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The Price of Anarchy for Matching Congestion Games

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Abstract

In this thesis we mainly study matching congestion games with identity cost functions and bipartite graphs. A congestion game is a game where a finite set of players individually choose a strategy which consists of a subset of a finite set of resources. The cost of a resource depends on the number of players using it and the cost of each player is the sum over the cost of every resource a player uses. In a matching congestion game resources are represented by the edges of a graph and the edges in every strategy form a perfect matching. We are interested in the price of anarchy; the total cost of the worst equilibrium when agents choose selfishly their own strategy relative to the total cost of a strategy profile that minimizes total cost. With p the number of players, we found that the upper bound is equal to $2 - 1/p$ for two, three and four players. For five or more players the best bound is still the bound which was already known, namely $5/2$. For a special case, when the graph allows as many disjoint matchings as there are players, our proof shows that the price of anarchy is at most $2 - 1/p$. For this special case this bound also holds for five or more players. We found a lower bound example, which also holds for the special case, where the price of anarchy is equal to $2 - 1/p$ for two players, which makes the bound tight.

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Chapter 1

Introduction

A congestion game, first introduced by Rosenthal [17], is a game where a number of selfish agents chooses simultaneously from a set of strategies. Each strategy consists of resources and the cost of a player depends on the number of players using the same resource. To our knowledge there is no study yet about matching congestion games, which we study in this thesis. In matching congestion games each game is represented in some bipartite graph $G = (V_1 \cup V_2, E)$ with p players and where every edge of the graph represents a resource. Every player chooses resources which form a perfect matching. Matching congestion games are a subclass of general congestion games.

The price of anarchy, first introduced by Koutsoupias and Papadimitriou [15], shows the inefficiency that can occur when selfish agents choose their own strategy (a set of resources). More precisely, the price of anarchy measures the efficiency of a game by comparing the worst Nash equilibrium with the optimal solution [3]. The higher the ratio, the the higher the potential loss of efficiency. It is interesting to know how high this ratio can be, because it is not always easy or without cost to let agents choose a strategy which leads to an optimal solution.

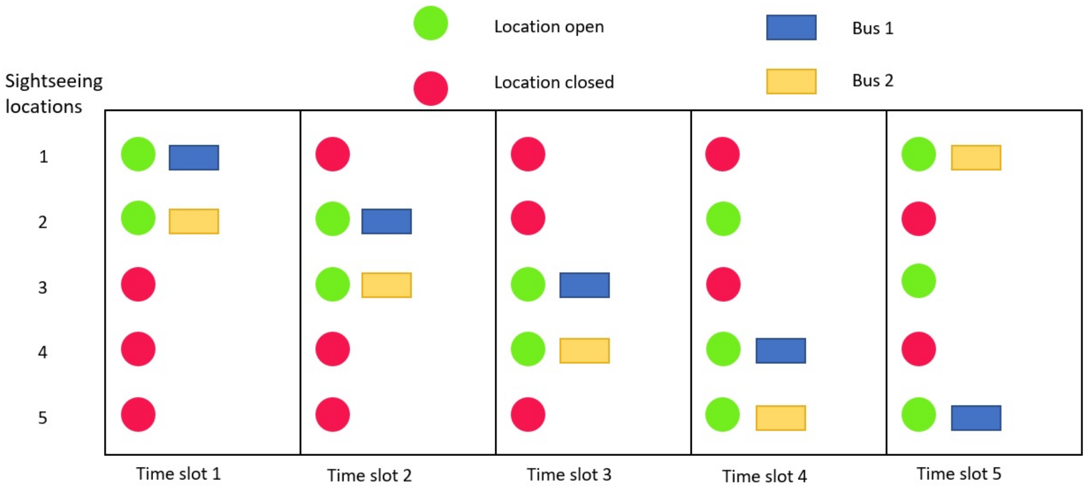
Christodoulou and Koutsoupias [3] proved that the price of anarchy for symmetric general congestion games with affine cost functions is $(5p-2)/(2p+1)$, with p being the number of players. A subclass of general congestion games are for example congestion games with restrictions on the strategy spaces such that every player needs to choose exactly n resources, this is called an n-uniform congestion game, this is studied by de Jong et al. [7], [6]. Another subclass of congestion games are network congestion games. For this type of congestion games, every player's strategy space consists of every path possible between a source and a sink node. In this thesis we study the price of anarchy for symmetric *matching congestion games*. Matching congestion games are defined as congestion games where the players (agents) choose from the edges of a bipartite graph such that the edges chosen by any player form a perfect matching (for server-matching or semi-matching congestion games, where every player is matched to one server, see Kothari et al. [13] and Harvey et al. [10]). Matching congestion games are a subclass of congestion games and therefore we know that the upper bound on the price of anarchy for matching congestion games is at most $(5p-2)/(2p+1)$. The main question we are studying here is whether the price of anarchy for symmetric matching congestion games is even smaller than for general symmetric congestion games and whether this bound is tight, i.e. if we can find a lower bound example such that the bound is tight.

Next, we motivate matching congestion games on the basis of an example.

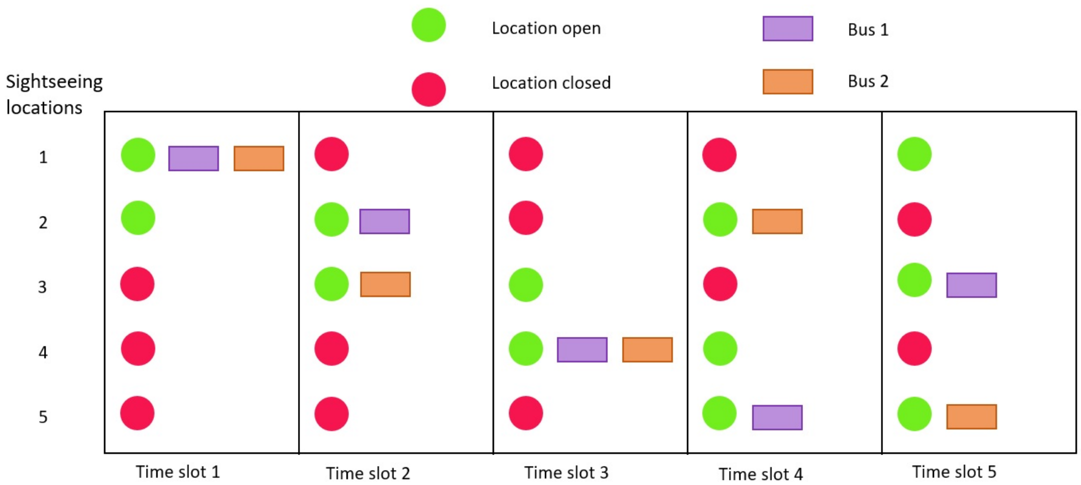
Example: Assume a cruise ship docks in a city and a number of buses are filled with tourists from the cruise ship. Those buses have to visit all the sight seeing locations of that city. Clearly, a bus can only be at one place at a time. The problem is described as follows: every bus has to choose at which time slot it will be at a certain location and the bus chooses this for all the sight seeing locations. All sight seeing locations need to be visited by all buses, therefore this can be seen as a matching congestion problem. Assuming that the number of time slots is equal to the number of places to visit, a solution where one bus visits all locations once is a perfect matching. This can be described as a bipartite graph where time slots are the nodes on one side of the graph and the locations are the nodes on the other side of the graph. The edges are the possibilities of a bus to visit a location at a certain time.

For illustration see Figure 1.1 where an example with two buses, five sightseeing locations and five time slots is shown. An optimal solution scheme and a Nash equilibrium is given in Figure 1.1a and in Figure 1.1b respectively. The situation mentioned above can be described by a bipartite graph, this is shown in Figure 1.1c. The graph in green indicates all different possibilities of a bus to visit a location at a certain time slot, where s are the different sightseeing locations and t the different time slots. The matchings in blue, yellow, purple and orange correspond to the matchings in the schemes of the buses in the same color.

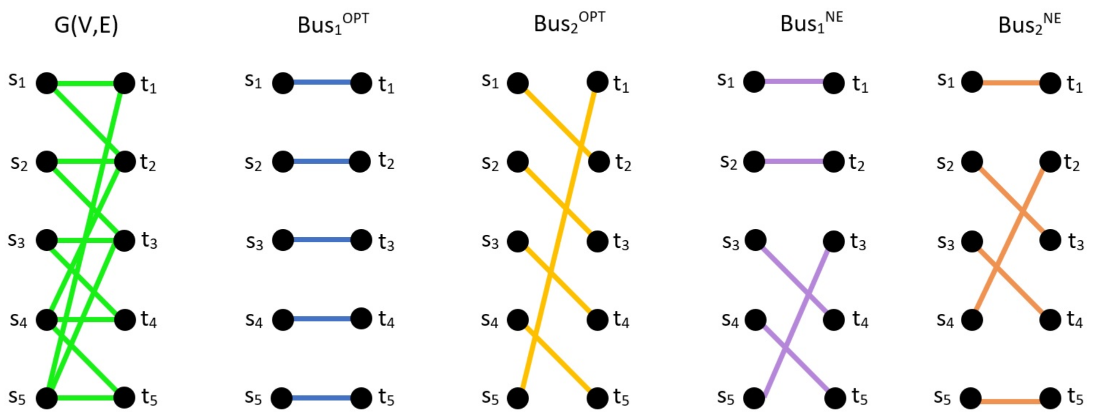
To see why Figure 1.1a and Figure 1.1b are an optimal solution and a Nash equilibrium respectively, the objective needs to be stated. A strategy profile is an optimal solution when the total cost of all buses is minimized. The cost for each bus can be measured in the total waiting time the bus needs to wait at every stop. When there are multiple buses at the same time at the same place, the time it takes to visit the locations increases. For simplicity, we assume when p buses are at the same location at the same time, every of those buses have to wait p time units. This means that the total cost of those p buses for that time slot is p^2 time units. If those p buses were each alone at different locations at the same time slot, the cost for each of those p buses at that time slot would be 1 time unit. In total at that time slot the cost for those buses would be p time units. It might happen when the buses choose selfishly their own schedule that they are in an equilibrium (no player wants to switch it's



(a) Scheme of an optimal solution



(b) Scheme of a Nash equilibrium



(c) Graph of the example (green) and all four different possible matchings (blue, yellow, purple and orange)

Figure 1.1: Example of a matching congestion game

strategy while other players do not change their strategy), but with higher total cost than when the bus schedules where chosen optimally. This means that not every equilibrium has to be optimal.
In Figure 1.1a, no buses are at the same location at the same time. However, in Figure 1.1b, there are two sightseeing locations where the buses are at the same time. This is at sightseeing locations 1 and 4 at time slots 1 and 3 respectively. When, for example bus 1 changes its location at time slot 1 from location 1 to location 2, it also has to change it's location at time slot 2, as it is again at location 2. It can be checked that it does not matter how bus 1 changes its schedule, it will always have overlap with bus 2 at two time slots. The same applies for bus 2, hence they are in an equilibrium, however it is not an optimal solution.
The results we found for the upper bound on the price of anarchy for matching congestion games with identity cost functions and a given bipartite graph is equal to $2 - 1/p$ for two, three and four players. For five or more players we were not able to find a better bound than $\frac{5p-2}{2p+1}$. We found a lower bound of $2 - 1/p$ for two players. For three or more players we were not able to find a lower bound higher than 1. This means that the bound of $2 - 1/p$ is only tight for two players. For a special case, when the graph allows as many disjoint matchings as there are players, we found also an upper bound of $2 - 1/p$ and this bound holds for any number of p players. The same lower bound holds for this special case and therefore also here the bound of $2 - 1/p$ is tight for two players.
The outline of this report is as follows; in Chapter 2 we explain some of the game theory definitions and show the mathematical notations used in this report. Furthermore, we state our results and discuss related work. Then in Chapter 3 we analyse the price of anarchy for n-uniform congestion games with identity cost functions. Subsequently, in Chapter 4 we study the price of anarchy for matching congestion games. Thereafter, in Chapter 5 we give the proof of finding a Nash equilibrium in matching congestion games in polynomial time. Finally, we end with Chapter 6 where we state the conclusions and ideas for further research.

Chapter 2

Preliminaries

In this chapter we will explain some important definitions that we use in this thesis. Next, we will give a list of important notations. We end this chapter with a brief section on our results and related work.

