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Basics for a polyhedron semiring

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Preface

Both this project and my whole Bachelor has been challenging and thought-provoking. I would not have been able to do all of this without:

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Finally, all figures in this paper are Made with Geogebra from GeoGebra GmbH at https://www.geogebra.org and I thank them for helping me visualise the various polyhedra described in this paper.

Basics for a polyhedron semiring

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Abstract

We explore the polyhedron semiring and show an approximation to solving matrix equalities over the polyhedron semiring, as well as show that some of the unbounded facets behave as a semiring and that the matrix equality can be reduced to individual inequalities. Finally, convex hull of vectors of polyhedra is introduced. This paper can be used to advance neural networks or to do calculations with intervals instead of average values.

Keywords: unbounded polyhedron, tropical semiring, system of equations, system of inequalities, pointed cone, simplicial cone

1 Introduction

In the mathematical field of algebra, many structures are taught, including groups, rings and fields [1]. They are defined by some simple axiomatic properties and from this other statements can be deduced. In this paper, we focus on a lesser-known structure called a semiring. As the name suggests, it is most similar to a ring, yet lacking the requirement that an additive inverse must exist. This gives more freedom to work with different objects than points, such as polyhedra. As this has not been explored yet, this paper focuses on simple properties that the polyhedra from the polyhedron semiring have. Nevertheless, many properties can be slightly modified and applied to other semirings or polyhedra. Some applications of the polyhedron semiring are neural networks [2, 3, 4] and interval arithmetic. When mathematics is applied to any problem where the values are taken from measurements, people often use the average measured value for calculations. However, the average value might not be representative of the sample itself - the average value cannot represent the whole sample and in that case the calculations might not be accurate. This can be solved by using intervals instead of precise values, such as 95% error intervals. However, then we run into some problems such as: what is the underlying structure of the calculations? It is clear that substituting a polyhedron in place of a single point can quickly stop being intuitive or even not be defined. This paper has the potential to improve the understanding of such calculations. The inspiration for this paper is taken from a paper by Speyer and Sturmfels [5], specifically the following research problem.

> The tropical semiring generalizes to higher dimensions: The set of convex polyhedra in $\mathbb{R}^n$ can be made into a semiring by taking $\odot$ as "Minkowski sum" and $\oplus$ as "convex hull of the union". [..] Develop linear algebra and algebraic geometry over these semirings.

One of the main results of this paper is an algorithm for approximating a solution of the equality $A + X = B$ for polyhedra, given that $A \leq B$ if the polyhedron A is contained in polyhedron B. When solving the corresponding inequality $A + X \leq B$, many solutions exist, so there is some maximal polyhedron X that solves the inequality to be as close to an equality as possible, and the presented algorithm makes use for that. Another result is the recursiveness of some facets of polyhedra: we work with unbounded polyhedra, and so some of their facets are unbounded as well, and we show that the structure of these facets is again a polyhedron semiring but in one less dimension. After that, we show that for this specific semiring, solving matrix inequalities is considerably easier than solving matrix inequalities for the ordinary real numbers, as it can be reduced to solving each summand separately. We also look at equalities and show a way to find a solution if it exists, as well as when no solution exists. We show how to approximate the solution using inequalities, and finally construct some convex hulls of vectors of polyhedra.

All results from Sections 3 to 7, as well as the proof of Theorem 2.13 are the first advances for working with the polyhedron semiring, they have not been presented before to the best of author's knowledge.

In Section 2 we will cover some preliminaries to familiarize the reader with the topic, such as how is a polyhedron defined and what properties does a semiring have. After that, we will introduce the specific polyhedron semiring and explore some special cases of the semiring in Section 3. Then in Section 4 we will present specific properties of polyhedra that extend infinitely in some fixed directions and propose the name "polyhedron semiring with a fixed recession cone". This set has not been explored before, at least to the author's knowledge. Section 5 focuses on solving $A+X \leq B$ for single polyhedron. Then in Section 6, which presents how multi-polyhedra vectors can be defined, we cover some properties of matrix equality for polyhedra $\mathbf{A}+\mathbf{X}=\mathbf{B}$, and demonstrate that an inequality solving algorithm is the best way to approximate the solution. Section 7 explores how tropically convex hulls of vectors of polyhedra can be defined and what is the analogy with the tropical semiring.

There is a clear relation between rings and semirings, with the latter missing one property. This means that some of the statements presented are generalized statements of rings. Similarly, many of the ideas covered in this paper could potentially be generalized by disregarding some of the necessary conditions. Relaxing the structure of the semiring might sometimes give a more general result of some statements, nevertheless, it can happen that an algorithm becomes unusable if the conditions are slightly changed.

The algorithms presented in this paper can be improved as they do not give the optimal solution yet, which is left for future research.

1.1 Related work

Many papers exist about interval semirings, for example Kandasamy [6], Speyer and Sturmfels [5], Hardouin et al. [7], and Itenberg, Mikhalin, and Shustin [8] have written about this. However, this paper generalizes a specific interval semiring for multiple dimensions and therefore gives broader statements, as well as the first algorithms for equality approximation. A property conserved by this generalization is idempotency, and more information on this can be found in a paper by Litvinov [9].

One of the polyhedron semiring applications [10] is in neural networks, where using a tropical polyhedra has the advantage of having fewer defining objects than an ordinary polyhedra [4]. This is often related to Newton polyhedra, which are polyhedra defined by tropical multivariate polynomials. As a polyhedron can be decomposed into a cone and a polytope using the Minkowski-Weyl theorem [11, 10, 12], the polytopes can be defined by some powers as well [13] which might give more insight into operations with the polyhedra, however, this paper does not go into this relation. We focus more on the polyhedral properties than on polynomial properties, though it could give insight on solving

$A + X = B$ as when a solution exists, it can be found by division [14], [8] Section 1.5., [3], or through integral decomposition [15].

It is given by Gao and Lauder [15] and Hertrich [2] that checking whether a convex integral two-dimensional polygon has an integral decomposition is NP-complete, and therefore the algorithms presented in this paper give better computing time for approximations instead. A paper that nevertheless tries to solve $A + X = B$ for non-convex bodies is written by Sugihara [16], and an older paper by Hadwiger [17]. The paper by Mamatov and Nuritdinov [18] gives many properties of Minkowski difference which can potentially be used to improve the algorithm given in this paper.

2 Preliminaries

This section collects theorems and definitions mostly without involved proofs which have occured previously in other papers and therefore experienced readers may skip over this section. If the reader has not encountered any of the definitions or theorems, plenty of other resources exist that elaborate on each statement.

2.1 Polyhedra properties

More information about the statements presented in this subsection can be found in the papers of "Basic Concepts and Simplest Properties of Convex Polyhedra" [19] and Schrijver [20].

Let us first define the extended real numbers. Note that while it is never explicitly said, only the positive infinity is used here, which is in contrast to high-school mathematics where both $+\infty$ and $-\infty$ exist.

Definition 2.1 (Extended real numbers). The set $\mathbb{R}^m \cup \{\infty\}$ is called $\mathbb{T}^m$, $m \in \mathbb{Z}$, $m \geq 1$. Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^m$, then the sum of the two vectors is defined in the ordinary elementwise sense: $(\mathbf{a} + \mathbf{b})_i = \mathbf{a}_i + \mathbf{b}_i$ for all $i = 1, 2, \ldots, m$. Operations with infinity are defined as follows:

$$\mathbf{a} + \infty = \infty + \mathbf{a} = \infty$$
$$\infty + \infty = \infty$$

Let us now focus on having a set of points that behaves nicely.

Definition 2.2 (Convexity). The convex hull of some points $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n \in \mathbb{R}^m$ contains all points in $\mathbb{R}^m$ that are located between the given set of points.

$$\operatorname{conv}(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n) = \{\lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 + \cdots + \lambda_n \mathbf{a}_n : 0 \leq \lambda_i \leq 1, \sum_{i=1}^{n} \lambda_i = 1\}.$$

If a set S is not convex, one can calculate the convex hull of such a set which is the smallest convex set which contains set S.

In Figure 1, one can see the gray line as the convex hull of the two points A and B.

The convex hull of a set of vectors intuitively can be seen as blowing up an infinitely stretchy balloon such that all points are completely inside the balloon, and then letting the balloon deflate. It will stretch and touch some of the points, and possibly not touch some other points, but it will not have any dents, nor it will become two balloons. Now the reader imagining a balloon might ask, "when does the balloon remain round and when does it become more rectangular?" For that we first need to define how to cut the space into halves, and then we are ready to explore the definition of polyhedra and polytopes.

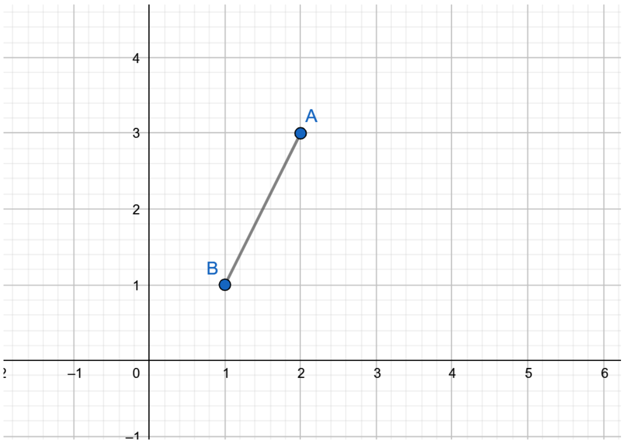

FIGURE 1: Convex hull of two points

Definition 2.3 (Hyperplane and halfspace). Let $\mathbf{a} = (a_1, a_2, \ldots, a_m) \in \mathbb{R}^m$ and $b \in \mathbb{R}$ be given. A hyperplane in $\mathbb{R}^m$ is a set of vectors $\{\mathbf{x} = (x_1, x_2, \ldots, x_m) : a_1x_1 + a_2x_2 + \cdots + a_mx_m = b\}$ and a halfspace is a set of vectors $\{\mathbf{x} = (x_1, x_2, \ldots, x_m) : a_1x_1 + a_2x_2 + \cdots + a_mx_m \leq b\}$.

