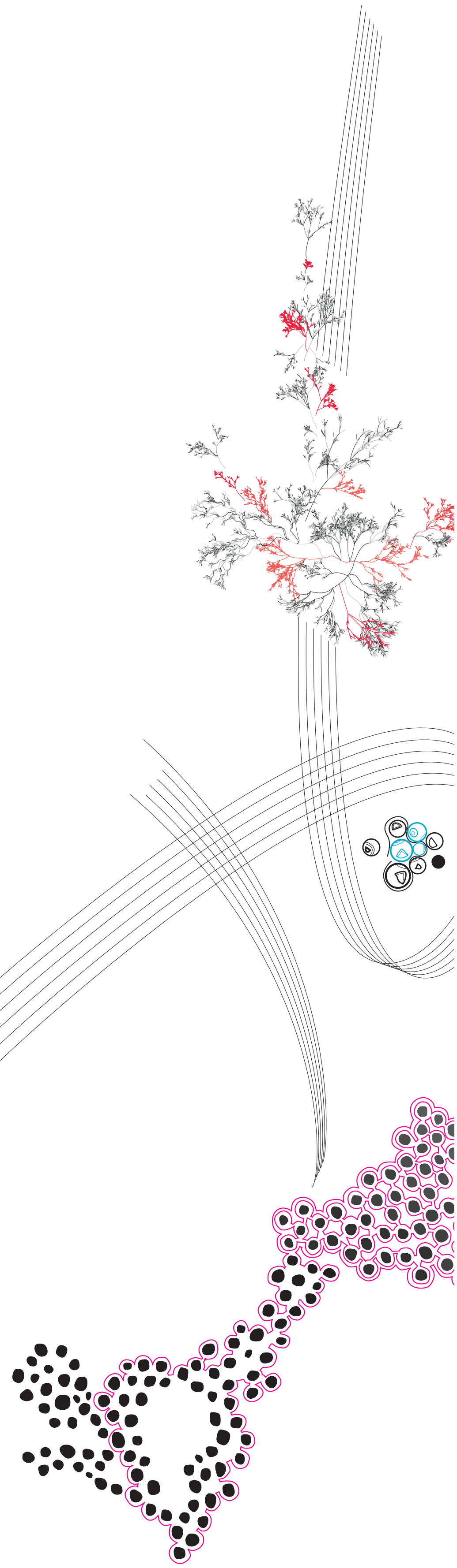

BSc Thesis Applied Mathematics

Fractal Art;
On transformations, the root-placement of butterfly-like fractals and image approximation.

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July 1, 2022

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Preface

This thesis, titled '*Fractal Art; On transformations, the root-placement of butterfly-like fractals and image approximation.*' is written to finalize my Bachelor's program at the University of Twente.

During my years in Dutch secondary school, I decided to not pursue an education in graphic design, but to study mathematics instead. Nevertheless for my thesis I get to combine my enthusiasm for art and my studies in mathematics.

Finally I like to thank F.H.C Bertrand for supervising the bachelors thesis and helping me formulate the research topic.

Tim Hut, Enschede, 2022

Fractal Art;
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Abstract

This thesis is concerned with fractal art obtained with the Newton algorithm. Three crucial results are obtained. First the relation between transformations on the roots of a polynomial and the transformation on its Newton fractal are proved. Second, a characterisation is defined and the placement of the roots of a polynomial that have that characterisation is determined.The characterisation defined are butterfly-like fractals for polynomials of degree four. Third, a method for the approximation of an image with Newton fractals using the least squares method is provided.

Keywords: Newton's method, Newton fractals, fractal art

1 Introduction

Fractal are geometric shapes containing detailed structure at arbitrarily small scales and can be created with mathematical algorithms. Those algorithms can be used to produce algorithmic art, i.e. visual art in which the design is generated by an algorithm. In this thesis, we restrict ourselves to the fractals created by applying the so-called Newton-Raphson method or simply Newton's method.

This methods was first formulated by Isaac Newton in '*De analysi per aequationes numero terminorum infinitas*', written most likely in 1669. There, the methods described how to find roots of polynomials. Later this formulation reappears in '*Principia Mathematica*' and '*De Methodis Serierum et Fluxionum*'. In 1687, the method was extended by Newton to find the roots of non polynomial functions in '*Philosophiæ Naturalis Principia Mathematica*' [9]. Independently from Newton, Joseph Raphson publishes in 1690 a similar method for solving polynomials equation, see [5].

In 1975, the term fractal was first introduced by American-french-polish mathematician Benoit B. Mandelbrot, in his book '*Les objets fractals : forme, hasard et dimension*'[4], Mandelbrot pointed out that fractals could be used in applied mathematics for modeling a variety of phenomena from physical objects to the behavior of the stock [1].

Some related works that deal with fractal art or that deal with Newton fractals are; '*Research on garment pattern design based on fractal graphics*' by Weijie Wang, Gaopeng

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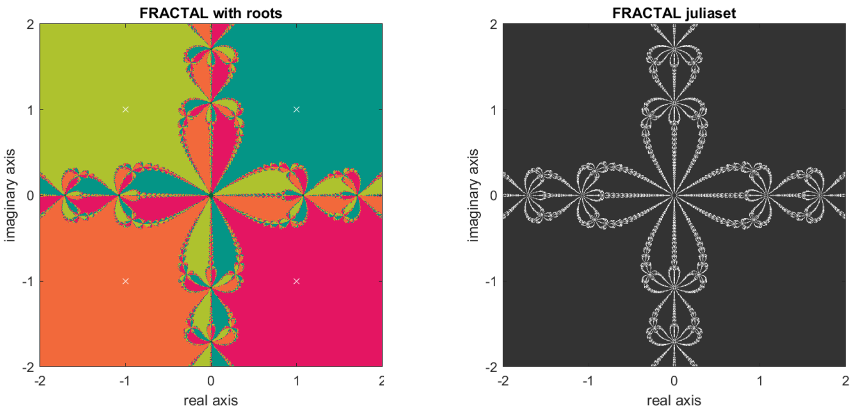


(A) Coloured basins of attraction (B) Julia set

FIGURE 1: Newton fractal of $p_4(z) = z^4 + 4$, which has roots at $\pm 1 \pm i$

Zhang, Luming Yang and Wei Wang, see [7], '*Graphical representations for the homogeneous bivariate Newton's method*' by José M. García Calcines, José M. Gutiérrez, Luis J. Hernández Paricio and M. Teresa Rivas Rodríguez, see [3], and '*On the iteration of a rational function: Computer experiments with Newton's method*' by James H. Curry, Lucy Garnett and Dennis Sullivan, [2].

The outline of this thesis is a follows: after introducing the Newton method in section 2, in section 3 the links between transformations on Newton fractals and transformations on the roots of n degree polynomials are explored. In section 4, a characterisation is defined and the placement of the roots are investigated such that characterisation appears in the newton fractals of fourth degree polynomials. In the final part of this article, we will pave the way towards an approximation algorithm capable of comparing Newton fractals to given images.

2 The Newton method

In this section, the concepts related to Newton's method are introduced. For $n \in \mathbb{N}$, let $p_n : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree n on the complex domain. The fundamental theorem of algebra states that every polynomial of degree n has n, not necessarily distinct, zeros or roots [6]. For $k \in \{1, 2, \ldots, n\}$ the kth root of p_n is denoted as ρ_k. Let $\{z_k\}_{k\in\mathbb{Z}_{\geq}}$ be a sequence where $z_0 \in \mathbb{C}$ and for $k = 1, 2, \ldots$; z_k is obtained using Newton's method;

$$z_k = z_{k-1} - \frac{p_n(z_{k-1})}{p_n'(z_{k-1})}. \tag{1}$$

Then in most cases this sequence will converge to a root of p_n. When for $z^* \in \mathbb{C}$ the sequence with $z_0 = z^*$ converges, z^* is assigned to a colour corresponding to the root it converges to. Doing this for all $z \in \mathbb{C}$ results in a fractal pattern as for instance in figure 1.

The colouring is denoted as $X(z) \subseteq \{1, 2 \ldots n\}$, for all $\chi \in \{1, 2 \ldots n\}$ if the sequence starting with $z_0 = z$ converges to ρ_χ then $\chi \in X(z)$. Hence z lies in the basin of attraction of ρ_χ[6]. This notation allows for roots with multiplicity $m \in \mathbb{N}^{\leq n}$, let

$\chi_1, \chi_2, \ldots, \chi_m \in \{1, 2, \ldots, n\}$ such that $\chi_1 < \chi_2 < \ldots < \chi_m$, and $\rho_{\chi_1} = \rho_{\chi_2} = \ldots = \rho_{\chi_m}$, we denote $X(z) = \{\chi_1, \chi_2, \ldots, \chi_m\}$ when z converges to ρ_{χ_1}.[1] When z does not converge this implies $X(z) = \emptyset$, however this is denoted as $X(z) = \{0\}$ instead.

On the boundary of the basins of attraction lies the Julia set. Formally this is defined as follows; the filled Julia set for a polynomial function $p_n(z)$ is the set of points that launch bounded orbits through iteration of p_n; the Julia set is the boundary of the filled Julia set [6].

3 Transformations on Newton Fractals

3.1 Translation

First the notation of translation is introduced. Let $T_\alpha : \mathbb{C} \to \mathbb{C}$ define a translation in the complex plane with complex number α. Then for all $z \in \mathbb{C}$, it follows $T_\alpha(z) = z + \alpha$.

A method to translate Newton fractals in the complex plane is defined. We start by observing the relation between a polynomial p_n and a polynomial $\hat{p}_n$ which has the same roots as p_n but translated.

Lemma 3.1. ***Translation of roots***
Let $p_n : \mathbb{C} \to \mathbb{C}$, be a polynomial of degree n; in factorised form $p_n(z) = \prod_{k=1}^{n}(z - \rho_k)$, with complex roots ρ_k, $k = 1, 2, ..., n$. Let α be a complex number.
If $\hat{p}_n(z)$ is a polynomial of degree n, with roots $\hat{\rho}_k = \rho_k + \alpha$ $(= T_\alpha(\rho_k))$ for $k = 1, 2, ..., n$. Then $\hat{p}_n(z)$ equals $p_n(z - \alpha)$ and its derivative $\hat{p}_n'(z)$ equals $p_n'(z - \alpha)$.

Proof.

$$\hat{p}_n(z) = \prod_{k=1}^{n}(z - (\rho_k + \alpha)) \qquad = \prod_{k=1}^{n}((z - \alpha) - \rho_k) \qquad = p_n(x - \alpha)$$

Using chain rule it follows that its derivative $\hat{p}_n'(z)$ equals $p_n'(z - \alpha)$. □

Let $z_0 \in \mathbb{C}$, we continue by comparing the sequences obtained from Newton's method on p_n and $\hat{p}_n$ with initial condition z_0 and $T_{0,\alpha}(z_0)$ respectively.

Theorem 3.2. ***Translation of Newton fractals***
Let $p_n : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree n, with with complex roots ρ_k, $k = 1, 2, ..., n$. Let $\{z_k\}_{k \in \mathbb{Z}^{\geq}}$ denote a sequence obtained from Newton's method on $p_n(z)$ that converges to root ρ_χ, $\chi \in \{1, 2, ..., n\}$ and let α be a complex number.
If $\hat{p}_n(z)$ is a polynomial of degree n, with roots $\hat{\rho}_k = \rho_k + \alpha$ for $k = 1, 2, ..., n$. Then the sequence obtained using Newton's method on $\hat{p}_n(z)$ with initial value $\hat{z}_0 = z_0 + \alpha$, implies $\{\hat{z}_k\}_{k \in \mathbb{Z}^{\geq}} = \{z_k + \alpha\}_{k \in \mathbb{Z}^{\geq}}$ and $\{\hat{z}_k\}_{k \in \mathbb{Z}^{\geq}}$ converges to root $\hat{\rho}_\chi = \rho_\chi + \alpha$.

