : \Delta_i \to \mathbb{R}^J$ for all $i \in \mathcal{A}$. We use the following notation:

$$F(\boldsymbol{\pi}) = -\nabla \hat{\boldsymbol{U}}(\boldsymbol{\pi})$$

and

$$F_i(\boldsymbol{\pi}) = -\nabla_{\boldsymbol{\pi}_i} \hat{\boldsymbol{U}}_i(\boldsymbol{\pi}) \quad \forall i \in A$$

The next proposition establishes the equivalence of a NE (and thus a QRE) with a Variational Inequality (VI) problem.

Proposition 4.3. *Suppose Assumptions 4.1 and 4.2 hold. Then $\boldsymbol{\pi}^* \in \Delta$ is a NE of Γ_R if and only if $\boldsymbol{\pi}^*$ satisfies:*

$$\langle \boldsymbol{\pi} - \boldsymbol{\pi}^*, F(\boldsymbol{\pi}^*) \rangle \geq 0 \quad \text{for all } \boldsymbol{\pi} \in \Delta \tag{4.1}$$

Proof. See Melo [34], Appendix A.6. □

Intuitively Proposition 4.3 makes sense: Let $\boldsymbol{\pi}^*$ be a Nash equilibrium of the game Γ_R and consider an agent i. If for some strategy $\boldsymbol{\pi}_i \neq \boldsymbol{\pi}_i^*$ we have that for some alternative j $\pi_{ij} > \pi_{ij}^*$, (4.1) requires that $F_{ij}(\boldsymbol{\pi}^*) = -\nabla_{\pi_{ij}} \hat{U}_{ij} \geq 0$. And hence the utility $\hat{\boldsymbol{U}}$ must be nonincreasing for increasing π_{ij}. The reverse holds for $\pi_{ij} < \pi_{ij}^*$ as well and we see that (4.1) resembles the definition of the Nash equilibrium.

Equation (4.1) is known as a Variational Inequality problem.

Definition 4.4 (Variational Inequality)**.** Consider a function $f : Y \subseteq X \to X^*$ where X^* denotes the dual space corresponding to X, then the *variational inequality* problem is the problem of finding the variables $x \in Y$ that solve

$$\langle f(x), y - x \rangle \geq 0 \qquad \forall y \in Y \tag{4.2}$$

Facchinei and Pang [14] proved that such a problem has a unique solution if the function f is either a strongly monotone operator or a uniform block-P function:

Definition 4.5 (Strongly monotone)**.** A function $f : X \to \mathbb{R}^n$ is called *strongly monotone* if there exists a $\mu > 0$ such that:

$$(f(x) - f(x'))^T (x - x') \geq \mu ||x - x'||_2^2 \quad \text{for all } x, x' \in X \tag{4.3}$$

Definition 4.6 (Uniform block P-function)**.** A function $\boldsymbol{f} = \left(f_i : X \to \mathbb{R}^k\right)_{i \in n}$ is called a *uniform block P-function* with respect to $X \subseteq \mathbb{R}^{k \times n}$ if there exists an $\eta > 0$ such that

$$\max_{i \in n} [f_i(x) - f_i(x')]^T [x_i - x_i'] > \eta ||x - x'||_2^2 \quad \text{for all } x, x' \in X \tag{4.4}$$

Lemma 4.1. *(Facchinei and Pang 2003 [14]) The VI problem* (4.2) *admits a unique solution under any of the following conditions*

(a) f is strongly monotone.

(b) f is a uniform block P-function with respect to X.

Proof. See Facchinei and Pang [14]. □

This means that in order to prove uniqueness of the NE in the game Γ_R and hence uniqueness of the QRE in the game Γ it suffices to prove that

$$F(\boldsymbol{\pi}) = -\nabla \hat{\boldsymbol{U}}(\boldsymbol{\pi})$$

is either a uniform block-P function or a strongly monotone operator. Since then the VI problem (4.1) has a unique solution.

Definition 4.7 (Strategic influence)**.** Given a game $\Gamma = (A, \{\Delta_i\}_{i=1}^n, \{\boldsymbol{U}_i\}_{i=1}^n)$ the *strategic influence* δ is defined as

$$\delta := \max\{|V_{ij}(\pi_{ij}, \boldsymbol{\pi}_{-i}) - V_{ij}(\pi_{ij}, \tilde{\boldsymbol{\pi}}_{-i})|\}$$

where the maximum is taken over all agents i, all pure strategies $\boldsymbol{\pi}_i$ and all pairs $\boldsymbol{\pi}_{-i}, \tilde{\boldsymbol{\pi}}_{-i}$ of opponents' pure strategies such that only one agent i' has $\pi_{i'j} \neq \tilde{\pi}_{i'j}$ for some $j \in \mathcal{J}$. The interpretation of this value δ is that it is the maximum impact that unilateral deviations have on agents' payoffs. If this $\delta \geq 0$ is finite, the game is said to be of bounded influence. In a game with no interaction, $\delta = 0$.

Lemma 4.2. *Consider the game* $\Gamma_R = (A, \{\Delta_i\}_{i=1}^n, \{\hat{\boldsymbol{U}}_i\}_{i=1}^n)$. *Suppose that Assumptions 4.1 and 4.2 hold and that, additionally,* $L < \frac{1}{\delta(n-1)}$ *where* δ *is the strategic influence of the corresponding game* $\Gamma = (A, \{\Delta_i\}_{i=1}^n, \{\boldsymbol{U}_i\}_{i=1}^n)$. *Then* $F(\boldsymbol{\pi})$ *is a uniform block P-function.*

Proof. See Melo [34] Proposition 11. □

Theorem 4.3. *Consider a game* $\Gamma = (A, \{\Delta_i\}_{i=1}^n, \{\boldsymbol{U}_i\}_{i=1}^n)$ *and let* δ *be the strategic influence of this game. If Assumptions 4.1 and 4.2 hold, there exists a unique QRE if additionally the following condition holds:*

$$L < \frac{1}{\delta(n-1)} \tag{4.5}$$

Proof. Following Melo [34], Appendix A.7: under Assumptions 4.1 and 4.2 and condition (4.5) by Lemma 4.2 $F(\boldsymbol{\pi})$ is a uniform block P-function and hence by Lemma 4.1 and Proposition 4.3 Γ_R has a unique NE and thus Γ has a unique QRE by Proposition 4.2. □

To illustrate the intuition behind these conditions and assumptions, in the next section the specific case of a QRE in a game where error terms are distributed according to a type I extreme value distribution is discussed, which is known as the Logit-QRE.

4.3 Logit-QRE

In the field of quantal response equilibria, the Logit-QRE is the main model used in the context of experimental work [19]. The uniqueness of the Logit-QRE plays a significant role in the field of econometrics in discrete games. For example in the work by Aradillas-López [1] the uniqueness of the Logit-QRE is discussed. An example where application of the Logit-QRE has been shown to be useful is in contest games[2] [26].

Definition 4.8 (Logit-QRE)**.** Consider a game $\Gamma = (\mathcal{A}, \{\Delta_i\}_{i=1}^n, \{\boldsymbol{U}_i\}_{i=1}^n)$ where the error terms ϵ_{ij} follow a type I extreme value distribution with cumulative distribution function Q_i:

$$Q_i(\epsilon_{ij}) = e^{-e^{-(\epsilon_{ij}-\mu_i)/\lambda_i}}$$

where $\lambda_i > 0$ is the scale parameter. In order to satisfy Assumption 4.1 we require $\mathbb{E}[\epsilon_{ij}] = 0$, and since the expected value of the type I extreme value distribution equals $\mathbb{E}[\epsilon_{ij}] = \mu_i + \lambda_i\gamma$ with γ Euler's constant[3], we define $\mu_i = -\lambda_i\gamma$ to obtain the distribution:

$$Q_i(\epsilon_{ij}) = e^{-e^{-\epsilon_{ij}/\lambda_i - \gamma}}$$

i.e. the type I extreme value distribution with mean zero and scale parameter λ_i. The standard deviation of this distribution is $\frac{\lambda_i \pi}{\sqrt{6}}$. If we assume ϵ_{ij} to be independent among agents and alternatives, the expected maximum utility can be expressed in closed form as:

$$\mathbb{E}\big[\max_{j\in\mathcal{J}}\{V_{ij} + \epsilon_{ij}\}\big] = \lambda_i \log\Big(\sum_{j\in\mathcal{J}} e^{V_{ij}(\pi_{-i})/\lambda_i}\Big)$$

which is derived in Appendix **??**. Differentiating this expression with respect to V_{ij} yields the QRE of this game (Theorem 4.2), which is called the Logit-QRE:

$$\pi_{ij} = \frac{e^{V_{ij}(\boldsymbol{\pi}_{-i})/\lambda_i}}{\sum_{j'\in\mathcal{J}} e^{V_{ij'}(\boldsymbol{\pi}_{-i})/\lambda_i}} \quad \text{for all } i \in \mathcal{A}, j \in \mathcal{J} \tag{4.6}$$

Note that this means that when $\lambda_i \to \infty$ (when the variance of the error terms approaches infinity) the choice probabilities for agent i converge to $\frac{1}{J}$. On the other hand, when $\lambda_i \to 0$ (when the error terms vanish) agent i chooses the alternative with maximum representative utility with probability 1. Hence the Logit-QRE for $\lambda_i \to 0$ is equivalent to the Nash Equilibrium. Unsurprisingly, the strategy (4.6) resembles the familiar logit formula (3.1). Hence, in a game with no interaction, the Logit-QRE is equivalent to the standard logit model from random utility theory. In general, the scale parameters $\lambda_i = \lambda$ are assumed to be equal for all agents $i \in \mathcal{A}$.

[2] A contest game is a game in which players compete with costly efforts to win a scarce prize.

[3] Euler's constant, or the Euler-Mascheroni constant is defined as $\gamma = \lim_{n\to\infty}\big(-\log n + \sum_{k=1}^n \frac{1}{k}\big) \approx 0.577$.

Proposition 4.4. *Consider a game with error terms as in* (4.8) *with scale parameter* λ. *Then the value of* L *in Assumption 4.2 and thus in Theorem 4.3 equals* $\frac{1}{\lambda}$.

Proof. See Melo [34], Section 3.1. □

This implies, by Theorem 4.3, that a sufficient condition for uniqueness of the Logit-QRE is $\lambda > \delta(n-1)$.

Example 4.1. Consider the game with payoffs as defined in Table 4.1. Let $\lambda \to 0$, then the agent chooses the alternative with the highest expected utility with probability 1, $L \to \infty$ and hence δ must equal zero in order to guarantee a unique Logit-QRE. In this case, the QRE converges to the Nash Equilibrium, the solutions that are bold in the figure. The condition $\delta = 0$ is clearly violated in this example, and there are multiple equilibria. Hence this is an example of when a violation of condition (4.5) results in a non-unique equilibrium. If, on the other hand, $\lambda \to \infty$, the variance in the error terms grows infinitely large, and the representative utility becomes negligible, and hence each alternative is perceived equally favourable. So, irregardless of the representative utilities $\boldsymbol{V}$ (and the value of δ), the QRE is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ for both agents, and hence unique.

Agent 1 strategy	Agent 2 strategy		
	Option A	Option B	Option C
Option A	0, 0	**25**, **40**	5, 10
Option B	**40**, **25**	0, 0	5, 15
Option C	10, 5	15, 5	**10**, **10**

Table 4.1: Example of a game, with in bold the Nash equilibria.

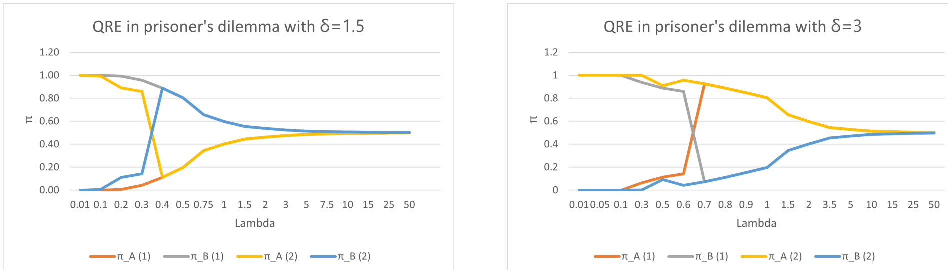
In this example, $\delta = 40$ and hence $\lambda > 40$ is a sufficient condition for uniqueness of the Logit-QRE, following Theorem 4.3.

We conclude the analysis of the Logit-QRE with an example of a prisoner's dilemma and analyze the (possible multiple) equilibria for different values of λ.