One should also note that the elements of the halfspace lie exactly on the hyperplane and on one side of the hyperplane. This makes intuitive sense as by looking at the plane of $z \geq 0$ in three dimensions, it is simply all the points with positive z coordinate, which are all on one side of the plane $z = 0$.

Definition 2.4 (Polyhedron and polytope). Let k halfspaces exist, $k \in \mathbb{Z}$, $k \geq 0$, and let the i^{th} halfspace be defined by $\mathbf{n}_i$ and b_i for $i = 1, 2, \ldots, k$. A polyhedron is a set which can be defined by the intersection of finitely many halfspaces. If $k = 0$, the polyhedron is the whole space. For $k \geq 1$, any point $\mathbf{x}$ in the polyhedron satisfies all given inequalities $\mathbf{n}_i^T\mathbf{x} \leq b_i$, where $\mathbf{n}_i$ and b_i define a halfspace for $i = 1, 2, \ldots, k$. A polytope is a bounded polyhedron, it does not have directions in which it extends infinitely.

A specific polyhedron is given below in Definition 2.5.

Definition 2.5 (Nonnegative orthant). In $\mathbb{R}^m$ the orthonormal basis vectors $\mathbf{e}_i$, $i = 1, 2, \ldots, m$ are defined as follows:

$$(\mathbf{e}_i)_j = \begin{cases} 1, & \text{for } j = i \\ 0, & \text{for } j = 1, 2, \ldots, i-1, i+1, \ldots, m. \end{cases}$$

The polyhedron in $\mathbb{R}^m$ which is given by all points $\mathbf{x}$ that satisfy $\mathbf{x}^T\mathbf{e}_i \geq 0$ for all basis vectors $\mathbf{e}_i$, $i = 1, 2, \ldots, m$ is called the nonnegative orthant.

Theorem 2.6 (Supporting hyperplane and maximal points). *Between each point $\mathbf{y}$ that is not in the polyhedron P and points in the polyhedron, one can find the **supporting hyperplane** with normal $\mathbf{n}$ such that $\mathbf{n}^T\mathbf{p} > \mathbf{n}^T\mathbf{x}$ for all $\mathbf{x} \in P$. Given this supporting hyperplane and the corresponding halfspace, let F be the points $\hat{\mathbf{x}}$ in P for which a finite maximum is achieved, $F = \{\hat{\mathbf{x}} \in P : \mathbf{n}^T\hat{\mathbf{x}} \geq \mathbf{n}^T\mathbf{x}, \forall \mathbf{x} \in P\}$. Then F consists of points of P that are **maximal** in the direction $\mathbf{n}$. In other words, points of F lie on the supporting hyperplane.*

One can see general properties of polyhedra in “Basic Concepts and Simplest Properties of Convex Polyhedra” [19] and supporting hyperplanes in the paper by Mcmullen [21]. In n dimensions, each hyperplane equation defines an equation in n variables. If we want to find a vertex of a polyhedron, which is a unique solution to some intersecting hyperplanes, at least n hyperplanes are needed. It can happen that the solution is not unique, then more hyperplanes are needed, but it can never be the case that less than n hyperplanes define a vertex. For example, in two dimensions two lines may intersect at a point, in three dimensions at least three planes are needed to define a point. Therefore if k hyperplanes already exist, then adding another hyperplane adds at most $\binom{k}{n-1}$ vertices and therefore as a polyhedron is defined by finitely many hyperplanes, it also has finitely many vertices.

The following definition is of a specific type of polyhedra that is important throughout the paper as it will be needed to specify them quite often.

Definition 2.7 (Cone). A convex polyhedral cone C is a polyhedron having the property: for $\mathbf{s}, \mathbf{r} \in C$, also $\lambda_1 \mathbf{s} + \lambda_2 \mathbf{r} \in C$, $\lambda_1, \lambda_2 \geq 0$.
A **pointed cone** is a cone which does not contain subspaces apart from the trivial subspace of $\{\mathbf{0}\}$, and a **simplicial cone** in n dimensions is a cone which is of the following form:

$$\left\{ \sum_{i=1}^{n} \lambda_i \mathbf{x}_i : \lambda_i \geq 0, i = 1, 2, \ldots, n, \quad \{\mathbf{x}_1, \ldots, \mathbf{x}_n\} \text{ are linearly independent} \right\}.$$

The vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are called the spanning vectors of the cone.

By relaxing the requirement that this set must be a polyhedron, a cone does not have to be convex or be defined by finitely many halfspaces. An example of a non-convex cone is the set $\{\lambda_1(1,0) : \lambda_1 \geq 0\} \cup \{\lambda_2(0,1) : \lambda_2 \geq 0\}$ which only contains two rays. A cone also does not have to be polyhedral by allowing it to have infinitely many spanning vectors, such as having a round cone with vectors $\{(x, y, z) : \sqrt{x^2 + y^2} = z\}$ contained in its boundary. However, this paper focuses on convex polyhedral cones and therefore the words "convex polyhedral" will be omitted from "convex polyhedral cone".

The definition of a pointed cone implies that there are no complete lines through zero that are completely contained in the cone, there are no nonzero points $s \in C$ such that $\lambda s \in C$ for both positive and negative λ.

Some examples for simplicial cones include the cone spanned by $\{(1,0), (1,1)\}$ in $\mathbb{R}^2$ as it includes two linearly independent vectors, or the cone spanned by $\{(1,0,0), (-1,0,0), (3,2,1), (0,0,1)\}$ which is not pointed. An important simplicial cone in $\mathbb{R}^m$ is the cone spanned by the basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m\}$. An example for non-simplicial cone is the ray spanned by $\{(1,0)\}$ in $\mathbb{R}^2$ as it does not contain two linearly independent vectors. One can see that intuitively a simplicial cone in n dimensions should have nonzero hyper-volume of n dimensions, so in two-dimensions any simplicial cone should have nonzero area and in three dimensions any simplicial cone should have nonzero volume. Finally, we point out that the nonnegative orthant is a simplicial pointed cone.

Let us now look at the boundary of polyhedra in more detail and define some useful concepts.

Definition 2.8 (Face, facet and vertex). A face of a polyhedron P is a subset F of points of the polyhedron for which there exists a vector $\mathbf{c}$ such that the set of points $\hat{\mathbf{p}} \in F$ attains the maximum of $\mathbf{c}^T \mathbf{p}$ over all $\mathbf{p} \in P$, provided this maximum is finite. In three dimensions, faces are vertices, edges and ordinary faces. A facet is a maximal face relative to inclusion that is not the polyhedron P itself. In two dimensions the equivalent of a facet is an edge, in three dimensions it is an ordinary face. One can observe that a facet has one less dimension than the polyhedron itself. Finally, a vertex is a one-dimensional face.

It might be confusing to the reader that face does not mean "a two-dimensional polygon that is the side of a three-dimensional object" but instead extends to any set of maximal points. The reader should familiarize themselves with these definitions as in higher dimensions the "sides" of polyhedra might not have specific words. One can see that faces are simply points in the polyhedron which belong to one or more of the polyhedron's defining hyperplanes. Similarly, introducing facets will be useful as they have one less dimension than the polyhedron itself and under the assumption that the polyhedron is full-dimensional, a facet can be defined by exactly one hyperplane.

Remark 2.9 (Faces are polyhedra). Each face of a polyhedron P is a polyhedron. Let the face be described by $c_{max} = \mathbf{c}^T\mathbf{p}$ for $p \in P$. Since P can be described as a set of linear inequalities, the face can be described by the same set of linear inequalities with an additional inequality $-\mathbf{c}^T\mathbf{p} \leq -c_{max}$ and therefore is a polyhedron.

2.2 Minkowski addition

A binary operator of sets called Minkowski addition is defined in this subsection, and some of its properties are presented afterwards. The addition of two sets is defined straightforwardly.

Definition 2.10 (Minkowski addition). For any two sets S_1, S_2 in the same space, their Minkowski addition can be defined as $S_1 + S_2 = \{a + b : a \in S_1, b \in S_2\}$.

Minkowski addition is quite important to understand as it is one of the two main operations we work with in this paper, along with convex hull. An example for Minkowski addition is in Figure 2.

Theorem 2.11 (Minkowski-Weyl Theorem). *Any polyhedron P can be written as the Minkowski addition $P = T + C$, where T is a polytope and C is a cone, called the* ***recession cone****. The recession cone is unique, whereas T might not be.*

For a proof, see Blair and Schrijver [12] Corollary 7.1b or Fukuda [11]. The former also indirectly proves the following corollary.

For any polyhedron, the recession cone contains the zero vector, namely if $\mathbf{p} \in P$, then also $\mathbf{p} + \mathbf{0} \in P$. For polytopes, the recession cone only contains the zero vector. If it did not, we could take any point $\mathbf{p}$ from the polytope and add an arbitrary large nonzero vector from C, so the polytope would not be bounded. We will primarily focus on unbounded polyhedra in this paper, and Corollary 2.12 gives some intuition for the representation of such polyhedra. If the recession cone is known, it can be disregarded as this corollary tells us how it can be quickly restored.

Corollary 2.12 (Finite representation). *Since any polyhedron P can be broken down in a polytope and a unique recession cone $\mathbb{O}$ by Theorem 2.11, the polytope has the minimal representation conv$(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ which is exactly convex hull of the vertices of P. In that case we write $P = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} + \mathbb{O}$ or $P = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ if the cone is clear from the context.*

A paper that explores Minkowski addition both in terms of vertices of polyhedra and in terms of intersecting halfspaces is by Gabidullina [22]. While each polyhedron has a minimum representation, in this paper it is not crucial that we use the minimum representation, any finite representation can be used which simplifies the notation as there is no need to check which vertices can be discarded in the convex hull. Nevertheless, in practical applications one should try to reduce the number of points from which the convex hull is calculated, and Fukuda [11] presents such an algorithm.