Proof. ***proof by induction***
base step; since $\hat{z}_0 = z_0 + \alpha$ and using lemma 3.1,

$$\hat{x}_1 = \hat{z}_0 - \frac{\hat{p}_n(\hat{z}_0)}{\hat{p}_n'(\hat{z}_0)} \qquad = \left(z_0 - \frac{p_n(z_0 + \alpha - \alpha)}{p_n'(z_0 + \alpha - \alpha)}\right) + \alpha$$
$$= x_1 + \alpha$$

[1] In the graphical representation of the fractal, z has the colour associated with χ_1

hypothesis step; now assume $\hat{z}_{k-1} = z_{k-1} + \alpha$
induction step; using the hypothesis step and lemma 3.1.

$$\hat{z}_k = \hat{z}_{k-1} - \frac{\hat{p}_n(\hat{z}_{k-1})}{\hat{p}_n'(\hat{z}_{k-1})} = \left(z_{k-1} - \frac{p_n(z_{k-1} + \alpha - \alpha)}{p_n'(z_{k-1} + \alpha - \alpha)} \right) + \alpha$$
$$= z_k + \alpha$$

Hence $\{\hat{z}_k\}_{k \in \mathbb{Z}^{\geq}} = \{z_k + \alpha\}_{k \in \mathbb{Z}^{\geq}}$ and $\lim_{k\to\infty} \hat{z}_k = \lim_{k\to\infty}(z_k + \alpha) = \rho_\chi + \alpha$ □

This implies that when the mapping of the roots are translated also the mapping of the Newton fractal is translated with the same value.

3.2 Rotation

Now the notation used for rotation is introduced. Let $R_{\alpha,\phi} : \mathbb{C} \to \mathbb{C}$ define a counterclockwise rotation in the complex plane around complex number α with angle ϕ. In this section only rotation around the origin is considered. Let be $z \in \mathbb{C}$, z is written in polar form; let $|z|$ be the modulus and φ the argument of z, thus $z = |z|\mathrm{e}^{i\varphi}$. Then after rotating with ϕ.

$$\hat{z} = |z|\mathrm{e}^{i(\varphi+\phi)} = |z|\mathrm{e}^{i\varphi}\mathrm{e}^{i\phi} = z\mathrm{e}^{i\phi}.$$

Therefore $R_{0,\phi}(z) = z\mathrm{e}^{i\phi}$.

A method to rotate Newton fractals in the complex plane is defined. Like before, we start by observing the relation between a polynomial p_n and a polynomial $\hat{p}_n$ which now has the same roots as p_n but rotated with ϕ with respect to the origin.

Lemma 3.3. ***Rotation of roots around the origin***
Let $p_n : \mathbb{C} \to \mathbb{C}$, be a polynomial of degree n; in factorised form $p_n(z) = \prod_{k=1}^{n}(z - \rho_k)$, with complex roots ρ_k, $k = 1, 2, ..., n$. Let ϕ be an angle between $-\pi$ and π.
If $\hat{p}_n(z)$ is a polynomial of degree n, with roots $\hat{\rho}_k = \rho_k \mathrm{e}^{i\phi}$ $(= R_{0,\phi}(\rho_k))$ for $k = 1, 2, ..., n$. Then $\hat{p}_n(z)$ equals $(\mathrm{e}^{i\phi})^{(n-1)} p_n'(z\mathrm{e}^{-i\phi})$ and its derivative $\hat{p}_n'(z)$ equals $(\mathrm{e}^{i\phi})^{(n-1)} p_n'(z\mathrm{e}^{-i\phi})$.

Proof.

$$\hat{p}_n(z) = \prod_{k=1}^{n}(z - \rho_k\mathrm{e}^{i\phi}) = \prod_{k=1}^{n} \mathrm{e}^{i\phi}(z\mathrm{e}^{-i\phi} - \rho_k)$$
$$= \mathrm{e}^{in\phi} p_n(z\mathrm{e}^{-i\phi})$$

Using the chain rule it follows that its derivative $\hat{p}_n'(z)$ equals $\mathrm{e}^{i(n-1)\phi} p_n'(z\mathrm{e}^{-i\phi})$. □

Let $z_0 \in \mathbb{C}$, we continue by comparing the sequences obtained from Newton's method on p_n and $\hat{p}_n$ with initial condition z_0 and $R_{0,\phi}(z_0)$ respectively.

Theorem 3.4. ***Rotation of Newton fractals around origin***
Let $p_n : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree n, with with complex roots ρ_k, $k = 1, 2, ..., n$. Let $\{z_k\}_{k \in \mathbb{Z}^{\geq}}$ denote a sequence obtained from Newton's method on $p_n(z)$ that converges to root ρ_χ, $\chi \in \{1, 2, ..., n\}$ and let ϕ be an angle between $-\pi$ and π.
If $\hat{p}_n(z)$ is a polynomial of degree n, with roots $\hat{\rho}_k = \rho_k\mathrm{e}^{i\phi}$ for $k = 1, 2, ..., n$. Then the sequence obtained using Newton's method on $\hat{p}_n(z)$ with initial value $\hat{z}_0 = z_0\mathrm{e}^{i\phi}$, implies $\{\hat{z}_k\}_{k \in \mathbb{Z}^{\geq}} = \{z_k\mathrm{e}^{i\phi}\}_{k \in \mathbb{Z}^{\geq}}$ and $\{\hat{z}_k\}_{k \in \mathbb{Z}^{\geq}}$ converges to root $\hat{\rho}_\chi = \rho_\chi\mathrm{e}^{i\phi}$.

Proof. ***proof by induction***
base step; since $\hat{z}_0 = z_0 e^{i\phi}$ and using lemma 3.3,

$$\hat{z}_k = \hat{z}_0 - \frac{\hat{p}_n(\hat{z}_0)}{\hat{p}_n'(\hat{z}_0)} = \left(z_0 - \frac{p_n(z_0 e^{i\phi - i\phi})}{p_n'(z_0 e^{i\phi - i\phi})}\right) e^{i\phi}$$

$$= z_k e^{i\phi}$$

hypothesis step; now assume $\hat{z}_{k-1} = z_{k-1} e^{i\phi}$
induction step; using the hypothesis step and lemma 3.3,

$$\hat{z}_k = \hat{z}_{k-1} - \frac{\hat{p}_n(\hat{z}_{k-1})}{\hat{p}_n'(\hat{z}_{k-1})} = \left(z_{k-1} - \frac{p_n(z_{k-1} e^{i\phi - i\phi})}{p_n'(z_{k-1} e^{i\phi - i\phi})}\right) e^{i\phi}$$

$$= z_k e^{i\phi}$$

Hence $\{\hat{z}_k\}_{k \in \mathbb{Z}^{\geq}} = \{z_k e^{i\phi}\}_{k \in \mathbb{Z}^{\geq}}$ and $\lim_{k\to\infty} \hat{z}_k = \lim_{k\to\infty} z_k e^{i\phi} = \rho_\chi e^{i\phi}$ □

This implies that when the mapping of the roots are rotated around the origin also the mapping of the Newton fractal is rotated around the origin with the same angle.

3.3 Reflection

In this last subsection, the notation for reflection is introduced. Let $r_l : \mathbb{C} \to \mathbb{C}$ define a reflection in the complex plane in line l. In this section only reflection in the real line is considered. Let $z \in Z$ and $x, y \in \mathbb{R}$, then z can be written as $z = x + yi$, when reflected in the real line

$$\hat{z} = x - yi = \bar{z}.$$

Therefore $r_{\mathbb{R}-axis}(z) = \bar{z}$.

A method to reflect Newton fractals in the complex plane is defined. Again, we start by observing the relation between a polynomial p_n and a polynomial $\hat{p}_n$ which now has the same roots as p_n but reflected in the real line.

Lemma 3.5. ***Reflection of roots in real line***
Let $p_n : \mathbb{C} \to \mathbb{C}$, be a polynomial of degree n; in factorised form $p_n(z) = \prod_{k=1}^{n}(z - \rho_k)$, with complex roots ρ_k, $k = 1, 2, ..., n$.
If $\hat{p}_n(z)$ is a polynomial of degree n, with roots $\hat{\rho}_k = \bar{\rho}_k$ $(= r_{\mathbb{R}-axis}(\rho_k))$ for $k = 1, 2, ..., n$. Then $\hat{p}_n(z)$ equals $\bar{p_n}(\bar{z})$ and its derivative $\hat{p}_n'(z)$ equals $\bar{p}_n'(\bar{z})$.

Proof.

$$\hat{p}_n(z) = \prod_{k=1}^{n}(z - \bar{\rho}_k) = \prod_{k=1}^{n}\overline{(\bar{z} - \rho_k)}$$

$$= \bar{p}_n(\bar{z})$$

Using chain rule it follows that its derivative $\hat{p}_n'(z)$ equals $\bar{p}_n'(\bar{z})$.

□

Let $z_0 \in \mathbb{C}$, we continue by comparing the sequences obtained from Newton's method on p_n and $\hat{p}_n$ with initial condition z_0 and $r_{\mathbb{R}-axis}(z_0)$ respectively.

Theorem 3.6. ***Reflection of Newton fractals in real line***
Let $p_n : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree n, with with complex roots ρ_k, $k = 1, 2, ..., n$. Let $\{z_k\}_{k \in \mathbb{Z}^{\geq}}$ denote a sequence obtained from Newton's method on $p_n(z)$ that converges to root ρ_χ, $\chi \in \{1, 2, ..., n\}$.
If $\hat{p}_n(z)$ is a polynomial of degree n, with roots $\hat{\rho}_k = \bar{\rho}_k$ for $k = 1, 2, ..., n$. Then the sequence obtained using Newton's method on $\hat{p}_n(z)$ with initial value $\hat{z}_0 = \bar{z}_0$, implies $\{\hat{z}_k\}_{k \in \mathbb{Z}^{\geq}} = \{\bar{z}_k\}_{k \in \mathbb{Z}^{\geq}}$ and $\{\hat{z}_k\}_{k \in \mathbb{Z}^{\geq}}$ converges to root $\hat{\rho}_\chi = \bar{\rho}_\chi$.

Proof. ***proof by induction***
base step; since $\hat{z}_0 = \bar{z}_0$ and using lemma 3.5

$$\begin{aligned} \hat{x}_1 = \hat{z}_0 - \frac{\hat{p}_n(\hat{z}_0)}{\hat{p}_n'(\hat{z}_0)} &= \bar{z}_0 - \frac{\bar{p}_n(\bar{\bar{z}}_0)}{\bar{p}_n'(\bar{\bar{z}}_0)} = \overline{\left(z_0 - \frac{p_n(z_0)}{p_n'(z_0)}\right)} \\ &= \bar{z}_1 \end{aligned}$$

hypothesis step; now assume $\hat{z}_{k-1} = \bar{z}_{k-1}$
induction step; using the hypothesis step and lemma 3.5,

$$\begin{aligned} \hat{z}_k = \hat{z}_{k-1} - \frac{\hat{p}_n(\hat{z}_{k-1})}{\hat{p}_n'(\hat{z}_{k-1})} &= \overline{\left(z_{k-1} - \frac{p_n(z_{k-1})}{p_n'(z_{k-1})}\right)} \\ &= \bar{z}_k \end{aligned}$$

Hence $\{\hat{z}_k\}_{k \in \mathbb{Z}^{\geq}} = \{\bar{z}_k\}_{k \in \mathbb{Z}^{\geq}}$ and $\lim_{k\to\infty} \hat{z}_k = \lim_{k\to\infty} \bar{z}_k = \bar{\rho}_\chi$ □

This implies that when the mapping of roots are reflected in the real line also the mapping of the Newton fractal is reflected in the real line.

3.4 General rotation and reflection

Using the methods of translation, rotation around zero and reflection in the real line for Newton fractals of polynomials, more general methods of rotation an reflection can be constructed.

Let z and α be complex and let ϕ be between $-\pi$ and π. When z is rotated around α with ϕ, it is perceived that the relative potion of z to α in the complex plane is the same as the relative potion of $\hat{z} = z - \alpha$ with the origin when rotating around the it with ϕ. Hence when z is first translated to have α be at the origin, then is rotated around 0 and finally is translated to have α at its original position, this is the same as rotation around α. Therefore

$$R_{\alpha,\phi}(z) = T_\alpha \circ R_{0,\phi} \circ T_{-\alpha}(z) = (z - \alpha)\mathrm{e}^{i\phi} + \alpha. \tag{2}$$

Then directly from theorems 3.2 and 3.4, the next corollary follows.