Example 4.2. Consider the prisoner's dilemma as in Table 4.2 with $\delta > 1$. Based on the introduced theory, we would expect that if $\lambda \to 0$ the QRE is not unique, since this game has two NE. Following condition (4.5) we would expect that $\lambda > \delta$ would be necessary to guarantee a unique equilibrium. In Figure 4.1 the different equilibria are shown for different values of λ. We see that for $\delta = 1.5$, $\lambda = 0.4$ is high enough to have a unique equilibrium. For $\delta = 3$, $\lambda = 0.7$ is high enough for both equilibria to be the same. Hence, indeed higher δ requires a higher value of λ, but definitely not as high as condition (4.5) would suggest. Furthermore, we see that as λ grows larger the unique QRE converges to $(\frac{1}{2}, \frac{1}{2})$ in both cases which we would expect, and when $\lambda \to 0$ the QRE's are the pure strategies for option A or B (which are the Nash equilibria of the game for $\delta > 1$).

Agent 1 strategy	Agent 2 strategy	
	Option A	Option B
Option A	δ, δ	0, 1
Option B	1, 0	1, 1

Table 4.2: Example of a prisoner's dilemma



(a) (b)

Figure 4.1: The QRE for different values of λ with in (a) $\delta = 1.5$ and in (b) $\delta = 3$.

4.4 Relaxations of the uniqueness conditions

Melo presented a relaxation of condition (4.5) in [34] as well, based on the concept of games of strategic substitutes:

Definition 4.9 (Strategic substitutes)**.** A game of strategic substitutes is a game $\Gamma = (\mathcal{A}, \{\Delta_i\}_{i=1}^n, \{\boldsymbol{U}_i\}_{i=1}^n)$ where the set of alternatives is extended with an *outside* alternative $j = 0$. The utilities follow the following linear structure for all $i \in \mathcal{A}$ and $\boldsymbol{\pi}_{-i} \in \Delta_{-i}$

$$V_{ij}(\boldsymbol{\pi}_{-i}) = \begin{cases} \kappa_j^{(1)} - \kappa_j^{(2)} \sum_{i' \neq i \in \mathcal{A}} \pi_{i'j} & \text{if } j = 1, \dots, J \\ 0 & \text{if } j = 0 \end{cases} \tag{4.7}$$

An example of such a game is a public goods game:

Definition 4.10 (Public goods game)**.** A public goods game is a game in which each player gets a certain amount of tokens that the player may either put into a pot for a public good, or keep. The payoff is determined by the number of tokens in a pot multiplied by a factor (between one and the number of players), which is then evenly divided over the players that put tokens into that pot. The payoff for keeping tokens is zero.

To abuse the linearity in payoffs for a less strict condition, the proof by Melo uses the concept of the Game Jacobian:

Definition 4.11 (Game Jacobian)**.** Consider a game $\Gamma = (\mathcal{A}, \{\Delta_i\}_{i=1}^n, \{\hat{\boldsymbol{U}}_i\}_{i=1}^n)$ with alternatives $\mathcal{J}$. Let $F(\pi) = -\nabla\hat{\boldsymbol{U}}(\boldsymbol{\pi})$. The *Game Jacobian* is defined as the $nJ \times nJ$ valued matrix ∇F with

$$\nabla_{\boldsymbol{\pi}} F(\boldsymbol{\pi}) = -\nabla^2_{\boldsymbol{\pi}\boldsymbol{\pi}} \hat{\boldsymbol{U}}(\boldsymbol{\pi}) \quad \text{for all } \boldsymbol{\pi} \in \Delta$$

In general we express

$$\nabla_{\boldsymbol{\pi}} F(\boldsymbol{\pi}) = D(\boldsymbol{\pi}) + W(\boldsymbol{\pi}) \quad \text{for all } \boldsymbol{\pi} \in \Delta$$

where $D(\boldsymbol{\pi})$ is a block diagonal matrix with blocks $\nabla_{\boldsymbol{\pi}_i} F_i(\boldsymbol{\pi}) := -\nabla^2_{\boldsymbol{\pi}_i\boldsymbol{\pi}_i} \hat{\boldsymbol{U}}_i(\boldsymbol{\pi})$ for $i \in \mathcal{A}$ and $W(\boldsymbol{\pi})$ is a block matrix with blocks $\nabla_{\boldsymbol{\pi}_{i'}} F_i(\boldsymbol{\pi}) := -\nabla^2_{\boldsymbol{\pi}_i\boldsymbol{\pi}_{i'}} \hat{\boldsymbol{U}}_i(\boldsymbol{\pi})$ for $i \neq i' \in \mathcal{A}$ and matrices with all entries zero on the diagonal.

This Game Jacobian thus consists of $n \times n$ blocks of size $J \times J$ describing the precise interaction between agents i and i' (and vice versa) captured by $W(\boldsymbol{\pi})$ and the second derivative of the utilities with respect to their own strategy captured by $D(\boldsymbol{\pi})$.

Lemma 4.3. *In the game $\Gamma = (\mathcal{A}, \{\Delta_i\}_{i=1}^n, \{\hat{\boldsymbol{U}}_i\}_{i=1}^n)$ we have the following with $F(\boldsymbol{\pi}) := -\nabla\hat{\boldsymbol{U}}(\boldsymbol{\pi})$*

(a) If $\nabla_{\boldsymbol{\pi}} F(\boldsymbol{\pi})$ is strongly positive definite, then $F(\boldsymbol{\pi})$ is strongly monotone.

(b) If $\nabla_{\boldsymbol{\pi}} F(\boldsymbol{\pi})$ is a P-matrix, then $F(\boldsymbol{\pi})$ is a uniform block P-function.

Proof. See Melo [34], Proposition 10. □

Recall that the function F being strongly monotone or a uniform block P-function implies that the underlying game has a unique QRE. Melo showed that for games of strategic substitutes, $L < \frac{1}{\delta}$ is a sufficient condition for $F(\boldsymbol{\pi})$ to be strongly monotone. In the remainder of this section a generalization to any linear payoff game is presented.

Relaxation to any linear payoff game

Definition 4.12 (Linear payoff game)**.** A *linear payoff game* is a game $\Gamma = (\mathcal{A}, \{\Delta_i\}_{i=1}^n, \{\boldsymbol{U}_i\}_{i=1}^n)$ where the representative utilities follow the following linear structure for all agents $i \in \mathcal{A}$ and opponents' strategies $\boldsymbol{\pi}_{-i} \in \Delta_{-i}$:

$$V_{ij}(\boldsymbol{\pi}_{-i}) = \kappa_{ij}^{(1)} - \sum_{j' \in \mathcal{J}} \left(\kappa_{ijj'}^{(2)} \sum_{i' \neq i \in \mathcal{A}} \pi_{i'j'}\right) \qquad \text{for all } j \in \mathcal{J} \tag{4.8}$$

To establish the relaxed uniqueness condition, we determine the Game Jacobian ∇F for the corresponding game Γ_R as defined in Definition 4.2. Subsequently, we proof a sufficient condition for ∇F to be strongly positive definite. Finally, we formally proof that this sufficient condition is also a sufficient condition for uniqueness of the QRE in the game Γ. Throughout the lemmas and proofs we use preliminary results on matrices that are presented in detail in Appendix A.

Lemma 4.4. *Consider a linear payoff game Γ. The Game Jacobian of the corresponding perturbed game Γ_R can be written as*

$$\nabla F(\boldsymbol{\pi}) = D(\boldsymbol{\pi}) + K(\boldsymbol{\pi})$$

with $D(\boldsymbol{\pi})$ a block-diagonal matrix with blocks $\nabla_{\boldsymbol{\pi}_i} F_i(\boldsymbol{\pi}) = -\nabla^2_{\boldsymbol{\pi}_i\boldsymbol{\pi}_i} \hat{\boldsymbol{U}}_i(\boldsymbol{\pi})$ for $i \in \mathcal{A}$ and

$$K(\boldsymbol{\pi}) := \begin{bmatrix} \mathbf{0} & K_1 & \dots & K_1 \\ K_2 & \mathbf{0} & \ddots & \vdots \\ \vdots & \ddots & \ddots & K_{n-1} \\ K_n & \dots & K_n & \mathbf{0} \end{bmatrix} \tag{4.9}$$

with $\mathbf{0} = [0]_{J \times J}$ and with

$$K_i := \begin{bmatrix} \kappa_{i11}^{(2)} & \kappa_{i12}^{(2)} & \dots & \kappa_{i1J}^{(2)} \\ \kappa_{i21}^{(2)} & \kappa_{i22}^{(2)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \kappa_{i,J-1,J}^{(2)} \\ \kappa_{iJ1}^{(2)} & \dots & \kappa_{i,J,J-1}^{(2)} & \kappa_{iJJ}^{(2)} \end{bmatrix} \tag{4.10}$$

Proof. Note that $\nabla F(\boldsymbol{\pi})$ is an $nJ \times nJ$ matrix consisting of $n \times n$ $J \times J$ matrices: a $J \times J$ matrix for each agent-agent interaction. On the diagonal are the matrices $\nabla_{\boldsymbol{\pi}_i} F_i(\boldsymbol{\pi}) = -\nabla_{\boldsymbol{\pi}_i\boldsymbol{\pi}_i} \hat{\boldsymbol{U}}_i$. These matrices constitute the block diagonal matrix $D(\boldsymbol{\pi})$.
Now we derive the matrix K: it consists of all blocks $\nabla_{\boldsymbol{\pi}_{i'}} F_i(\boldsymbol{\pi})$ for $i \neq i'$. Consider an arbitrary player i and another arbitrary player i'. The block, $K_{i,i'}$ is a $J \times J$ matrix, consider an entry j, j' of this matrix, we have:

$$\begin{aligned} \left(\nabla_{\boldsymbol{\pi}_{i'}} F_i(\boldsymbol{\pi})\right)_{j,j'} &= -\nabla^2_{\boldsymbol{\pi}_{ij}\boldsymbol{\pi}_{i'j'}} \hat{\boldsymbol{U}}_i \\ &= -\nabla_{\boldsymbol{\pi}_{i'j'}} \left(\kappa_{ij}^{(1)} - \sum_{j' \in \mathcal{J}} \left(\kappa_{ijj'}^{(2)} \sum_{i' \neq i \in \mathcal{A}} \tilde{\pi}_{i'j'}\right) - \nabla R_i(\boldsymbol{\pi}_i)\right) \end{aligned}$$

$\nabla R_i(\boldsymbol{\pi}_i)$ is a function of $\boldsymbol{\pi}_i$ only so it vanishes. Moreover, $\kappa_{ij}^{(1)}$ is a constant so it vanishes, too. Hence $K_{ii'}$ is a diagonal matrix with entries:

$$K_{ii'} := \begin{bmatrix} \kappa_{i11}^{(2)} & \kappa_{i12}^{(2)} & \dots & \kappa_{i1J}^{(2)} \\ \kappa_{i21}^{(2)} & \kappa_{i22}^{(2)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \kappa_{i,J-1,J}^{(2)} \\ \kappa_{iJ1}^{(2)} & \dots & \kappa_{i,J,J-1}^{(2)} & \kappa_{iJJ}^{(2)} \end{bmatrix}$$

Note that $K_{ii'}$ does not depend on i' so we may write $K_i := K_{ii'}$ for all agents $i' \neq i$. Now, if we let $\mathbf{0} = [0]_{J \times J}$ we have that

$$K(\boldsymbol{\pi}) = K := \begin{bmatrix} \mathbf{0} & K_1 & \dots & K_1 \\ K_2 & \mathbf{0} & \ddots & \vdots \\ \vdots & \ddots & \ddots & K_{n-1} \\ K_n & \dots & K_n & \mathbf{0} \end{bmatrix}$$

and it is clear to see that $D(\boldsymbol{\pi}) + K(\boldsymbol{\pi}) = \nabla F(\boldsymbol{\pi})$. ∎

To prove that $\nabla F(\boldsymbol{\pi})$ is strongly positive definite, we need to prove that $\frac{\nabla F(\boldsymbol{\pi})+\nabla F(\boldsymbol{\pi})^T}{2} \succ \alpha I$ for some $\alpha > 0$ by definition of strongly positive definite (Definition A.3(c)). Since $D(\boldsymbol{\pi})$ is block-diagonal and $K(\boldsymbol{\pi})$ is a block matrix (with same sized blocks as $D(\boldsymbol{\pi})$) with zero matrices on the diagonal we have

$$(D(\boldsymbol{\pi}) + K(\boldsymbol{\pi}))^T = D(\boldsymbol{\pi})^T + K(\boldsymbol{\pi})^T$$

and hence we may write

$$\frac{\nabla F(\boldsymbol{\pi}) + \nabla F(\boldsymbol{\pi})^T}{2} = \frac{D(\boldsymbol{\pi}) + D(\boldsymbol{\pi})^T}{2} + \frac{K(\boldsymbol{\pi}) + K(\boldsymbol{\pi})^T}{2}$$

Lemma 4.5. *$D(\boldsymbol{\pi})$ as defined in Lemma 4.4 is $\frac{1}{L}$-strongly positive definite.*

Proof. See Melo [34], Appendix A.3. ☐

Lemma 4.6. *Let λ^* be the minimal eigenvalue of $\frac{K+K^T}{2}$ with K defined as in Lemma 4.4. Then if $\lambda^* + \frac{1}{L} > 0$, the Game Jacobian $\nabla F(\boldsymbol{\pi})$ is strongly positive definite.*

Proof. Let

$$\lambda^* := \lambda_{min}\Big(\frac{K + K^T}{2}\Big)$$

Then we have

$$\frac{K + K^T}{2} \succcurlyeq \lambda^* I_{nJ}$$

by Lemma A.1. Then, using Lemma 4.5 we have

$$\frac{\nabla_{\boldsymbol{\pi}} F(\boldsymbol{\pi}) + \nabla_{\boldsymbol{\pi}} F(\boldsymbol{\pi})^T}{2} \succ \frac{1}{L} I_{nJ} + \lambda^* I_{nJ} = (\lambda^* + \frac{1}{L}) I_{nJ}$$

and hence if $\frac{1}{L} + \lambda^* > 0$ $\nabla F(\boldsymbol{\pi})$ is strongly positive definite. ☐

Theorem 4.4. *If, on top of Assumptions 4.1 and 4.2 it also holds that Γ is a linear payoff game, condition* (4.5) *in Theorem 4.3 can be relaxed to*

$$\lambda^* > -\frac{1}{L}$$

with λ^* as defined in Lemma 4.6.