Theorem 2.13 (Vertices of Minkowski sum). *Let P be the Minkowski sum of convex sets P_1 and P_2. This is equal to the statement that the maximal points of P in the direction of any $\mathbf{d}$ are ordinary sums of maximal points of P_1 and P_2 in the direction of $\mathbf{d}$.*

Proof. The structure of the proof goes as follows: we begin by recalling properties about maximal points and Minkowski sum. Then we assume that the maximal property of P and $P_1 + P_2$ does not hold, and show that $P \neq P_1 + P_2$. After that, we assume that the Minkowski sum does not hold and show that there is a direction in which the maximal property does not hold by looking at two cases.

The maximal points of P_1 and P_2 in the direction $\mathbf{d}$ are points that lie on the supporting hyperplanes of P_1 and P_2 correspondingly, with normal $\mathbf{d}$. Such points have the restriction $\mathbf{d}^T\mathbf{p} = c_i$ for some finite constants c_1, c_2, that may differ in P_1 and P_2, and all other points $\mathbf{s}_1 \in P_1$, $\mathbf{s}_2 \in P_2$ have that $\mathbf{d}^T\mathbf{s}_1 < c_1$ and $\mathbf{d}^T\mathbf{s}_2 < c_2$. Let $\mathbf{p}_1 \in P_1$ and $\mathbf{p}_2 \in P_2$ that are maximal in direction $\mathbf{d}$. The sum of the maximal points will result in $\mathbf{d}^T(\mathbf{p}_1 + \mathbf{p}_2) = \mathbf{d}^T\mathbf{p}_1 + \mathbf{d}^T\mathbf{p}_2 = c_1 + c_2$.

Assume that there is a point $\mathbf{p}$ in P that is maximal in direction $\mathbf{d}$ but cannot be represented as the sum of any of the corresponding maximal points $\mathbf{p}_1$, $\mathbf{p}_2$, where $\mathbf{p}_1 \in P_1$, $\mathbf{p}_2 \in P_2$. Then $\mathbf{d}^T\mathbf{p} \neq \mathbf{d}^T\mathbf{p}_1 + \mathbf{d}^T\mathbf{p}_2$. There are two cases: if $\mathbf{d}^T\mathbf{p} < \mathbf{d}^T\mathbf{p}_1 + \mathbf{d}^T\mathbf{p}_2$, then there are points $\mathbf{p}_1$, $\mathbf{p}_2$ whose sum is not in the set P, in which case $P_1 + P_2 \neq P$. The other case is $\mathbf{d}^T\mathbf{p} > \mathbf{d}^T\mathbf{p}_1 + \mathbf{d}^T\mathbf{p}_2$, in which case there are no two points $\mathbf{s}_1 \in P_1$, $\mathbf{s}_2 \in S_2$ such that $\mathbf{s}_1 + \mathbf{s}_2 = \mathbf{p}$ because $\mathbf{d}^T\mathbf{s}_1 + \mathbf{d}^T\mathbf{s}_2 \leq \mathbf{d}^T\mathbf{p}_1 + \mathbf{d}^T\mathbf{p}_2 < \mathbf{d}^T\mathbf{p}$. Therefore if the maximality condition does not hold, then $P \neq P_1 + P_2$.

Let us now prove that if the Minkowski sum does not hold, there exists a direction in which the maximal property does not hold. Assume P_1, P_2, P are convex sets, $P_1 + P_2 \neq P$. There are two possible cases: either there is a point in P that is not the sum of any two points of P_1 and P_2 respectively, or there is a sum $\mathbf{p}_1 + \mathbf{p}_2$, where $\mathbf{p}_1 \in P_1$, $\mathbf{p}_2 \in P_2$ such that $\mathbf{p}_1 + \mathbf{p}_2 \notin P$. These cases can overlap.

Assume that there is a point $\mathbf{p}_0$ in P that is not the sum of any two points of P_1 and P_2. Find the Euclidean distance from this point to the set $P_1 + P_2$, and call the direction of this distance $\mathbf{d}$. Then in the direction of $\mathbf{d}$, the maximal points $\mathbf{x}$ of P have their constant $\mathbf{d}^T\mathbf{x} \geq \mathbf{d}^T\mathbf{p}_0$, but $\mathbf{d}^T(\mathbf{p}_1 + \mathbf{p}_2) < \mathbf{d}^T\mathbf{p}_0 \leq \mathbf{d}^T\mathbf{x}$ for any $\mathbf{p}_1 \in P_1$, $\mathbf{p}_2 \in P_2$. Therefore there is a point $\mathbf{x}$ of P that cannot be expressed as the Minkowski sum of any two points of P_1, P_2 respectively.

Let us now assume that there exist $\mathbf{p}_1 \in P_1$, $\mathbf{p}_2 \in P_2$ such that the sum $\mathbf{p}_1 + \mathbf{p}_2$ that is not in P. By Theorem 2.6, there exists a supporting hyperplane with normal $\mathbf{d}$ between $\mathbf{p}_1 + \mathbf{p}_2$ and P. Let $\mathbf{p}$ be the closest point to $\mathbf{p}_1 + \mathbf{p}_2$ that is in P. Then by Definition 2.3, $\mathbf{d}^T(\mathbf{p}_1 + \mathbf{p}_2) > \mathbf{d}^T\mathbf{p}$. Therefore in the direction of $\mathbf{d}$, the maximal points of P_1 and P_2 do not sum up to the points of P. □

The theorem is also covered by Barki, Denis, and Dupont [23] and Gao and Lauder [15], along with other properties of maximal points of Minkowski sums, however, these papers do not give explicit proofs for this statement. This theorem implies that the vertices of a polyhedron P are exactly sums of vertices of P_1 and P_2 and no other vertices exist. Even more, the number of vertices of P must be larger or equal to the number of vertices of P_1 and larger or equal to the number of vertices of P_2. It also works the other way around, as when for all directions d, the maximal points of polyhedron P are exactly the sums of maximal points of P_1 and P_2, then $P = P_1 + P_2$.

It is clear that a polyhedron in $\mathbb{P}_m$ with only one vertex $\mathbf{v}$ is a translated cone, since the existence of $\mathbb{O}$ implies that the cone starting from $\mathbf{v}$ must be in the polyhedron. At the same time, no point outside of this translated cone is in the polyhedron. Similarly, one can see that for a polyhedron $P \in \mathbb{P}_m$ executing $P \odot \{\mathbf{v}\}$ simply translates all the vertices by Theorem 2.13. If two polytopes P_1 and P_2 are added, their Minkowski sum can be found by calculating all possible $\mathbf{v}_1 + \mathbf{v}_2$ for all vertices $\mathbf{v}_1 \in P_1$ and all vertices $\mathbf{v}_2 \in P_2$, and taking their convex hull.

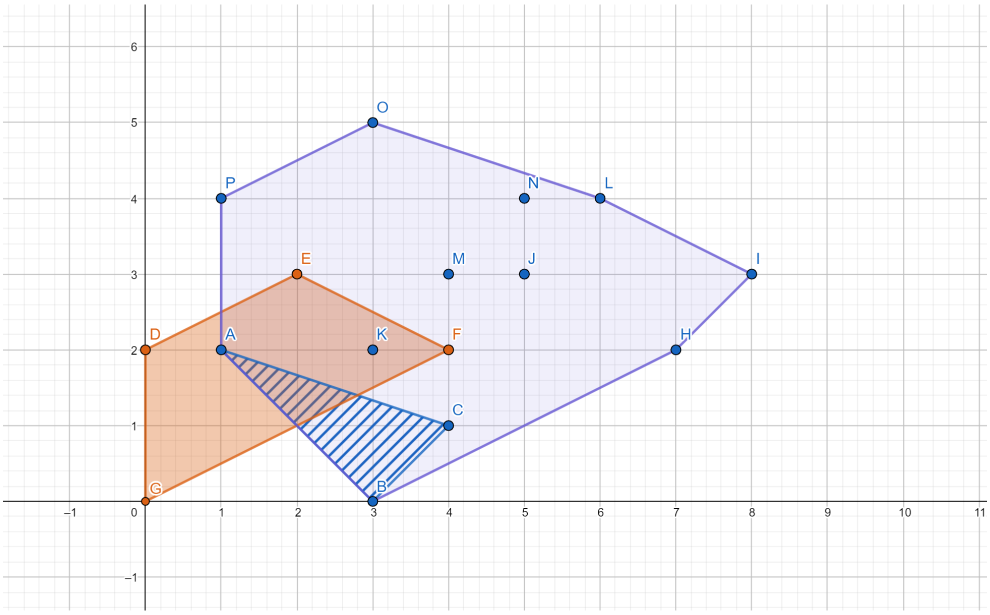

FIGURE 2: Minkowski sum of the striped triangle and dark quadrangle

One can see an example of Minkowski sum in Figure 2 where the larger polyhedron is the sum of the striped triangle and dark quadrangle. Other sums of vertices are also portrayed in the figure as points, for example, point M is the sum of points C and D, however, it does not lie on the boundary.

2.3 Semiring and special cases

We now continue to introduce an idea that is most likely **not** introduced in high-school mathematics and therefore the reader should make sure they understand the topic. The semiring can have parallels with rings or fields, with the exception of some properties that are missing here. Almost any problem where rings or fields are used, can have semirings substituted in them to produce a slightly different problem which can in turn lead to greatly different solutions and provoke thought. A paper that goes into more details and draws these parallels is presented by Gaubert and Katz [10].