Corollary 3.7. ***Rotation of Newton Fractals***
Let $p_n : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree n, with with complex roots ρ_k, $k = 1, 2, ..., n$. Let $\{z_k\}_{k \in \mathbb{Z}^{\geq}}$ denote a sequence obtained from Newton's method on $p_n(z)$ that converges to root ρ_χ, $\chi \in \{1, 2, ..., n\}$, let α be a complex number and let ϕ be an angle between $-\pi$ and π.
If $\hat{p}_n(z)$ is a polynomial of degree n, with roots $\hat{\rho}_k = (\rho_k - \alpha)\mathrm{e}^{i\phi} + \alpha$ for $k = 1, 2, ..., n$. Then the sequence obtained using Newton's method on $\hat{p}_n(z)$ with initial value $\hat{z}_0 = (z_0 - \alpha)\mathrm{e}^{i\phi} + \alpha$, implies $\{\hat{z}_k\}_{k \in \mathbb{Z}^{\geq}} = \{(z_k - \alpha)\mathrm{e}^{i\phi} + \alpha\}_{k \in \mathbb{Z}^{\geq}}$ and $\{\hat{z}_k\}_{k \in \mathbb{Z}^{\geq}}$ converges to root $\hat{\rho}_\chi = (\rho_\chi - \alpha)\mathrm{e}^{i\phi} + \alpha$.

Proof. Using theorem 3.2 twice and theorem 3.4, it follows

$$\{\hat{z}_k\}_{k\in\mathbb{Z}^{\geq}} = \{z_k^* + \alpha\}_{k\in\mathbb{Z}^{\geq}} = \{z_k^{**}\mathrm{e}^{i\phi} + \alpha\}_{k\in\mathbb{Z}^{\geq}} = \{(z_k - \alpha)\mathrm{e}^{i\phi} + \alpha\}_{k\in\mathbb{Z}^{\geq}},$$

and $\lim_{k\to\infty} \hat{z}_k = \lim_{k\to\infty}((z_k - \alpha)\mathrm{e}^{i\phi} + \alpha) = \rho_\chi \mathrm{e}^{i\phi}$. □

For this reason it can be concluded that when the mapping of roots are rotated around α in the complex plane, the fractal is also rotated around α in the complex plane.

Let z and α be the same as previously mentioned, furthermore let $t \in \mathbb{R}$, $\beta \in \mathbb{C}$ and let line l be defined as $l(t) = \beta t + \alpha$, then in a similar manner as general rotation. It is observed that the relative position of z to α when reflected in l is the same as the relative distance of $\hat{z} = (z - \alpha)\mathrm{e}^{-i\mathrm{Arg}(\beta)}$ to the origin rotated with the argument of β when reflected in the real axis. Hence when z is first translated to have α be at the origin, rotated clockwise with the argument of β such that l equals the reals axis, then reflected in the real axis and finally rotated and translated such that α and l are in their original position, this equals reflection in l. therefore

$$\begin{aligned} r_l(z) = T_\alpha \circ R_{0,\mathrm{Arg}(\beta)} \circ r_{\mathbb{R}-axis} \circ R_{0,-\mathrm{Arg}(\beta)} \circ T_{-\alpha}(z) \quad &= \overline{(z-\alpha)\mathrm{e}^{-i\mathrm{Arg}(\beta)}}\mathrm{e}^{i\mathrm{Arg}(\beta)} + \alpha \\ = (\bar{z} - \bar{\alpha})\mathrm{e}^{2i\mathrm{Arg}(\beta)} + \alpha \end{aligned}$$

Then directly from theorems 3.2, 3.4 and 3.6 the next corollary follows.

Corollary 3.8. ***Reflection of Newton Fractals***
Let $p_n : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree n, with with complex roots ρ_k, $k = 1, 2, ..., n$. Let $\{z_k\}_{k\in\mathbb{Z}^{\geq}}$ denote a sequence obtained from Newton's method on $p_n(z)$ that converges to root ρ_χ, $\chi \in \{1, 2, ..., n\}$, let α be a complex number and let ϕ be an angle between $-\pi$ and π.
If $\hat{p}_n(z)$ is a polynomial of degree n, with roots $\hat{\rho}_k = (\bar{\rho}_k - \bar{\alpha})\mathrm{e}^{2i\phi} + \alpha$ for $k = 1, 2, ..., n$. Then the sequence obtained using Newton's method on $\hat{p}_n(z)$ with initial value $\hat{z}_0 = (\bar{z}_0 - \bar{\alpha})\mathrm{e}^{2i\phi} + \alpha$, implies $\{\hat{z}_k\}_{k\in\mathbb{Z}^{\geq}} = \{(\bar{z}_k - \bar{\alpha})\mathrm{e}^{2i\phi} + \alpha\}_{k\in\mathbb{Z}^{\geq}}$ and $\{\hat{z}_k\}_{k\in\mathbb{Z}^{\geq}}$ converges to root $\hat{\rho}_\chi = (\bar{\rho}_\chi - \bar{\alpha})\mathrm{e}^{2i\phi} + \alpha$.

Proof. Using theorems 3.2 and 3.4 twice and theorem 3.6, it follows

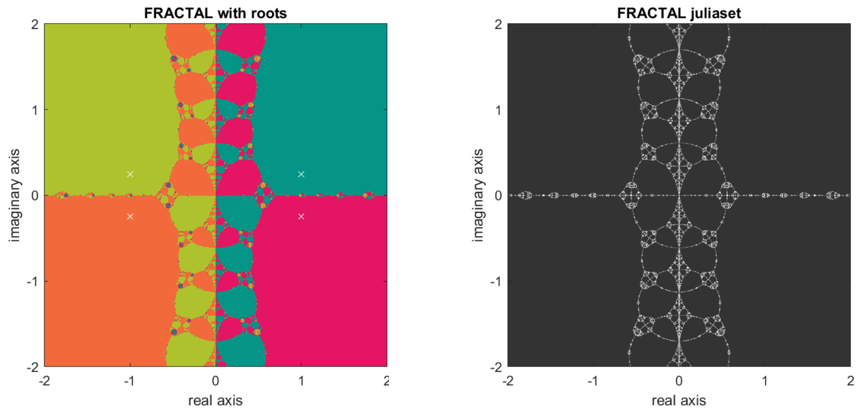
$$\begin{aligned} \{\hat{z}_k\}_{k\in\mathbb{Z}^{\geq}} &= \{z_k^* + \alpha\}_{k\in\mathbb{Z}^{\geq}} = \{z_k^{**}\mathrm{e}^{2i\phi} + \alpha\}_{k\in\mathbb{Z}^{\geq}} = \{\bar{z}_k^{***}\mathrm{e}^{2i\phi} + \alpha\}_{k\in\mathbb{Z}^{\geq}} \\ &= \{(\bar{z}_k - \bar{\alpha})\mathrm{e}^{2i\phi} + \alpha\}_{k\in\mathbb{Z}^{\geq}}, \end{aligned}$$

and $\lim_{k\to\infty} \hat{z}_k = \lim_{k\to\infty}((\bar{z}_k - \bar{\alpha})\mathrm{e}^{2i\phi} + \alpha) = (\bar{\rho}_\chi - \bar{\alpha})\mathrm{e}^{2i\phi} + \alpha$. □

Thus it can be concluded that when the mapping of the roots are reflected in line l in the complex plane, the fractal is also reflected in line l in the complex plane.

4 Characterization, symmetries and root placement

One of the main goals is determining a characterisation for some fractal and then defining the placement of the roots of a polynomial in relation to each other such that after performing Newton's method a fractal with that characterisation is created.



(A) Coloured basins of attraction (B) Julia set

FIGURE 2: Newton fractal of $p_4(z) = z^4 + \frac{30}{16}z^2 + \frac{289}{256}$, which has roots at $\pm 1 \pm \frac{i}{4}$.

4.1 Characterization and symmetries in Julia set

To find a characterization many fractals of fourth degree polynomials were created[2]. After comparing the fractals, those with a butterfly-like appearance were chosen to be further investigated. An example of such a fractal is the fractal of $p_4(z) = z^4 + \frac{30}{16}z^2 + \frac{289}{256}$ shown in figure 2.

Apart from that these fractals present a recognizable figurative repetition, as opposed by the abstract shapes present in other fractals, butterflies have appeared throughout art history symbolizing various things. Such as the precariousness of nature and beauty and as result also hope, but also as a warning symbol and representing the human soul [8].

Let $\alpha, \beta, \beta^* \in \mathbb{C}$ and let for $t \in \mathbb{R}$, $l(t) = \beta t + \alpha$ and $l^*(t) = \beta^* t + \alpha$ be two lines such that l and l^* are orthogonal and intersect in α, the fractals with a butterfly-like appearance can be characterized as the Julia set of such fractals is observed to have symmetry group $\{R_{\alpha,0}, R_{\alpha,\pi}, r_l, r_{l^*}\}$. Furthermore when the fractal (with coloured basins of attractions) is rotated with π or reflected in l or l^*, the colours are shifted according to some permutations.

4.2 Root placement

Let $X(p_4(z)) = \{0, 1, 2, 3, 4\}$ be the set of colours in the fractal of a fourth degree polynomial. Arbitrary choose one root of p_4 to be root 1, then let the other roots of p_4 be placed such that clockwise starting at root 1 the roots are ρ_1, ρ_2, ρ_3 and ρ_4. Then let $\gamma_1 = (0)(1\ 3)(2\ 4)$ be a permutation on $X(p_4(z))$ in cycle notation, and therefore for all $z \in \mathbb{C}$,

$$X(R_{\alpha,\pi}(z)) = \gamma_1(X(z)) \tag{3}$$

describes when the Julia set has a rotational symmetry with π, and when rotating with π the colours are shifted.

Let for $k = 1, 2, 3, 4$, $\hat{p}_4$ denote the polynomial corresponding to the roots $\hat{\rho}_k = (\rho_k - \alpha)e^{i\pi} + \alpha$. Let $\{z_k\}_{k \in \mathbb{Z}^{\geq}}$ denote a sequence obtained from Newton's method on $p_4(z)$

[2]For those interested in the MATLAB code for creating fractals, and various fractals, I would like to direct you to appendix A.1, A.2 and B

that converges to root ρ_χ, $\chi \in \{1,2,3,4\}$. Then according to equation 3 the sequence $\{R_{\alpha,\pi}(z_k)\}_{k\in\mathbb{Z}^{\geq}}$ has to converge to $\rho_{\gamma_1(\chi)}$. From corollary 3.7 follows that the sequence $\{R_{\alpha,\pi}(z_k)\}_{k\in\mathbb{Z}^{\geq}}$ obtained from Newton's method on $\hat{p}_4$ converges to $\hat{\rho}_\chi = (\rho_\chi - \alpha)\mathrm{e}^{i\pi} + \alpha$. Then the fractals of p_4 and $\hat{p}_4$ are the same if $\rho_{\gamma_1(\chi)} = \hat{\rho}_\chi = (\rho_\chi - \alpha)\mathrm{e}^{i\pi} + \alpha$. Since $\mathrm{e}^{i\pi} = -1$, it can therefore be derived that

$$\rho_3 = -(\rho_1 - \alpha) + \alpha = -\rho_1 + 2\alpha \qquad \text{and} \qquad \rho_4 = -\rho_2 + 2\alpha \tag{4}$$