2.1 Definitions

In this thesis we study congestion games. A *congestion game*, first introduced by Rosenthal [17], is a game where a finite number of p players choose selfishly simultaneously from a finite set of resources E and with P the set of all players. For each player $i \in P$, a subset $\mathcal{S}_i \subseteq 2^E$ of strategies is given. When players can all choose from the same set of strategies, the game is called *symmetric*. When not all players have the same set of strategies to choose from, i.e., $\mathcal{S}_i \neq \mathcal{S}_j$ for $i \neq j$, the game is called *asymmetric*.

Definition 2.1.1 (Symmetric game)**.** *A symmetric game is a game where all players have access to the same set of strategies;* $\mathcal{S}_i = \mathcal{S}_j \; \forall i, j \in P$

Definition 2.1.2 (Asymmetric game)**.** *An asymmetric game is a game where there are players which do not have access to the same set of strategies;* $\mathcal{S}_i \neq \mathcal{S}_j$ *for some* $i \neq j \in P$

Definition 2.1.3 (Strategy profile)**.** *A Strategy profile is a vector of the strategies chosen by each player:* $S = (S_1, S_2, \ldots, S_p)$*.*

Every strategy S_i is a subset of *resources*, $S_i \subseteq E$ and because the resources are often the edges of a graph they are denoted as $e \in E$, where E is the set of all resources. When all players have chosen their strategy the cost can be calculated. The cost function of a resource depends on the number of players using this resource. We define $x_e(S) = |\{i \in P : e \in S_i\}|$ as the number of players that have chosen resource e for strategy profile S. The cost functions can for example be identity, linear or affine.

Definition 2.1.4 (Identity cost function)**.** *Identity cost functions are cost functions of the form:* $c_e(x_e) = x_e$*, with* $e \in E$*.*

Definition 2.1.5 (linear cost function)**.** *Linear cost functions are cost functions of the form:* $c_e(x_e) = a_e x_e$*, with* $e \in E$ *and where* a_e *is nonnegative.*

Definition 2.1.6 (affine cost function)**.** *Affine cost functions are cost functions of the form:* $c_e(x_e) = a_e x_e + b_e$*, with* $e \in E$ *and where* a_e *and* b_e *are nonnegative.*

Identity cost functions are a subclass of linear cost functions and linear cost functions are a subclass of affine cost functions. Using the cost of the resources for a strategy profile S the cost for each player can be determined. The cost for each player for a strategy profile S is equal to the sum of the costs of the resources it uses, $c_i(S) = \sum_{e \in S_i} c_e(x_e(S))$, $\forall i \in P$. For the total or social cost of the strategy profile S we sum over all players' individual cost, $cost(S) = \sum_{i=1}^{p} c_i(S) = \sum_{i=1}^{p} \sum_{e \in S_i} c_e(x_e(S))$. Now we can give a more formal definition of a congestion game which is defined by multiple aspects and can therefore be defined in a tuple.

Definition 2.1.7 (Congestion game)**.** *A congestion game* M *can be defined as a tuple* $M = (P, E, (\mathcal{S}_i)_{i \in P}, (c_e)_{e \in E})$*, where* $P = \{1, 2, \ldots, p\}$ *denotes the players,* $E = \{1, 2, \ldots, r\}$ *the set of resources,* $\mathcal{S}_i \subseteq 2^E$ *denotes the set of strategies for player* $i \in P$*, where each strategy* $S_i \in \mathcal{S}_i$ *is a set of resources and finally,* c_e *is the cost function for resource* $e \in E$*.*

There are different classes of congestion games, for example, there are network, n-uniform and singleton congestion games.

Definition 2.1.8 (Network congestion games)**.** *A Network congestion game is a congestion game where the resources are given by the edges of a given directed graph,* $D = (V, A)$ *with* A *the set of all directed edges. For every player a source and sink node are given,* $s_i, t_i \in V$ *and* $i \in P$*. The set of strategies for every player* i*,* $\mathcal{S}_i$*, consists of all directed* (s_i, t_i)*-paths in* D*, where every strategy consists of a different* (s_i, t_i)*-path. When every player has the same source and sink node, it is a symmetric network congestion game.*

Definition 2.1.9 (n-uniform congestion games)**.** *An* n*-uniform congestion game is a congestion game where every player has to choose any n resources out of the total set of resources* E*;* $\mathcal{S}_i = \{S_i \subseteq E : |S_i| = n\}$ *.*

Definition 2.1.10 (Singleton congestion games)**.** *A singleton congestion game is a congestion game where players only choose one resource out of the set of all resources* E*;* $\mathcal{S}_i = \{S_i \subseteq E : |S_i| = 1\}$*.*

Note that singleton congestion games are a subclass of the n-uniform congestion games. Another congestion game which is the main subject of this thesis are the *matching congestion games* where each player chooses a strategy where the resources must form a perfect matching.

Definition 2.1.11 (Matching congestion games)**.** *A matching congestion game is a congestion game where the resources of every strategy of a player form a perfect matching in a given undirected graph* $G = (V, E)$*.*

In this thesis we use *bipartite* graphs and we only use graphs $G(V_1 \cup V_2, E)$ with $|V_1| = |V_2| = n$. This means for matching congestion games, that every player has to choose n resources which form a perfect matching. Because the resources need to form a perfect matching, a player cannot choose from every subset of n resources. Therefore, matching congestion games are not a subclass of n-uniform congestion games.
Furthermore, we study the price of anarchy for matching congestion games which is the ratio of the total cost of the worst Nash equilibrium compared to the total cost of a socially optimal solution. First, a Nash equilbrium is defined and later a socially optimal solution is defined.

Definition 2.1.12 (Nash equilibrium)**.** *A Nash equilibrium is defined as a strategy profile* $S = (S_1, S_2, \dots, S_p)$ *where no player has an incentive to deviate from their strategy, this means* $c_i(S) \leq c_i(S'_i, S_{-i})$*, for all* $S'_i \in \mathcal{S}_i$ *and all* $i \in P$*.*

Here (S'_i, S_{-i}) means that only player i switches from strategy S_i to some other strategy $S'_i \in \mathcal{S}_i$ for a given instance M. A Nash equilibrium strategy profile is denoted by $S^{NE}(M)$ and the set of all profiles which are a Nash equilibrium of a given instance M is denoted by $\mathcal{A}^{NE}(M)$.
A socially optimal solution can be defined in different ways. Christodoulou and Koutsoupias [3] used the maximum and the average player costs as the social cost, however for the average player costs they used for simplicity the sum of all player costs (which is equal to p times the average costs). The minimum total players cost is what we use in this thesis as a socially optimal solution, which we refer to as an optimal solution. The socially optimal strategy profile for a given instance M is defined as $S^{OPT}(M)$, note that $cost(S^{OPT}(M)) \leq cost(S(M))$, for all strategy profiles S of the instance M.
We will now define the price of anarchy, however we have to distinguish between the price of anarchy of a single game M and the price of anarchy of the class of all matching congestion games $\mathcal{M}$.

Definition 2.1.13 (Price of anarchy of congestion game M)**.** *The price of anarchy of a congestion game is defined as the ratio of the cost of the worst Nash equilibrium to the cost of the optimal solution of the game,*

$$PoA(M) := \max_{S \in \mathcal{A}^{NE}(M)} \frac{cost(S)}{cost(S^{OPT}(M))}.$$

Definition 2.1.14 (Price of anarchy of class of congestion games $\mathcal{M}$)**.** *The Price of anarchy of a given class of congestion games $\mathcal{M}$ is the highest price of anarchy over all congestion games in that class. In other words, the highest ratio of the cost of a Nash equilibrium to the cost of an optimal solution over all instances of the given class of congestion games.*

$$PoA(\mathcal{M}) := \sup_{M \in \mathcal{M}} PoA(M).$$

In the next section we will give a brief overview in the form of a list of the notations we defined here.

2.2 List of notations

In this section we summarize the important symbols with the definitions. Which can be of later use while reading this report.

$P = \{1, 2, \cdots, p\}$	Denotes the set of agents/players,
$E = \{1, 2, \cdots, r\}$	Denotes the set of resources,
$(\mathcal{S}_i)_{i \in P} \subseteq 2^E$	Is the set of pure strategies for player i,
$S_i \in \mathcal{S}_i$	Set of resources chosen by player i; for matching congestion games they form a perfect matching,
$S = (S_1, S_2, \cdots, S_p)$	A strategy profile which is a vector consisting of the strategy chosen by each player,
$(c_e)_{e \in E}$	The cost functions per resource $e \in E$,
$c_i(S) := \sum_{e \in S_i} c_e(x_e(S))$	The cost of player i, for a strategy profile $S = (S_1, S_2, \dots, S_p)$,

$x_e(S)$	Denotes the number of players using resource e in strategy profile S,
$cost(S) := \sum_{i \in P} c_i(S)$	Is the total cost of strategy profile S,
(S_i, S_{-i})	Strategy profile where player i plays S_i and all other players play $S_{-i} = (S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_p)$,
S^{NE}	Denotes a strategy profile which is a Nash equilibrium; no single player has an incentive to deviate from S^{NE},
S^{OPT}	Denotes a strategy profile which is socially optimal meaning that it has lowest total cost; $cost(S^{OPT}) \leq cost(S)$ for any strategy profile S,
$\mathcal{A}^{NE}(M)$	The set of all strategy profiles which are a Nash equilibrium of a given instance M,
$PoA(M) := \max_{S \in \mathcal{A}^{NE}(M)} \frac{cost(S)}{cost(S^{OPT}(M))}$	Denotes the price of anarchy of a congestion game M,
$PoA(\mathcal{M}) := \sup_{M \in \mathcal{M}} PoA(M)$	Denotes the price of anarchy for $\mathcal{M}$, a given class of congestion games.

2.3 Our results

We mainly studied the price of anarchy for symmetric matching congestion games with identity cost functions. First we proved that for symmetric n-uniform congestion games with identity cost functions the price of anarchy is always equal to 1. In matching congestion games every player chooses a subset of n resources. However, since resources of every strategy set in matching congestion games need to form a matching, this upper bound of 1 on the price of anarchy will not necessarily hold for matching congestion games. In our proofs on the upper bound on price of anarchy for matching congestion games we actually did not use the restriction that resources in every strategy need to form a matching and therefore the upper bound on the price of anarchy also holds for n-uniform congestion games. However, the upper bound we proved is higher than the upper bound of 1 for symmetric n-uniform congestion games with identity cost functions.
We also studied a special case for matching congestion games. For this special case the graph allows as many disjoint matchings as there are players. For this special case we proved that the upper bound on the price of anarchy is $2 - 1/p$, with p the number of players. We also showed that this bound is asymptotically tight for $p = 2$. However for $p \geq 3$ we did not find any example with a lower bound significantly higher than 1. In the general case, where an optimal solution with p disjoint matchings in the bipartite graph does not necessarily exist, the upper bound on the price of anarchy is also $2 - 1/p$ for two, three and four players. For five or more players we were not able to prove a better upper bound than the previously known bound proven by Christodoulou and Koutsoupias [3] which is $(5p - 2)/(2p + 1)$. The same lower bounds holds and therefore the bound of $2 - 1/p$ is asymptotically tight for two players for the special case as well.