Definition 2.14 (Semiring). A semiring R is a set on which two binary operations $\odot$, $\oplus$ are defined such that:

- $(R, \oplus)$ is a commutative monoid with additive identity 0:
  - $(a \oplus b) \oplus c = a \oplus (b \oplus c)$
  - $0 \oplus a = a = a \oplus 0$
  - $a \oplus b = b \oplus a$
- $(R, \odot)$ is a monoid with multiplicative identity 1:
  - $(a \odot b) \odot c = a \odot (b \odot c)$

– $1 \odot a = a = a \odot 1$

- $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$
- $(a \oplus b) \odot c = (a \odot c) \oplus (b \odot c)$
- $a \odot 0 = 0 = 0 \odot a$

Some examples of semirings include the real numbers with ordinary addition and multiplication, or the set $\mathbb{Z}_2 = \{0, 1\}$, where the addition is the OR operation and the multiplication is the AND operation. We will focus on a slightly different semiring, where minimum is used.

Definition 2.15 (Idempotency). Let a semiring R be given. If for any $a \in R$, $a \oplus a = a$, the set R is called an idempotent semiring.

More information about idempotency and idempotent semirings can be found in Litvinov [9].

Proposition 2.16 (Tropical semiring). *The set of the real numbers together with infinity* $\mathbb{R} \cup \{\infty\} = \mathbb{T}$ *is a semiring with operations* $a \oplus b = \min(a, b)$ *and* $a \odot b = a + b$. *The additive identity is* ∞ *and the multiplicative identity is* 0, *and the semiring itself is called the tropical semiring.*

The reader is invited to check that the tropical semiring indeed satisfies all of the properties above. Other literature uses maximum and $-\infty$ instead of minimum and ∞, nevertheless this change is simple as both semirings are isomorphic in the sense that multiplying every element by -1 will produce a set that behaves like the tropical semiring defined above. The paper by Kandasamy [6] uses intervals of the form $[0, a]$ which is again isomorphic to this tropical semiring by transforming each interval to $[\frac{1}{a-1}, \infty]$ for $0 < a < 1$ and $[a - 1, \infty]$ for $a \geq 1$. Additionally, since $\min(a, a) = a$, the tropical semiring is idempotent. A good paper for basics of tropical mathematics is Speyer and Sturmfels [5], which is the basis for this paper. Finally, Itenberg, Mikhalin, and Shustin [8] goes into great detail about tropical geometry and explores different curves and shapes under a logarithmic transformation which produces sets with tropical properties.

Example 2.17. Let us calculate a simple expression in the tropical semiring.

$$\begin{aligned}
(3 \oplus 4) \odot (4 \oplus 10) &= \\
\min(3, 4) \odot \min(4, 10) &= \\
3 \odot 4 &= \\
3 + 4 &= 7
\end{aligned}$$

3 Polyhedron semiring

In the previous section we worked with the tropical semiring, which has been explored previously. It can be generalized to tropical geometry, where each entry of a vector comes from a tropical semiring, or by multivariate polynomials and their corresponding Newton polytopes [14, 3]. However, we now define a different generalization of the tropical semiring. One can observe that this generalization preserves many properties of the tropical semiring, such as idempotency.

Proposition 3.1 (Polyhedra with a pointed recession cone semiring). *The set of polyhedra with vertices in $\mathbb{R}^m$ and the nonnegative orthant (see Definition 2.5) of m dimensions as the recession cone $\mathbb{O}$ together with the infinity polyhedron is a semiring with $\oplus$ as the convex hull and $\odot$ as the Minkowski addition. The semiring in m dimensions is called a polyhedron semiring $\mathbb{P}_m$. For this semiring, the point $\{\infty\}$ is the additive identity as defined below and the cone $\mathbb{O}$ itself is the multiplicative identity. The nonnegative orthant is both a pointed cone and a simplicial cone and is m-dimensional.*

The convex hull of two polyhedra P_1, P_2 whose corresponding polytopes are bounded is defined as such: $P_1 \odot P_2 = \{\lambda \mathbf{p}_1 + (1-\lambda)\mathbf{p}_2 : 0 \geq \lambda \geq 1, \mathbf{p}_1 \in P_1, \mathbf{p}_2 \in P_2\}$. The convex hull of $\{\infty\}$ with itself is $\{\infty\}$, and the convex hull of $\{\infty\}$ and any polyhedra P with bounded corresponding polytope is the polyhedra P itself.

The infinity polyhedron $\{\infty\}$ is defined as being contained in the intersection of any two polyhedra of the polyhedron semiring, and it interacts with a polyhedron from the semiring in the following manner:

$$
\begin{aligned}
P \oplus \{\infty\} &= P \\
P \odot \{\infty\} &= \{\infty\} \\
\{\infty\} \oplus \{\infty\} &= \{\infty\} \\
\{\infty\} \odot \{\infty\} &= \{\infty\}
\end{aligned}
$$

Proof. The cone is preserved under each of the operations. Let us look at the convex hull of $P_1 \oplus P_2$ first, $P_1, P_2 \in \mathbb{P}_m$.

Let us prove that the cone of $P_1 \oplus P_2$ contains the nonnegative orthant cone. A point $\mathbf{p}$ in $P_1 \oplus P_2$ is a convex combination of a point $\mathbf{p}_1$ from P_1 and a point $\mathbf{p}_2$ from P_2, $\mathbf{p} = \lambda\mathbf{p}_1 + (1-\lambda)\mathbf{p}_2$, where $0 \leq \lambda \leq 1$. Then for any vector $\mathbf{c}$ from the cone $\mathbb{O}$, we can look at $\lambda(\mathbf{p}_1 + \mathbf{c}) + (1-\lambda)(\mathbf{p}_2 + \mathbf{c})$. Since $\mathbf{c}$ is in the cones of P_1 and P_2, also $\lambda\mathbf{c}$ and $(1-\lambda)\mathbf{c}$ is in the cone, so $\mathbf{p}_1 + \mathbf{c} \in P_1$ and $\mathbf{p}_2 + \mathbf{c} \in P_2$. Therefore the point $\mathbf{p} + \mathbf{c}$ is a convex combination of points of P_1 and P_2, which means that the cone of $P_1 \oplus P_2$ contains the cone $\mathbb{O}$.

Now let us prove that any vector outside of the nonnegative orthant is not contained in the recession cone of $P_1 \oplus P_2$. Take some direction vector $\mathbf{v}$ that does not lie in the nonnegative orthant, then from any point $\mathbf{p}_1$ in P_1 there is some finite scaled version of this vector $\gamma_1\mathbf{v}$, with $\gamma_1 \geq 0$, that can be added such that $\mathbf{p}_1 + \gamma_1\mathbf{v}$ is in the set P_1. Similarly for P_2, from any point the half ray in direction of $\mathbf{v}$ is not fully contained in P_2. Therefore in $P_1 \oplus P_2$, from any point $\lambda(\mathbf{p}_1+\gamma_1\mathbf{v})+(1-\lambda)(\mathbf{p}_2+\gamma_2\mathbf{v}) = \lambda\mathbf{p}_1+(1-\lambda)\mathbf{p}_2+(\gamma_1\lambda+(1-\lambda)\gamma_2)\mathbf{v}$, the half-ray is not contained in $P_1 \oplus P_2$ for arbitrarily large scaled $\mathbf{v}$, therefore $\mathbf{v}$ is not in the cone of $P_1 \oplus P_2$.

Let us now focus on the Minkowski addition and whether it preserves the cone, we will start with proving that the cone of $P_1 \odot P_2$ contains the nonnegative orthant cone.

Take an arbitrary point $\mathbf{p} \in P_1 \odot P_2$, and take an arbitrary vector $\mathbf{c} \in \mathbb{O}$. By definition, the point $\mathbf{p}$ can be broken down as $\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2$ for some $\mathbf{p}_1 \in P_1$ and $\mathbf{p}_2 \in P_2$. It is known that $\mathbf{p}_1 + \mathbf{c} \in P_1$ as the recession cone of P_1 contains $\mathbf{c}$, and therefore $\mathbf{p} = (\mathbf{p}_1 + \mathbf{c}) + \mathbf{p}_2$. As the point $\mathbf{p} + \mathbf{c}$ can be expressed as the sum of two points from P_1 and P_2 respectively, the point is in the Minkowski sum of the two polyhedra.

Now let us prove that any direction vector that is not in the nonnegative orthant, is not contained in the recession cone of $P_1 \odot P_2$. Take an arbitrary direction vector $\mathbf{v}$ that does not point in the nonnegative orthant, then for an arbitrary point $\mathbf{p}_1 \in P_1$ there is some maximal γ such that $\mathbf{p}_1 + \gamma_1\mathbf{v}$ is contained in P_1, $\gamma_1 \geq 0$. Similarly the half ray from an arbitrary point of P_2 parallel to $\mathbf{v}$ is not fully contained in P_2. Then

$(\mathbf{p}_1 + \gamma_1 \mathbf{v}) + (\mathbf{p}_2 + \gamma_2 \mathbf{v}) = \mathbf{p}_1 + \mathbf{p}_2 + (\gamma_1 + \gamma_2)\mathbf{v}$ does not extend infinitely in the direction of $\mathbf{v}$ and therefore $\mathbf{v}$ is not in the cone of $P_1 \odot P_2$. $\square$

One can see an example of addition and multiplication in Figure 3, where A is the polyhedron starting from the vertex A, and B is the polyhedron starting from vertex B. The light gray polyhedron encompasses both A and B, as it is the tropical sum of A and B, and the dark polyhedron starting from H is the multiplication of A and B, as it is the Minkowski sum of both.