Proof. If $\lambda^* > -\frac{1}{L}$ then by Lemma 4.6 the Game Jacobian $\nabla F(\boldsymbol{\pi})$ corresponding to the perturbed game Γ_R is strongly positive definite. Hence, by Lemma 4.3, $F(\boldsymbol{\pi})$ is strongly monotone. Thus, by Lemma 4.1 the VI problem (4.1) has a unique solution. Finally, by Proposition 4.2, Γ has a unique QRE. ☐

An interesting observation is that for games of strategic substitutes $\lambda^* = -\delta$ and the sufficient condition $L < \frac{1}{\delta}$ as found by Melo, follows from Theorem 4.4, too.

Although this result generalizes the work by Melo by allowing for both heterogeneous parameters and interaction among different alternatives, the size of the matrix K may be very large for realistic situations, since it depends on the number of agents. On top of this, in for example a destination choice model, the number of alternatives may be 1000 or even more. It is possible to save time by using an algorithm like the Lanczos algorithm, since only the smallest eigenvalue is relevant. However, for computational purposes, it is beneficial to find some good bounds on these eigenvalues. Two of those bounds are presented in the remainder of this chapter.

Theorem 4.5 (Gerschgorin circle theorem [18])**.** *Let A be a complex $n \times n$ matrix with entries a_{ij}. For $i \in \{1, \dots, n\}$ let r_i be the sum of the absolute values of the non-diagonal entries in the i-th row: $r_i = \sum_{j \neq i} |a_{ij}|$. Define the disk $D(a_{ii}, r_i) \subseteq \mathbb{C}$. Then every eigenvalue of A lies within at least one of the disks.*

Proof. See Gershgorin [18]. ☐

Corollary 4.1. *The eigenvalues of A must also lie within the disks corresponding to the columns of A.*

Proof. See Gershgorin [18]. ☐

In a negative linear payoff game a row of the K-matrix (indexed by agent i and alternative j, indicating it is row number $(i-1)J+j$) has sum:

$$\text{rowsum}_{ij} = (n-1)\sum_{j'\in\mathcal{J}} |\kappa_{ijj'}|$$

A column of the K-matrix (indexed by agent i and alternative j, indicating it is column $(i-1)J+j$) has sum:

$$\text{colsum}_{ij} = \sum_{i'\neq i\in\mathcal{A}} \sum_{j'\in\mathcal{J}} |\kappa_{ij'j}|$$

Since the row sums equal the column sum for the symmetric matrix $\frac{K+K^T}{2}$ we have that the uniqueness condition is:

$$\max_{i\in\mathcal{A},j\in\mathcal{J}} \left\{ \frac{n-1}{2} \sum_{j'\in\mathcal{J}} |\kappa_{ijj'}| + \frac{1}{2} \sum_{i'\neq i\in\mathcal{A}} \sum_{j'\in\mathcal{J}} |\kappa_{ij'j}| \right\} < \frac{1}{L} \tag{4.11}$$

In Corollary 4.2 it is shown that for games where the utility of some alternative only depends on how many other agents choose that alternative (and thus not on how many times other alternatives are chosen), the relaxation in Theorem 4.4 combined with the Gershgorin circle theorem will always give a less strict condition than (4.5), the condition proven by Melo. For many games, this independence of other alternatives is realistic, and thus this is a powerful result.

Corollary 4.2. *Consider a linear payoff game with nonnegative parameters $\kappa^{(2)}$ where additionally $\kappa^{(2)}_{ijj'} = 0$ if $j \neq j'$, then the condition found by combining Theorem 4.4 and the Gershgorin circle theorem is always less strict than the condition in Theorem 4.3.*

Proof. Since $\kappa_{ijj} \leq \delta$ for all agents i and alternatives j we have from condition (4.11) that

$$\begin{aligned}
&\max_{i\in\mathcal{A},j\in\mathcal{J}} \left\{ \frac{n-1}{2} \sum_{j'\in\mathcal{J}} |\kappa_{ijj'}| + \frac{1}{2} \sum_{i'\neq i\in\mathcal{A}} \sum_{j'\in\mathcal{J}} |\kappa_{ij'j}| \right\} \\
&= \max_{i\in\mathcal{A},j\in\mathcal{J}} \left\{ \frac{n-1}{2} |\kappa_{ijj}| + \frac{1}{2} \sum_{i'\neq i\in\mathcal{A}} |\kappa_{ijj}| \right\} \\
&\leq \frac{n-1}{2}\delta + \frac{n-1}{2}\delta = (n-1)\delta
\end{aligned}$$

Resulting in the condition (4.5) if and only if all coefficients κ equal δ and a less strict condition otherwise. □

Another Theorem regarding bounds on eigenvalues was presented and proven by Hoffman in [22]:

Theorem 4.6. *For any real matrix $A \in \mathbb{R}^{n\times n}$ define*

$$P_i := \sum_j a_{ij} - (n \max_{j\neq i} a_{ij})_+$$

$$Q_i := \sum_j a_{ij} - (n \min_{j\neq i} a_{ij})_-$$

Then for every real eigenvalue λ of A it holds that:

$$\lambda \in \bigcup_i [P_i, Q_i]$$

Proof. See [22]. □

Following this theorem $\min_i P_i$ can be used as a lower bound for the eigenvalue (note that i denotes a row number here not an agent). Again, denoting row $(i-1)J+j$ of the K-matrix by ij, and hence the row of $\frac{K+K^T}{2}$ by the row of K combined with the corresponding column of K divided by two, we obtain the following value of P for row ij of $\frac{K+K^T}{2}$:

$$P_{ij} = \frac{\text{rowsum}_{ij} + \text{colsum}_{ij}}{2} - nJ \max_{j'\in\mathcal{J}} \kappa_{ijj'}$$

Hence the corresponding uniqueness condition is:

$$\min_{i\in\mathcal{A},j\in\mathcal{J}}\left\{\frac{n-1}{2}\sum_{j'\in\mathcal{J}}|\kappa_{ijj'}|+\frac{1}{2}\sum_{i'\neq i\in\mathcal{A}}\sum_{j'\in\mathcal{J}}|\kappa_{ij'j}|-nJ\max_{j'\in\mathcal{J}}\kappa_{ijj'}\right\}>-\frac{1}{L} \tag{4.13}$$

Consider a linear payoff game with all coefficients equal. Then the bound (4.13) is:

$$(n-1)J\kappa-nJ\kappa>-\frac{1}{L}\quad\Rightarrow\quad -J\kappa>-\frac{1}{L}\quad\Rightarrow\quad \kappa<\frac{1}{LJ}$$

And, since in most cases $J << n$ this is a less strict condition than (4.5) as well.

In a case study (Section 7.1) we show the use/relevance of both the general relaxation of the uniqueness condition (Theorem 4.4) and the computationally more efficient bounds that may be used for this condition.

Relaxation to games with non-linear utility

In principle, the relaxation of the uniqueness condition is not restricted to games with linear utility. As long as the second derivatives of the functions $\hat{\boldsymbol{U}}_i$ are well-defined, the Game Jacobian is well-defined, and positive definiteness of the Game Jacobian can be verified via eigenvalue calculation.

Chapter 5

Application to systems of discrete choice models

The theory developed in Chapter 4 concerns games where the equilibrium is described by one (mixed) strategy. This theory can immediately be applied in a situation with one discrete choice model in which the utilities of the different alternatives for some agent depend on the choices of the other agents. However, in a scenario where there are multiple sequential (or parallel) discrete choice models with different forms of interaction between them the developed theory can not directly be applied. In this section the gap between the one-stage game and a system of discrete choice model is bridged. First of all the necessary assumptions to be able to analyze a system of discrete choice models as a one-stage game are presented. Then the methodology to setup this game is presented. Finally, the evaluation of the uniqueness conditions from Chapter 4 for systems of discrete choice models is presented.

5.1 Assumptions

Consider a system of discrete choice models $(\mathcal{A}, \mathcal{M})$ with alternative sets $\mathcal{J}^{(m)}$ and models $\mathcal{M} = \{m_1, \ldots, m_M\}$.

Assumption 5.1. *If an agent makes a decision $j \in \mathcal{J}^{(m_i)}$ this necessarily implies the next decision the agent makes is in some model m_k. Note that the next model for some agent i does not have to be the same as the next model for some other agent i'.*

This assumption states that, given that an agent makes a certain decision, the next discrete choice model that the agent makes a decision in, is known with probability 1. This movement through the system does not have to be the same for all agents. One may for example define different mode choice models for agents that own cars and agents that do not (since these groups of agents will have a different set of alternatives).

Assumption 5.2. *There exists an ordering $m_1, \ldots, m_M$ of all models in $\mathcal{M}$ such that for all agents, choices in m_i are made prior to choices in models m_k $\forall k > i$. Note that this ordering must be the same for all agents.*

Although this assumption seems restrictive, that is not necessarily the case: consider a situation where some agents first choose their mode, and subsequently their destination, whereas other agents first choose their destination ans subsequently their mode. One can setup the model as in Figure 5.1

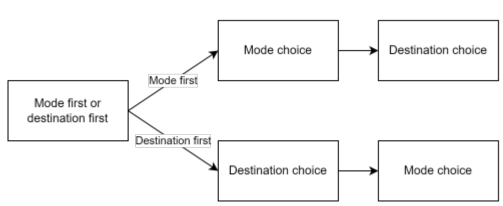


Figure 5.1: Example of modelling a different order of choices among agents.

This system now satisfies Assumption 5.2.

Assumption 5.3. *The utilities of the agents in any discrete choice model are independent of their own choices in previous and later models.*

This may also seem restrictive: for example, the destination choice may depend heavily on the previously chosen purpose of a trip: a person that goes grocery shopping is less likely to travel far than a person that travels to work. This can be resolved by creating two different destination choice models: one for grocery shopping trips, and one for work trips.

Assumption 5.4. *The error terms ϵ that determine the perceived utilities $\boldsymbol{U}$ in each model, are identically independently distributed among agents, alternatives and models following a type I extreme value distribution with scale parameter λ.*

This is an assumption that is relatively restrictive, but strictly necessary for the developed method for analyzing a system of discrete choice models as a one-stage game.
All in all, a lot of restrictions can be overcome by restructuring the system of discrete choice models, however the system of discrete choice model is restricted to be built from discrete choice models derived from the same (logit) random utility model.

5.2 The corresponding game

Given that all of the assumptions hold, a system of discrete choice models can be transformed into a game, meaning that it can be analyzed using the results from Chapter 4. Firstly, using the ordering from Assumption 5.2 a system of discrete choice models can be transformed into a sequence of discrete choice models. To illustrate this method, consider the example in Figure 5.2

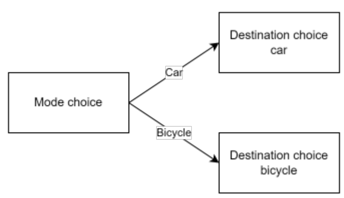


Figure 5.2: Example of a system of discrete choice models.