2.4 Related work

The price of anarchy was first introduced by Koutsoupias and Papadimitriou [15]. They studied simple network routing congestion games with m parallel links from a source to a target node with affine functions and a different, namely min-max social objective. They showed the lower bound for this problems to be $\Omega(\log m/\log\log m)$ and this bound is proved to be $\Theta(\log m/\log\log m)$ by Czumaj and Vöcking [5] and at the same time by Koutsoupias et al. [14].
From the study by Christodoulou and Koutsoupias [3] we know that price of anarchy for finite congestion games with affine cost functions and minimal total cost as social objective, is equal to $5/2$ for asymmetric congestion games with $p \geq 3$ and equal to 2 for $p = 2$. For symmetric congestion games with affine cost functions they proved

the price of anarchy equal to $(5p-2)/(2p+1)$. For symmetric network routing congestion games the bound $(5p-2)/(2p+1)$ was proved to be tight by Correa et al. [4] and for asymmetric network routing congestion games Christodoulou and Koutsoupias [3] showed that the bound of $5/2$ for the price of anarchy holds tight for $p \geq 3$. Moreover, in the study of de Jong et al. [7] the upper bound on the price of anarchy in symmetric uniform congestion games was proven to be bounded from above by $28/13 \approx 2.154$ and they found a lower bound of $7-4\sqrt{2} \approx 1.343$ for p (the number of players) large enough, so the bound $(5p-2)/(2p+1)$ is not tight for these games. This means that there is still a gap which can be improved.
For asymmetric singleton congestion games the upper bound of $5/2$ from Christodoulou and Koutsoupias [3] is proven to be tight by Caragiannis et al. [2]. For asymmetric singleton congestion games with identity cost functions Suri et al. [18] found a lower bound of approximately 2.0120669 and an upper bound of $(1+2\sqrt{3}) \approx 2.1547$. However, Caragiannis et al. [2] found an upper bound of approximately 2.012067 which is almost tight to the lower bound in the study of Suri et al. [18]. For symmetric singleton congestion games, Lücking et al. [16] proved the price of anarchy is equal to $4/3$.
In this thesis we also address the algorithmic complexity of computing a Nash equilibrium. We quickly summarize what is known. For general congestion games, symmetric congestion games and asymmetric network congestion games Fabrikant et al. [9] proved that finding a Nash equilibrium is PLS-complete. PLS (polynomial-time local search) is defined by Papadimitriou and Yannakakis [12]. They define a problem to be PLS when local optimality can be verified in polynomial time. Ieong et al. [11] proved that for singleton congestion games Nash equilbria can be found in polynomial time. Ackermann et al. [1] went further and proved that for matroid congestion games the best response sequences are polynomially bounded in the number of players and resources. A matroid congestion game is defined as a congestion game where the strategy set of each player corresponds to the set of bases of a matroid. Moreover, they showed that the matroid property of the players' strategy sets is necessary and sufficient to guarantee a polynomial time convergence to a Nash equilibrium. We show in Chapter 5 that a Nash equilibrium of a matching congestion game can be computed in polynomial time. We use the idea of Fabrikant et al. [9] and transfer it to matching congestion games. The idea of Fabrikant et al. that we use also here, is reducing the problem of finding a social optimal solution in a symmetric network congestion game to a min-cost flow problem. They did this by replacing every edge in a symmetric network congestion game by p edges with the cost of the edges non decreasing in p and they added a source and sink node. Del Pia et al. [8] also showed that finding a Nash equilibrium for symmetric matching congestion games can be done in polynomial time. However, they reduced the

problem to a different problem as the strategy sets of the players are defined as matchings and not necessarily perfect matchings.

Chapter 3

PoA for n-uniform congestion games with identity cost functions

In this chapter we give our main results on the price of anarchy of symmetric n-uniform congestion games with identity cost functions.

We begin with a simple lemma that states that in equilibrium, an n-uniform congestion game with identity cost functions, the number of players on resources are as equally spread as possible. Subsequently, we show, by using this lemma, that the price of anarchy is always equal to 1 for these games.

Lemma 3.0.1. *In a symmetric n-uniform congestion game with identity cost functions, r resources and p players, in every Nash equilibrium all resources are used by $\lceil \frac{np}{r} \rceil$ or $\lfloor \frac{np}{r} \rfloor$ players, with $r, p, n \in \mathbb{Z}_{>0}$.*

Proof. We are going to prove Lemma 3.0.1 by means of a contradiction. First we prove that there cannot be a resource used by more than $\lceil \frac{np}{r} \rceil$ players in a Nash equilibrium of an n-uniform congestion game and we also prove that there cannot be a resource which is used by less than $\lfloor \frac{np}{r} \rfloor$ players. We show that for a strategy profile in which there is a resource with more than $\lceil \frac{np}{r} \rceil$ or less than $\lfloor \frac{np}{r} \rfloor$ players using this resource, there always exists a player who wants to switch from strategy. Therefore, the strategy profile cannot be a Nash equilibrium.
Consider a Nash equilibrium strategy profile S^{NE}, r resources and p players. Furthermore, assume that there is a resource r' which is used by more than $\lceil \frac{np}{r} \rceil$ players. This means

$$x_{r'}(S^{NE}) > \left\lceil \frac{np}{r} \right\rceil .$$

In a symmetric n-uniform congestion game every player has access to the same set of strategies and every strategy contains n resources. Therefore, the sum of the number of players using a resource e over all resources is equal to a total of np. The mean of the number of players over all resources except resource r' is the total number of players on all resources subtracted by the number of players on resource r' and divided by the remaining $r-1$ resources. This means that the mean occupation rate of all other resources is

$$\begin{aligned}
\frac{1}{r-1}\sum_{e\in E\setminus r'} x_e(S^{NE}) &= \frac{np - x_{r'}(S^{NE})}{r-1} \\
&< \frac{np - \lceil \frac{np}{r} \rceil}{r-1} \\
&\leq \frac{np - \frac{np}{r}}{r-1} \\
&= \frac{(r-1)np}{r(r-1)} \\
&= \frac{np}{r} \\
&\leq \left\lceil \frac{np}{r} \right\rceil .
\end{aligned}$$

The first inequality comes from the assumption that the number of players using resource r' is strictly more than $\lceil \frac{np}{r} \rceil$. The second and last inequality hold because $\frac{np}{r} \leq \lceil \frac{np}{r} \rceil$ by definition of a ceiling function.
We proved that the mean of the number of players on a resource is strictly less than $\lceil \frac{np}{r} \rceil$. Hence, there must be a resource, r'', which is used by strictly less than $\lceil \frac{np}{r} \rceil$ players. This means that $x_{r''}(S^{NE}) \leq \lceil \frac{np}{r} \rceil - 1$. Therefore, the number of players on resource r' is strictly larger than the number of players on resource r'' plus one player,

$$x_{r'} > \left\lceil \frac{np}{r} \right\rceil \geq x_{r''}(S^{NE}) + 1.$$

Consequently, there is a player which uses resource r' and not resource r''. As the game is symmetric, this player can switch from r' to r'' and the cost of this player decreases with at least 1. Hence, this violates the assumption of S^{NE} being a Nash equilibrium. $\lightning$

We still have to prove that a resource cannot have less than $\lfloor \frac{np}{r} \rfloor$ players using this resource. Consider another Nash equilibrium strategy profile A^{NE} in a symmetric n-uniform congestion game. Furthermore, assume now that there is a resource which is used by less than $\lfloor \frac{np}{r} \rfloor$ players, call this resource r^*. This means

$$x_{r^*}(A^{NE}) < \left\lfloor \frac{np}{r} \right\rfloor .$$

The mean occupation rate of all other resources is now

$$\begin{aligned}\frac{1}{r-1}\sum_{e\in E\setminus r^*} x_e(A^{NE}) &= \frac{np - x_{r^*}}{r-1}\\ &> \frac{np - \lfloor\frac{np}{r}\rfloor}{r-1}\\ &\geq \frac{np - \frac{np}{r}}{r-1}\\ &= \frac{(r-1)np}{r(r-1)}\\ &= \frac{np}{r}\\ &\geq \left\lfloor\frac{np}{r}\right\rfloor.\end{aligned}$$

The first inequality comes from the assumption that the number of players using resource r^* is strictly less than $\lfloor\frac{np}{r}\rfloor$. The second and last inequality holds by definition of a floor function ($\frac{np}{r} \geq \lfloor\frac{np}{r}\rfloor$).
We just proved that the mean occupation rate of all other resources is strictly larger than $\lfloor\frac{np}{r}\rfloor$. Therefore, there must be a resource r^{**}, which is used by strictly more than $\lfloor\frac{np}{r}\rfloor$ players,

$$x_{r^{**}}(A^{NE}) \geq \lfloor\frac{np}{r}\rfloor + 1$$

and because $x_{r^*}(A^{NE}) < \lfloor\frac{np}{r}\rfloor$ the following hold

$$\begin{aligned} x_{r^*}(A^{NE}) < \left\lfloor\frac{np}{r}\right\rfloor \leq x_{r^{**}}(A^{NE}) - 1\\ \Rightarrow\ x_{r^{**}}(A^{NE}) > x_{r^*}(A^{NE}) + 1.\end{aligned}$$

The the number of players using resource r^{**} is strictly larger than the number of players using resource r^* plus one player. This means that there is a player that is using resource r^{**} and not resource r^* and when this player switches from resource r^{**} to resource r^* the cost of this player decreases with at least 1. Hence, this violates the assumption of A^{NE} being a Nash equilibrium. $\lightning$
Consequently, resources can only be used by $\lfloor\frac{np}{r}\rfloor$ or by $\lceil\frac{np}{r}\rceil$ players. This proves Lemma 3.0.1. $\square$

As the result of 3.0.1 it is easy to see that all Nash equilibria are equal to each other because of symmetry and hence are all optimal. We use this to prove our main theorem, namely that the price of anarchy is always 1.

Theorem 3.0.2. *The price of anarchy for symmetric n-uniform congestion games with identity cost functions is always equal to 1.*

Proof. It is easy to see that a solution where all resources are used by the same amount of players or one less (which is described above) is an optimal solution. The total cost cannot be lower than when each player chooses n resources and for all players together this is equally divided over all resources. Hence, all Nash equilibria in symmetric n-uniform congestion games with identity cost functions with a total of r resources and p players are always an optimal solution and therefore the price of anarchy is always equal to 1. $\square$

In the next sections we study the price of anarchy of symmetric matching congestion games. As matching congestion games are not a subclass of n-uniform congestion games, because not every set of n resources form a perfect matching which is required for a matching congestion game. This means that the lower bound examples do not hold for n-uniform congestion games. For the upper bound proofs no matching constraint is used. Therefore, these proofs do hold for both n-uniform as for matching congestion games. However, we already proved an upper bound of 1 (see Theorem 3.0.2) for the symmetric n-uniform congestion games (with identity cost functions). So the upper bound proves are redundant for n-uniform congestion games. We also show that the lower bound examples of matching congestion games do not necessarily hold for n-uniform congestion games, see Observation 4.1.2.

Chapter 4

PoA for matching congestion games with identity cost functions

In this chapter we will study the lower and upper bound on the price of anarchy for matching congestion games with identity cost functions. In Section 4.1 we will give lower bound examples for two players. Next, in Section 4.2 we will study the upper bound with two players and lastly, in Section 4.3 we will study the upper bound with p players.

4.1 Lower bound on the PoA for two players

In this section we will give a few examples of a matching congestion game with a given bipartite graph. We will show that when n, the size of the graph, goes to infinity, the price of anarchy goes to $3/2$.

Lemma 4.1.1. *The lower bound on the price of anarchy for matching congestion games with a bipartite graph, two players and identity cost functions goes asymptotically to* $2 - \frac{1}{p} = \frac{3}{2}$ *when* n *goes to infinity.*

Proof. Assume a bipartite graph $G = (V_1 \cup V_2, E)$, with $|V_1| = |V_2| = n$. The cost functions are identity, therefore $c_e(x_e) = x_e \ \forall e \in E$. We will give three examples, namely for $n = 5$, $n = 9$ and $n = 13$ and use these examples to provide a general example.
First, we give an example for $n = 5$, see Figure 4.1. It is easy to check that OPT^a, OPT^b, NE^a and NE^b are the only perfect matchings in $G(V, E)$.
Next, we show that the Nash condition holds for $NE = (NE^a, NE^b)$. In Figure 4.1 in NE the players have two edges in common and these edges have therefore cost 2. Both players have three edges not in common and these edges have therefore cost 1. Consequently, the cost for each player in NE is equal to 7. The same cost for

each player holds when player 1 switches to OPT^a or to OPT^b (note, when player 1 switches to NE^b the cost for both players increases).

$$\begin{aligned} c_1(NE) &= \sum_{e \in E} c_e(x_e(NE)) = 7 \\ &= c_1(OPT^a, NE_{-1}) \\ &= c_1(OPT^b, NE_{-1}). \end{aligned}$$

Because of symmetry, the same holds for $c_2(NE)$. This means that the Nash conditions are fulfilled for strategy profile NE. The price of anarchy for this example is:

$$\begin{aligned} PoA = \frac{cost(NE)}{cost(OPT)} &= \frac{\sum_{i=1}^{2} c_i(NE)}{\sum_{i=1}^{2} c_i(OPT)} \\ &= \frac{\sum_{i=1}^{2} \sum_{e \in NE_i} c_e(x_e(NE))}{\sum_{i=1}^{2} \sum_{e \in OPT_i} c_e(x_e(OPT))} = \frac{14}{10} = 1.4 \end{aligned}$$

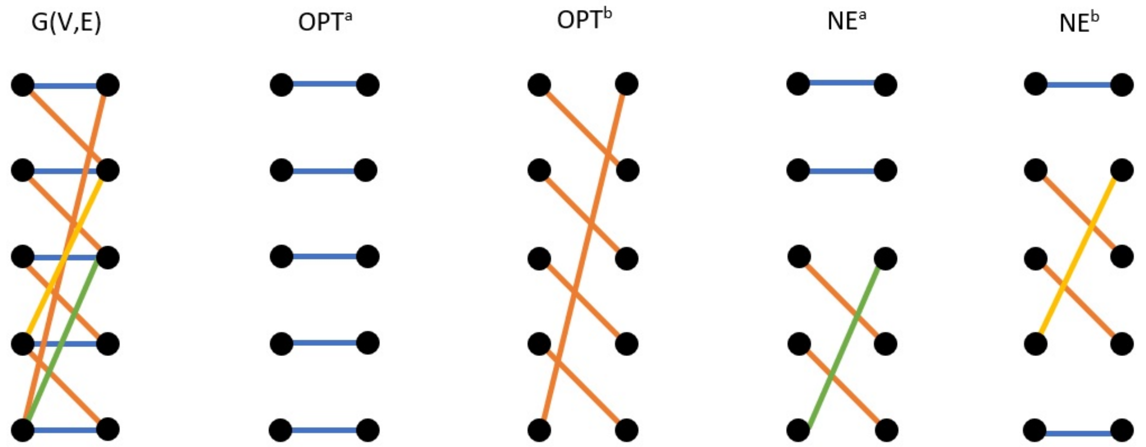


Figure 4.1: Lower bound example on the price of anarchy in a matching congestion game for 2 players, $n = 5$

We use this lower bound example to show that the lower bound examples of matching congestion games do not necessarily hold for n-uniform congestion games.