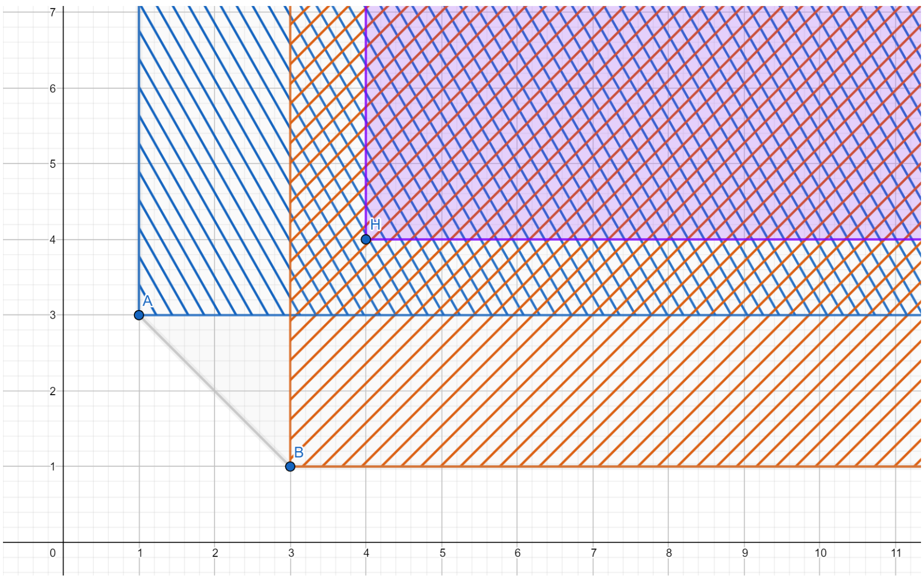

FIGURE 3: Sum and multiplication of two polyhedra in $\mathbb{P}_2$

Proposition 3.2 (Change of basis). *The set of polyhedra with vertices in $\mathbb{T}^m$ and any pointed simplicial recession cone O is a semiring with $\oplus$ as the convex hull and $\odot$ as the Minkowski addition.*

Since the cone is simplicial, it is spanned by m different vectors. Use change of basis to convert the space to the polyhedron semiring with the nonnegative orthant as the recession cone, and revert the change of basis when the original points are needed.

The above proposition means that all the possible simplicial pointed recession cones can be transformed into one another. Therefore all statements where $\mathbb{P}_m$ is used, the cone can be slightly different and a change of basis can be applied to find the corresponding statement for a recession cone that is not the nonnegative orthant. Because of that, when we now use a specific pointed recession cone, we always talk about the nonnegative orthant. Even so, the reader may still apply an analogous statement to their own work.

Proposition 3.3 (Convex sets with a pointed recession cone). *The set of convex sets (as opposed to polyhedra) in $\mathbb{T}^m$ and the nonnegative orthant as cone $\mathbb{O}$ is a semiring. The corresponding addition $\oplus$ is the convex hull and the corresponding multiplication $\odot$ is the Minkowski addition.*

One can see that this is a supersemiring of the polyhedron semiring, as it includes all polyhedra from $\mathbb{P}_m$ while also having elements that are not in the polyhedron semiring. For example, the ball with radius 1 around origin plus the cone of nonnegative orthant is included in this set.

From this point on, when the word "addition" is used, it refers to tropical addition unless specified as "Minkowski addition" or "ordinary addition". Similarly, the word "multiplication" refers to tropical multiplication unless specified as "ordinary multiplication". The multiplication symbol for tropical multiplication is $\odot$, whereas the ordinary multiplication symbol is $\cdot$ or concatenation. One should note, though, that the symbols 0 and 1 are used to represent the ordinary zero and unity.

Example 3.4 (Special case of polyhedron semiring). The semiring $\mathbb{P}_1$ from Proposition 3.1 is isomorphic to the tropical semiring from Proposition 2.16 since the polyhedra in this case are simply intervals with the largest endpoint being ∞. It can be seen that the smallest endpoints behave like the tropical semiring:

$$\begin{aligned}[a,\infty]\oplus[b,\infty] &= \mathrm{conv}([a,\infty],[b,\infty]) = [\min(a,b),\infty] \\ a\oplus b &= \min(a,b) \\ [a,\infty]\odot[b,\infty] &= [a,\infty]+[b,\infty] = [a+b,\infty] \\ a\odot b &= a+b.\end{aligned}$$

Example 3.5 (Integer polyhedron semiring). The subset of polyhedron semiring $\mathbb{P}_m$ with vertices with all integer coordinates is a subsemiring.

The convex hull of polyhedron from $\mathbb{P}_2$ with vertices $\{(4,0),(1,1),(0,4)\}$ and polyhedron with vertices $\{(3,0),(0,3)\}$ has vertices $\{(3,0),(1,1),(0,3)\}$ and their Minkowski sum has vertices $\{(7,0),(4,1),(1,4),(0,7)\}$. A visualisation of this can be seen in Figure 4.

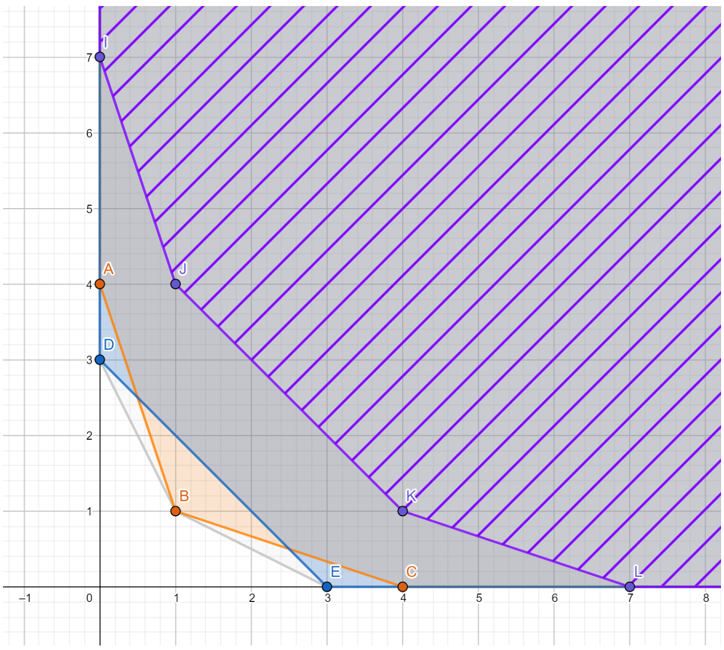

FIGURE 4: Example of polyhedra from $\mathbb{P}_2$ with integer vertices

Example 3.6 (Symmetric polyhedron semiring). Let the polyhedron semiring $\mathbb{P}_m$ be given, $m \geq 2$. Let the i^{th} coordinate of each vertex be equal to the j^{th} coordinate of the respective vertex. Then this set of restricted polyhedra is a subsemiring of the polyhedron semiring. It is also possible that more than two coordinates are be restricted this way. For example in $\mathbb{P}_3$, if all vertices of any polyhedra is of the form (a,a,b), then the Minkowski sum will also contain vertices of this form and so will the convex hull because no new vertices appear in the convex hull.

Example 3.7 (Nonnegative polyhedron semiring). The subset of polyhedron semiring $\mathbb{P}_m$ with vertices with all nonnegative coordinates is a subsemiring. The convex hull gains no new vertices and the Minkowski sum of two vertices still has all entries nonnegative.

These subsemirings of Examples 3.4, 3.5, 3.6 and 3.7 are not proven here as properties of later sections are needed for easier proofs, and are more observations rather than functional statements.

4 Properties of the polyhedron semiring

We now explore the polyhedron semiring from Proposition 3.1 and its properties. Let us first try to order the polyhedra. An idea that could be used is inclusion: polyhedron A is smaller than polyhedron B if it is inside B. However, this does not work for arbitrary two polyhedra, what if they only partially overlap? For such a set of polyhedra, we cannot always compare two objects, so only partial order exists.

Definition 4.1 (Partial order). Polyhedra can be partially ordered by inclusion: for polyhedra A and B, if $A \oplus B = A$, it is then said that $B \leq A$. Similarly, two polyhedra A and B are called equal, $A = B$, if both $A \leq B$ and $B \leq A$.

Lemma 4.2. *The equality $A \oplus B = A$ holds if and only if all of B is contained in A, $A \subseteq B$.*

While partial order exists, we can try to compare the objects which partially overlap by trying to find the minimum and maximum of the two polyhedra. In such a sense we define a lattice.

Definition 4.3 (Lattice). Let a, b be elements of S, and let $a \wedge b = c$ be the largest element in S such that $a \geq c$ and $b \geq c$. Similarly, let $a \vee b = d$ be the smallest element in S such that $a \leq d$ and $b \leq d$.

Now it can be seen that for the polyhedron semiring $\mathbb{P}_m$, the smallest element that includes any two polyhedra P_1 and P_2 is the convex hull of the polyhedra, as taking away some point in this convex hull will either remove a point that is in P_1 or P_2, or will destroy the convexity property of this set. Similarly, the largest element that is contained in both P_1 and P_2 is the intersection of the two polyhedra. Here we can see that the infinity polyhedron appears naturally as it is contained in all intersections of polyhedra, and such it is the smallest element of all polyhedra from $\mathbb{P}_m$.