This system can be transformed into a sequence of discrete choice models as depicted in Figure 5.3

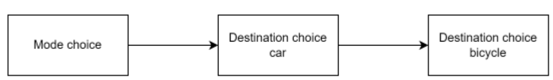


Figure 5.3: The system from Figure 5.2 turned into a sequence of discrete choice models.

From this sequence, to recover the actual probability that an agent chooses to go by car to a certain destination the following formula can be used

$$P_{j_1,j_2} = \sum_{j_3 \in J^{(m_3)}} P_{j_1,j_2,j_3} \quad \forall j_1, j_2 \in J^{(m_1)} \times J^{(m_2)}$$

where m_1, m_2 and m_3 denote the mode choice, destination choice car and destination choice bicycle models respectively.
To create such a sequence, Assumptions 5.1 and 5.2 are necessary. To create a game from this sequence, in addition Assumptions 5.3 and 5.4 are necessary. In Theorem 5.1 the equivalence between a sequence of discrete choice models and a game is formalized.

Theorem 5.1. *Consider a system of discrete choice models $(\mathcal{A}, \mathcal{M})$ rewritten as a sequence satisfying Assumptions 5.1, 5.2, 5.3 and 5.4. This model can be written as one game $\Gamma = (\mathcal{A}, \{\Delta_i\}_{i=1}^n, \{\boldsymbol{U}_i\}_{i=1}^n)$ with $V_{ij} = \sum_{i=1}^M V_{ij_{m_i}}^{(m_i)}$, i.e. the utilities of the sequence of alternatives can be added up and considered as the utility of the complete choice of the combination of alternatives.*

Proof. For simplicity of notation, only the result for the scale parameter $\lambda = 1$ is shown. Consider any system of discrete choice models, with representative utilities $\boldsymbol{V}^{(m)}$ for each model and i.i.d. error terms $\epsilon, \epsilon', \epsilon_m, \epsilon'_m$ for all $m \in \mathcal{M}$ and for each agent i and alternative $j \in \mathcal{J}^{(m)}$. Note that for any player i

$$\mathbb{P}\Big(V_{ij_1}^{(1)} + \cdots + V_{ij_M}^{(M)} + \epsilon > V_{ij'_1}^{(1)} + \cdots + V_{ij'_M}^{(M)} + \epsilon' \quad \forall (j'_1, \ldots, j'_M) \neq (j_1, \ldots, j_M) \in \mathcal{J}^{(m_1)} \times \cdots \times \mathcal{J}^{(m_M)}\Big) \tag{5.1a}$$

by the logit formula

$$= \frac{e^{V_{j_1}^{(1)} + \cdots + V_{j_M}^{(M)}}}{\sum_{(j'_1, \ldots j'_M) \in (J^{(1)} \times \cdots \times J^{(M)})} e^{V_{j'_1}^{(1)} + \cdots + V_{j'_M}^{(M)}}} \tag{5.1b}$$

and that

$$\Pi_{m=1}^{M} \mathbb{P}\big(V_{j_m}^{(m)} + \epsilon_m > V_{j'_m}^{(m)} + \epsilon'_m\big) = \Pi_{m=1}^{M} \frac{e^{V_{j_m}^{(m)}}}{\sum_{j'_m \in J^{(m)}} e^{V_{j'_m}^{(m)}}} \tag{5.1c}$$

Note that the numerators in equation (5.1b) and (5.1c) are equivalent. To see that the denominators are equivalent note that

$$\sum_{j'_1 \in J^{(1)}} e^{V_{j'_1}^{(1)}} * \sum_{j'_2 \in J^{(2)}} e^{V_{j'_2}^{(2)}} = \sum_{(j'_1, j'_2) \in (J^{(1)}, J^{(2)})} e^{V_{j'_1}^{(1)} + V_{j'_2}^{(2)}} \tag{5.1d}$$

Hence, the probability that the sum of utilities of a certain sequence of alternatives plus some error term is larger than the sum of utilities of the other sequences of alternatives plus some error term, is exactly equal to the probability that the utility of each alternative plus some error terms in this sequence of alternatives is larger than the utilities of the other alternatives plus some error terms in that model. Hence, the probability of a certain sequence of alternatives being chosen is exactly equal to the multiplied probabilities of the individual alternatives being chosen, which is due to independence[1] indeed the probability of the combination of alternatives being chosen. □

Concluding, any system of discrete choice models satisfying Assumptions 5.1, 5.2, 5.3 and 5.4 can be transformed into a sequence of discrete choice models which in turn can be transformed into a game.

5.3 Uniqueness conditions

For systems of discrete choice models, verifying the uniqueness conditions requires some more effort than for a single discrete choice model. The value of L in Theorem 4.3 remains the scale parameter λ as long as the system of discrete choice models satisfies Assumptions 5.3 and 5.4 (Theorem 5.1). The number of agents n is a known input variable. Hence, when using Theorem 4.3 for determining the uniqueness, it remains to determine the value of the strategic influence δ. For a certain player i the strategic influence δ_i can be calculated in the following manner: denote by $\mathcal{J}^*$ ($J^* := |\mathcal{J}^*|$) the set of all possible combinations of alternatives $(j_{m_1}, j_{m_2}, \ldots, j_{m_M})$ for a sequence of discrete choice models $\mathcal{M}$. Furthermore, denote by j_m the alternative from model m in the combined alternative $j \in \mathcal{J}^*$. Then

$$\delta_i := \max_{j, j' \in J^*} \sum_{m \in \mathcal{M}} \sum_{m' \in \mathcal{M}} \kappa_{ij_m j'_{m'}}^{(2)} \tag{5.2}$$

where $\kappa_{ij_m j'_{m'}}$ denotes, as in Section 4.4, the linear coefficient of influence between alternatives j_m and $j_{m'}$. δ can be found by taking the maximum over these δ_i's. An upper bound for the value of δ is hence

$$\delta \leq \tilde{\delta} := \sum_{m \in \mathcal{M}} \sum_{m' \in \mathcal{M}} \max_{i,j,j'} |\kappa_{ij_m j'_{m'}}^{(2)}| \tag{5.3}$$

which is the sum of the maximal coefficients between two models for all combinations of models.
If Theorem 4.3 is not sufficient to establish uniqueness of the solution, the relaxation from Theorem 4.4 can be investigated. However, since computing eigenvalues of a large matrix is computationally expensive, an initial step would be checking the bounds found via the Gershgorin circle theorem and the Hoffman bounds (Equations (4.11) and (4.13)). The $K-$matrix used in Theorem 4.4 is set up as a $nJ^* \times nJ^*$

[1] Note that this independence is only true because of Assumption 5.3.

matrix K with the same structure as the $K-$matrix in (4.9). However the entries of K_i are now defined as:

$$(K_i)_{j,j'} = \sum_{m\in\mathcal{M}} \sum_{m'\in\mathcal{M}} \kappa^{(2)}_{ij_m j'_{m'}} \tag{5.4}$$

which is the sum of all coefficients between the different alternatives. Note that dependent on the type of relations, most of these values of κ will be zero. For ease of notation define

$$\kappa^*_{ijj'} := \sum_{m\in\mathcal{M}} \sum_{m'\in\mathcal{M}} \kappa_{ij_m j'_{m'}} \qquad \forall i \in \mathcal{A} \quad \forall j, j' \in \mathcal{J}^* \times \mathcal{J}^* \tag{5.5}$$

This yields the following two sufficient conditions for uniqueness:

$$\max_{i\in\mathcal{A},j\in\mathcal{J}} \left\{ \frac{n-1}{2} \sum_{j'\in\mathcal{J}} |\kappa^*_{ijj'}| + \frac{1}{2} \sum_{i'\neq i\in\mathcal{A}} \sum_{j'\in\mathcal{J}} |\kappa^*_{ij'j}| \right\} < \frac{1}{L} \tag{5.6}$$

or

$$\min_{i\in\mathcal{A},j\in\mathcal{J}} \left\{ \frac{n-1}{2} \sum_{j'\in\mathcal{J}} |\kappa^*_{ijj'}| + \frac{1}{2} \sum_{i'\neq i\in\mathcal{A}} \sum_{j'\in\mathcal{J}} |\kappa^*_{ij'j}| - nJ \max_{j'\in\mathcal{J}} \kappa^*_{ijj'} \right\} > -\frac{1}{L} \tag{5.7}$$

If neither of these conditions is satisfied, an additional method for checking uniqueness would be to calculate or approximate the smallest eigenvalue of the $\frac{K+K^T}{2}$ matrix.

Chapter 6

Solution algorithms

A quantal response equilibrium can be calculated by applying a best-response algorithm, which is a similar method to the well-known solution method for finding a fixed-point as stated in the Banach fixed-point theorem [5]:

Theorem 6.1 (Banach fixed-point theorem)**.** *Let (X, d) be a non-empty metric space with a contraction mapping $T : X \to X$. Then T admits a unique fixed-point x^* in X, furthermore, define the sequence $(x_n)_{n\in\mathbb{N}}$ by $x_n = T(x_{n-1})$ with an arbitrary element $x_0 \in X$. Then $\lim_{n\to\infty} x_n = x^*$.*

In Section 6.1 the best-response algorithm is introduced. In Section 6.2 a variant of this algorithm using Monte Carlo simulation is introduced. Section 6.3 discusses the validation of this Monte Carlo algorithm, and Section 6.4 discusses the possible computational benefits of this algorithm, as well as an analysis of the dependencies of the runtime on the different input parameters.

6.1 Best-response algorithm

In [34], Melo presents an asynchronous best-response algorithm to find the unique QRE. The correctness proof follows from Theorem 10 in Scutari et al. [44]. In the best-response algorithm, in each iteration, the agents determine their strategy based on the outcome of the previous iteration. In the asynchronous version of this algorithm not all agents update their strategy in each iteration. The essence of the synchronous best-response algorithm is repetitively using the output of the algorithm as the input in the next iteration, similar to the Banach fixed-point theorem.

Algorithm 1: Iterative (synchronous) best-response algorithm to solve QRE

1: Input: utility function $\boldsymbol{V}_i : \Delta \to \mathbb{R}^J$ for all agents i, scale parameter λ and an initial set of strategies $\boldsymbol{\pi}^0$

2: Repeat until $||\boldsymbol{\pi}^n - \boldsymbol{\pi}^{n-1}||_\infty <$ tol:

- $\boldsymbol{V}^n = \boldsymbol{V}(\boldsymbol{\pi}^{n-1})$
- Calculate $\boldsymbol{\pi}^n$

$$\pi_{ij}^n = \frac{\exp\left(V_{ij}^n(\boldsymbol{\pi}^{n-1})/\lambda\right)}{\sum_{j'\in\mathcal{J}} \exp\left(V_{ij'}^n(\boldsymbol{\pi}^{n-1})/\lambda\right)} \quad \forall (i,j) \in \mathcal{A} \times \mathcal{J}$$

3: Return: $\boldsymbol{\pi}_n$

The computational complexity of this algorithm is $\mathcal{O}(nJ)$. The convergence condition used in this algorithm is that the maximal difference in the probability that some agent chooses some alternative changes with less than "tol", which is a predetermined tolerance, over two iterations. From the definition (Definition 3.5) a QRE is a point where, for some set of utilities, the strategy is exactly the same as the probability distribution of any of the alternatives having the highest perceived utility. If $||\boldsymbol{\pi}^n - \boldsymbol{\pi}^{n-1}||_\infty <$tol then the maximal difference between a strategy and the probability distribution over the alternatives given that strategy is tol and hence the definition of QRE is (almost) satisfied. It is also possible to use the same convergence condition with $\boldsymbol{V}$ instead of $\boldsymbol{\pi}$, there are however two reasons to prefer $\boldsymbol{\pi}$:

- The probability distribution (and similar, the strategy) always consists of values between 0 and 1 and sums up to 1 for each agent, whereas the absolute value of some utility V_{ij} does not necessarily have a meaning. The meaning is derived from the difference in utility between two alternatives. Hence, using one tolerance tol could result in significantly different probability distributions, when the absolute values of the utilities increase whilst the relative differences remain the same.
- When using SNET, the choice allocation technique used in Octavius, a population of agents is grouped into segments, groups of agents with the same probability distribution. If such a segment has a size s we know for that the average probability of choosing any alternative is $\frac{1}{sJ}$ and we can use this to determine an appropriate tolerance for the equilibrium probability distribution, to ensure that running more iterations would not change the allocated choices.