Observation 4.1.2. *In Figure 4.2 two strategies are shown which players can choose from if the example in Figure 4.1 is an n-uniform congestion game. It is easy to see that, if for example player 1 chose NE^a and player 2 chose NE^b (see 4.1) and when player 1 switches to strategy S'' or if player 2 switches to strategy S' (see Figure 4.2 for strategies S' and S''), the cost decreases of the player who switched strategy. Therefore, in an n-uniform congestion game $NE = (NE^a, NE^b)$ is no longer a Nash equilibrium. Other strategies can be found with lower cost than (S'', NE^b)*

and (NE^a, S'). Hence, a Nash equilibrium for a matching congestion game does not necessarily hold for an n-uniform congestion game.

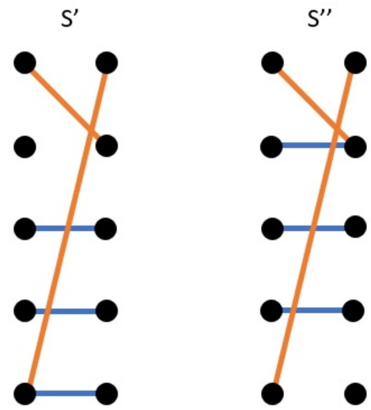


Figure 4.2: Example of two different strategies in an n-uniform congestion game, where a player of the example in Figure 4.1 can switch to.

Next, we give an example with $n = 9$, see Figure 4.3. There are two perfect matchings that are disjoint in the graph and two other perfect matchings that form a Nash equilibrium (NE). These four matchings (OPT^a,OPT^b, NE^a and NE^b) are the only perfect matchings in the graph. In NE the players have four edges in common, which have therefore cost 2 and the players have five edges not in common, which have therefore cost 1. So the total cost for each player in NE is equal to 13. The same cost for each player holds when player 1 switches to OPT^a or OPT^b (note, when player 1 switches to NE^b the cost for both players increases).

$$\begin{aligned} c_1(NE) &= \sum_{e \in E} c_e(x_e(NE)) = 13 \\ &= c_1(OPT^a, NE_{-1}) \\ &= c_1(OPT^b, NE_{-1})). \end{aligned}$$

Because of symmetry the same holds for $c_2(NE)$. Hence, the Nash conditions hold for strategy profile NE. The price of anarchy for Example 4.3 is as follows:

$$\begin{aligned} PoA &= \frac{cost(NE)}{cost(OPT)} = \frac{\sum_{i=1}^{2} c_i(NE)}{\sum_{i=1}^{2} c_i(OPT)} \\ &= \frac{\sum_{i=1}^{2} \sum_{e \in NE_i} c_e(x_e(NE))}{\sum_{i=1}^{2} \sum_{e \in OPT_i} c_e(x_e(OPT))} = \frac{26}{18} = \frac{13}{9} \approx 1.444 \end{aligned}$$

Lastly, we give an example for $n = 13$, see Example 4.4. Again, the four matchings (OPT^a,OPT^b, NE^a and NE^b) that are shown in the example are the only four possible matchings. This is still easy to check, because most nodes have a degree of two

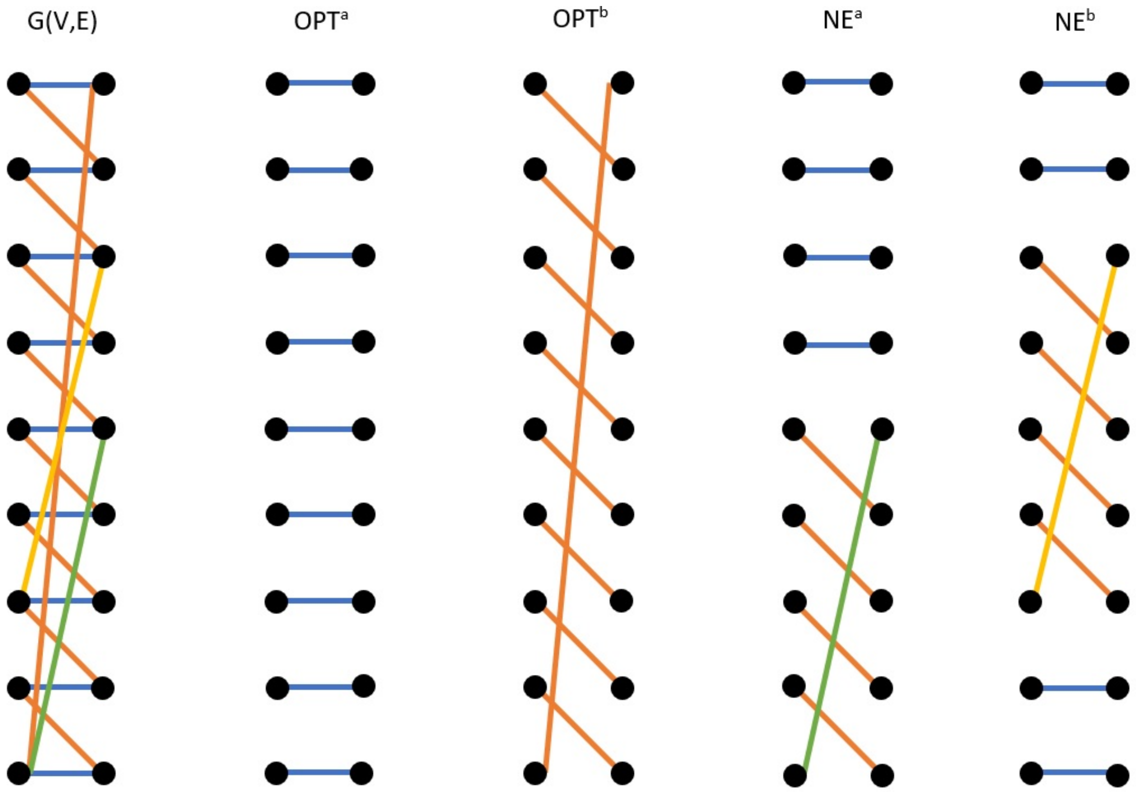


Figure 4.3: Lower bound example on the price of anarchy in a matching congestion game for 2 players, $n = 9$

and therefore, if one edge is chosen, most other edges also need to be chosen to form a perfect matching. In NE the players have six edges in common, which have therefore cost 2. Both players have seven edges not in common and these edges have therefore cost 1. Consequently, the cost for each player in NE is equal to 19. Again the same cost hold for each player when player 1 switches to OPT^a or OPT^b (note, when player 1 switches to NE^b the cost for both players increases).

$$\begin{aligned} c_1(NE) &= \sum_{e \in E} c_e(x_e(NE)) = 19 \\ &= c_1(OPT^a, NE_{-1}) \\ &= c_1(OPT^b, NE_{-1}). \end{aligned}$$

Because of symmetry the same holds for $c_2(NE)$. Therefore, the Nash conditions hold for the strategy profile NE. The price of anarchy for this example is as follows:

$$\begin{aligned} PoA &= \frac{cost(NE)}{cost(OPT)} = \frac{\sum_{i=1}^{2} c_i(NE)}{\sum_{i=1}^{2} c_i(OPT)} \\ &= \frac{\sum_{i=1}^{2} \sum_{e \in NE_i} c_e(x_e(NE))}{\sum_{i=1}^{2} \sum_{e \in OPT_i} c_e(x_e(OPT))} = \frac{38}{26} = \frac{19}{13} \approx 1.462 \end{aligned}$$

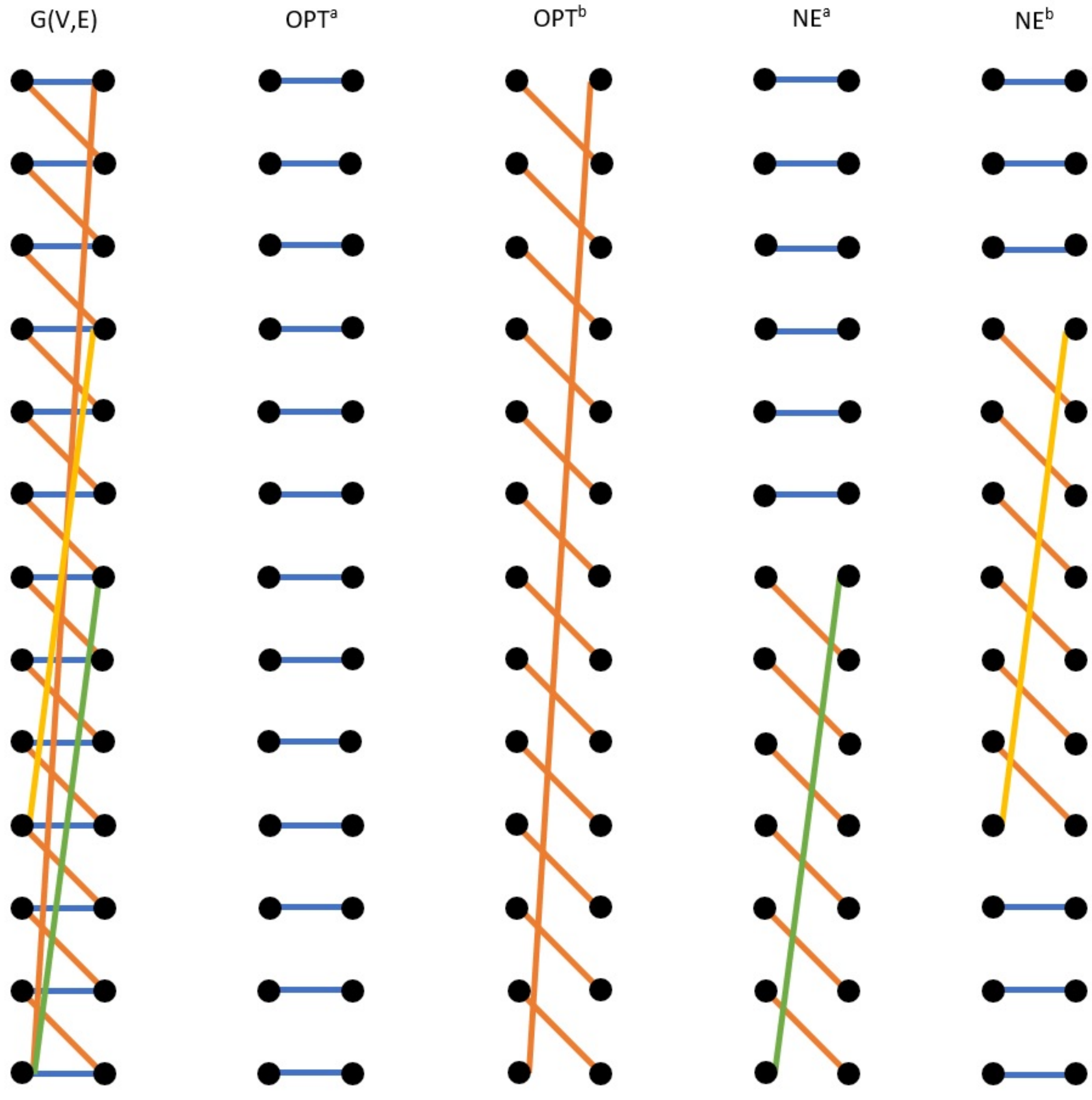


Figure 4.4: Lower bound example on the price of anarchy in a matching congestion game for 2 players, $n = 13$