Theorem 4.4 (Union and intersection). *Let $A, B \in \mathbb{P}_m$ and assume the inequality $A \odot X \leq B$ has two solutions $X_1, X_2 \in \mathbb{P}_m$. Then also $X_1 \wedge X_2 = X_1 \cap X_2$ and $X_1 \vee X_2 = conv(X_1 \cup X_2)$ are solutions to $A \odot X \leq B$.*

Proof. To prove the theorem for union, add the two inequalities $A \odot X_1 \leq B$ and $A \odot X_2 \leq B$:

$$\begin{aligned} & (A \odot X_1) \oplus (A \odot X_2) \leq B \oplus B \\ \iff \quad & A \odot (X_1 \oplus X_2) \leq B \\ \iff \quad & A \odot \mathrm{conv}(X_1 \cup X_2) \leq B. \end{aligned}$$

Now let us prove the theorem for intersection. It can be seen that $X_1 \cap X_2 \leq X_1$. Therefore $A \odot (X_1 \cap X_2) \leq A \odot X_1 \leq B$. □

Corollary 4.5. *If $X_1 = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\} + \mathbb{O}$ and $X_2 = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_h\} + \mathbb{O}$ are solutions to $A \odot X \leq B$, then also $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_h\} + \mathbb{O} = conv(X_1 \cup X_2)$ is a solution to $A \odot X \leq B$.*

Lemma 4.6 (Comparing two translated cones). *Let $P = \{\mathbf{p}\} + \mathbb{O}$ and $Q = \{\mathbf{q}\} + \mathbb{O}$, $P, Q \in \mathbb{P}_m$. Let $\mathbf{p} = (p_1, p_2, \ldots, p_m)$ and $\mathbf{q} = (q_1, q_2, \ldots, q_m)$ be the coordinates of the two vertices. If for all $i = 1, 2, \ldots, m$ it holds that $p_i \geq q_i$, then $P \leq Q$.*

Proof. Let $\mathbf{c} = \mathbf{p} - \mathbf{q}$. It can be seen that each entry is nonnegative, $p_i - q_i \geq 0$. But this means that this is a vector from the nonnegative orthant, $\mathbf{c} \in \mathbb{O}$. Therefore $\mathbf{p} = \mathbf{q} + \mathbf{c}$, the point $\mathbf{p}$ can be expressed as the point $\mathbf{q}$ plus a vector from the recession cone of Q, and therefore the vertex of P is contained in Q. By Lemma 4.2, $P \leq Q$. □

The statements below are interesting observations that do not have specific uses in this paper; nevertheless they are left here to provoke the reader to find more special properties of similar semirings.

Theorem 4.7 (Minkowski sum $P + P$). *The Minkowski sum of a polyhedron P with itself is an ordinary scaled version of P.*

Proof. Let P_1 be the sum of polyhedra P and P. Fix any vertex of P_1 and find a direction $\mathbf{d}$ in which the vertex is the only maximal value. Then by Theorem 2.13, it is a sum of unique points of P, so it is the sum of two vertices of P. Even more, it must be the sum of a vertex $\mathbf{v}$ with itself since the unique maximal points of the first P and the second P in direction of $\mathbf{d}$ must be the same vertex. Therefore the fixed vertex of P_1 is simply $2\mathbf{v}$. Similarly, the facets of P_1 must be parallel to facets of P, and since all vertices are of the form $2\mathbf{v}$, all facets of P_1 are simply scaled versions of facets of P. □

Remark 4.8 (Facet normals). Let a polyhedron in the polyhedron semiring be given. Under the restriction that the normal of each facet of the polyhedron must point inside the polyhedron, all entries of the normal vector are nonnegative as the vector points in the nonnegative orthant. Additionally, for any face the orthogonal vectors of it pointing inside the cone are convex combinations of the normals of the facets that determine the face.

Proof. The first statement is trivial. To prove the second statement, observe that the negative of an orthogonal vector of a face is equivalent to the direction in which the face is maximal. Slightly changing the direction such that it is still contained in the interior of the convex hull of negative normals of the nearby facets will not change the maximal set of points.

Let a face V be given which is the intersection of $n \geq 2$ facets which are maximal in directions $\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_n$, respectively. . Now observe that V is a subset of each of the given facets and therefore is maximal in all directions $\mathbf{d}_1, \ldots, \mathbf{d}_n$ and therefore is also maximal in any convex combination of said directions. □

Remark 4.9 (Zonotope representation). While one option for the finite representation of finite polytopes from $\mathbb{P}_m$ from Corollary 2.12 is the convex hull of the vertices, another one is adding additional vertices so as to have the polytope's facets be centrally symmetric. For a given $P \in \mathbb{P}_m$, find a point $\mathbf{c} = d \cdot (1, 1, \ldots, 1)$ such that d is larger than any entry of any vertex of P. Then for any vertex $\mathbf{v}$ of P, add the point $\mathbf{v}' = \mathbf{c} + (\mathbf{c} - \mathbf{v})$ to the set of vertices of the polytope. Now the convex hull of all $\mathbf{v}$ and $\mathbf{v}'$ has the property that the facets are centrally symmetric around $\mathbf{c}$, and this convex hull with such a property is also known as a zonotope.

By setting the representations to be zonotopes, operating with these sets preserves many properties [24], and an application of zonotopes is mentioned in [3].

Theorem 4.10 (Conical facets). *Let $m \geq 2$ be given, let $P \in \mathbb{P}_m$, $P \neq \{\infty\}$. There are exactly m facets of P with the normals of $\mathbf{e}_i$, $i = 1, \ldots, m$ and those facets are isomorphic to some polyhedra in $\mathbb{P}_{m-1}$ for $m \geq 1$. Call such facets* ***conical facets****.*

Proof. We will prove this by induction. For $m = 1$ dimension, the only facet is the lower endpoint of the interval which has a normal in the positive direction.

Assume that the hypothesis holds for $m = k$ dimensions. Now let us check whether $m = k + 1$ dimensions hold.

It is known that the polyhedron $P \in \mathbb{P}_{k+1}$ is $(k + 1)$-dimensional since the cone is $(k+1)$-dimensional and therefore its facets are k-dimensional polyhedra by Definitions 2.7, 2.8. Let us iterate over the basis vectors for the cone, $\mathbf{e}_1, \ldots, \mathbf{e}_{k+1}$.

Take the basis vector $\mathbf{e}_i$ and look at the face V that is maximal in the direction of $-\mathbf{e}_i$. This face exists as otherwise the vector $-\mathbf{e}_i$ is in the cone of P. One can note that this face must be dimension k since any other basis vector $\mathbf{e}_j$ is orthogonal to $\mathbf{e}_i$ and therefore for any point $\mathbf{p}$ on the face, also $\mathbf{p} + \mathbf{e}_j$ is on the face. To summarize, if $\mathbf{p} \in V$ and $\mathbf{d}_j^{\mathbf{e}} = 0$, then $\mathbf{p} + \mathbf{d} \in V$. Therefore V is a facet and it has at least k linearly independent vectors spanning its cone.

By removing the i^{th} entry of all points of V, it becomes a polyhedra with a recession cone spanned by all the unit basis vectors in k dimensions, as all the previous unit vectors except $\mathbf{e}_i$ are still orthogonal unit vectors after removing the i^{th} entries.

Let us now prove that all vectors in the recession cone of V have nonnegative entries. Let $\mathbf{p} \in V$ and $\mathbf{v} \in \operatorname{rcone}(V)$, where $\operatorname{rcone}(V)$ is the recession cone of the polyhedra V. Since V is a subset of P, the recession cone of V cannot exceed the reccession cone of P. Since all vectors in the recession cone of P have nonnegative entries, also all vectors in the recession cone of V must have nonnegative entries. □

5 Linear inequality in polyhedron semiring

Usually solving inequalities or equalities with a single variable is quite easy to do. Because subtraction is allowed, solving $3 + x = 10$ in real numbers is straightforward, whereas in semirings subtraction is not defined, and one can see that in the tropical semiring from Proposition 2.16 with minimum as the tropical addition, the equality $3 \oplus x = 10$ is not solvable at all. On the other hand, solving $3 \odot x = 10$ can be done in the tropical semiring with integer coordinates from Examples 3.4 and 3.5, whereas if $x \in \mathbb{Z}$, the equality $3 \cdot x = 10$ cannot be solved. We now focus on solving a deceptively simple inequality in the polyhedron semiring with finite polyhedra.

Theorem 5.1 (Linear inequality in polyhedron semiring). *Let A and B be polyhedra from $\mathbb{P}_m$. Then the inequality $A \odot X \leq B$ can always be solved for $X \in \mathbb{P}_m$.*

Proof. The proof presented is a constructive proof, see Algorithm 1.

Step 3 always terminates: let the largest entry of any vertex of B be b, and let the smallest entry of A be a. Then by letting $X = \{(b - a) \cdot \mathbb{1}\} + \mathbb{O}$, clearly each entry of each vertex of $A \odot X$ is larger than each entry of each vertex of B and by Lemma 4.6 and Theorem 4.4, the convex hull of vertices of B keeps the inequality for any vertex of $A \odot X$, and then by taking the convex hull of vertices of $A \odot X$, the inequality $A \odot X \leq B$ holds. One can see that step 3 takes discrete increments and therefore might have a solution $X = \{\lambda \cdot \mathbb{1}\} + \mathbb{O}$, where $\lambda \notin \mathbb{Z}$, however, clearly any larger λ will also solve the inequality. □

Algorithm 1 Inequality solver

Require: $A, B \in \mathbb{P}_m$

Ensure: $A \odot X_1 \leq B$

if $B \neq \{\infty\}$ **then**
  $X_0 = \{\mathbf{0}\} + \mathbb{O}$
  while $A \odot X \nleq B$ **do**
    $X_0 = X_0 \odot (\{\mathbb{1}\} + \mathbb{O})$
  end while
  $X_1 = X_0$
else
  $X_1 = \{\infty\}$
end if

Now Algorithm 1 always solves the inequality, however, we might want to solve the inequality such that it is as close to equality as possible. The solution X in this case is called a maximal equation and there always is a finite maximal solution by Corollary 2.12. If two solutions X_1, X_2 exist, we may always take their convex hull $X = \operatorname{conv}(X_1, X_2)$ to get a larger solution: $X \geq X_1$, $X \geq X_2$. Therefore there exists a maximal solution that is larger than any other solution. Algorithm 2 presents a potential approximation for the maximal solution.