When the Julia set has a line symmetry and when reflecting in that line the colours are shifted, this is described by the permutation $\gamma_2 = (0)(1\ 2)(3\ 4)$ and when for all $z \in \mathbb{C}$,

$$X(r_l(z)) = \gamma_2(X(z)). \tag{5}$$

Let for $t \in \mathbb{R}$, the line $l(t)$ be described as in section 4.1 and let for $k = 1,2,3,4$, p_4^* denote the polynomial corresponding to the roots $\rho_k^* = (\bar{\rho}_k - \bar{\alpha})\mathrm{e}^{2i\mathrm{Arg}\beta} + \alpha$. Let $\{z_k\}_{k\in\mathbb{Z}^{\geq}}$ denote a sequence obtained from Newton's method on $p_4(z)$ that converges to root ρ_χ, $\chi \in \{1,2,3,4\}$. Then according to equation 5 the sequence $\{r_l(z_k)\}_{k\in\mathbb{Z}^{\geq}}$ has to converge to $\rho_{\gamma_2(\chi)}$. From corollary 3.8 follows that the sequence $\{r_l(z_k)\}_{k\in\mathbb{Z}^{\geq}}$ obtained from Newton's method on $\hat{p}_4$ converges to $\rho_\chi^* = (\bar{\rho}_\chi - \bar{\alpha})\mathrm{e}^{2i\mathrm{Arg}\beta} + \alpha$. Then the fractals of p_4 and $\hat{p}_4$ are the same if $\rho_{\gamma_2(\chi)} = \rho_\chi^* = (\bar{\rho}_\chi - \bar{\alpha})\mathrm{e}^{2i\mathrm{Arg}\beta} + \alpha$. Therefore

$$\rho_2 = (\bar{\rho}_1 - \bar{\alpha})\mathrm{e}^{2i\mathrm{Arg}\beta} + \alpha \qquad \text{and} \qquad \rho_4 = (\bar{\rho}_3 - \bar{\alpha})\mathrm{e}^{2i\mathrm{Arg}\beta} + \alpha. \tag{6}$$

Substituting equations 4 in equations 6 results in

$$\rho_4 = (\overline{-\rho_1 + 2\alpha} - \bar{\alpha})\mathrm{e}^{2i\mathrm{Arg}\beta} + \alpha \qquad = -(\bar{\rho}_1 - \bar{\alpha})\mathrm{e}^{2i\mathrm{Arg}\beta} + \alpha.$$

This is the same as describing the line symmetry in the Julia set in the line orthogonal to the line described previously; let for $t \in \mathbb{R}$, the lines $l(t)$ and l^* be described as in section 4.1, since l and l^* are orthogonal, therefore $|\mathrm{Arg}\beta - \mathrm{Arg}\beta^*| = \frac{\pi}{2}$, and this results in,

$$\begin{aligned} r_l \circ R_{\alpha,\pi}(z) &= ((\overline{(z-\alpha)\mathrm{e}^{i\pi} + \alpha}) - \bar{\alpha})\mathrm{e}^{2i\mathrm{Arg}\beta} + \alpha \\ &= (\bar{z} - \bar{\alpha})\mathrm{e}^{i(2\mathrm{Arg}\beta^* \pm 2\frac{\pi}{2} - \pi)} + \alpha \qquad = r_{l^*}(z) \end{aligned}$$

Altogether, let $\zeta, \alpha, \beta \in \mathbb{C}$ and let $\rho_1 = \zeta$, a fractal with rotational symmetry with permutation γ_1 around α and line symmetry with permutation γ_2 in $l(t) = \beta t + \alpha$ follow when the roots are placed such that

$$\begin{cases} \rho_1 &= \zeta \\ \rho_2 &= \alpha + (\bar{\zeta} - \bar{\alpha})\mathrm{e}^{2i\mathrm{Arg}\beta} \\ \rho_3 &= 2\alpha - \zeta \\ \rho_4 &= \alpha - (\bar{\zeta} - \bar{\alpha})\mathrm{e}^{2i\mathrm{Arg}\beta} \end{cases} \tag{7}$$

Fractals according to the root placement in 7 are created, for $a = 0, 0.25, 0.5, ..., 1$ and $b = 0, 0.25, 0.5, ..., 1$, such that $\zeta = a + bi$. In table 1, $\alpha = 0$ and $\beta = 1$ is taken and in table 2, $\alpha = 0.25$ and $\beta = 1 + 2i$ is taken.

Currently when the roots are placed according to equation 7, two things are observed from table 1 and 2. Namely when the Julia set has a rotational symmetry with $\frac{\pi}{2}$ or when a root has a multiplicity bigger than 1, the fractal does not have a butterfly-like appearance, therefore it will be useful to define what symmetries are not in the Julia set of the fractal.

Table 1: Fractals with roots according to the equations 7 where $\zeta = a + bi$, rotational around 0 and line symmetric in real axis

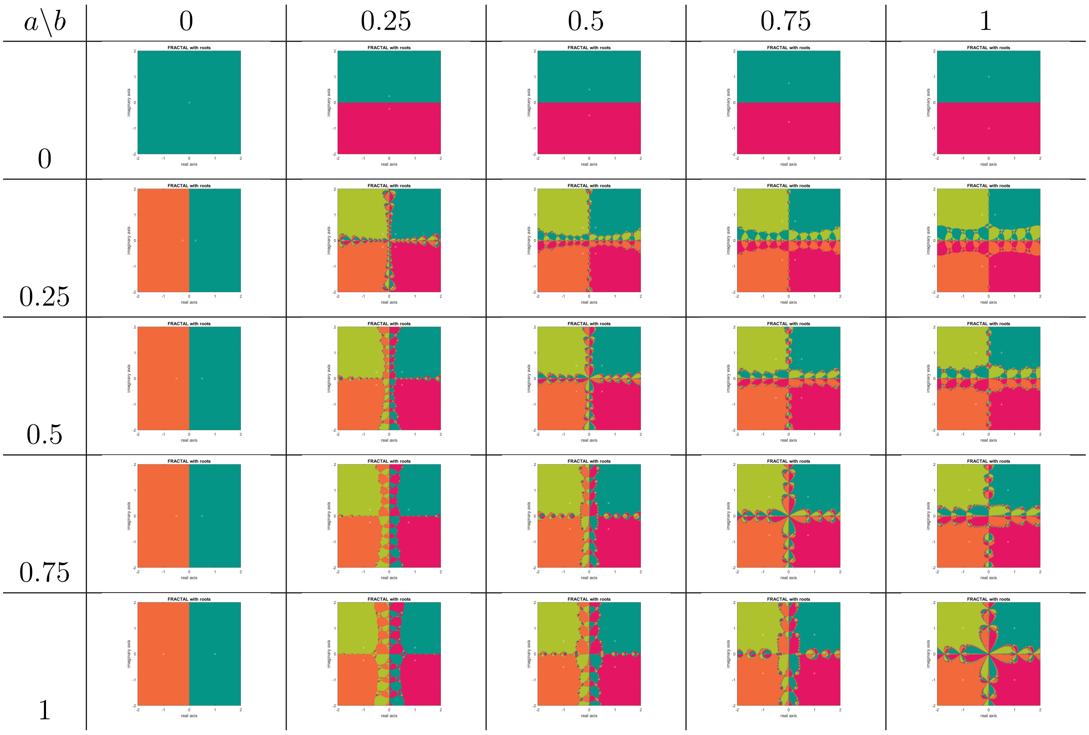
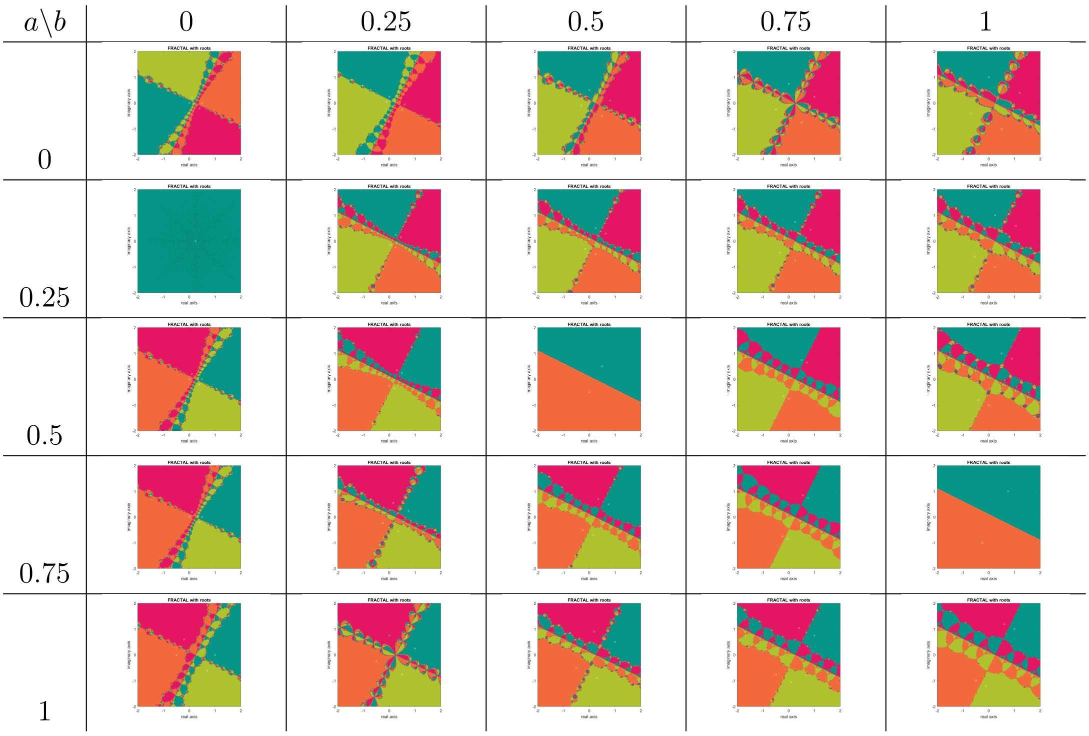


Table 2: Fractals with roots according to the equations 7 where $\zeta = a + bi$, rotational around 0.25 and line symmetric $l(t) = 0.25 + (1 + 2i)t$



First we look when the Julia set does not have a rotational symmetry with $\frac{\pi}{2}$. Let $\gamma_3 = (0)(1\ 2\ 3\ 4)$ be a permutation on $X(p_4(z))$, then this can be described as for some $z \in \mathbb{C}$,

$$X(R_{\alpha,\frac{\pi}{2}}(z)) \neq \gamma_3(X(z)) \tag{8}$$

Let for $k = 1, 2, 3, 4$, $\hat{p}_4$ denote the polynomial corresponding to the roots $\hat{\rho}_k = (\rho_k - \alpha)e^{i\frac{\pi}{2}} + \alpha$. Let $\{z_k\}_{k\in\mathbb{Z}^{\geq}}$ denote a sequence obtained from Newton's method on $p_4(z)$ that converges to root ρ_χ, $\chi \in \{1, 2, 3, 4\}$. Then if $X(R_{\alpha,\frac{\pi}{2}}(z)) = \gamma_3(X(z))$ would hold the sequence $\{R_{\alpha,\pi}(z_k)\}_{k\in\mathbb{Z}^{\geq}}$ has to converge to $\rho_{\gamma_3(\chi)}$. From corollary 3.7 follows that the sequence $\{R_{\alpha,\frac{\pi}{2}}(z_k)\}_{k\in\mathbb{Z}^{\geq}}$ obtained from Newton's method on $\hat{p}_4$ converges to $\hat{\rho}_\chi = (\rho_\chi - \alpha)e^{i\frac{\pi}{2}} + \alpha$. Then from equation 8, follows $\rho_{\gamma_3(\chi)} \neq \hat{\rho}_\chi = (\rho_\chi - \alpha)e^{i\frac{\pi}{2}} + \alpha$. Therefore,

$$\begin{aligned} &\rho_4 \neq (\rho_1 - \alpha)e^{i\frac{\pi}{2}} + \alpha = \rho_1 i + (1-i)\alpha, && && \rho_1 \neq \rho_2 i + (1-i)\alpha, \\ &\rho_2 \neq \rho_3 i + (1-i)\alpha && \text{and} && \rho_3 \neq \rho_4 i + (1-i)\alpha. \end{aligned} \tag{9}$$

Furthermore any root should not have a multiplicity bigger than 1, for this reason it follows that

$$\rho_1 \neq \rho_2 \neq \rho_3 \neq \rho_4. \tag{10}$$

Let $\zeta, \alpha, \beta \in \mathbb{C}$, then combining equations 9 and 10 with the placement in 7 results in the placement of the roots where the fractal has a butterfly-like appearance.

$$\begin{cases} \rho_1 &= \zeta & \text{, but } \rho_1 \neq \alpha + (\bar{\zeta} - \bar{\alpha})e^{i(2\mathrm{Arg}\beta + \frac{\pi}{2})} \\ \rho_2 &= \alpha + (\bar{\zeta} - \bar{\alpha})e^{2i\mathrm{Arg}\beta} & \text{, but } \rho_2 \neq (1+i)\alpha - \zeta i \\ \rho_3 &= 2\alpha - \zeta & \text{, but } \rho_3 \neq \alpha - (\bar{\zeta} - \bar{\alpha})e^{i(2\mathrm{Arg}\beta + \frac{\pi}{2})} \\ \rho_4 &= \alpha - (\bar{\zeta} - \bar{\alpha})e^{2i\mathrm{Arg}\beta} & \text{, but } \rho_4 \neq (1-i)\alpha + \zeta i \\ \rho_1 &\neq \rho_2 \neq \rho_3 \neq \rho_4 & \end{cases} \tag{11}$$

5 Image approximation

In the final part of this article the approximation of an image with 'newton fractals is discussed. Given an image, the goal is to obtain the fractal that approximates the image the best, the accomplish this least squares method is used.