Subsequently, we will use the examples above as a base to provide a more general example. The idea for these three examples is as follows: we want two perfect matchings which are disjoint and two perfect matchings which form NE. When the matchings of OPT are disjoint then both matchings of NE can have with both the matchings in OPT a maximum of $\lfloor n \rfloor$ edges in common. Because of the Nash condition also the matchings in NE can have a maximum of $\lfloor n \rfloor$ edges in common with each other. In a matching congestion game with two players, eight nodes extra (or $|V_1| = |V_2| = n + 4$) in the bipartite graph enables for both strategies in NE to have two extra edges in common with both the matchings of OPT. Therefore, the strategies in NE can also have two extra edges in common without violating the Nash conditions. So with eight extra nodes the two strategies in NE can have two extra edges in common such that one of those two edges is also used in one of the strategies in OPT and the other is also used in the other strategy in OPT. We define m such that when n increases with 4, m increases with 1 and when $n = 5$, $m = 1$. This means that $m := \frac{1}{4}(n-1)$. We define m this way because now a general example can be given for which PoA is exact when $m \in \mathbb{Z}_{>0}$. Here below we show that when n increases with 4, m increases with 1.

$$\begin{aligned} m(n+4) - m(n) &= \frac{1}{4}(n+4-1) - \frac{1}{4}(n-1) \\ &= \frac{3}{4} + \frac{1}{4} = 1 \end{aligned}$$

So we have:

$$\begin{array}{ccc} m = 1 & \Leftrightarrow & n = 5 \\ m = 2 & \Leftrightarrow & n = 9 \\ m = 3 & \Leftrightarrow & n = 13 \\ m = 4 & \Leftrightarrow & n = 17 \\ \vdots & & \vdots \end{array}$$

Based on the pattern we detected in the three examples provide above, we created a more general example, see Figure 4.5. This example holds for $n = 4m + 1$ with

$m \in \mathbb{Z}_{>0}$. In this general example the total cost for OPT and for NE are as follows:

$$\begin{aligned}
cost(OPT) &= 2n = 8m + 2 \\
cost(NE) &= \sum_{i=1}^{2} \left(\left(\frac{n+3}{4} - 1 \right) + \left(\frac{3n+1}{4} - 1 - \left(\frac{n+1}{2} - 1 \right) \right) \right) \cdot 2 \\
&\quad + \sum_{i=1}^{2} \left(n - \left(\frac{n+3}{4} - 1 + \frac{3n+1}{4} - 1 - \left(\frac{n+1}{2} - 1 \right) \right) \right) \cdot 1 \\
&= \sum_{i=1}^{2} \left(\frac{2n-2}{4} \cdot 2 \right) + \sum_{i=1}^{2} \left(\frac{2n+2}{4} \cdot 1 \right) \\
&= \sum_{i=1}^{2} \left(\frac{2n-2}{2} + \frac{2n+2}{2} \right) \\
&= \sum_{i=1}^{2} \left(\frac{3n-1}{2} \right) \\
&= 3n - 1 \\
&= 12m + 2.
\end{aligned}$$

Therefore, a lower bound on the price of anarchy is as follows:

$$\begin{aligned}
PoA &= \frac{12m+2}{8m+2} \\
\Rightarrow \lim_{m \to \infty} PoA &= \lim_{m \to \infty} \frac{12m+2}{8m+2} = \frac{12}{8} = \frac{3}{2}.
\end{aligned}$$

When $m = \frac{1}{4}(n-1)$ goes to infinity then the price of anarchy goes to $\frac{3}{2}$, which proves Lemma 4.1.1. $\square$

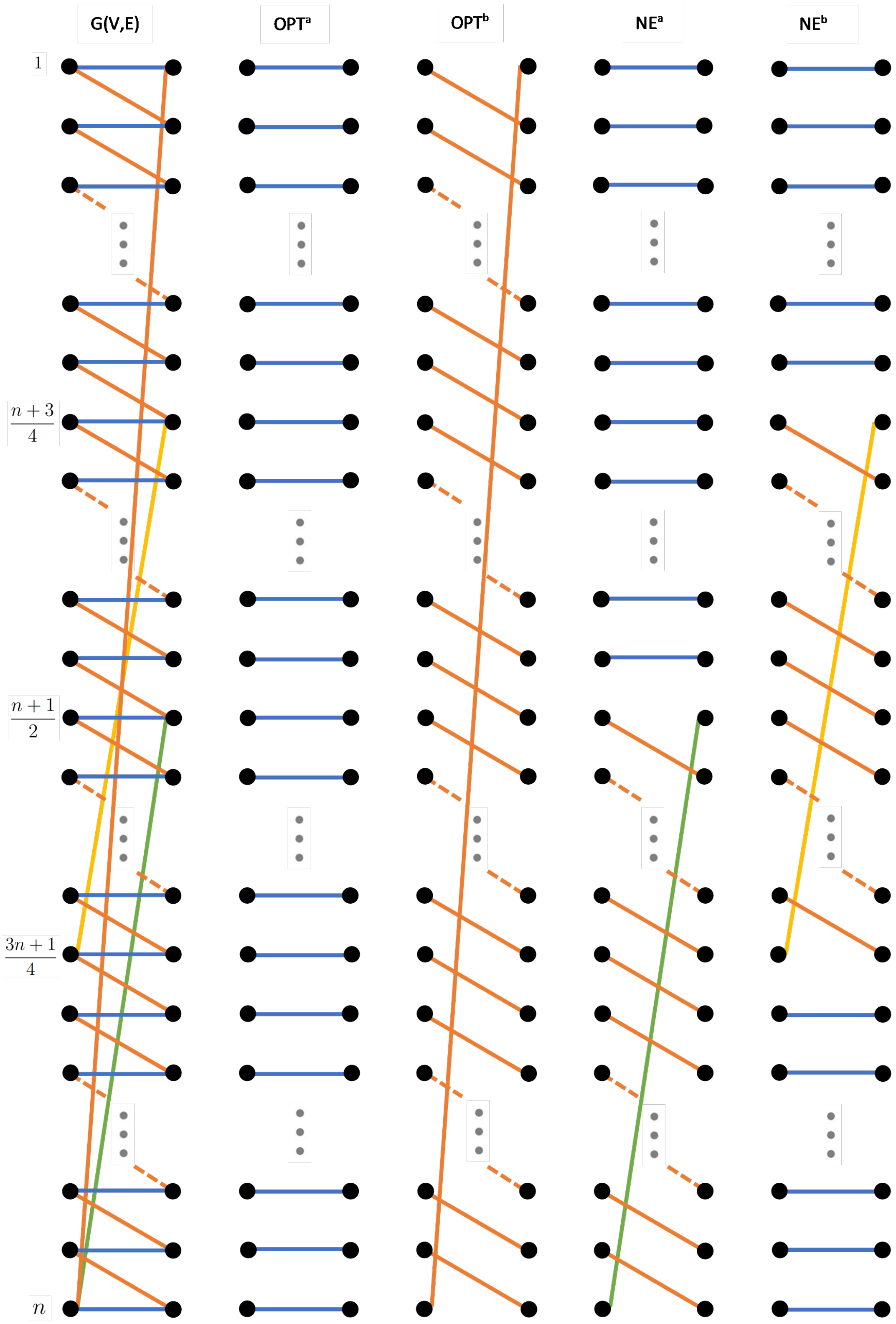


Figure 4.5: General lower bound example on the price of anarchy in a matching congestion game for 2 players

In this section we showed that the lower bound on the price of anarchy for two players asymptotically goes to $3/2$. For three or more players we were not able to find an example which has an price of anarchy significantly higher than 1. When we follow the two players example for $p = 3$ and $n = 8$ there are already over 100 different strategies which form a perfect matching.
In the next section, we will analyse the upper bound on the price of anarchy for two players in bipartite graphs.

4.2 Upper bound on the PoA for two players

First, we introduce some extra notations that we use in the proofs to follow. Namely, we denote by $OPT = (S_1^*, S_2^*)$ the optimal solution, and by $NE = (S_1, S_2)$ the worst-case Nash equilibrium. Where S_i^* and S_i are the strategies of player i in the optimal solution and the worst-case Nash equilibrium respectively. Furthermore, we call $cost(NE)$ and $cost(OPT)$ the total cost of the worst Nash equilibrium and the optimal solution respectively. Lastly, we denote by $d^* = |S_1^* \cap S_2^*|$ the total overlap in edges between S_1^* and S_2^*, and by $d = |S_1 \cap S_2|$ the total overlap in edges between S_1 and S_2.
Now that we have set the notation we are going to investigate the upper bound on the price of anarchy for two players in a matching congestion game with bipartite graph $G = (V_1 \cup V_2, E)$ with $|V_1| = |V_2| = n$ and with identity cost functions $c_e(x_e) = x_e$, for all $e \in E$.

Theorem 4.2.1. *In a symmetric matching congestion game with two players, a bipartite graph $G = (V_1 \cup V_2, E)$ where $|V_1| = |V_2| = n$ and identity cost functions, the price of anarchy is at most $\frac{3}{2}$. Specifically,*

$$PoA = \frac{cost(NE)}{cost(OPT)} \leq \frac{3}{2} - \frac{d^*}{n + d^*}.$$

Where d^* equals the number of edges that are used by both players simultaneously in the optimal solution.

Proof. We have a matching congestion game with a bipartite graph $G = (V_1 \cup V_2, E)$, two players and identity cost functions $c_e(x_e) = x_e$, for all $e \in E$. The edges have identity cost functions, therefore the cost for each of the two players using a common edge is 2 for each of the common edges and the cost is 1 for all other edges. There are d edges which are used by both players in NE and therefore $n - d$ edges are only used by one player. Note, this holds for both players. Summing the costs for both players, the total cost for NE is equal to,

$$\begin{aligned} cost(NE) &= \sum_{i=1}^{2} (d \cdot 2 + 1 \cdot (n - d)) \\ &= 4d + 2(n - d) \\ &= 2n + 2d \end{aligned}$$

The same holds for $cost(OPT)$ only now the cost depends on d^* in stead of d.

$$\begin{aligned} cost(OPT) &= \sum_{i=1}^{2} (2 \cdot d^* + 1 \cdot (n - d^*)) \\ &= 4d^* + 2(n - d^*) \\ &= 2n + 2d^* \end{aligned}$$

We want to bound d from above in terms of d^* and n, such that the price of anarchy is also bounded from above in terms of d^* and n. The cost of OPT is not higher than the cost of any other strategy profile, by definition of an optimal solution. Hence, the cost of OPT is not higher than the cost of NE.

$$\begin{aligned} & cost(OPT) \leq cost(NE) \\ \Rightarrow \quad & 2n + 2d^* \leq 2n + 2d \\ \Rightarrow \quad & d^* \leq d \end{aligned}$$

Because of the Nash condition the following holds:

$$d = |S_1 \cap S_2| \leq |S_1 \cap S_1^*|$$

Otherwise, player 2 could choose S_1^* in stead of S_2. Because of symmetry, the same argument holds for the following:

$$\begin{aligned} d &\leq |S_1 \cap S_2^*| \\ d &\leq |S_1^* \cap S_2| \\ d &\leq |S_2^* \cap S_2| \end{aligned}$$

We define $c_1 := |S_1 \cap S_1^* \cap S_2^*|$ and $c_2 := |S_2 \cap S_1^* \cap S_2^*|$. We know that $c_1 \leq d^*$ and $c_2 \leq d^*$, because $d^* = |S_1^* \cap S_2^*|$. Furthermore, all strategies sets in G have n edges and thus $|S_1| = |S_2| = n$. Therefore the following holds (see also Figure 4.6),

$$\begin{aligned} & n \geq |S_1 \cap S_1^*| + |S_1 \cap S_2^*| - |S_1 \cap S_1^* \cap S_2^*| \\ \Rightarrow \quad & |S_1 \cap S_1^*| + |S_1 \cap S_2^*| \leq n + c_1. \end{aligned} \tag{4.1}$$