A method to determine the initial strategies can for example be running the model as though the interactions were not there, or using an educated guess for the average choices of agents.

6.2 Monte Carlo algorithm

In Chapter 5 the theory behind the equivalence of a system of discrete choice models and a game was described. Using this theory, any system of discrete choice models can be transformed into a game, and Algorithm 1 can be applied. However, this requires enumerating all possible combinations of alternatives, which requires a large amount of memory in a situation with a large system of discrete choice models. Hence, we propose to not enumerate the alternatives, but instead simulate the complete system model by model. This means that some form of discrete simulation is necessary. The most common method for using discrete simulation to approximate deterministic results is Monte Carlo simulation. In such a model, discrete events are sampled using random sampling, and the desired measure (in this case, the equilibrium probability distribution) is averaged over the iterations. An alternative method of sampling discrete events that was developed for Octavius, SNET, could also be used, but after some initial tests, it turned out that repeatedly running SNET with or without averaging, did not result in convergence towards the solution found with the synchronous best-response algorithm, which was proven to be correct in [34]. Therefore, we propose to use Monte Carlo simulation with averaging. This algorithm is presented below:

Algorithm 2: Monte Carlo simulation to solve QRE

1: Input: utility function $\boldsymbol{V}_i^{(m)} : \Delta^{(m)} \to \mathbb{R}^{J^{(m)}}$ for all agents i, scale parameter λ and an initial set of strategies $\boldsymbol{\pi}_0^{(m)}$ for all models $m \in \mathcal{M}$.

2: Repeat until $||\bar{\boldsymbol{\pi}}_n^{(m)} - \bar{\boldsymbol{\pi}}_{n-1}^{(m)}||_\infty <$tol $\forall m \in \mathcal{M}$

- **for** $m \in \mathcal{M}$ **do**
  1. $\boldsymbol{V}^{(m)} = \boldsymbol{V}(\boldsymbol{\pi}_{n-1}^{(m)})$
  2. Calculate
  $$\boldsymbol{\pi}_n^{(m)} = \left(\frac{\exp\left(V_{ij}^{(m)}/\lambda\right)}{\sum_{j' \in \mathcal{J}^{(m)}} \exp\left(V_{ij'}^{(m)}/\lambda\right)} \right)_{(i,j)\in\mathcal{A}\times\mathcal{J}^{(m)}}$$
  3. Update $\bar{\boldsymbol{\pi}}_n^{(m)} = \frac{1}{n}\boldsymbol{\pi}_n^{(m)} + \frac{n-1}{n}\bar{\boldsymbol{\pi}}^{n-1}$
  4. Randomly sample a choice for each agent according to the probability distribution $\boldsymbol{\pi}_n^{(m)}$.

3: Return: $(\bar{\boldsymbol{\pi}}_n^{(m)})_{m\in\mathcal{M}}$

For a single discrete choice model, the running time is still $\mathcal{O}(nJ)$ since each calculation requires $\mathcal{O}(1)$ time. However, in a situation with multiple discrete choice models, when enumerating the alternatives, the computational complexity of the best-response algorithm is $\mathcal{O}(n\Pi_{m\in\mathcal{M}}J^{(m)})$, whereas running the system model per model has computational complexity $\mathcal{O}(n\sum_{m\in\mathcal{M}} J^{(m)})$. Moreover, the amount of information that needs to be stored is only one choice per agent per model, instead of a complete probability distribution over the alternatives per agent per model, which saves a lot of memory usage.

6.3 Validation

To validate the Monte Carlo algorithm, we plot the absolute average difference (in probabilities) between the Monte Carlo and the best-response algorithm for different tolerances between 0 and 0.001. We do this for models 1 and 2 as shown in Figure 6.1.

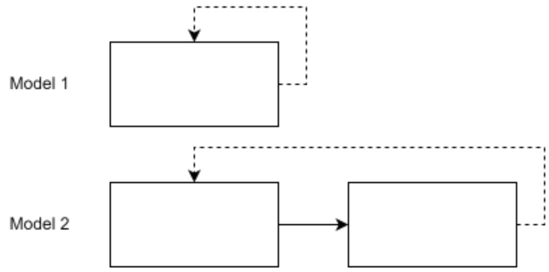


Figure 6.1: Models 1 and 2 used for validation.

The results are shown in Figure 6.2.

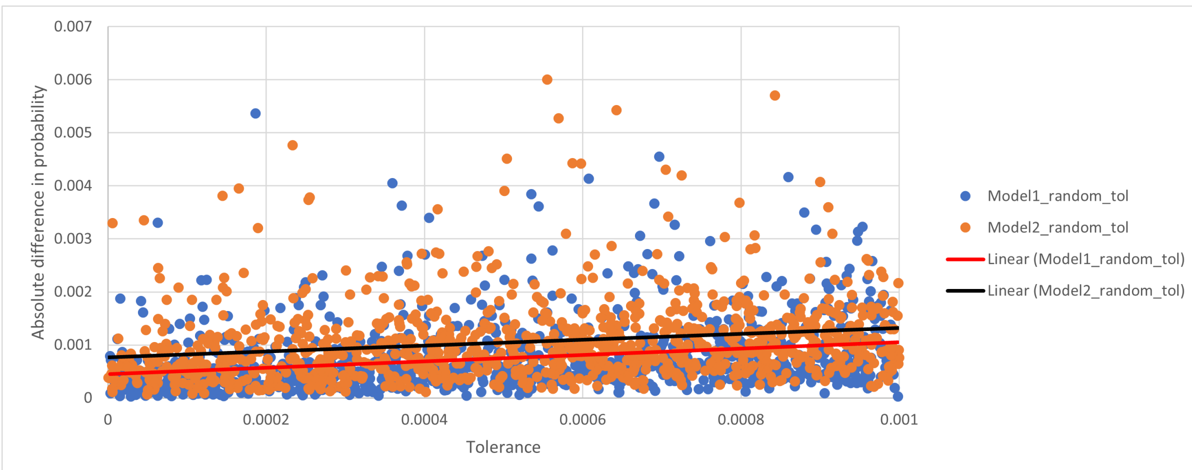


Figure 6.2: Average difference between the Monte Carlo and best-response algorithm plotted against the tolerance.

There are three conclusions we draw from this analysis: First of all, the average difference between the continuous and Monte Carlo best-response algorithm is relatively small (in most cases not more than two times the tolerance). Moreover, we see, using linear trend lines, that a larger tolerance leads to a larger difference, but with a small tolerance, still, the average difference may be relatively large. Finally, we see that for the larger model (model 2) the average difference is slightly higher. In Figure 6.3 we present a similar analysis, but now for a fixed tolerance of 0.001.

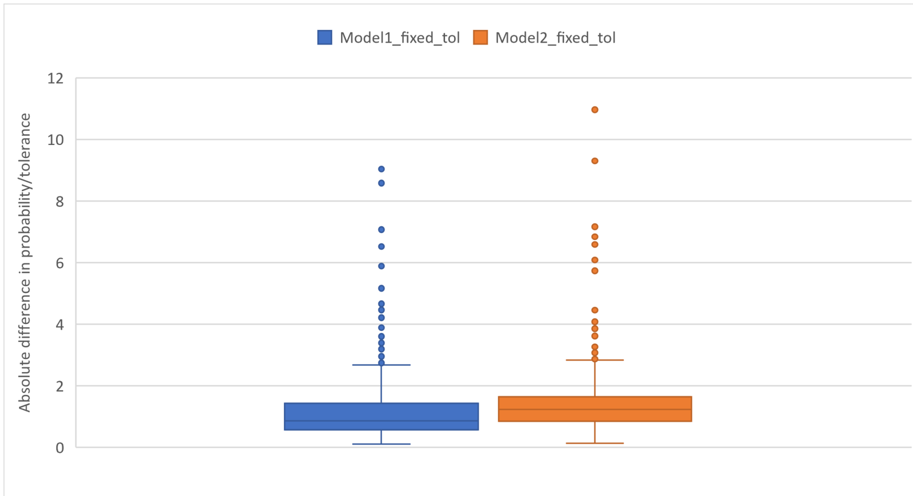


Figure 6.3: Boxplot for the ratio between the difference and the tolerance for fixed tolerance 0.001.

In this boxplot, the box represents the range of 25-75% of the data points. The error bars represent the minimal and maximal values, excluding outliers. Outliers are defined as anything more than 1.5 times the length of the 25-75% box outside of the box.
Using the boxplot we see that indeed the average difference is smaller than 0.002 for a tolerance of 0.001. Moreover, we see that a tolerance of 0.001 almost always guarantees that the probability distributions on average differ not more than 0.01 from the unique solution.

6.4 Computational analysis

The computational analysis consists of two parts. Firstly, we present an analysis of the different input parameters on the runtime. Secondly, we present a comparison of the runtimes of the Monte Carlo and the best-response algorithm.

6.4.1 Effects of parameters on runtime

For this analysis, consider model 2 as introduced in Figure 6.1. We analyze the influence of the following parameters on the runtime of the model:

1. n: The number of agents.
2. J: The number of alternatives.
3. tol: The tolerance.
4. S: The number of different property-combinations in the population (the heterogeneity).
5. $\delta(n-1)$: The strength of the interaction.

The population, alternatives and parameters are uniformly randomly sampled: the population consists of 2 to 10000 agents, with 2 to 5 alternatives (per discrete choice model), 2 to 100 different properties (with the population uniformly randomly divided over these properties). The discrete choice models are modelled as logit models with scale parameter $\lambda = 1$, hence $L = 1$ in the uniqueness condition (4.5). Therefore, we sampled the parameters describing the interaction (κ) uniformly between 0 and $\frac{1}{n-1}$, ensuring that the uniqueness condition from Theorem 4.3 is satisfied. For testing the effect of each parameter on the runtime we fix the other parameters to $n = 1000, J = 5 * 5, \text{tol} = 0.001, S = 5$ and $\delta(n-1) \approx 1$. The first analysis is the influence of the tolerance, which is set randomly between 0 and 0.001.

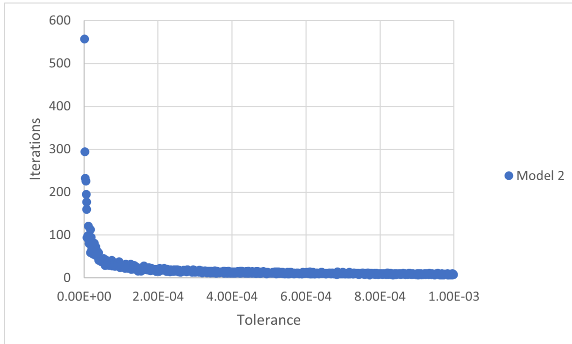


Figure 6.4: Number of iterations for different values of the tolerance. One particular run with tolerance $1.04 * 10^{-7}$ and 3867 iterations is omitted from the figure.

In Figure 6.4 we see that the number of iterations does increase for smaller tolerance.
The second parameter we analyze is the number of agents (n). Based on the computational complexity of the model, we expect a linear dependency of the runtime on the number of agents.

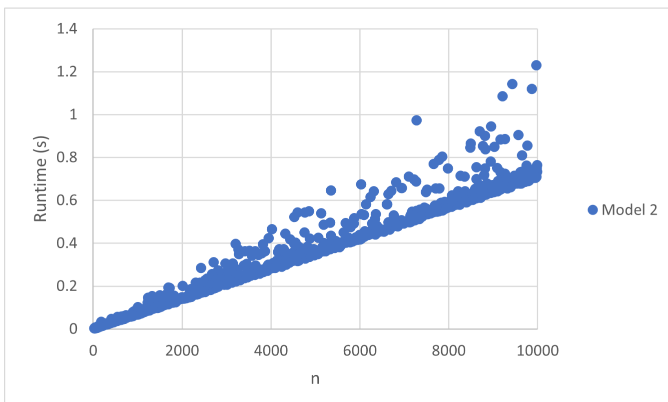


Figure 6.5: Runtime results for different values of n.

Similar to the number of agents, we expect the runtime to be linearly dependent on the number of alternatives as well. In Figure 6.6 we see that this is indeed the case.

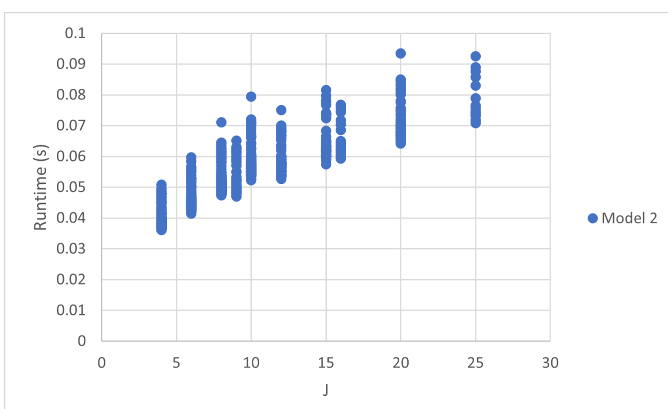


Figure 6.6: Runtime results for different number of alternatives.