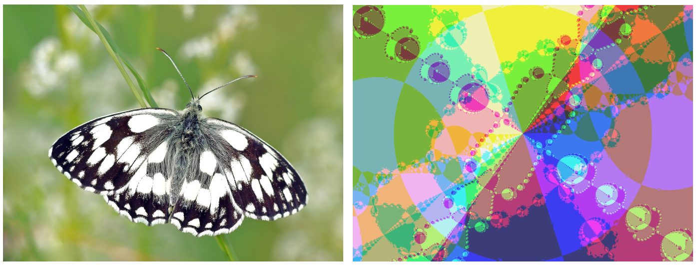
Since an image is pixellated and each pixel is coloured using a combination of the additive primary colours: red, green and blue. Let an $m \times n$ pixel image be described by a $n \times m \times 3$ matrix I. For $p \in \{0, 1, \ldots, m\}$ and $q \in \{0, 1, \ldots, n\}$, then for p, qth pixel of the image; $I(q, p, 1)$ describes the amount of red, $I(q, p, 2)$ describes the amount of green and $I(q, p, 3)$ describes the amount of blue. Furthermore for all p, q and $k = 1, 2, 3$ holds $I(q, p, k) \in \{0, 1, 2, \ldots, 255\}$.

To account for the pixels. Given X a colouring corresponding to a newton fractal, let F be a $n \times m$ matrix be the pixellation of that fractal on domain $0 \leq \mathrm{Re}z \leq 1,\ 0 \leq \mathrm{Im}z \leq \frac{n}{m}$, then

$$F(q, p) = X(\frac{1}{m}(p - 1 + (q-1)i)).$$

Note that F inverts the fractal horizontally.

(A) image of Peacock butterfly and approximation



(B) image of Marbled white (butterfly) and approximation

FIGURE 3: images and their least squares approximation

The fractals that are considered are restricted to fractals with a butterfly-like appearance as discussed in previous section. These fractals are obtained according to the placement in equation 11. The fractals are dependent on three variables; ζ, α and β. Furthermore the actual colours associated with the colouring also influence how the fractal looks. An fractal approximation is determined for the red, green and blue values of the image separately. For each value $\chi \in \{0, 1, 2, 3, 4\}$ in F has to correspond to a value c in $\{0, 1, \dots, 255\}$. To accomplish this, the values in F are multiplied by a scalar $s \in \mathbb{N}$, such that $c = s\chi$. To also consider colourings in a non clockwise manner, the values of χ are changed with different permutations $\gamma \in \Gamma_{(0)}$, the group of permutations where 0 remains 0.

Then the least squares method can be described as, for $k = 1, 2, 3$,

$$L_k = \min_{\zeta,\alpha,\beta\in\mathbb{C},\gamma\in\Gamma_{(0)}} \left(\sum_{p=1}^{n} \sum_{q=1}^{m} I(q,p,k) - c\gamma(F(q,p)) \right)$$

Using this equation, the combination of fractals in figure 3 are determined as best least squares approximations. To reduce computation time, α is fixed as $\frac{1}{2} + \frac{n}{2m}$, for $r = \frac{1}{10}, \frac{2}{10} \dots 1$, $\phi = 0, \frac{1}{10}\pi, \frac{2}{10}\pi, \dots \frac{1}{2}\pi$ and $\varphi = 0, \frac{1}{10}\pi, \frac{2}{10}\pi, \dots \frac{1}{2}\pi$ over $\beta = \mathrm{e}^{i\phi}$ and $\zeta = (r\mathrm{e}^{i\varphi} - \alpha)\mathrm{e}^{-i\phi} + \alpha$ is iterated.

When comparing the image to the fractal in figure 3a, the minimum errors squared are $0.8163 \cdot 10^{10}$, $0.7502 \cdot 10^{10}$ and $1.8456 \cdot 10^{10}$, or 3936.6, 3618.1 and 8900.6 per pixel, for the value of red, green and blue respectively. It suggests that the fractals are mainly influenced by the green of the background and the white in the flowers, as those are the most dominant colours on the image. When comparing the image to the fractal in figure 3b, the minimum errors squared are $1.7894 \cdot 10^{10}$, $1.7308 \cdot 10^{10}$ and $2.0953 \cdot 10^{10}$, or 6454, 6242.9 and 7557.5 per pixel, for the value of red, green and blue respectively. Although less clear than in figure 3a, by the spread of green it also suggest that the most dominant colour in the image influences the chosen fractals.

6 Conclusion

To summarize, for polynomials it is shown that when translation, rotation and reflection are performed on the image of the roots in the complex plane the same transformations are performed on the Newton fractal. Secondly fractals with a butterfly-like appearance are created when the roots of a fourth degree polynomial are placed according to equation 11. Finally the least squares approximation between these fractals and some images is investigated.

To reflect, because this paper was restricted to the use of polynomials, the transformations were exclusively proved for polynomials. Therefore when considering other types of functions these results have to be shown independently or the proves could be extended to include other types of function. Furthermore the placement in equation 11 sufficiently places the roots such that the chosen characterization appears. Although the formulation greatly depends on the relative order of the roots, if a non clockwise ordering is sought after simply taking a permutation on the roots suffices. Currently this characterization is only constructed for polynomials of degree four, however it could be interesting to investigate if ensuring the same symmetries and asymmetries in the Julia set for fractals of higher degree polynomials or other types of functions still produces the same characterisation. Although the approximation algorithm is not yet perfect, admittedly restricting the fractals to the butterfly-like fractals was an artistic choice. For future research performing least squares on various types of fractals with substantially more roots is advised.

7 Acknowledgements

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8 Table of variables

Table 3: Table of Variables, in order of appearence

symbol	description	page	notes
z	complex number, $z = x + yi$, with $x, y \in \mathbb{R}$		
$p_n(z)$	polynomial of degree n, with variable z	2	
ρ_k	kth root of $p_n(z)$	2	from 'ρίζα' = root
χ	colour	2	from 'χρώμα' =colour
$X(z)$, $X(p_n(z))$	colouring of z, colouring/fractal of $p_n(z)$	2	
$T_\alpha(z)$	translation of z with α	3	
$R_{\alpha,\phi}(z)$	rotation of z around α with ϕ	4	
$r_l(z)$	reflection of z in line l	5	
$l(t)$	line in complex plane, with $t \in \mathbb{R}$	5	
$\gamma(S)$	permutation on set S	8	
I	image in matrix form	11	
F	inverted pixellation of fractal in matrix form	11	
L	value obtained from least squares method	12	

A MATLAB code

A.1 newtonfractal4.m

```
function Z = newtonfractal4(pol_roots, xrange, yrange, pix, show_fig,
    juliaset)
%NEWTONFRACTAL4 Creates a fractal of polynomial of degree 4 by inserting 4
%complex roots.
% code adapted from Jeffrey Chasnov (11 Feb 2021), Coding the Newton
    Fractal | Lecture 19 |
% Numerical Methods for Engineers [Video], https://youtu.be/_FrpXPbP-zk

% define function, derivative and roots
cf = poly(pol_roots);
f = @(Z) Z.^4 + cf(2)*Z.^3 + cf(3)*Z.^2 + cf(4)*Z + cf(5);
fp = @(Z) 4*Z.^3 + 3*cf(2)*Z.^2 + 2*cf(3)*Z + cf(4);

nx = (xrange(2)-xrange(1))*pix; ny = (yrange(2)-yrange(1))*pix;

x = linspace(xrange(1), xrange(2), nx); y = linspace(yrange(1), yrange(2),
    ny);
[X,Y] = meshgrid(x,y);
Z = X + 1i*Y;

% Newton method
nit = 40;
for n = 1:nit
    Z = Z - f(Z) ./ fp(Z);
end

% determining to which root and accounting for multiplicity
eps = 0.001;
Z1 = abs(Z - pol_roots(1)) < eps; Z2 = abs(Z - pol_roots(2)) < eps & ~Z1;
Z3 = abs(Z - pol_roots(3)) < eps & ~(Z1 + Z2);
Z4 = abs(Z - pol_roots(4)) < eps & ~(Z1 + Z2 + Z3);
ZN = ~(Z1 + Z2 + Z3 +Z4);
Z = (Z1 + 2*Z2 + 3*Z3 + 4*Z4 + 5*ZN);

%% plot figure
if show_fig == true
    figure;
    map = [0.0196    0.5882    0.5294;
        0.8980    0.0824    0.3882;
        0.9529    0.4118    0.2314;
        0.6824    0.7608    0.1843;
        0.0235    0.3961    0.2392];
    colormap(map);
    image(xrange, yrange, Z);
    set(gca,'Ydir','normal');

    hold on
    plot(real(pol_roots), imag(pol_roots), 'wx')
    hold off

    axis equal; axis tight; axis([xrange, yrange])
    set(gca, 'Xtick', linspace(xrange(1), xrange(2), 5),'Ytick',linspace(
    yrange(1), yrange(2),5))
    xlabel('real axis')
    ylabel('imaginary axis')
    title('FRACTAL with roots')
end
```

```matlab

%% Juliaset
if juliaset == true
    J = zeros(ny,nx);
    J(1,1) = (Z(1,1) ~= Z(1,2)) + (Z(1,1) ~= Z(2,1)) + 1;
    J(ny,nx) = (Z(ny,nx) ~= Z(ny,nx-1)) + (Z(ny,nx) ~= Z(ny-1,nx)) + 1;
    for dx = 2:ny-1
        J(dx,1) = (Z(dx,1) ~= Z(dx,2)) + (Z(dx,1) ~= Z(dx+1,1)) + (Z(dx,1) ~= Z(dx-1,1)) + 1;
        J(dx,ny) = (Z(dx,ny) ~= Z(dx,nx-1)) + (Z(dx,nx) ~= Z(dx+1,nx)) + (Z(dx,nx) ~= Z(dx-1,nx)) + 1;
        for dy = 2:nx-1
            J(1,dy) = (Z(1,dy) ~= Z(1,dy+1)) + (Z(1,dy) ~= Z(1,dy-1)) +...
                (Z(1,dy) ~= Z(2,dy)) + 1;
            J(ny,dy) = (Z(ny,dy) ~= Z(ny,dy+1)) + (Z(ny,dy) ~= Z(ny,dy-1)) +...
                (Z(ny,dy) ~= Z(ny-1,dy)) + 1;
            J(dx,dy) = (Z(dx,dy) ~= Z(dx,dy+1)) + (Z(dx,dy) ~= Z(dx,dy-1)) +...
                (Z(dx,dy) ~= Z(dx+1,dy)) + (Z(dx,dy) ~= Z(dx-1,dy)) + 1;
        end
    end
    figure;
    mapj = [0.2 0.2 0.2; 0.4 0.4 0.4; 0.6 0.6 0.6; 0.8 0.8 0.8; 1 1 1];
    colormap(mapj);
    image(xrange, yrange, J);
    set(gca,'Ydir','normal');

    axis equal; axis tight; axis([xrange, yrange])
    set(gca, 'Xtick', linspace(xrange(1), xrange(2), 5),'Ytick',linspace(yrange(1), yrange(2),5))
    xlabel('real axis')
    ylabel('imaginary axis')
    title('FRACTAL juliaset')
end
end
```