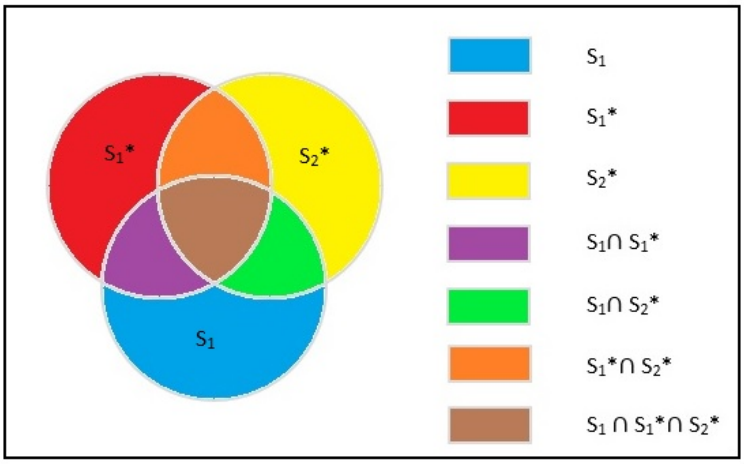


Figure 4.6: Venn diagram of S_1, S_1^* and S_2^*

and,

$$
\begin{aligned}
n &\geq |S_2 \cap S_1^*| + |S_2 \cap S_2^*| - |S_2 \cap S_1^* \cap S_2^*| \\
\Rightarrow \quad & |S_2 \cap S_1^*| + |S_2 \cap S_2^*| \leq n + c_2 \qquad (4.2)
\end{aligned}
$$

As mentioned above, $d \leq |S_i \cap S_j^*|$ for all $i, j \in \{1, 2\}$, if we now sum over i and j we get the following,

$$
\begin{aligned}
4d &\leq |S_1 \cap S_1^*| + |S_1 \cap S_2^*| + |S_2 \cap S_1^*| + |S_2 \cap S_2^*| \\
&\leq c_1 + n + c_2 + n \qquad (4.3) \\
&\leq 2d^* + 2n \qquad (4.4) \\
\Rightarrow d &\leq \frac{1}{2}d^* + \frac{1}{2}n. \qquad (4.5)
\end{aligned}
$$

The inequality in 4.3 holds because of 4.1 and 4.2. The inequality of 4.4 holds because $c_1 \leq d^*$ and $c_2 \leq d^*$ as mentioned earlier. We showed that d is bounded from above in terms of d^* and n. Therefore, $cost(NE)$ is bounded in terms of d^* and n as follows:

$$
\begin{aligned}
cost(NE) &= 2n + 2d \\
&\leq 2n + 2(\frac{1}{2}d^* + \frac{1}{2}n) \\
&= 3n + d^*.
\end{aligned}
$$

We used 4.5 for the inequality above. This means that for two players the price of anarchy is:

$$\begin{aligned} PoA = \frac{cost(NE)}{cost(OPT)} &\leq \frac{3n + d^*}{2(n + d^*)} \\ &= \frac{3n + 3d^* - 2d^*}{2(n + d^*)} \\ &= \frac{3}{2} - \frac{2d^*}{2(n + d^*)} \\ &= \frac{3}{2} - \frac{d^*}{n + d^*} \end{aligned}$$

Because n and d^* are nonnegative we see in the last equality that PoA is maximized when d^* = 0, therefore, PoA is bounded from above by $\frac{3}{2}$. This proves Theorem 4.2.1. □

4.3 Upper bound on the PoA for p players

For finding an upper bound on the price of anarchy for matching congestion games with p players with a given bipartite graph and identity cost functions we start with using the same steps as for two players. Only now $OPT = (S_1^*, S_2^*, \dots, S_p^*)$ an optimal strategy profile and $NE = (S_1, S_2, \dots, S_p)$ the worst-case Nash equilibrium. We start with finding the cost for a strategy profile S, then we bound this cost by n and d^*. Where d^* is now the sum of the intersections of every pair of strategies S_i^* and S_j^* $(\forall i \neq j \in P)$ in the strategy profile OPT.

$$\begin{aligned} d^* &= |S_1^* \cap S_2^*| + |S_1^* \cap S_3^*| + \ldots + |S_{p-1}^* \cap S_p^*| \\ &= \sum_{\substack{i<j \\ i,j \in P}} |S_i^* \cap S_j^*| \end{aligned}$$

Subsequently, we find an upper bound on the price of anarchy in terms of n, p and d^*.

We start with the cost of a strategy profile S. Because of the identity cost function every edge that is used by r players has cost r^2. First, in S every player i uses n edges in the strategy S_i and therefore

$$\sum_{i=1}^{p} |S_i| = pn \tag{4.6}$$

This is the total cost only if there is no edge which is used by more than one player. Because if for example an edge is used in both S_i and in S_j the cost of this edge is equal to 2 in 4.6. However, the cost of edges used by two players, is equal to

$2^2 = 4$. So when the cost of a strategy S is equal to pn, the cost of an edge which is used by r players is only equal to r in stead of r^2. This is only correct if there is no overlap between the strategies of the players. However, if there is some overlap of edges between the strategies of different players this cost function is not correct. We observe that the cost function of a strategy profile S is as follows:

Lemma 4.3.1. *In a matching congestion game with identity cost functions and strategy profile S the total cost for all players is equal to:*

$$cost(S) = pn + 2d_s$$

Where, d_s is the sum of the intersections of every pair of strategies S_i and S_j ($\forall i \neq j \in P$) in the strategy profile $S = (S_1, S_2, \dots, S_p)$.

$$\begin{aligned} d_s &= |S_1 \cap S_2| + |S_1 \cap S_3| + \dots + |S_{p-1} \cap S_p| \\ &= \sum_{\substack{i<j \\ i,j \in P}} |S_i \cap S_j| \end{aligned}$$

.

Proof. We prove this by checking the cost of the edges which are used by r players. Note that the cost of those edges should be equal to r^2. In $\sum_{i=1}^{p} |S_i| = pn$ an edge e which is used by r players is counted r times with cost 1 and therefore, this edge has cost r. However, the cost of e in $2d_s$ has to be added. There are $\binom{r}{2}$ pairs of intersections of the r strategies in d_s with cost 2. When we sum the cost of e in pn and $2d_s$ we see that it is exactly equal to r^2, which is required.

$$\begin{aligned} r + 2 \cdot \binom{r}{2} &= r + 2 \cdot \frac{r!}{(r-2)!2!} \\ &= r + \frac{r(r-1)(r-2)!}{(r-2)!} \\ &= r + r(r-1) \\ &= r + r^2 - r \\ &= r^2 \end{aligned}$$

This means that the cost of S is as follows,

$$\begin{aligned} cost(S) &= |S_1| + |S_2| + \dots + |S_p| + 2|S_1 \cap S_2| + \dots 2|S_{p-1} \cap S_p| \\ &= pn + 2d_s. \end{aligned}$$

This proves Lemma 4.3.1. □

We know that in case of a Nash equilibrium, the cost will be $cost(NE) = pn + 2d$ and in case of an optimum solution, the cost will be $cost(OPT) = pn + 2d^*$. Next, d needs to be bounded from above by the terms d^*, n and p, such that the upper bound on the price of anarchy is also bounded from above by d^*, n and p.

Lemma 4.3.2. *d is bounded from above by d^*, n and p:*

$$d \leq \frac{p-1}{2}n + \frac{1}{p}(2^{p-1} - 1)d^*$$

Proof.

$$d = \sum_{\substack{i<j \\ i,j \in 1,\dots,p}} |S_i \cap S_j| \tag{4.7}$$
$$d^* = \sum_{\substack{i<j \\ i,j \in 1,\dots,p}} |S_i^* \cap S_j^*|$$

Because of the Nash condition we know that in a Nash equilibrium when one player changes its strategy S_i to any other strategy, the cost of player i must be at least as high as its cost when choosing strategy S_i. When player 1 switches from strategy S_1 to strategy S_1^* means that the following must hold:

$$|S_1 \cap S_2| + |S_1 \cap S_3| + \ldots + |S_1 \cap S_p| \leq |S_1^* \cap S_2| + |S_1^* \cap S_3| + \ldots + |S_1^* \cap S_p|$$

The same is true for every strategy in NE that is changed to any strategy in OPT:

$$\sum_{j \in -i} |S_i \cap S_j| \leq \sum_{j \in -i} |S_k^* \cap S_j|, \qquad \forall\, i,k \in P \tag{4.8}$$

Because $i, k \in \{1, \dots, p\}$, there are p^2 such inequalities. We define a term of d (with d defined in 4.7) as $|S_i \cap S_j|$ with $i < j$ and $i, j \in P$. Every element of d in 4.8 occurs twice for every $k \in \{1, \dots, p\}$ on the left hand side, namely for player i and also for player j. Therefore, in total every term of d occurs $2p$ times. On the right hand side of the inequalities, every cardinality of the intersection of a strategy of NE with a strategy of OPT occurs $(p-1)$ times. This is true, because from the p^2 inequalities, one times a player i switches from strategy S_i to strategy $S_k^*, \forall k \in P$ and the cardinality of $|S_k^* \cap S_j|$ occurs on the right hand sight $(p-1)$ times. The minus one comes from the fact that in the inequality where player j switches to S_k^* there is no $|S_k^* \cap S_j|$ term on the right hand side. Therefore, the following holds:

$$\Rightarrow \qquad 2pd \leq (p-1) \sum_{k \in P} \sum_{j \in P} |S_k^* \cap S_j|$$
$$\Rightarrow \quad \frac{2pd}{p-1} \leq \sum_{k \in P} \sum_{j \in P} |S_k^* \cap S_j| \tag{4.9}$$

Furthermore, we know that $|S_i| = n \; \forall i \in P$. Therefore, the sum of the intersections of S_i with every other matching of OPT minus the edges that are counted multiple times, is bounded from above by n:

$$\begin{aligned} n \quad & \geq \sum_{j\in P} |S_i \cap S_j^*| - \sum_{j<k} |S_i \cap S_j^* \cap S_k^*| + \sum_{j<k<l} |S_i \cap S_j^* \cap S_k^* \cap S_l^*| - \dots \\ & (-1)^{p-1} |S_i \cap S_1^* \cap S_2^* \cap \dots \cap S_p^*|, \qquad \forall\, i \in 1,\dots,p \\ \Rightarrow \sum_{j\in P} |S_i \cap S_j^*| \quad & \leq n + \sum_{j<k} |S_i \cap S_j^* \cap S_k^*| - \dots \\ & (-1)^{p} |S_i \cap S_1^* \cap S_2^* \cap \dots \cap S_p^*|, \qquad \forall\, i \in 1,\dots,p \end{aligned} \tag{4.10}$$

If we now take the sum over all $i \in P$ on the left side of (4.10) we get the same as the right side of (4.9). Furthermore, a plus term of intersections on the right side of (4.10) is smaller equal than the term with one of those intersections less. For example,

$$|S_i \cap S_j^* \cap S_k^*| \leq |S_j^* \cap S_k^*|.$$

The term on the right side of the inequality is a part of d^*. For all plus parts on the right side of 4.10 holds that it is smaller or equal to a part of d^*. The total number of those plus parts is,

$$\binom{p}{2} + \binom{p}{4} + \binom{p}{6} + \dots$$

Using the sum of the binomial coefficients, we know that:

$$\begin{aligned} (1+x)^p &= \binom{p}{0} + \binom{p}{1}x + \binom{p}{2}x^2 + \dots + \binom{p}{p}x^p \\ (1-x)^p &= \binom{p}{0} - \binom{p}{1}x + \binom{p}{2}x^2 + \dots (-1)^p\binom{p}{p}x^p \end{aligned}$$

Adding these two equation together and setting $x = 1$, we get:

$$\begin{aligned} (1+x)^p + (1-x)^p &= 2[\binom{p}{0} + \binom{p}{2}x^2 + \binom{p}{4}x^4 + \dots] \\ \Rightarrow 2^p &= 2[\binom{p}{0} + \binom{p}{2} + \binom{p}{4} + \dots] \\ \Rightarrow 2^{p-1} &= \binom{p}{0} + \binom{p}{2} + \binom{p}{4} + \dots \\ \Rightarrow 2^{p-1} - 1 &= \binom{p}{2} + \binom{p}{4} + \binom{p}{6} + \dots \end{aligned}$$

So we have $2^{p-1}-1$ parts of d^*, and d^* consist of $\binom{p}{2}$ parts. So putting everything together using 4.9, 4.10 and that the total sum of the plus parts in 4.10 on the right side is smaller equal to $\frac{2^{p-1}-1}{\binom{p}{2}}d^*$ with all the negative terms in (4.10) set to 0, we get:

$$\begin{aligned}
\frac{2pd}{p-1} \leq \sum_{k\in P}\sum_{j\in P}|S_k^* \cap S_j| &\leq pn + pd^*\frac{2^{p-1}-1}{\binom{p}{2}} \\
&= pn + \frac{p}{\left(\frac{p\cdot(p-1)\cdot(p-2)!}{(p-2)!\cdot 2}\right)}d^*(2^{p-1}-1) \\
&= pn + \frac{2}{p-1}d^*(2^{p-1}-1) \\
&= pn + \frac{1}{p-1}d^*(2^p-2)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow d &\leq \frac{p-1}{2}\left(n + \frac{1}{p(p-1)}(2^p-2)d^*\right) \\
&= \frac{p-1}{2}n + \frac{1}{p}(2^{p-1}-1)d^*
\end{aligned}$$

This proves Lemma 4.3.2. □

Now that d is bounded in terms of d^* and n and p, we can calculate an upper bound on the price of anarchy.