Another parameter that influences the runtime is the interaction; if there is no strategic interaction the model converges after two iterations, since in the second iteration nothing has changed and the convergence criterion is satisfied. We expect that with stronger interaction, more iterations are required to meet the convergence criterion. We sampled interactions to attain the value of $\delta(n-1)$ between 0 and 1, where close to 1 indicates that the uniqueness condition from Theorem 4.3 is almost violated, and close to 0 means that there is almost no interaction. Indeed we see that the number of iterations increases for larger values of $\delta(n-1)$.

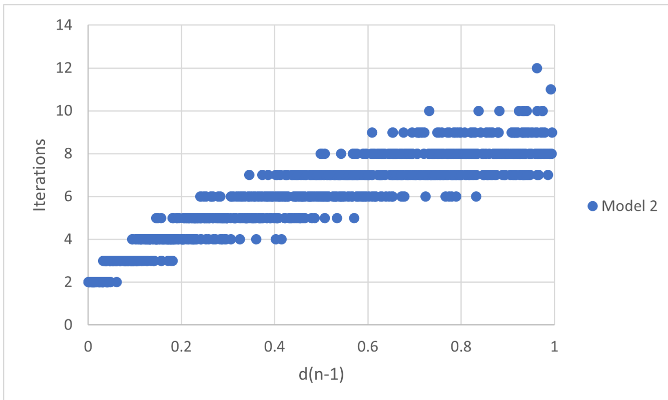


Figure 6.7: Number of iterations for different values of $\delta(n-1)$.

A final dependency we investigate is the dependency of the runtime on the heterogeneity of the population: we do not calculate the utility and probability per agent, but instead per group of agents, or segment, with similar characteristics. Hence, in terms of arithmetic operations, there should be a linear dependency on the number of different sets of properties within a population. In Figure 6.8 we see that there is indeed a linear dependency, however this dependency is less strong than the dependency on the number of agents and number of alternatives. This is likely due to the fact that although within Octavius calculations are performed per segment, for this specific equilibrium model we had to store all information per agent and check convergence per agent.

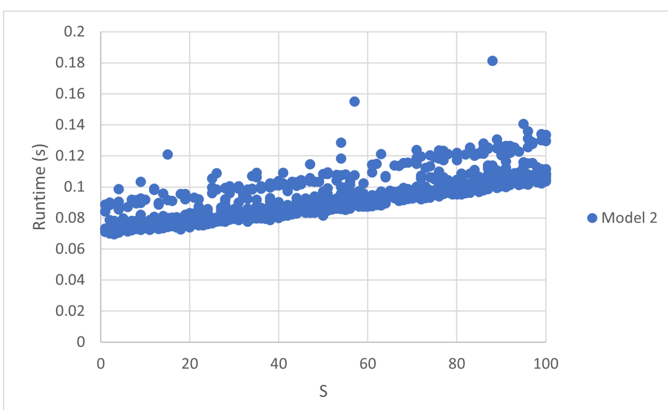


Figure 6.8: Runtime results for different number of possible sets of properties existing within the population.

6.4.2 Comparison Monte Carlo and best-response algorithm

Based on the results of the previous section, we know that there are three main factors influencing the runtime: the number of agents, the number of alternatives and the strength of the interaction ($\delta(n-1)$). Earlier in this section we derived that the difference between the computational complexity of the Monte Carlo best-response algorithm lies mainly in the factors n and J. Hence, in Figure 6.9 the influence of

$nJ := n(\Pi_m J^{(m)})$ on the runtime of these algorithms is shown for models 1 and 2 as introduced in Figure 6.1.

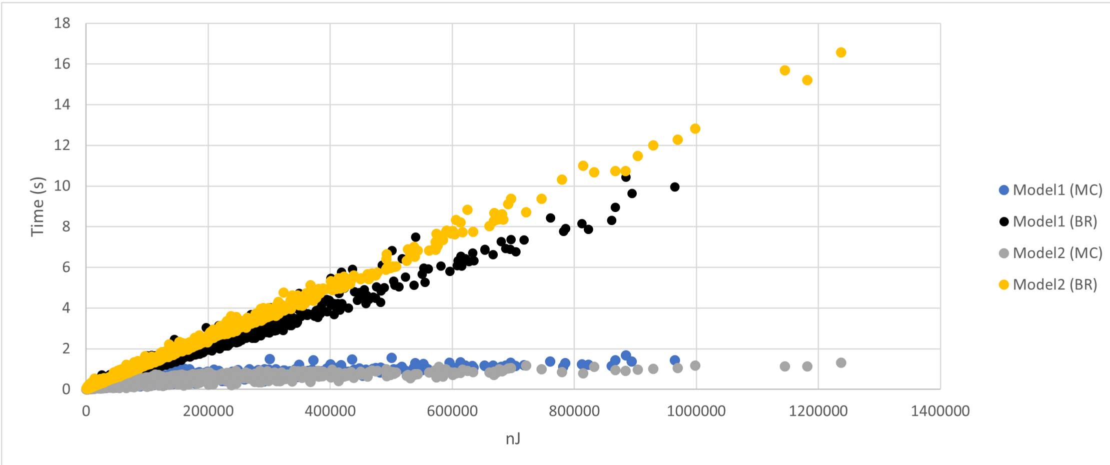


Figure 6.9: Runtime plotted against nJ. MC denotes the result from the Monte Carlo algorithm, and BR the results from the best-response algorithm.

It is clear that the runtime of the best-response algorithm increases more rapidly with increasing nJ. Moreover, where in the continuous best-response algorithm model 2 seems to perform worse than model 1 with the same amount of enumerated alternatives, the Monte Carlo algorithm does not seem to suffer from a longer sequence of models (and scales similarly for the same amount of enumerated alternatives).

Chapter 7

Case study

In this chapter we show the relevance of the results obtained in this research, both from a theoretical and practical point of view. First of all, the relevance of the relaxations of the uniqueness conditions is studied in games with linear payoff. We show a situation in which the uniqueness condition from Theorem 4.3 can be significantly relaxed by using Theorem 4.4. Secondly, we investigate the practical use of the Monte Carlo algorithm by creating a feedback loop between Octavius and a route choice model and discuss the possibilities of analyzing the uniqueness of the resulting solution.

7.1 Case study 1: The relaxed uniqueness condition

To show the relevance of the relaxed uniqueness condition, we present an analysis of uniqueness guarantees in games of strategic substitutes, as introduced in Definition 4.9. However, we introduce heterogeneous parameters, relaxing the assumption that the values of $\kappa^{(1)}$ and $\kappa^{(2)}$ are the same for all agents. In Corollary 4.2 we already saw that the uniqueness condition from Theorem 4.4 is always as strict or less strict as the condition from Theorem 4.3. We know that, with the uniqueness condition found by Melo from this latter Theorem, we would have to ensure that $\delta = \max_{ij} \kappa_{ij}^{(2)} < \frac{1}{n-1}$ if we consider a logit model with scale parameter 1. Hence we investigate for what values of δ the eigenvalue calculation still provides a uniqueness guarantee, which we describe by a δ-factor: the extend to which we can increase the maximal interaction whilst guaranteeing uniqueness. For example, if we guarantee uniqueness for a δ-factor of 10 this means that when $\max_{ij} \kappa_{ij}^{(2)} < \frac{10}{n-1}$ we guarantee uniqueness. The analysis is based on uniformly random sampling of the values of κ between 0 and the set maximum (δ-factor times $\frac{1}{n-1}$). We say that the model guarantees uniqueness for a certain δ-factor, if at least 95/100 times of random sampling κ values with this δ-factor yields a uniqueness guarantee.

We analyze the influence of three parameters on the maximal δ-factor: the number of agents, the number of alternatives and the heterogeneity. The goal of this analysis is to see if increasing/decreasing any of these variables leads to a substantially different maximal δ-factor for which uniqueness can be guaranteed. The first result is obtained with 20 agents, consisting of 5 groups of 4 agents with the same parameters. The number of alternatives ranges from 20 to 60. For this analysis we found that for 20 up to and including 40 alternatives, we could guarantee uniqueness for a δ-factor of up to 9. For 45 up to and including 60 alternatives, the guarantee was a δ-factor of 8.5.

For the second analysis we fix the number of agents to 40, the number of alternatives to 20 and vary the heterogeneity between 2,5,10,20 and 40 different sets of parameters. Here we see larger differences between the runs, which are plotted in Figure 7.1.

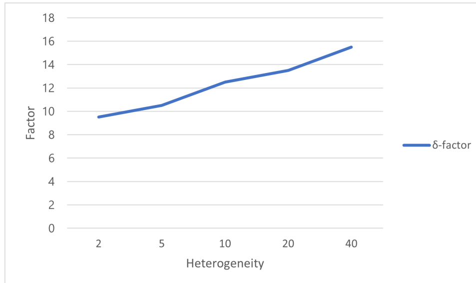


Figure 7.1: Maximal δ-factor for various levels of heterogeneity.

Finally we analyze the effects of the number of agents on the maximal δ-factor. The results of this are shown in Figure 7.2. Similar to the heterogeneity, there seems to be an increasing relation between the maximal δ-factor and the number of agents. However for 35 up to 50 agents the maximal δ-factor is the same.

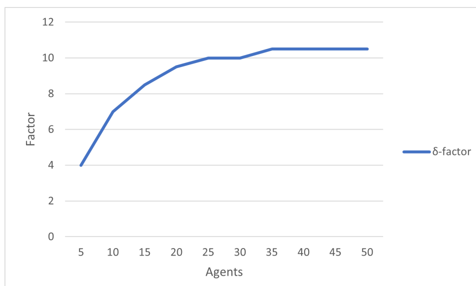


Figure 7.2: Maximal δ-factor for increasing number of agents with 5 different sets of properties and 20 alternatives.

It is clear that using the relaxed uniqueness condition allows for a stronger interaction between the agents in the model. However we found that this uniqueness can only be verified using eigenvalue calculation and not via the Gershgorin or Hoffman bound on the eigenvalue. When performing the same tests for these bounds we found that in some cases we could have a maximal δ-factor of about 1.5 to 2 for these bounds, but in most cases they did not improve on the uniqueness condition as presented by Melo. Hence, relaxing the uniqueness condition can be computationally expensive. However, increasing the number of agents or the heterogeneity only seemed to increase the maximal δ-factor and increasing the number of alternatives did not seem to decrease the maximal δ-factor too much. Hence, setting up a small variant of the desired model and analyzing the uniqueness of the smaller variant to conclude something about the uniqueness of the desired model may be a valid approach.

7.2 Case study 2: Route assignment

To test the practical relevance of the developed methodology and algorithm, we implement the algorithm in the Octavius software in OmniTRANS. A microscopic, logit-based, route choice model does not exist within the OmniTRANS software, hence we use static traffic assignment with volume averaging. The volume averaging traffic assignment is a generalization of the Method of Successive Averages (MSA), which is described in detail in [38], Section 10.5.4. The volume averaging traffic assignment is described briefly in Chapter 3 of [36]. In essence, the method consists of repetitive all-or-nothing assignments (assignments in which every traveler chooses the fastest route), and recalculating the costs of the routes. In recalculating the costs, a bpr function is used. This bpr function describes the relation between the actual travel time (T), the "free flow" travel time (T_0), the volume of cars on a road (V) and the capacity of that road (C). The formula describing this relation is

$$T = T_0(1 + \alpha(V/C)^{\beta}) \tag{7.1}$$

where α and β are estimated parameters. A larger value of α results in significant delays for lower values of V/C. β describes the rate at which the delay grows; a larger value of beta means a steeper bpr function. The role of α and β is visualized in Figure 7.3

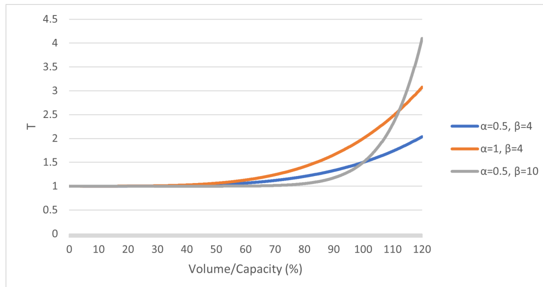


Figure 7.3: Bpr function for different values of α and β for $T_0 = 1$.