Algorithm 2 Improved inequality

Require: $A, X_1, B \in \mathbb{P}_m$, $A \odot X_1 \leq B$, $B \neq \{\infty\}$
Ensure: $A \odot X \leq B$, $X \geq X_1$
1: $X = X_1$
2: fix $V = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ for $\mathbf{v}_i \notin \mathbb{O}, i = 1, \ldots, n$
3: $S = \{$all permutations of $V\}$
4: **for** perm in S **do**
5:   $Y = X_1$
6:   **for** $\mathbf{v}$ in perm **do**
7:     $Y = Y \odot (\{\lambda_0 \cdot \mathbf{v}\} + \mathbb{O})$ where $A \odot (Y \odot (\{\lambda \cdot \mathbf{v}\} + \mathbb{O})) \leq B$ for $\lambda \leq \lambda_0$ and $A \odot (Y \odot (\{\lambda_0 \cdot \mathbf{v}\} + \mathbb{O})) \nleq B$ for $\lambda > \lambda_0$
8:   **end for**
9:   $X = \operatorname{conv}(X, Y)$
10: **end for**

A possible option for V is all the negative basis vectors from $\mathbb{P}_m$. The vectors point outside the cone and therefore moving A in their directions would eventually move it outside of B as the recession cone only contains nonnegative vectors. Therefore step 7 always terminates. Clearly all found Y's are solutions and therefore by Theorem 4.4, their convex hull is also a solution.

If basis vectors are used, this algorithm needs to go over all possible permutations of $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m\}$, some simplifications can be made if more memory is available: when comparing permutations P_1 and P_2 where the two last entries of the permutations are swapped, clearly only the last two iterations of P_2 will be different from all of the iterations of P_1 and therefore not all of the steps need to be repeated. By arranging all of the vertices in a tree graph with root as the empty set and each node in level k having $m - k$ children, which are the basis vectors that have not been iterated over yet, one can see that any leaf corresponds to a permutation and any two permutations who have common edges, may take the starting polyhedra of X_1 already shifted to the position that corresponds to the last common node.

The algorithm needs to check in step 7 whether the convex hull of two polyhedra is exactly one of the polyhedra. An algorithm for this is presented by Fukuda [11] 2.19.

One can also see that X_1 might not be the maximal solution in the sense that $X_1 \geq X_2$ for any solution X_2. An example of this is Figure 5 where the vertex $(1, 1)$ cannot be reached and thus the solution will not give equality. However, solving for the maximal solution is a complex problem that is out of the scope of this paper. One can see papers on a similar problem go into great detail [25, 16]. If the reader has a different algorithm on hand, they are encouraged to follow the remaining sections with their algorithm instead. Importantly as per Theorem 4.4, taking the convex hull of the solution presented above and the reader's solution can only increase the likelihood of finding the maximum solution and therefore cooperation and collaboration are key in this problem. Some inspiration can potentially be taken from the paper by Mamatov and Nuritdinov [18] where the Minkowski difference is studied, however, the given statements or the proofs need to be modified to be presented as finite algorithms, as well as restrict X to have a specific recession cone.

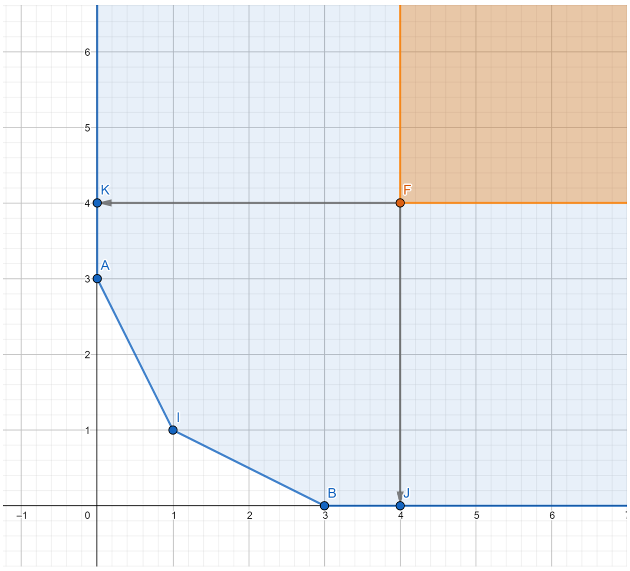

FIGURE 5: Algorithm of Theorem 5.1 does not produce maximal solution

6 Equality over polyhedron semiring

6.1 Single dimension

Previously, we solved the inequality $A \odot X \leq B$ in the polyhedron semiring $\mathbb{P}_m$. The Algorithm 2 showed an approximation of how a maximal solution can be found. Since finding a maximal solution is similar to finding a solution such that the inequality is as close to equality as possible, in this section we focus on whether this equality indeed holds and present a shortcut for checking single equalities of the form $A \odot X = B$ which uses Minkowski addition as part of the algorithm.

Theorem 6.1 (Equality algorithm). *Let $A = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_l\} \in \mathbb{P}_m$ and $B = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k\} \in \mathbb{P}_m$. Then the vertices of the solution to $A \odot X = B$ are a subset of all of the* *points* $X_{i,j} = \{\mathbf{b}_j - \mathbf{a}_i\}$*,* $i = 1, 2, \ldots, l$ *and* $j = 1, 2, \ldots, k$*. Potentially no solution exists and this subset is the empty set.*

Proof. Assume the solution X contains a vertex $\mathbf{v}$. Since $\mathbf{v}$ is a vertex of X, there is a direction $\mathbf{d}$ such that the vertex $\mathbf{v}$ is the unique maximal point of X in the direction $\mathbf{d}$. In this direction the face of A is also a polyhedron by Remark 2.9 and therefore has a nonempty set of vertices. By Theorem 2.13, the face of B in direction of $\mathbf{d}$ is the sum of $\mathbf{v}$ and the face of A. The vertices of the aforementioned face of B are of the form $\mathbf{v}+\mathbf{a}_i$, where $\mathbf{a}_i$ are vertices of A. Since this maximal polyhedron is a face of B, its vertices are also vertices of B. Therefore we have shown that all vertices $\mathbf{v}$ of a solution X to $A \odot X = B$ satisfy $\mathbf{v} + \mathbf{a}_i = \mathbf{b}_j$ for some vertices $\mathbf{a}_i \in A$ and $\mathbf{b}_j \in B$. □

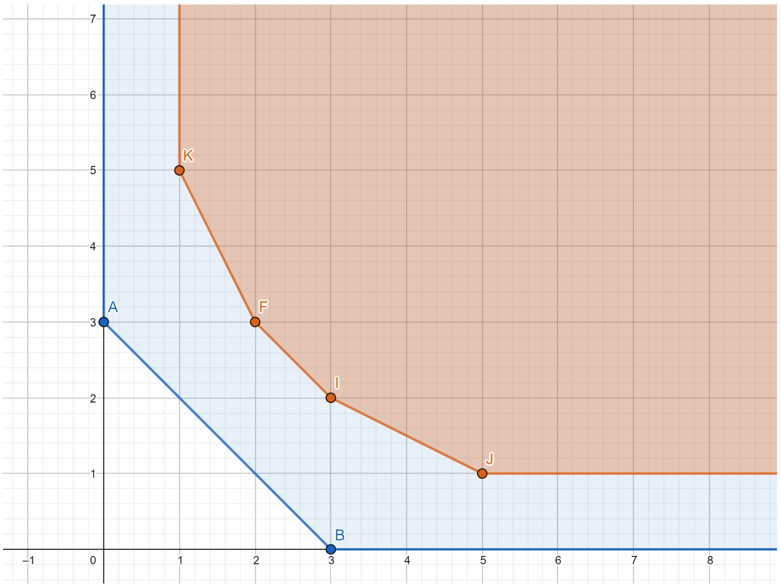

FIGURE 6: Two polyhedra where equality cannot be achieved

An easy way to check for equality is therefore check whether each vertex difference $\hat{X} + \mathbb{O}$ satisfies $A \odot X \leq B$ and taking the convex hull of the valid inequality solutions is the only possible candidate for solution of $A \odot X = B$. If this algorithm does not produce a valid solution, there is no solution to equality at all.

However, it is also possible that none of the differences of vertices $\mathbf{b}_j - \mathbf{a}_i$ are solutions to the inequality, and in that case we have to resort to the known algorithms to have any approximation to the solution at all.

Example 6.2. An example is in Figure 6, where $A = \{(5,1), (3,2), (2,3), (1,5)\}$ is the darker polyhedron and $B = \{(3,0), (0,3)\}$ is the lighter one. The subtraction of vertices gives the set of possible solutions $\{(-2,1), (0,-2), (1,-3), (2,-5), (-5,2), (-3,1), (-2,0), (1,-2)\}$. Now let us see why each point fails to be a solution; remember that to satisfy the inequality, the sum of vertex X and arbitrary vertex $\mathbf{a}$ of polyhedron A has to lie inside of

B.

$$
\begin{aligned}
(-2,1)+(1,5)=&(-1,6)\\
(0,-2)+(5,1)=&(5,-1)\\
(1,-3)+(5,1)=&(6,-2)\\
(2,-5)+(5,1)=&(7,-4)\\
(-5,2)+(1,5)=&(-4,7)\\
(-3,1)+(1,5)=&(-2,6)\\
(-2,0)+(1,5)=&(-1,5)\\
(1,-2)+(5,1)=&(6,-1)
\end{aligned}
$$

Each result has a negative entry, while all points in B have nonnegative entries. As a result, none of the possible solutions actually solve the inequality and thus we cannot find a single solution for inequality this way.

Remark 6.3 (Finiteness of solution). It can be seen that if $A, B \in \mathbb{P}_m$, then the solution X of $A \odot X \leq B$ will be a polyhedron. By Theorem 2.13, the maximal points of B in some direction $\mathbf{d}$ are the maximal points of A summed with maximal points of X in the direction $\mathbf{d}$. For convex sets, Theorem 2.13 can also be interpreted as the boundary of B is the sum of the boundaries of A and X. Therefore, if X has infinitely many faces, then B must also have infinitely many faces. In any direction, the face of B must be larger or equal to the face of X in the same direction.

6.2 Multiple dimensions

As we have already seen, a semiring can be seen as "one-dimensional", as we can add and multiply elements similarly as for the field of real numbers or integers, and inequalities can be solved. Interestingly, this does mean that the polyhedron semiring in this sense is also one-dimensional. Therefore the next logical step is to extend this to multiple dimensions. The paper by Butkovič [26] approaches the matrix equality for points, in other words, for polyhedra from the polyhedron semiring $\mathbb{P}_1$, however, we focus on a polyhedron semiring with arbitrary amount of dimensions.