Theorem 4.3.3.

$$PoA = \frac{cost(NE)}{cost(OPT)} \leq 2 + \frac{\frac{1}{p}(2^p-4p-2)d^*-n}{pn+2d^*}$$

Proof. Using Lemma 4.3.2, the cost of NE is:

$$\begin{aligned}
cost(NE) &= pn + 2d \\
&\leq pn + 2\left(\frac{p-1}{2}n + \frac{1}{p}(2^{p-1}-1)d^*\right) \\
&= pn + pn - n + \frac{1}{p}(2^p-2)d^* \\
&= (2p-1)n + \frac{1}{p}(2^p-2)d^*
\end{aligned}$$

The cost of OPT is as follows:

$$cost(OPT) = pn + 2d^*$$

Therefore the price of anarchy is:

$$\begin{aligned} PoA = \frac{cost(NE)}{cost(OPT)} &\leq \frac{(2p-1)n + \frac{1}{p}(2^p - 2)d^*}{pn + 2d^*} \\ &= \frac{2pn + 4d^* + \frac{1}{p}(2^p - 2)d^* - 4d^* - n}{pn + 2d^*} \\ &= 2 + \frac{\frac{1}{p}(2^p - 4p - 2)d^* - n}{pn + 2d^*} \end{aligned}$$

Which proves Theorem 4.3.3. □

Unfortunately, this bound is not very useful for the general case, because of the 2^p term. However, it does immediately result in an upper bound on the price of anarchy for 2, 3, 4 players and when no player in the optimal strategy share an edge with another player, i.e. $d^* = 0$.

Theorem 4.3.4. *If there exists a solution where no player uses the same edge as another player in an optimal solution, i.e. $d^* = 0$, then the price of anarchy in a matching congestion game with identity cost functions is bounded from above by*

$$PoA = \frac{cost(NE)}{cost(OPT)} \leq 2 - \frac{1}{p}.$$

Proof. Using Theorem 4.3.3 and with $d^* = 0$ the proof immediately yields.

$$\begin{aligned} PoA = \frac{cost(NE)}{cost(OPT)} &\leq 2 + \frac{\frac{1}{p}(2^p - 4p - 2)d^* - n}{pn + 2d^*} \\ &= 2 - \frac{n}{pn} \\ &= 2 - \frac{1}{p}. \end{aligned}$$

□

Now we state four theorems, each for a different number of players. In these theorems no assumption about d^* is made.

Theorem 4.3.5. *In a matching congestion game with two players and with identity cost functions the price of anarchy is bounded from above by*

$$PoA = \frac{cost(NE)}{cost(OPT)} \leq \frac{3}{2}.$$

Proof. Using Theorem 4.3.3 and set $p = 2$ we get

$$\begin{aligned} PoA = \frac{cost(NE)}{cost(OPT)} &\leq 2 + \frac{\frac{1}{p}(2^p - 4p - 2)d^* - n}{pn + 2d^*} \\ &= 2 + \frac{\frac{1}{2}(2^2 - 4 \cdot 2 - 2)d^* - n}{2n + 2d^*} \\ &= 2 + \frac{\frac{1}{2} \cdot (-6)d^* - n}{2n + 2d^*} \\ &= 2 - \frac{n + 3d^*}{2n + 2d^*} \\ &= 2 - \frac{(n + d^*) + 2d^*}{2n + 2d^*} \\ &= 2 - \frac{1}{2} - \frac{2d^*}{2n + 2d^*} \\ &= \frac{3}{2} - \frac{d^*}{n + d^*} \\ &\leq \frac{3}{2} \end{aligned}$$

The last inequality holds, because n, $d^* \geq 0$. □

Theorem 4.3.6. *In a matching congestion game with three players and with identity cost functions the price of anarchy is bounded from above by*

$$PoA = \frac{cost(NE)}{cost(OPT)} \leq \frac{5}{3}.$$

Proof. Using Theorem 4.3.3 and set $p = 3$ we get

$$\begin{aligned} PoA = \frac{cost(NE)}{cost(OPT)} &\leq 2 + \frac{\frac{1}{p}(2^p - 4p - 2)d^* - n}{pn + 2d^*} \\ &= 2 + \frac{\frac{1}{3}(2^3 - 4 \cdot 3 - 2)d^* - n}{3n + 2d^*} \\ &= 2 - \frac{2d^* + n}{3n + 2d^*} \\ &= 2 - \frac{(n - \frac{2}{3}d^*) - \frac{4}{3}d^*}{3n + 2d^*} \\ &= \frac{5}{3} - \frac{\frac{4}{3}d^*}{3n + 2d^*} \\ &\leq \frac{5}{3} \end{aligned}$$

The last inequality holds, because n, $d^* \geq 0$. □

Theorem 4.3.7. *In a matching congestion game with four players and with identity cost functions the price of anarchy is bounded from above by*

$$PoA = \frac{cost(NE)}{cost(OPT)} \leq \frac{7}{4}.$$

Proof. Using Theorem 4.3.3 and set $p = 4$ we get

$$\begin{aligned} PoA = \frac{cost(NE)}{cost(OPT)} &\leq 2 + \frac{\frac{1}{p}(2^p - 4p - 2)d^* - n}{pn + 2d^*} \\ &= 2 + \frac{\frac{1}{4}(2^4 - 4 \cdot 4 - 2)d^* - n}{4n + 2d^*} \\ &= 2 - \frac{\frac{1}{2}d^* + n}{4n + 2d^*} \\ &= 2 - \frac{1}{4} \\ &= \frac{7}{4} \end{aligned}$$

□

We see that this upper bound on the price of anarchy for four players does not depend on n or d^*.

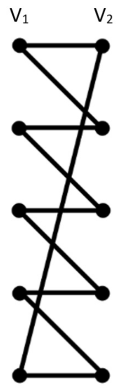
Chapter 5

Computation time of optimal solutions and Nash equilibria in bipartite graphs

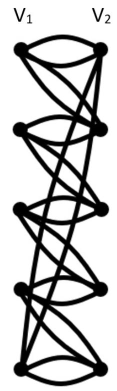
In this chapter we prove that the computation time of finding an optimal solution in a symmetric matching congestion game can be done in polynomial time. We do this by using the idea of Fabrikant et al. [9]. They reduced the problem of finding an optimal solution of a symmetric network congestion game to a minimum-cost flow problem. However, instead of a network congestion game, we reduce the problem of finding an optimal solution of a matching congestion game to a minimum-cost flow problem. We also prove that every optimal solution in these problems is always a Nash equilibrium and hence, a Nash equilibrium can be found in polynomial time.
Call the problem of finding an optimal solution in symmetric matching congestion games with a given bipartite graph $G = (V_1 \cup V_2, E)$ with $|V_1| = |V_2| = n$ and identity cost functions, Π^{OPT}. We state the following lemma:

Lemma 5.0.1. *Solving Π^{OPT} can be done in polynomial time.*

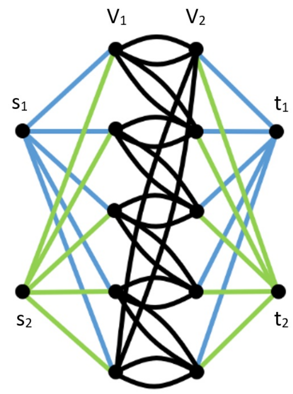
Proof. To prove this we reduce an instance of Π^{OPT} to an instance of a min-cost flow problem. There are multiple algorithms known that solve an instance of a min-cost flow problem in polynomial time and therefore it will also solve an instance of Π^{OPT} in polynomial time. A min-cost flow problem consist of a network graph $G(V^{flow}, E^{flow})$ with a source node and a sink node, such that flow can go from the source node to the sink node. All edges have a cost function a_k with $k \in E^{flow}$.
Π^{OPT} contains a bipartite graph $G(V_1 \cup V_2, E)$ and the cost functions are identity, so $c_e(x_e) = x_e, \ \forall e \in E$. To transform an instance of Π^{OPT} to an instance of the min-cost flow problem, we first replace every edge $e \in E$ by p parallel edges. This means that edge e is now replaced by $\{e_1, e_2, \dots, e_p\}$, see Figure 5.1b. The capacity of these edges are 1 and with non-decreasing cost a_{e_i} in i, which we define as



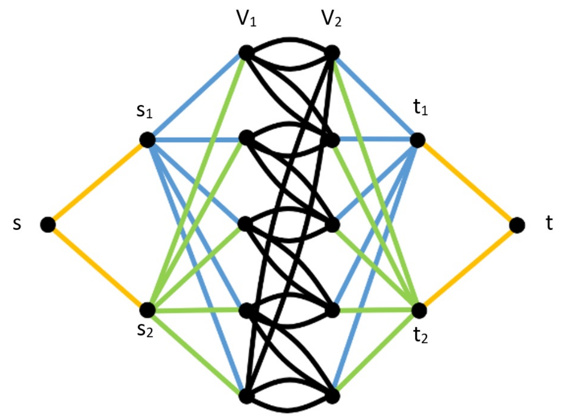
(a) A bipartite graph $G(V_1 \cup V_2, E)$



(b) Replacing every edge $e \in E$ by p edges (here $p = 2$)



(c) Adding a source node s_i and a sink node t_i for every player $i \in P$ with edges to every $u \in V_1$ and $v \in V_2$ respectively.



(d) Adding an s and t node connecting with every s_i and t_i respectively. All the arcs are directed from left to right

Figure 5.1: Transformation of an instance of Π^{OPT} to a min-cost flow network

$a_e(i) := i^2 - (i-1)^2 = 2i - 1$ with $i \in P$. So the following holds,

$$a_{e_1} \leq a_{e_2} \leq \ldots \leq a_{e_p} \qquad\qquad \forall\, e \in E.$$

Next, we add a source node, s_i, and a target node, t_i, for every player $i \in P$. Every source node s_i is connected with every node $v \in V_1$ and every target node t_i is connected with every node $v \in V_2$. These edges have capacity 1 and zero cost, see Figure 5.1c. Lastly, we add a super source node s which is connected with every source node s_i $\forall i \in P$ and a super target node t which is connected with every target node t_i $\forall i \in P$. These edges have capacity n and also zero cost, see Figure 5.1d.

To make sure there are p perfect matchings in Π, the required flow from s to t is equal to np. Furthermore, the required flow from s to every s_i is equal to n $\forall i \in P$

. Same holds for every flow from t_i to t. However, this is redundant because of the capacity constraints and the required flow from s tot t. We define the ingoing and outgoing arcs of a node v as follows:

$$\delta_{in}(v) = \text{ingoing arcs of node } v$$
$$\delta_{out}(v) = \text{outgoing arcs of node } v$$

Minimizing over the total cost gives the following integer program:

$$\begin{array}{llll}
\min & \displaystyle\sum_{e\in E}\sum_{i\in P} a_{e_i}\cdot f_{e_i} & & \\
s.t. & f(u,v)\leq 1 & \forall u\in V_1, v\in V_2 & \textit{(capacity constraints)}\\
 & f(s_i,v)\leq 1 & \forall i\in P, v\in V_1 & \textit{(capacity constraints)}\\
 & f(v,t_i)\leq 1 & \forall i\in P, v\in V_2 & \textit{(capacity constraints)}\\
 & \displaystyle\sum_{e\in\delta_{in}(v)} f(e) = \sum_{e\in\delta_{out}(v)} f(e) & \forall v\neq s,t & \textit{(Flow conservation)}\\
 & \displaystyle\sum_{e\in\delta_{out}(s)} f(e) = n\cdot p & & \textit{(required flow)}
\end{array}$$