A.2 CreateFractals.m

```
%% No export
rho1 = -1; rho2 = 1;
rho3 = -1/3 - 1i; rho4 = 1/3 + 1i;
newtonfractal4([rho1, rho2, rho3, rho4], [-2, 2], [-2, 2],250,1,1);


%% 4th degree: Rotational Symmetric Roots
clear
for r = 0:0.1:1
    for phi = 0:0.05*pi:0.5*pi
        rho1 = -1; rho2 = r*exp(1i*(-pi + phi));
        rho3 = 1; rho4 = r*exp(1i*(phi));
        newtonfractal4([rho1, rho2, rho3, rho4], [-2, 2], [-2, 2],250,1,0);
        % EXPORT FILE
        Folder = 'C:\Users\Tim Hut\Desktop\Bachelor Assignement\Fractals\
    Rotational';
        filename = sprintf('R(r,phi)(%0.1f,%0.2f).png', r, phi/pi);
        file     = fullfile(Folder, filename);
        saveas(figure(1), file);
        close(figure(1))
    end
end
%% 4th degree; Lines, Squares, Rectangles and Trapezoids
clear
for r = 0:0.05:1
    for s = 0:0.1:1
        rho1 = 1 + r*1i; rho2 = s - r*1i;
        rho3 = -1 + r*1i; rho4 = -s - r*1i;
        newtonfractal4([rho1, rho2, rho3, rho4], [-2, 2], [-2, 2],250,1,0);
        % EXPORT FILE
        Folder = 'C:\Users\Tim Hut\Desktop\Bachelor Assignement\Fractals\
    Line-Square-Trapezoid';
        filename = sprintf('r(x,y)(1;%0.1f,%0.2f).png', s, r);
        file     = fullfile(Folder, filename);
        saveas(figure(1), file);
        close(figure(1))
    end
end
%% butterflies I
clear
alpha = 0;   %0.25;
beta = 1;  %1 + 2i;
for a = 0:0.25:1
    for b = 0:0.25:1
        zs = a + b*1i;
        r1 = zs; r2 = alpha + (conj(zs)-conj(alpha))*exp(2i*angle(beta));
        r3 = 2*alpha - zs; r4 = alpha - (conj(zs)-conj(alpha))*exp(2i*angle
    (beta));
        newtonfractal4([r1, r2, r3, r4], [-2, 2], [-2, 2],250,1,0);
        % EXPORT FILE
        Folder = 'C:\Users\Tim Hut\Desktop\Bachelor Assignement\Fractals\
    Butterflies';
        filename = sprintf('butterfly(zs = %0.2f + %0.2f i)(in %0.1f + %0.1
    f t).png', a, b, alpha, beta);
        file     = fullfile(Folder, filename);
        saveas(figure(1), file);
        close(figure(1))
    end
end
```

```
%% butterflies II; julia set
clear
alpha = 0;  %0.25;
beta = 1;   %1 + 2i;
for a = 0:0.25:1
    for b = 0:0.25:1
        zs = a + b*1i;
        r1 = zs; r2 = alpha + (conj(zs)-conj(alpha))*exp(2i*angle(beta));
        r3 = 2*alpha - zs; r4 = alpha - (conj(zs)-conj(alpha))*exp(2i*angle(beta));
        newtonfractal4([r1, r2, r3, r4], [-2, 2], [-2, 2],250,0,1);
        % EXPORT FILE
        Folder = 'C:\Users\Tim Hut\Desktop\Bachelor Assignement\Fractals\Butterflies';
        filename = sprintf('julia(zs = %0.2f + %0.2f i)(in %0.1f + %0.1f t).png', a, b, alpha, beta);
        file     = fullfile(Folder, filename);
        saveas(figure(1), file);
        close(figure(1))
    end
end
```

A.3 perm_newtonfractal4.m

```
function X = perm_newtonfractal4(Z)
%   PERM_NEWTONFRACTAL4 based on a matrix of a fractal (created from
%   NEWTONFRACTAL4) creates all permuations of the colors (exceps for the color assigned to the basin of no convergence)

Z1 = (Z == ones(size(Z))); Z2 = (Z == 2*ones(size(Z)));
Z3 = (Z == 3*ones(size(Z))); Z4 = (Z == 4*ones(size(Z)));
ZN = (Z == 5*ones(size(Z)));

% mappings
X = zeros(size(Z));
X(:,:,1) = (1*Z1 + 2*Z2 + 3*Z3 + 4*Z4 + 5*ZN); %1
X(:,:,2) = (2*Z1 + 3*Z2 + 4*Z3 + 1*Z4 + 5*ZN);
X(:,:,3) = (3*Z1 + 4*Z2 + 1*Z3 + 2*Z4 + 5*ZN);
X(:,:,4) = (4*Z1 + 1*Z2 + 2*Z3 + 3*Z4 + 5*ZN);

X(:,:,5) = (1*Z1 + 2*Z2 + 4*Z3 + 3*Z4 + 5*ZN); %2
X(:,:,6) = (2*Z1 + 3*Z2 + 1*Z3 + 4*Z4 + 5*ZN);
X(:,:,7) = (3*Z1 + 4*Z2 + 2*Z3 + 1*Z4 + 5*ZN);
X(:,:,8) = (4*Z1 + 1*Z2 + 3*Z3 + 2*Z4 + 5*ZN);

X(:,:,9) = (1*Z1 + 3*Z2 + 2*Z3 + 4*Z4 + 5*ZN); %3
X(:,:,10) = (2*Z1 + 4*Z2 + 3*Z3 + 1*Z4 + 5*ZN);
X(:,:,11) = (3*Z1 + 1*Z2 + 4*Z3 + 2*Z4 + 5*ZN);
X(:,:,12) = (4*Z1 + 2*Z2 + 1*Z3 + 3*Z4 + 5*ZN);

X(:,:,13) = (1*Z1 + 3*Z2 + 4*Z3 + 2*Z4 + 5*ZN); %4
X(:,:,14) = (2*Z1 + 4*Z2 + 1*Z3 + 3*Z4 + 5*ZN);
X(:,:,15) = (3*Z1 + 1*Z2 + 2*Z3 + 4*Z4 + 5*ZN);
X(:,:,16) = (4*Z1 + 2*Z2 + 3*Z3 + 1*Z4 + 5*ZN);

X(:,:,17) = (1*Z1 + 4*Z2 + 2*Z3 + 3*Z4 + 5*ZN); %5
X(:,:,18) = (2*Z1 + 1*Z2 + 3*Z3 + 4*Z4 + 5*ZN);
X(:,:,19) = (3*Z1 + 2*Z2 + 4*Z3 + 1*Z4 + 5*ZN);
X(:,:,20) = (4*Z1 + 3*Z2 + 1*Z3 + 2*Z4 + 5*ZN);

```

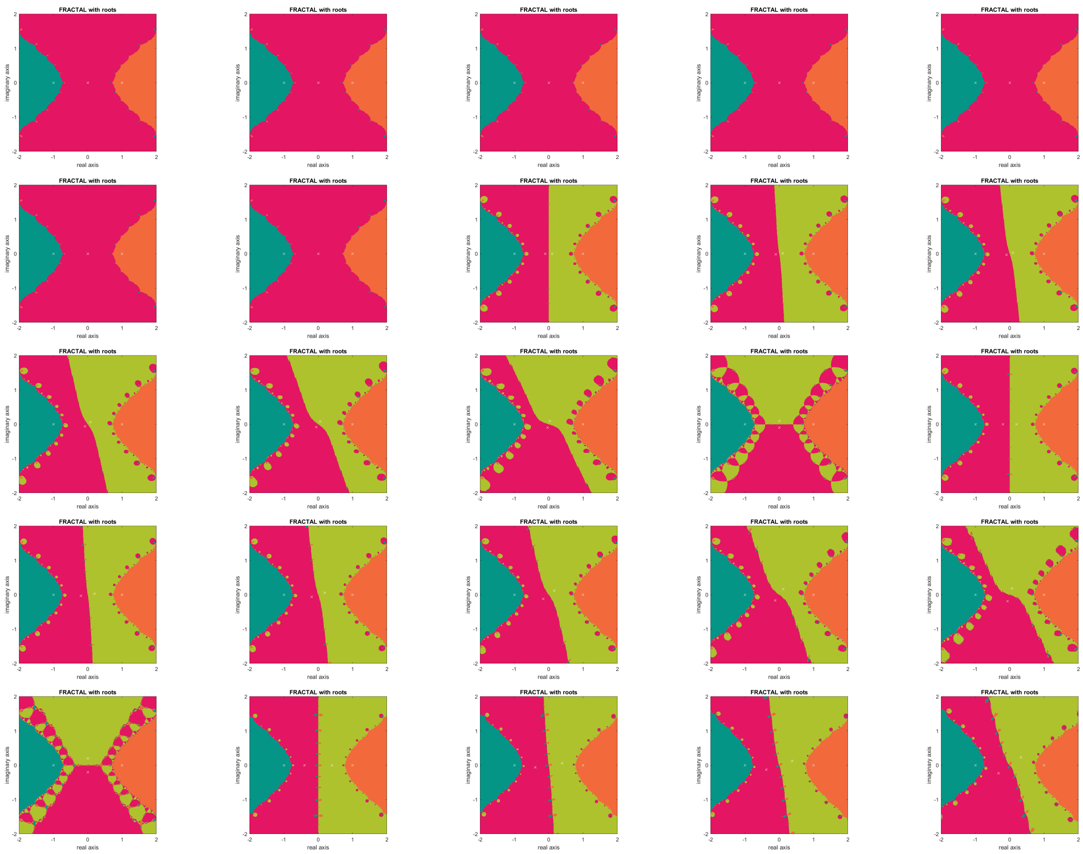
```
X(:,:,21) = (1*Z1 + 4*Z2 + 3*Z3 + 2*Z4 + 5*ZN); %6
X(:,:,22) = (2*Z1 + 1*Z2 + 4*Z3 + 3*Z4 + 5*ZN);
X(:,:,23) = (3*Z1 + 2*Z2 + 1*Z3 + 4*Z4 + 5*ZN);
X(:,:,24) = (4*Z1 + 3*Z2 + 2*Z3 + 1*Z4 + 5*ZN);
end
```