This explicit function for the delay can help us in analyzing the uniqueness. The maximal increase in travel time (and thus, the maximal interaction) on a road section caused by one other traveler equals $\frac{\alpha T_0}{C^\beta}$. However, the total increase in travel time for a certain destination depends both on the origin of the trip of some agent and on the chosen route, making it more difficult to analyze this interaction explicitly. Moreover, the value of α depends on the link type: a highway is less likely to be congested at a given volume/capacity ratio than a road through the city. And the values of T_0 and C differ per road section, too. The value of β is from experience, according to the OmniTRANS manual, 4. In Appendix C the values of α per link type that are normally used for route assignment with Octavius are shown. For our results we multiplied the value of α by a certain factor (referred to as the α-factor) to increase/decrease the congestion effects. Before we present the results, we present the model in detail.

With the route assignment model established, we set up the feedback loop. Since the population synthesizer and the choices made by the agents in the tourgenerator and the destination choice for non-car modes are, in Octavius, not affected by increasing car travel times, the results from these models are assumed to be fixed. The other input consists of a bpr function per link type and fixed freight and external demand; for this application, we only model the area in and around the city of Zwolle with Octavius, and all other traffic passing through the city is assumed to be constant. Moreover, freight traffic is in general not modelled using Octavius. In Figure 7.4 the system is visualized. Following Algorithm 2, we run the models for destination choice mode car, mode choice and route choice in sequence. The output of an iteration of this sequence is a new set of travel times, which is the input for the next iteration.

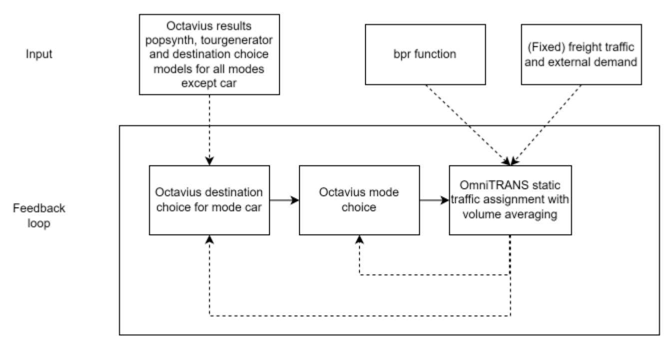


Figure 7.4: Figure depicting the implemented feedback loop.

The first iteration is set up as follows: the input is the same as in Figure 7.4, but there is an additional input of “free flow” travel times, which are essentially the travel times in a situation where the demand from Octavius is zero. Such a “free flow” travel time can be obtained by giving an empty population as input for the traffic assignment model. After this first iteration, the model is run for up to 50 iterations following the feedback loop.

The convergence criteria we analyze is the change in (running) average probability distribution over the course of the iterations. To save computational time, we only considered the probability distribution for the main destination choice (the destination choice for the main purpose in the first tour of each agent). Although this is not the exact convergence criterion as presented in Algorithm 2, it should still work. If the probabilities in one of the destination choice models have converged, this implies directly that the travel times have converged, due to the fact that the travel times are the only variable in the utility functions for the destination choice models in this feedback loop (the rest of the representative utilities remains constant over the iterations). And, if the travel times have converged, then since the travel times are the only variable in any of the choice models within Octavius in the feedback loop, all other choice models should have converged as well. Another change from Algorithm 2 is that we analyze the convergence of the probability distribution for segments, rather than agents. A segment s is a group of agents, which have the same probability distribution in one choice model. It is clearly more efficient to only calculate the probability distribution for each segment instead of each agent. Let $\mathcal{S}$ be the set of all segments and denote by $s \in \mathcal{S}$ a segment. Denote by $\bar{\pi}_{sj}^n$ the probability for segment s for alternative j averaged over the first n iterations. The average difference in probability distribution is then defined as

$$\frac{\sum_{sj} |\bar{\pi}_{sj}^n - \bar{\pi}_{sj}^{n-1}|}{|\mathcal{S}||\mathcal{J}|}$$

and the maximal difference by

$$\max_{s \in \mathcal{S}, j \in \mathcal{J}} \left\{ |\bar{\pi}_{sj}^n - \bar{\pi}_{sj}^{n-1}| \right\}$$

In Figure 7.5 we plot both the average and maximal difference against the iterations.

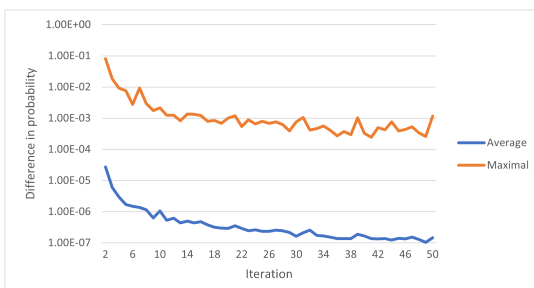


Figure 7.5: Convergence behaviour of the feedback loop model with α-factor 0.25 and $\beta = 2$.

There is clear convergence in probability distributions. The average difference in probability distribution is, after 50 iterations, around $1 * 10^{-7}$ which is sufficient for guaranteeing that a QRE was found: by definition (Definition 3.5), if the output is used as input and yields (almost) the same output again, a fixed point is found.

The relatively large difference in probability distribution between the first and second iteration can be explained best by looking at the results on a road section level. In Figure 7.6 a difference plot is shown between the results with the running average probabilities after iteration 2 and iteration 1 (which is based on free flow travel times).

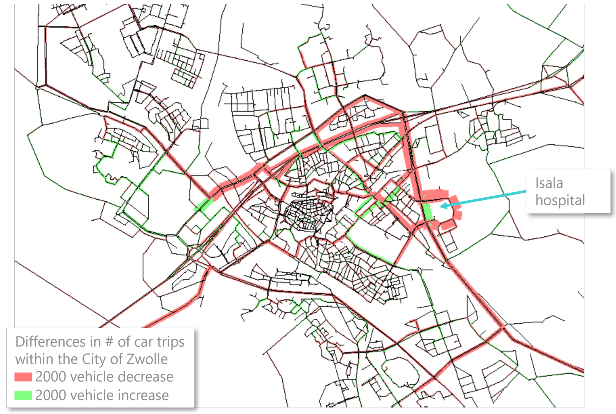


Figure 7.6: Difference plot on road section level between the second and the first (free flow) iteration with α-factor 0.25 and β=2.

Other than some minor differences throughout the network, one area attracts attention: around the Isala hospital, there is a large decrease in vehicles after iteration 2. The reason for this is that the expected number of trips with purpose work to this location is around 6800, whilst the road capacity is only 3200 for the morning rush hour. Of course, in reality, hospital workers do not all arrive during the morning rush hour, but in the assignment model a large percentage of these trips does (recall that Octavius does not model the departure times). Exceeding the capacity of the road by this much results in congested travel times of hundreds and possibly thousands of minutes, following the bpr functions (Figure 7.3). This is also the main reason why we chose to use a relatively low α-factor and β value, to prevent the effects arond this area to be even larger. And although this overshadows the general effects of taking into account congestion, it does show that the model with feedback loop is capable of representing (extreme) congestion effects.

In Figure 7.7 we look at the effect of taking into account congestion effects on the total vehicle hours, which is the sum of all trips times the travel time of these trips (i.e. the total time spent by people in a car in the modelled area on a given day).

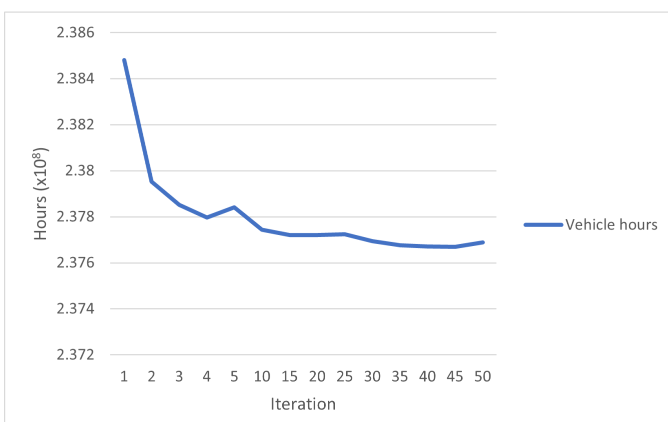


Figure 7.7: Total vehicle hours over the iterations for the feedback loop model with α-factor=0.25, $\beta = 2$.

The decrease in total vehicle hours is caused by two factors: a change in destination and a change in mode choice. In Figure 7.8 the change in mode choice is illustrated for the model with $\beta = 2$ and α-factor 0.25: the number of trips by car decreases by about 0.55% between the first and 50th iteration.

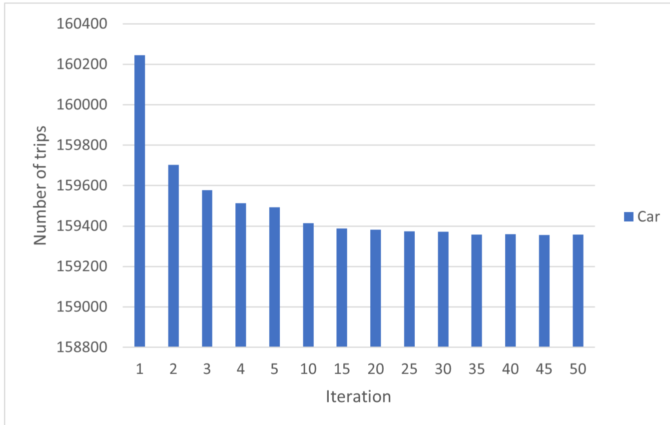


Figure 7.8: Mode choice results for $\beta = 2$, α-factor $= 0.25$.

Within the destination choice the changes were significantly larger:

- An increase of 6.45% in trips between 0.0 and 2.75 kilometres.
- An increase of 0.73% in trips between 2.75 and 5.5 kilometers.
- A decrease in longer distance trips, with the largest decreases in trips between 12.5 and 37.5 kilometres.

There are a couple of conclusions we can draw from this first attempt at implementing a feedback loop within Octavius. First of all, the algorithm as developed (Algorithm 2) works and is also efficient for a large scale model. Moreover, the effects of congestion have a significant effect on the choices agents make: shorter distances are preferred and there is a slight modal shift towards other modes than car. And although the extreme (unrealistic) congestion occurring at the Isala hospital had a large influence on the results, this is also a showcase of the fact that the model can adjust for these congestion effects and finds an equilibrium between the attraction of such a congested region, and the travel time towards the region.

Chapter 8

Conclusion and discussion

8.1 Conclusion

The literature research provided a lot of insight into the most common and best practices in microscopic travel demand modelling. The practical relevance of developing microscopic travel demand models was confirmed by this literature research. Moreover, we found that, although many of the existing models are successfully used and implemented, they are inherently different than the framework that Octavius is. The main conclusion from the literature research is that Octavius provides a lot of advantages when it comes to desired properties of strategic microscopic travel demand models. Therefore we concluded that integrating an existing travel demand model to extend Octavius with the ability of imposing equilibrium conditions may lead to losing these desired properties, and we specified our research goal to extending the framework Octavius with the ability of imposing equilibrium conditions.
From a mathematical perspective, the research goal was to analyze systems of discrete choice models with (strategic) interaction. The Quantal Response Equilibrium (QRE) fits this mathematical framework well, due to it being consistent with random utility modelling, which the discrete choice models in Octavius are derived from. We showed that the uniqueness conditions of the QRE, as presented by Melo [34], can be relaxed, specifically for games with linear payoff. Furthermore we developed a list of assumptions required for systems of discrete choice models to be analyzed using the uniqueness conditions. Finally we developed, implemented and tested an efficient solution algorithm based on Monte Carlo simulation. Computationally, the runtime of the algorithm depends linearly on the number of agents and number of alternatives, and the number of iterations required depends on the tolerance and the strength of the interaction. From the validation of the Monte Carlo algorithm we saw that a decreased tolerance leads to a better performance, and that the implemented stopping criterion with this tolerance is an appropriate measure for ensuring the deviation from the unique solution is within a certain range.
In a case study we showed that the techniques can be applied in a practical model, imposing equilibrium conditions on the interaction between Octavius and a route choice model. Unfortunately, with the currently developed theory, we can not prove uniqueness of the solution. However, the Monte Carlo algorithm converged to a solution which satisfies the fixed-point conditions of the QRE. Concluding, for practical implementations, some additional research into relaxing some of the assumptions regarding systems of discrete choice models (as presented in Chapter 5) is necessary to proof uniqueness. However, the developed theory and analyses provides a good basis for this and moreover already can give a good sense of whether uniqueness is satisfied.