This program can be seen as a min-cost flow program and the optimal solution can be computed in polynomial time by known minimum-cost flow algorithms. A solution to this program will give the total minimum cost with the corresponding flow over every edge. The sum of f_{e_i} over all players i and is equal to the number of players using edge $e \in E$, so $\sum_{i\in P} f_{e_i} = x_e$. Because of the capacity and the required flow constraints we have p perfect matchings on the nodes in $V_1 \cup V_2$. We can ignore the edges from s to s_i $\forall i \in P$ and the edges from every node s_i to all the nodes in V_1 and also we can ignore the edges from nodes in V_2 to every node t_i and the edges from every node t_i to t, because those edges have zero cost and therefore it does not matter how many players use these edges for the solution of Π^{OPT}. Next, we only have to show that the cost of the edges e_i summed over all $i \in P$ is equal to the cost of edge $e \in E$. If there is a flow of q players on edges $e_1 \cup e_2 \cup \ldots \cup e_p$ then we show that $\sum_{e_i} a_{e_i} \cdot f_{e_i} = x_e^2 = q^2$. Because of the non-decreasing cost functions in i of the edges, in an optimal solution only the first q edges will be used, therefore $f_{e_k} = 0$ for $k \in \{q+1, \ldots, p\}$ and $f_{e_k} = 1$ for $k \in \{1, 2, \ldots, q)$ because of the capacity constraints. This means

$$\begin{aligned}\sum_{k\in P} a_{e_k}\cdot f_{e_k} &= \sum_{k=1}^{q} a_{e_k}\cdot f_{e_k} + \sum_{k=q+1}^{p} a_{e_k}\cdot f_{e_k}\\ &= \sum_{k=1}^{q}(2k-1)\cdot 1 + \sum_{k=q+1}^{p}(2k-1)\cdot 0\\ &= \sum_{k=1}^{q}(2k-1)\\ &= 2\cdot\sum_{k=1}^{q} k - \sum_{k=1}^{q} 1\\ &= 2(\frac{q}{2}(q+1)) - q\\ &= q^2.\end{aligned}$$

All together, this means that an optimal solution to this transformed instance yields an optimal solution of the original problem Π^{OPT}. $\square$

Lemma 5.0.2. *An optimal solution of a matching congestion game with identity cost functions is always a Nash equilibrium.*

Proof. Let S^* be an optimal solution in a matching congestion game. For the sake of a contradiction assume, S^* is not a Nash equilibrium. This means that there is a player, say player i, that has an incentive to switch from strategy S_i^* to another strategy, say S_i, with S_i^*, $S_i \in \mathcal{S}_i$. Then the cost of player i in the strategy profile S^* must be strictly larger than the cost of player i when switching to strategy S_i; $c_i(S^*) > c_i(S_i, S_{-i}^*)$. Furthermore, S^* is an optimal solution, meaning that the total cost of S^* is not higher than the total cost when player i switches to strategy S_i; $cost(S^*) \leq cost(S_i, S_{-i}^*)\ \forall\ S_i \in \mathcal{S}_i$. Define $Y^* := S_i^*\backslash S_i$, all the edges that player i uses in S_i^* but not in S_i and define $Y := S_i\backslash S_i^*$, all the edges that player i uses in S_i, but did not use in S_i^*. The total number of edges in each strategy is the same, because a strategy in a matching congestion game must contain a perfect matching and therefore, $|Y^*| = |Y|$. We know, because of the identity cost functions that:

$$cost(S^*) = \sum_{e\in E} x_e^2(S^*)$$

Only player i changes its strategy, therefore $cost(S_i, S_{-i}^*)$ only differ from $cost(S^*)$ on the edges in Y^* and in Y. The cost of the edges in Y^* changes from $\sum_{e\in Y^*} x_e^2(S^*)$ to $\sum_{e\in Y^*}(x_e(S^*)-1)^2$ and the cost of the edges in Y changes from $\sum_{e\in Y} x_e^2(S^*)$ to

$\sum_{e\in Y}(x_e(S^*)+1)^2$. So,

$$\begin{aligned}
cost(S_i, S^*_{-i}) &= cost(S^*) + \left(\sum_{e\in Y^*}(x_e(S^*)-1)^2 - \sum_{e\in Y^*} x_e^2(S^*)\right) + \\
&\left(\sum_{e\in Y}(x_e(S^*)+1)^2 - \sum_{e\in Y} x_e^2(S^*)\right) \\
&= cost(S^*) + \sum_{e\in Y^*}(x_e^2(S^*) - 2x_e(S^*) + 1 - x_e^2(S^*)) + \\
&\sum_{e\in Y}(x_e^2(S^*) + 2x_e(S^*) + 1 - x_e^2(S^*)) \\
&= cost(S^*) - \sum_{e\in Y^*}(2x_e(S^*)-1) + \sum_{e\in Y}(2x_e(S^*)+1).
\end{aligned}$$

As stated above, $cost(S_i, S^*_{-i}) \geq cost(S^*)$ this means that:

$$\begin{aligned}
&\sum_{e\in Y}(2x_e(S^*)+1) - \sum_{e\in Y^*}(2x_e(S^*)-1) \ \geq 0 \\
\Rightarrow \quad & 2\sum_{e\in Y} x_e(S^*) + |Y| - 2\sum_{e\in Y^*} x_e(S^*) + |Y^*| \geq 0 \\
\Rightarrow \quad & \sum_{e\in Y} x_e(S^*) + |Y| - \sum_{e\in Y^*} x_e(S^*) \geq 0 \qquad (|Y^*| = |Y|) \qquad (5.1)
\end{aligned}$$

Furthermore, the cost $c_i(S_i, S^*_{-i})$ only differ from $c_i(S^*)$ in the edges in Y^* and in Y, because those are the only edges that change for player i.

$$c_i(S_i, S^*_{-i}) = c_i(S^*) - \sum_{e\in Y^*} x_e(S^*) + \sum_{e\in Y}(x_e(S^*)+1)$$

We know, as stated above that $c(S_i^*) > c(S_i)$. It follows that:

$$\begin{aligned}
&\sum_{e\in Y}(x_e(S^*)+1) - \sum_{e\in Y^*} x_e(S^*) < 0 \\
\Rightarrow \quad & \sum_{e\in Y} x_e(S^*) + |Y| - \sum_{e\in Y^*} x_e(S^*) < 0 \qquad (5.2)
\end{aligned}$$

Comparing (5.1) and (5.2) we see that we have a contradiction. No player can switch to a strategy with lower cost without lowering the total cost. This means that for $S*$ the Nash condition holds, hence S^* is a Nash equilibrium. This proves Lemma 5.0.2. □

Finally, we proof that a Nash equilibrium in a symmetric matching congestion game can be computed in polynomial time.

Lemma 5.0.3. *A Nash equilibrium of a symmetric matching congestion game with identity cost functions can be computed in polynomial time.*

Proof. Lemma 5.0.1 shows that an optimal solution of a symmetric matching congestion game can be computed in polynomial time. Moreover, Lemma 5.0.2 shows that an optimal solution of a symmetric matching congestion game with identity cost functions is always a Nash equilibrium. Hence, a Nash equilibrium of a symmetric matching congestion game with identity cost functions can be computed in polynomial time. □

We proved that a Nash equilibrium and an optimal solution of a matching congestion game with a bipartite graph and identity cost functions can be computed in polynomial time. However, this does not mean that we always have the worst Nash equilibrium and therefore we cannot claim to be able to compute the price of anarchy in polynomial time.

Chapter 6

Conclusions and recommendations

6.1 Conclusions

In this research, the price of anarchy of symmetric matching congestion games with identity cost functions was studied. We showed in Chapter 3 that the price of anarchy for symmetric n-uniform congestion games with identity cost functions is always equal to 1. Matching congestion games are not a subclass of n-uniform congestion games as players do choose a strategy with n resources, however the resources of every strategy need to form a perfect matching and therefore not every subset of n resources is available for a player. Hence, the price of anarchy of 1 for n-uniform congestion games does not hold for matching congestion games.
We studied the price of anarchy when the underlying graph G is a bipartite graph. We showed that the price of anarchy for two players is tight and equal to $3/2$. For three and four players the upper bound is equal to $2 - 1/p$ too, however we were not able to find a lower bound example which has a price of anarchy significantly higher than 1. A lower bound example for three or more players was harder than expected. When following the example for two players, small examples ($n = 8$) already have over 100 perfect matchings. We also could not find a better upper bound for five or more players better than the bound of $\frac{5p-2}{2p+1}$, which is the upper bound by Cristodoulou and Koutsoupias [3].
We also studied the price for anarchy of symmetric congestion games with identity cost functions when there exists a solution where no player shares a resource ('disjoint optimal solution'). We proved that an upper bound on the price of anarchy for this special case is indeed $2 - 1/p$. For the lower bound we could use the same examples as for the general situation, where there does not necessarily exist a disjoint optimal solution. In these lower bound examples a disjoint optimal solution exists, hence these lower bound examples also hold for the special case. The bound is only tight for two players, however.
In Chapter 5, we proved that an optimal solution of a symmetric congestion game

with identity cost functions with a given bipartite graph, is always a Nash equilibrium. Furthermore, we used the idea of Fabrikant et al. [9] and transferred it to matching congestion games. The idea of Fabrikant et al. that is reducing the problem of finding a socially optimal solution in a symmetric network congestion game to a min-cost flow problem. Using this reduction we showed that an optimal solution and hence, a Nash equilibrium, can be computed in polynomial time for matching congestion games.

6.2 Recommendations

We tried to prove a better upper bound with five and more players by using a quadratic program. We looked at the proof of Christodoulou and Koutsoupias [3], using the LP we tried to find different values of μ and λ than $\mu = \frac{p-1}{3p}$ and $\lambda = \frac{5p-2}{3p}$ which were used in their proof. They found that $cost(NE) \leq \frac{p-1}{3p}cost(NE) + \frac{5p-2}{3p}cost(OPT)$, which results in the upper bound of $PoA \leq \frac{5p-2}{2p+1}$. We specifically tried to find different coefficients λ and μ, such that the price of anarchy would be lower by using a quadratic program with an objective of $(1-\mu)cost(NE) - \lambda cost(OPT)$ for a given value of μ and λ. We used p, n, λ and μ as input. However, as we proved in Chapter 3 without modeling the fact that strategies must be perfect matchings, the price of anarchy is always equal to 1. Therefore, such an approach might only work if a given bipartite graph is used. When using such an approach for different examples of bipartite graphs and for different p, n, λ and μ there are too many cases to enumerate over to get an idea of some λ and μ that might give a lower price of anarchy. As the program is quadratic, this is too time consuming.
It might be more interesting to study symmetric matching congestion games with affine cost functions, which is a natural expansion. We found an example for two players which also has a price of anarchy of $2 - 1/p = 3/2$, see Figure 6.1. There, every thick edge has cost ax_e and every dotted edge has cost 1.

$$
\begin{aligned}
cost(OPT) &= cost(OPT^a, OPT^b) = 4a + 6 \\
cost(NE) &= cost(NE^a, NE^b) = 2^2a + 2a + 5 = 6a + 5 \\
cost(OPT^a \cap NE^a) &= cost(OPT^a \cap NE^b) \\
&= cost(OPT^b \cap NE^a) \\
&= cost(OPT^b \cap NE^b) \\
&= 2^2a + 2a + 5 = 6a + 5 \\
\Rightarrow \qquad PoA &= \frac{cost(OPT)}{cost(NE)} = \frac{6a+5}{4a+6}
\end{aligned}
$$

The price of anarchy goes to $3/2$ when $a \to \infty$.

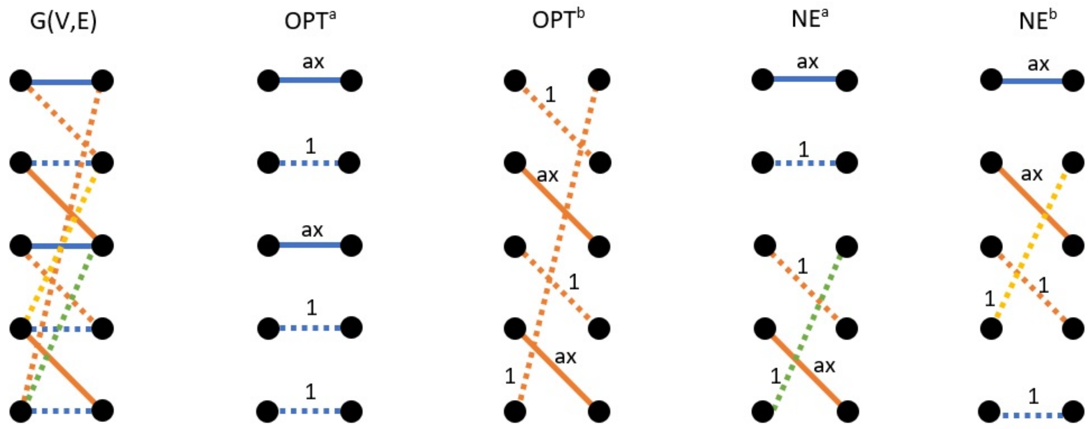


Figure 6.1: Example of symmetric matching congestion game with affine cost functions with a price of anarchy equal to $3/2$

Another direction which is interesting to study, is asymmetric matching congestion games, because an upper bound would also yield the same bound for the symmetric case.

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