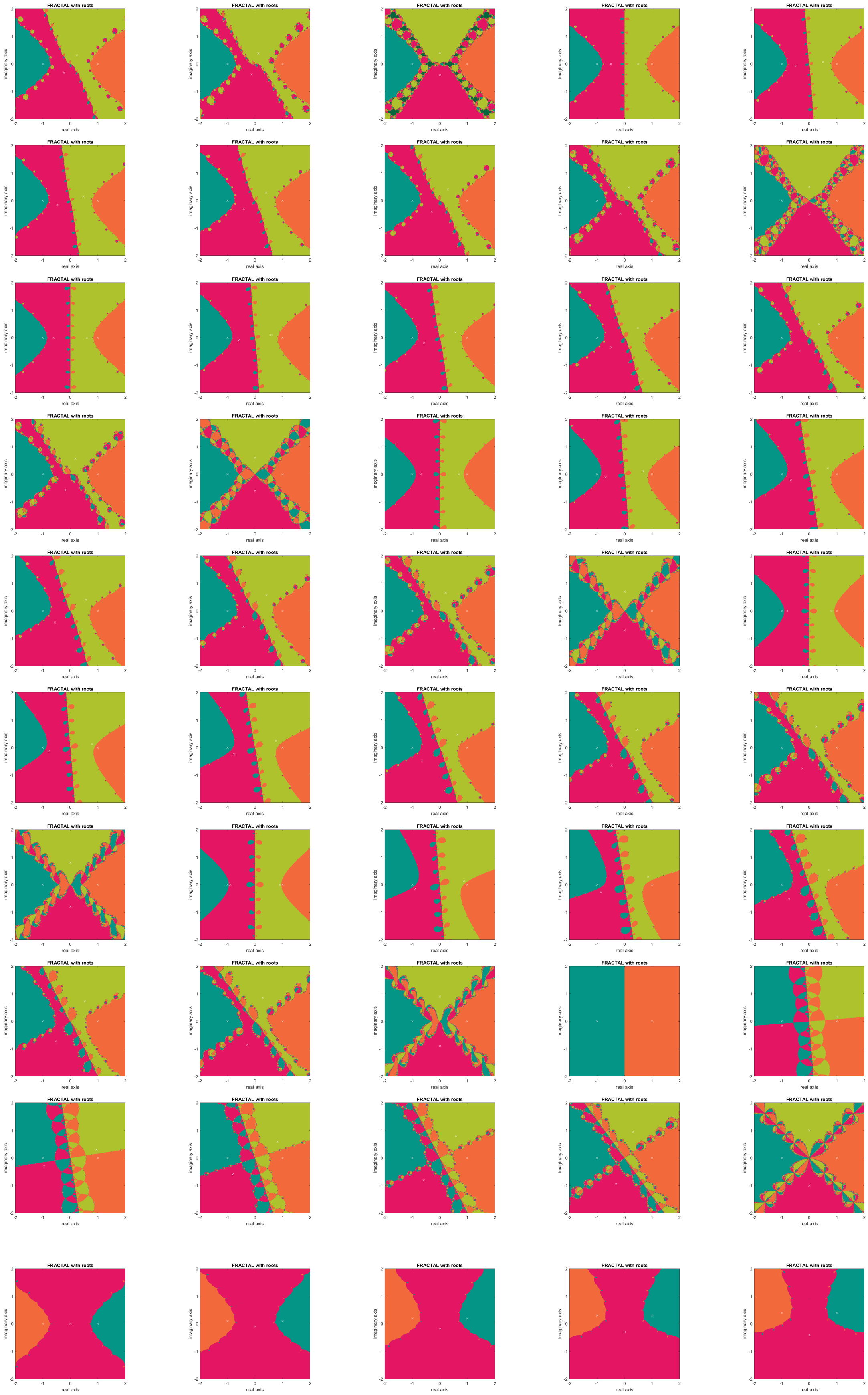
A.4 least_square.m

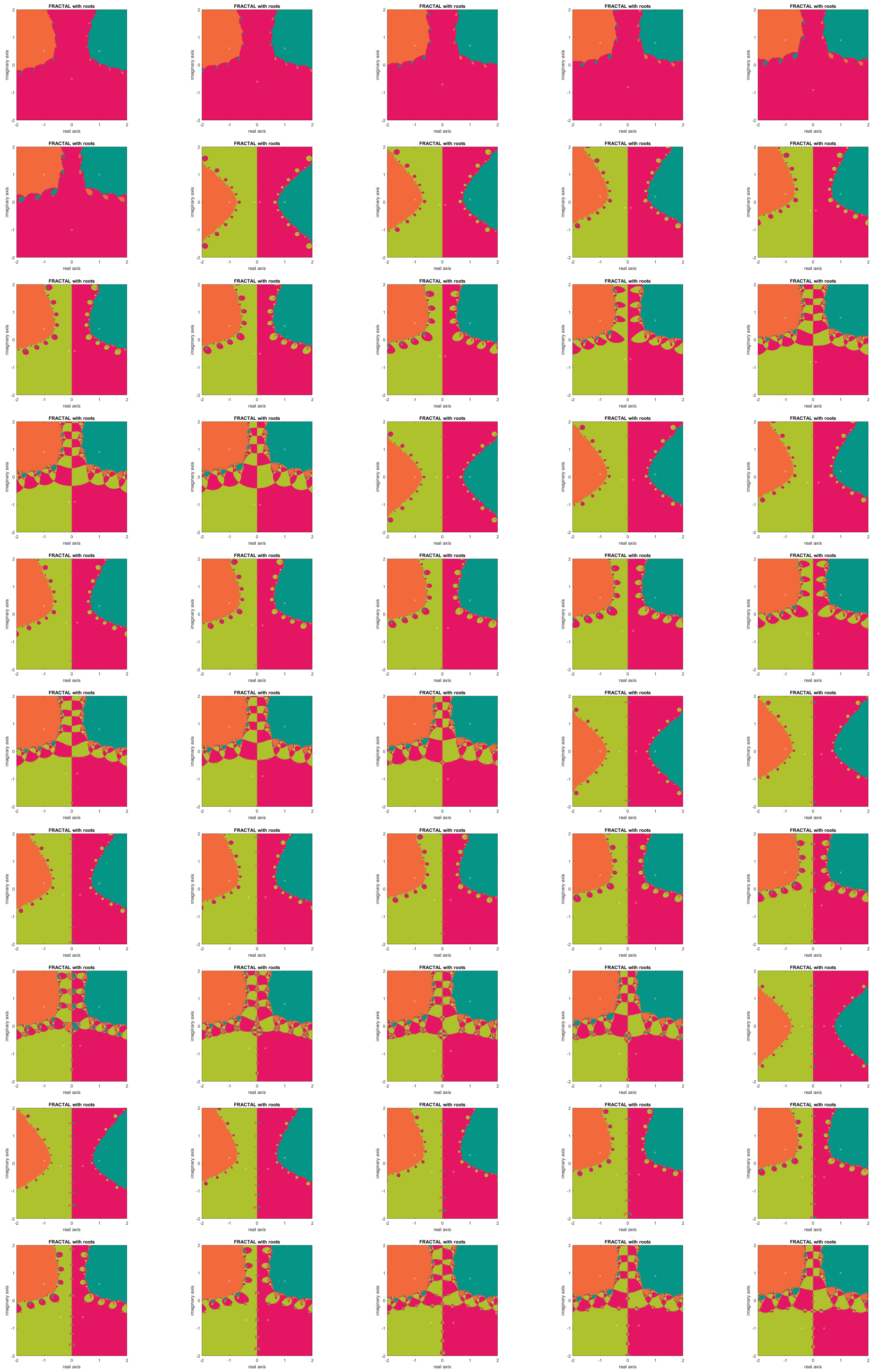
```
clear
% Image by <a href="https://pixabay.com/users/kie-ker-2367988/?utm_source=link-attribution&amp;utm_medium=referral&amp;utm_campaign=image&amp;utm_content=1526939">kie-ker</a> from <a href="https://pixabay.com/?utm_source=link-attribution&amp;utm_medium=referral&amp;utm_campaign=image&amp;utm_content=1526939">Pixabay</a>
filen = 'approx\peacock-butterfly.jpg'; %'approx\cornflower.jpg';%'approx\checkerboard.jpg'; %
A = imread(filen);
A_double = cast(A, "double");
A_size = size(A_double);
nx = A_size(2); ny = A_size(1);

%least square
l = [inf inf inf];
L = zeros(ny, nx, 3);
rts = zeros(3,4);
R = [inf inf inf];

epsilon = 0.001;
alpha = 0.5 + 0.5*ny/nx*1i;

for r = 0.1:0.1: 1
    for argb = 0:0.1*pi:0.5*pi
        for phi = 0:0.1*pi:0.5*pi
            r1 = (r*exp(1i*phi)-alpha)*exp(-1i*argb)+alpha;
            r2 = alpha + (conj(r1)-conj(alpha))*exp(2i*argb);
            r3 = 2*alpha - r1;
            r4 = alpha - (conj(r1) - conj(alpha))*exp(2i*argb);
            if (r1 - r2)> epsilon
                if (r1 - r3) > epsilon
                    if (r1 - r4) > epsilon
                        Z = newtonfractal4([r1, r2, r3, r4], [0,1],[0,ny/nx],nx, 0,0);
                        T = perm_newtonfractal4(Z);
                        for n = 1:3
                            for m = 1:1:24
                                t = sum((A_double(:,:,n)- 60*mod(T(:,:,m),5)).^2,'all');
                                if t < l(n)
                                    l(n) = t;
                                    L(:,:,n) = 60*mod(T(:,:,m),5);
                                    rts(n,:) = [r1, r2, r3, r4];
                                    R(n) = r;
                                end
                            end
                        end
                    end
                end
            end
        end
    end
```

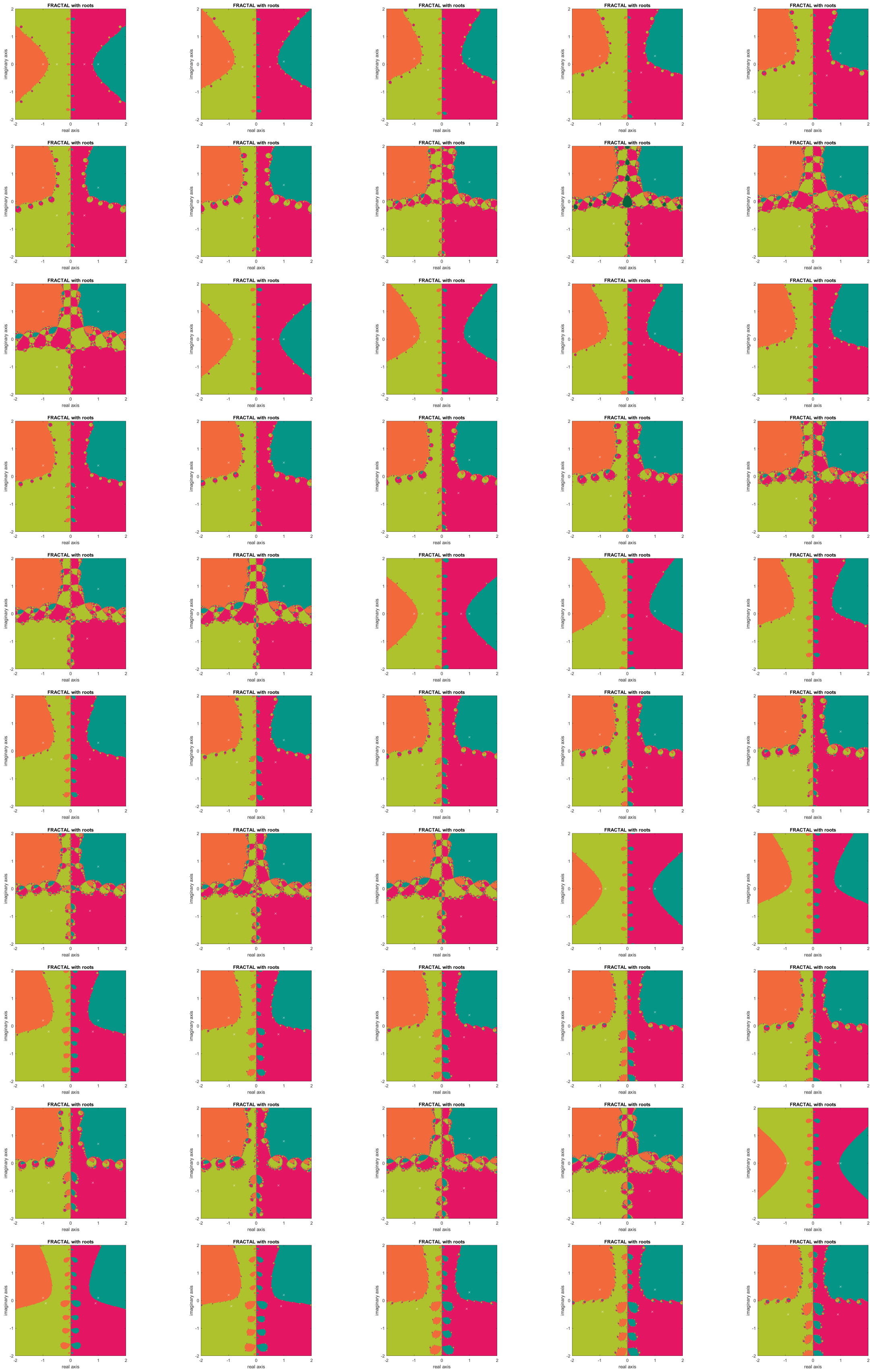
```
end

L_uint8 = cast(L, 'uint8');
Lr = zeros(ny, nx, 3); Lr(:,:,1) = L(:,:,1); Lr = cast(Lr, 'uint8');
Lg = zeros(ny, nx, 3); Lg(:,:,2) = L(:,:,2); Lg = cast(Lg, 'uint8');
Lb = zeros(ny, nx, 3); Lb(:,:,3) = L(:,:,3); Lb = cast(Lb, 'uint8');


figure(1)
imshow(L_uint8)
%%
figure(2)
subplot(1,3,1)
imshow(Lr)
subplot(1,3,2)
imshow(Lg)
subplot(1,3,3)
imshow(Lb)
%%
figure(3)
subplot(1,2,1)
imshow(A)
subplot(1,2,2)
imshow(abs(A-L_uint8))
```