8.2 Discussion and recommendations

The theoretical results in this thesis are relatively general: the relaxed uniqueness conditions can be applied in any game with perceived or perturbed utilities. Although most of the theory was developed strictly for linear dependencies, non-linear dependencies can be analyzed using this theory as well. One practical issue with the developed theory is the requirement of calculating the minimal eigenvalue of a matrix that scales with both the number of agents and the number of alternatives, growing large quite fast. Some well-known bounds on eigenvalue of matrices can be used to more efficiently determine the uniqueness condition. However, we found in the case study (Section 7.1) that two of these bounds in

general do not significantly improve on the more simple uniqueness condition as presented by Melo [34]. Therefore, a research into determining better bounds on these eigenvalues will likely prove useful. In the case study we attempted to analyze the effects of increasing the number of agents and alternatives and the heterogeneity of the population. There were definitely some effects which should be investigated in further detail. Based on the first impressions of these effects, using a small-scaled version of a large model to analyze uniqueness of that larger model can be appropriate.
The goal of the research, extending Octavius with the ability to impose equilibrium conditions, has succeeded. However, the methodology for analyzing the uniqueness conditions for (large) systems of discrete choice models requires quite some assumptions, which are for example not satisfied by most route assignment models. Especially requiring all models to be logit models with the same scale parameter is quite strict. Ideally, the uniqueness can be analyzed per interaction, instead of for the system as a whole. This would significantly decrease the size of the matrix for which eigenvalues would have to be calculated or approximated. This requires some further research.
The computational performance of the algorithm is relatively good. The Monte Carlo simulation is efficient, and there are also some improvements that can still be made. The main issue, which is a general issue for microscopic models, is the required amount of information that needs to be stored. An example of an improvement would be storing the parts of the representative utilities that are constant over the iterations to save computational time (by not recalculating them each and every iteration).
We are confident, based on the validation and the obtained results in the case study, that the Monte Carlo simulation approximates the unique solution in any model that has a unique solution. However, a formal, mathematical, analysis of this should still be done.
The capability of using equilibrium conditions as a modelling technique to add attraction, availability or capacity constraints to Octavius has not been investigated in great detail. These are all constraints that could be imposed using feedback loops, which can be analyzed with the developed theory and which can be implemented using Monte Carlo simulation.
To make imposing equilibrium conditions more relevant, a microscopic departure time choice model should be developed for Octavius. This could have resolved the problems occurring in the case study and created a more realistic solution. In principle, a microscopic route assignment model would be advantageous as well, especially for analyzing uniqueness, however this is less relevant.
A final investigation that might prove useful is quantifying the effects of using SNET instead of Monte Carlo simulation for choice allocation during the iterative procedure. Since the output of SNET is less stochastic than Monte Carlo, it might converge faster. The main problem is that since SNET is a recently developed method at DAT.Mobility and unknown outside DAT.Mobility, there is no scientific research in using it in a stochastic simulation setting for approximating deterministic results.

Concluding, with this thesis we extended the mathematical framework of Octavius with the notion of quantal response equilibria. We relaxed uniqueness conditions for the quantal response equilibrium and determined an efficient algorithm to compute the equilibrium. And although there are some possible improvements and extensions, the developed methodology is already sufficiently tested and validated to add feedback loops to Octavius where applicable.

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Appendix A

Preliminaries

This Appendix consists of some additional preliminary knowledge and theoretical background.

A.1 Convex functions

Definition A.1 (Notions of convexity)**.** Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function. Then f is said to be

a) convex over X iff

$$f(y) \geq f(x) + \langle \nabla f(x), y - x\rangle \quad \text{for any } x, y \in X$$

b) strictly convex over X iff

$$f(y) > f(x) + \langle \nabla f(x), y - x\rangle \quad \text{for any } x, y \in X$$

c) σ-strongly convex over X with respect to a norm $||\cdot||$ iff

$$f(y) > f(x) + \langle \nabla f(x), y - x\rangle + \frac{\sigma}{2}||y-x||^2 \quad \text{for any } x, y \in X$$

Definition A.2 (Legendre-Fenchel conjugate)**.** For a convex function $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$, $f^* : X^* \to \mathbb{R} \cup \{-\infty, \infty\}$ is called the convex conjugate or Legendre-Fenchel conjugate function of f, where X^* denotes the dual space corresponding to X:

$$f^*(x^*) := \sup_{x \in X}\{\langle x^*, x\rangle - f(x)\}$$

where $\langle x, y\rangle$ is defined as $\sum_i x_i y_i$.

Theorem A.1 (Fenchel's (in)equality)**.** *For a function f and its convex conjugate f^*, Fenchel's inequality holds:*

$$f(x) + f^*(y) \geq \langle x, y\rangle$$

Suppose f is a differentiable convex function, then $y = \nabla f(x)$ if and only if

$$f(x) + f^*(y) = \langle x, y\rangle$$

which is known as Fenchel's equality

Proof. First of all, note that Fenchel's inequality holds trivially because of the definition of the conjugate. Now for equality;
Because of convexity and differentiability we have

$$y = \nabla f(x) \iff f(z) \geq f(x) + \langle y, z - x\rangle \quad \forall z$$

hence, by additivity

$$\begin{aligned} &\iff f(z) \geq f(x) + \langle y, z\rangle - \langle y, x\rangle \quad \forall z \\ &\iff \langle y, x\rangle - f(x) \geq \langle y, z\rangle - f(z) \quad \forall z \end{aligned}$$

$$\iff \langle y, x\rangle - f(x) = \sup_z\{\langle y, z\rangle - f(z)\}$$

$$\iff \langle y, x\rangle - f(x) = f^*(y)$$

and thus

$$\iff f(x) + f^*(y) = \langle x, y\rangle$$

□

Theorem A.2 (Baillon-Haddad Theorem [4]). *The following statements are equivalent:*

(i) $f : X \to \mathbb{R}$ *is convex and differentiable with gradient* ∇f *which is Lipschitz continuous with respect to* $||\cdot||_X$ *with constant* $L > 0$.

(ii) The convex conjugate $f^* : X^* \to (-\infty, \infty]$ *is* $\frac{1}{L}$*-strongly convex with respect to the dual norm* $||\cdot||_X^*$

Proof. See [4]. □

A.2 Matrices

Definition A.3 (Positive definite). Let $A \in \mathbb{R}^{n\times n}$. The matrix A is said to be

a) Positive semidefinite, denoted by $A \succcurlyeq 0$ if and only if $x^T Ax \geq 0$ for all $x \in \mathbb{R}^n$

b) Positive definite, denoted by $A \succ 0$ if and only if $x^T Ax > 0$ for all $x \neq 0 \in \mathbb{R}^n$

c) Strongly positive definite if and only if there exists an $\alpha > 0$ such that $\frac{A+A^T}{2} \succ \alpha I$

Lemma A.1. *Let* $A \in \mathbb{R}^{n\times n}$ *b a symmetric matrix and denote by* $\lambda_{min}(A)$ *the minimal eigenvalue of* A*, then*

$$A \succcurlyeq \lambda_{min}(A)I \succ \alpha I$$

for any $\alpha < \lambda_{min}(A)$.

Proof. Consider the matrix $A - \lambda_{min}(A)I$. Any eigenvalue λ and corresponding eigenvector v of this matrix must satisfy

$$(A - \lambda_{min}(A)I)v = \lambda v$$

Hence we have

$$Av - \lambda_{min}(A)Iv = Av - \lambda_{min}(A)v = \lambda v$$
$$Av = (\lambda + \lambda_{min}(A))v$$

So $\lambda + \lambda_{min}$ is an eigenvalue of A and thus

$$\lambda + \lambda_{min} \geq \lambda_{min} \Rightarrow \lambda \geq 0$$

which holds for all eigenvalues of $A - \lambda_{min}(A)I$. Hence, all eigenvalues of $A - \lambda_{min}(A)I$ are nonnegative and thus

$$A - \lambda_{min}(A)I \succcurlyeq 0$$

or, equivalently

$$A \succcurlyeq \lambda_{min}(A)I$$

The second statement is trivial by noting that

$$\beta I \succ 0$$

for any $\beta > 0$. □

Definition A.4 (Block (diagonal) matrix). A block matrix is defined as a matrix $M \in \mathbb{R}^{n\times n}$ partitioned into $q \times q$ blocks $M_{ij} \in \mathbb{R}^{p\times p}$ (so $pq = n$) such that $M = (M_{ij})_{i,j\in q\times q} = (m_{ij})_{i,j\in n\times n}$. A block diagonal matrix is a matrix partitioned such that $M_{ij} = [0]_{p\times p}$ for all $i \neq j$.

Definition A.5 (Fiedler and Plak 1962 [16]). A matrix $M \in \mathbb{R}^{n\times n}$ is called a P-matrix if every principal minor[1] of M is positive.

[1] A principal minor of a matrix M is the determinant of a square submatrix of M obtained by deleting a set of rows and a set of columns with the same indices.

We denote by I the identity matrix (with appropriate dimensions if not specifically defined). The following definitions and properties hold for any vector $x \in \mathbb{R}^n$ and matrix $A \in \mathbb{R}^{n \times n}$:

1. $||x||_A^2 := x^T A x$ for $A \succ 0$
2. $||x||_I = ||x||_2 = \sqrt{\sum_{i=1}^n x_i^2}$ (the standard Euclidean or l^2-norm)
3. $||x||_1 := \sum_{i=1}^n |x_i|$ (the standard l^1-norm)
4. $||A||_2 := \sqrt{\rho(A^T A)}$ where $\rho(M)$ denotes the largest eigenvalue of a matrix M (spectral norm)
5. $||A||_\infty := \max_i \sum_{j=1}^n |[A]_{ij}|$ (max row sum)
6. $||A||_2 = ||A^T||_2$

Appendix B

Derivation of social surplus for logit

For player i Denote $X_i = \max_{j\in J}\left\{V_{ij} + \epsilon_{ij}\right\}$ with $Q_i(\epsilon_{ij}) = e^{-e^{-\epsilon_{ij}/\lambda_i - \gamma}}$ the cdf of ϵ_{ij}. Then

$$\mathbb{P}(X_i \leq x) = \Pi_{j\in J}\mathbb{P}(V_{ij} + \epsilon_{ij} \leq x)$$

by independence of the error terms amongst the alternatives. Taking the logarithm on both sides yields

$$\begin{aligned}
\log \mathbb{P}(X_i \leq x) &= \log \Pi_{j\in J}\mathbb{P}(V_{ij} + \epsilon_{ij} \leq x) \\
&= \sum_{j\in j} \log\left(\mathbb{P}(V_{ij} + \epsilon_{ij} \leq x)\right) \\
&= \sum_{j\in J} \log\left(\mathbb{P}(\epsilon_{ij} \leq x - V_{ij})\right) \\
&= \sum_{j\in J} \log\left(e^{-e^{-(x-V_{ij})/\lambda_i - \gamma}}\right) \\
&= \sum_{j\in J} -e^{-(x-V_{ij})/\lambda_i - \gamma} \\
&= -e^{-x/\lambda_i - \gamma}\sum_{j\in J} e^{V_{ij}/\lambda_i}
\end{aligned}$$

Let $\mu^* = \log\left(\sum_{j\in J} e^{V_{ij}/\lambda_i}\right)$, also known as the log-sum, then

$$\mathbb{P}(X_i \leq x) = e^{-e^{-x/\lambda_i - \gamma + \mu^*}} = e^{-e^{-(x-\lambda_i\mu^*)/\lambda_i - \gamma}}$$

and hence, X_i has a type I extreme value distribution with mean $\lambda_i \log\left(\sum_{j\in J} e^{V_{ij}/\lambda_i}\right)$ which concludes the derivation.

Appendix C

Route assignment model

As explained in Section 7.2, the bpr function is described by parameters α and β, where the value of α depends on the link type. Link types describe for example which modes are allowed to use a certain link, and link types are relevant for congestion effects. You can imagine that congestion effects differ between motorway roads and city roads with a cycleway. In Table C.1 the standard alpha values for the different link types are presented. The abbreviation GOW, in Dutch gebiedsontsluitingsweg, describes a main road that is used for leading traffic into and away a certain area. ETW, in Dutch erftoegangsweg, describes an access road to a property.

Link type	Maximum speed	α
Highway	130	0.5
Highway	100	1.0
GOW	80	1.5
ETW	60/45	1.5
GOW	70	4.0
GOW city	50	4.0
GOW district	50	6.0
Industrial	30/50	6.0
ETW	30	8.0
Residence area	15	8.0
Parking lot	10	30
Ferry	10	30
Other	10	0

Table C.1: Alpha values used in the bpr function.