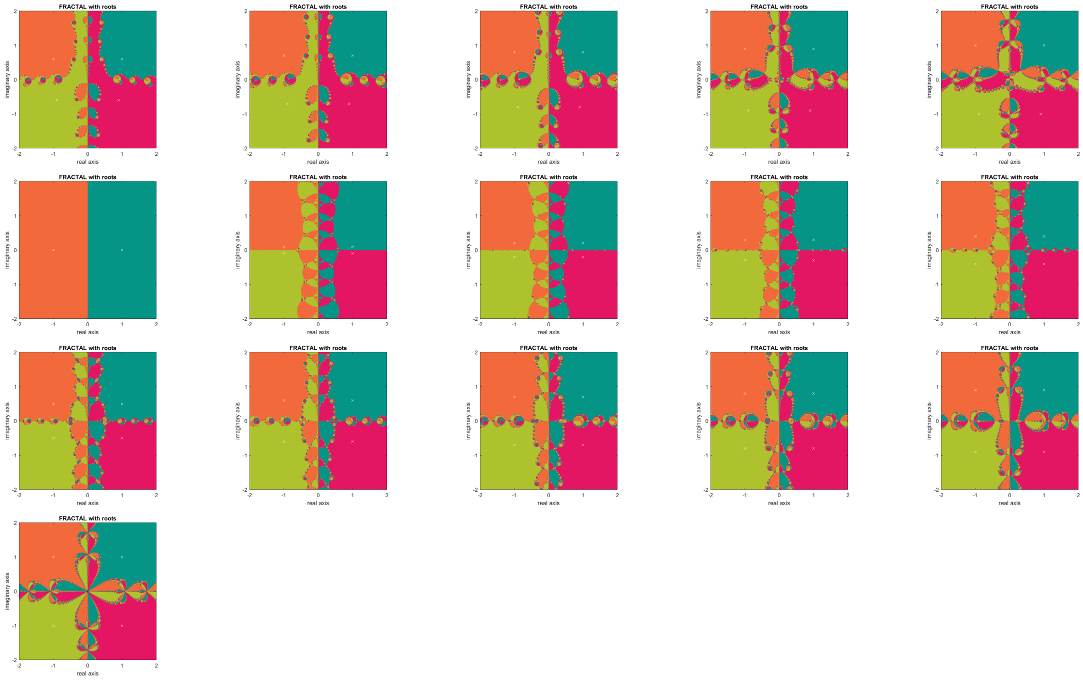
B Other Fractals



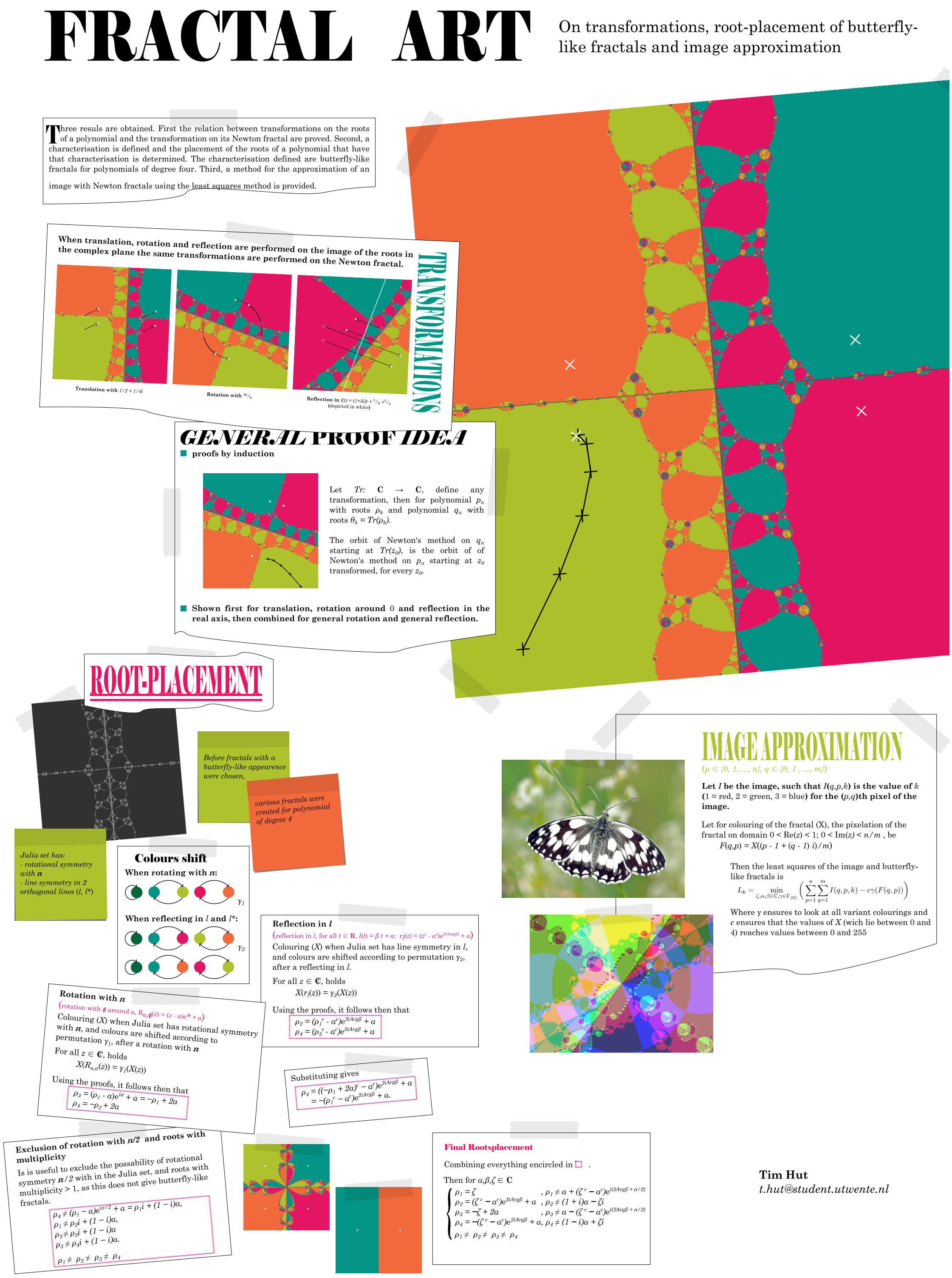

FRACTAL with roots
imaginary axis
real axis
-2
-1
0
1
2


FRACTAL with roots
imaginary axis
real axis


FRACTAL with roots
imaginary axis
real axis
-2
-1
0
1
2


FRACTAL with roots
imaginary axis
real axis

C Poster



FRACTAL ART
On transformations, root-placement of butterfly-like fractals and image approximation
Three resuls are obtained. First the relation between transformations on the roots of a polynomial and the transformation on its Newton fractal are proved. Second, a characterisation is defined and the placement of the roots of a polynomial that have that characterisation is determined. The characterisation defined are butterfly-like fractals for polynomials of degree four. Third, a method for the approximation of an image with Newton fractals using the least squares method is provided.
When translation, rotation and reflection are performed on the image of the roots in the complex plane the same transformations are performed on the Newton fractal.
TRANSFORMATIONS
Translation with $1/2 + 1/4i$
Rotation with $^{2\pi}/_5$
Reflection in $l(t) = (1+3i)t + {}^1/_2 + {}^1/_4$ (depicted in white)
GENERAL PROOF IDEA
proofs by induction
Let Tr: $\mathbf{C} \to \mathbf{C}$, define any transformation, then for polynomial p_n with roots ρ_k and polynomial q_n with roots $\theta_k = Tr(\rho_k)$.
The orbit of Newton's method on q_n starting at $Tr(z_0)$, is the orbit of of Newton's method on p_n starting at z_0 transformed, for every z_0.
Shown first for translation, rotation around 0 and reflection in the real axis, then combined for general rotation and general reflection.
ROOT-PLACEMENT
Before fractals with a butterfly-like appearence were chosen,
various fractals were created for polynomial of degree 4
Julia set has:
- rotational symmetry with π
- line symmetry in 2 orthogonal lines (l, l^*)
Colours shift
When rotating with π:
γ_1
When reflecting in l and l^*:
γ_2
Reflection in l
(reflection in l, for all $t \in \mathbf{R}$, $l(t) = \beta t + a$; $r_l(z) = (z^c - a^c)e^{2iArg(\beta)} + a$)
Colouring (X) when Julia set has line symmetry in l, and colours are shifted according to permutation γ_2, after a reflecting in l.
For all $z \in \mathbb{C}$, holds
$X(r_l(z)) = \gamma_2(X(z))$
Using the proofs, it follows then that
$\rho_2 = (\rho_1{}^c - a^c)e^{2iArg\beta} + a$
$\rho_4 = (\rho_3{}^c - a^c)e^{2iArg\beta} + a$
Rotation with π
(rotation with ϕ around a, $R_{a,\phi}(z) = (z - a)e^{i\phi} + a$)
Colouring (X) when Julia set has rotational symmetry with π, and colours are shifted according to permutation γ_1, after a rotation with π
For all $z \in \mathbb{C}$, holds
$X(R_{a,\pi}(z)) = \gamma_1(X(z))$
Using the proofs, it follows then that
$\rho_3 = (\rho_1 - a)e^{i\pi} + a = -\rho_1 + 2a$
$\rho_4 = -\rho_2 + 2a$
Substituting gives
$\rho_4 = ((-\rho_1 + 2a)^c - a^c)e^{2iArg\beta} + a$
$= -(\rho_1{}^c - a^c)e^{2iArg\beta} + a.$
Exclusion of rotation with $\pi/2$ and roots with multiplicity
Is is useful to exclude the possability of rotational symmetry $\pi/2$ with in the Julia set, and roots with multiplicity > 1, as this does not give butterfly-like fractals.
$\rho_4 \neq (\rho_1 - a)e^{i\pi/2} + a = \rho_1 i + (1 - i)a,$
$\rho_1 \neq \rho_2 i + (1 - i)a,$
$\rho_2 \neq \rho_3 i + (1 - i)a$
$\rho_3 \neq \rho_4 i + (1 - i)a.$
$\rho_1 \neq \rho_2 \neq \rho_3 \neq \rho_4$
Final Rootsplacement
Combining everything encircled in .
Then for $a,\beta,\zeta \in \mathbf{C}$
$\rho_1 = \zeta$, $\rho_1 \neq a + (\zeta^c - a^c)e^{i(2Arg\beta + \pi/2)}$
$\rho_2 = (\zeta^c - a^c)e^{2iArg\beta} + a$, $\rho_2 \neq (1 + i)a - \zeta i$
$\rho_3 = -\zeta + 2a$, $\rho_3 \neq a - (\zeta^c - a^c)e^{i(2Arg\beta + \pi/2)}$
$\rho_4 = -(\zeta^c - a^c)e^{2iArg\beta} + a$, $\rho_4 \neq (1 - i)a + \zeta i$
$\rho_1 \neq \rho_2 \neq \rho_3 \neq \rho_4$
IMAGE APPROXIMATION
($p \in \{0, 1, \ldots, n\}$, $q \in \{0, 1, \ldots, m\}$)
Let I be the image, such that $I(q,p,k)$ is the value of k (1 = red, 2 = green, 3 = blue) for the (p,q)th pixel of the image.
Let for colouring of the fractal (X), the pixelation of the fractal on domain $0 < \mathrm{Re}(z) < 1$; $0 < \mathrm{Im}(z) < n/m$, be
$F(q,p) = X((p - 1 + (q - 1)\, i)/m)$
Then the least squares of the image and butterfly-like fractals is
$L_k = \min_{\zeta,\alpha,\beta\in\mathbb{C},\gamma\in\Gamma_{(0)}} \left(\sum_{p=1}^{n}\sum_{q=1}^{m} I(q,p,k) - c\gamma(F(q,p)) \right)$
Where γ ensures to look at all variant colourings and c ensures that the values of X (wich lie between 0 and 4) reaches values between 0 and 255
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