In this section we primarily want to solve a set of equations over the polyhedron semiring. Let n and k be positive integers, $A_{i,j} \in \mathbb{P}_m$, $B_i \in \mathbb{P}_m$ for $i = 1, \ldots, k$ and $j = 1, \ldots, n$.

$$
\begin{aligned}
A_{1,1} \odot X_1 \oplus A_{1,2} \odot X_2 \oplus \cdots \oplus A_{1,n} \odot X_n &= B_1\\
A_{2,1} \odot X_1 \oplus A_{2,2} \odot X_2 \oplus \cdots \oplus A_{2,n} \odot X_n &= B_2\\
&\cdots\\
A_{k,1} \odot X_1 \oplus A_{k,2} \odot X_2 \oplus \cdots \oplus A_{k,n} \odot X_n &= B_k
\end{aligned}
$$

The system can also be written as $\mathbf{A} \odot \mathbf{X} = \mathbf{B}$, defined similarly to ordinary matrix multiplication: $(\mathbf{A} \odot \mathbf{X})_i = \bigoplus_{l=1}^{n} A_{i,l} \odot X_l$.

This system might not have a solution at all, for instance looking at Figure 6 again and setting $k = 1$, $n = 1$ there is no equality possible.

Now, the curious reader might wonder as to why we were trying to solve inequalities so much if in the end we want to focus on matrix equality. Theorem 6.7 gives an answer: not all of the solutions need to have equality, it might be enough that inequality holds individually and therefore it might be enough to focus on finding a maximal solution to $\mathbf{A} \odot \mathbf{X} \leq \mathbf{B}$ instead of solving $\mathbf{A} \odot \mathbf{X} = \mathbf{B}$ outright. In fact, finding a solution to matrix equality might be more likely than finding a solution to a single equality. However, first two other theorems are needed to show that the matrix inequality can be solvable by simply looking at each summand $A_{i,j} \odot X_j$ separately.

Proposition 6.4 (Inequality inclusion). *For $\hat{\mathbf{X}} \in \mathbb{P}_m^n$ to be a solution to $\mathbf{A} \odot \mathbf{X} \leq \mathbf{B}$, all vertices of $A_{i,j} \odot \hat{X}_j$ for $j = 1, \ldots, n$ must be contained in B_i for $i = 1, \ldots, k$.*

Proof. Assume that some vertex of $A_{i,j} \odot \hat{X}_j$ lies outside of B_i. Then the convex hull of $\{A_{i,1} \odot \hat{X}_1, A_{i,2} \odot \hat{X}_2, \ldots, A_{i,n} \odot \hat{X}_n\}$ also has a vertex that lies outside of B_i by Corollary 4.5. But the aforementioned convex hull is simply the expression $A_{i,1} \odot \hat{X}_1 \oplus A_{i,2} \odot \hat{X}_2 \oplus \cdots \oplus A_{i,n} \odot \hat{X}_n$ and therefore $\mathbf{A} \odot \hat{\mathbf{X}} \not\leq \mathbf{B}$ by Lemma 4.2. □

Corollary 6.5. *For fixed $i = 1, \ldots, k$, solving the inequality $A_{i,1} \odot \hat{X}_1 \oplus A_{i,2} \odot \hat{X}_2 \oplus \cdots \oplus A_{i,n} \odot \hat{X}_n \leq B_i$ is equivalent to solving inequalities $A_{i,j} \odot X_j \leq B_i$ separately for all $j = 1, \ldots, n$.*

Proposition 6.6 (Reducing the number of inequalities). *Let k be a positive integer, $A_i, B_i \in \mathbb{P}_m$ and let k not necessarily the same inequalities $A_i \odot X \leq B_i$ be given for $i = 1, \ldots, k$. If $X_i \in \mathbb{P}_m$ is a solution for inequality $A_i \odot X \leq B_i$, then $X = \bigcap_{i=1}^k X_i$ is a solution for the system $A_i \odot X \leq B_i$, $i = 1, \ldots, k$ at the same time.*

Proof. By Theorem 4.4, $X \leq X_i$, so $A_i \odot X \leq A_i \odot X_i \leq B_i$ for all $i = 1, \ldots, k$. □

Because of Corollaries 6.5 and 6.6, a vector $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ to a system of inequalities $A_{i,1} \odot \hat{X}_1 \oplus A_{i,2} \odot \hat{X}_2 \oplus \cdots \oplus A_{i,n} \odot \hat{X}_n \leq B_i$ for $i = 1, \ldots, k$ is a feasible solution if and only if $A_{i,j} \odot X_j \leq B_i$ for all $i = 1, \ldots, k$ and $j = 1, \ldots, l$.

Finally, let us look at when exactly the solution to inequality $\mathbf{A} \odot \mathbf{X} \leq \mathbf{B}$ gives a solution to the equality $\mathbf{A} \odot \mathbf{X} = \mathbf{B}$.

Theorem 6.7 (Optimal solution). *Let $\mathbf{A} \in \mathbb{P}_m^{k \times n}$, $\mathbf{B} \in \mathbb{P}_m^k$ and $\hat{\mathbf{X}} \in \mathbb{P}_m^n$ such that $\mathbf{A} \odot \hat{\mathbf{X}} \leq \mathbf{B}$. Then the two statements below are equivalent:*

- $\mathbf{A} \odot \hat{\mathbf{X}} = \mathbf{B}$
- *for all $i = 1, \ldots, k$, every vertex of B_i is contained in at least one of $A_{i,j} \odot \hat{X}_j$ for $j \in \{1, \ldots, n\}$.*

Proof. First assume that each vertex of B_i appears in the convex hull $\bigoplus_{j=1}^n A_{i,j} \odot \hat{X}_j$ as a vertex of some $A_{i,j} \odot \hat{X}_j$, and let $\mathbf{b}_l$ be the vertices of B_i, $l = 1, \ldots, w$. Let $\mathbf{c}$ be an arbitrary vector from the cone. Then also any convex combination of the vertices of B_i appear in the convex hull $\bigoplus_{j=1}^n A_{i,j} \odot \hat{X}_j$, and since $\mathbf{b}_l + \mathbf{c}$ must be in the convex hull $\bigoplus_{j=1}^n A_{i,j} \odot \hat{X}_j$ by the property that each $A_{i,j}$ has the same cone, then also the convex combination of $\mathbf{b}_l + \mathbf{c}$ will be in the convex hull. Therefore all points of B_i are in $\bigoplus_{j=1}^n A_{i,j} \odot \hat{X}_j$ and $\bigoplus_{j=1}^n A_{i,j} \odot \hat{X}_j \geq B_i$. Therefore, as $\mathbf{A} \odot \hat{\mathbf{X}} \geq \mathbf{B}$ and $\mathbf{A} \odot \hat{\mathbf{X}} \leq \mathbf{B}$, $\mathbf{A} \odot \hat{\mathbf{X}} = \mathbf{B}$.

Now assume some vertex $\mathbf{b}$ of B_i does not appear in the convex hull $\bigoplus_{j=1}^n A_{i,j} \odot \hat{X}_j$ as any vertex of any $A_{i,j} \odot \hat{X}_j$. First assume the vertex $\mathbf{b}$ is fully contained in $\bigoplus_{j=1}^n A_{i,j} \odot \hat{X}_j$. That implies that there is an open ball with radius $\epsilon > 0$ around the point $\mathbf{b}$ completely in $\bigoplus_{j=1}^n A_{i,j} \odot \hat{X}_j$. However, this ball is not completely in B_i as $\mathbf{b}$ is maximal in some direction. Therefore $\mathbf{b}$ must belong to a face of $\bigoplus_{j=1}^n A_{i,j} \odot \hat{X}_j$.

Now assume that $\mathbf{b}$ lies on a line segment $[\mathbf{a}_1, \mathbf{a}_2]$ completely contained in this face, where $\mathbf{a}_1 \neq \mathbf{b} \neq \mathbf{a}_2$. However, in B_i this line segment is not contained as $\mathbf{b}$ is a vertex and therefore does not lie on any line segments. This contradicts $\mathbf{A} \odot \hat{\mathbf{X}} \leq \mathbf{B}$.

Therefore $\mathbf{b}$ lies outside of $\bigoplus_{j=1}^n A_{i,j} \odot \hat{X}_j$ and thus $\mathbf{A} \odot \hat{\mathbf{X}} \not\geq \mathbf{B}$. □

While Theorem 6.7 is not directly used in constructing the optimal solution for the system of inequalities, the reader can now see why we wanted to have some maximal solution to the inequalities: then we are more likely to have the vertices overlap and such have greater possibility to solve the equality. One can also observe that adding an extra variable and constraints for it will not restrict the existing solution at all: if the equality was already achieved with the previous variables, then the extra variable only needs to satisfy the individual inequalities for it and has no need to be minimized. If the inequality was not achieved previously, then adding the new variable can only improve the existing solution.

7 Tropically convex sets

As observed above, the polyhedra can be seen as analogous to single-dimensional variable, and multiple of them can be stacked together to form vectors of polyhedra. Then by taking multiple vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ where $\mathbf{v}_i \in \mathbb{P}^n_m$, one can find the convex set of them under some restrictions.

$$\operatorname{conv}(\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}) := \left\{ \bigoplus_{i=1}^{k} \lambda_i \odot \mathbf{v}_i : \bigoplus_{i=1}^{k} \lambda_i = \mathbb{O}, \lambda_i \in \mathbb{P}^n_m \text{ for } i = 1, 2, \ldots, k \right\}$$

One can see that the scalars are not real numbers but similar polyhedra, however, this should be the case as the scalars should be from the same set as the vector entries.

In case $m = 1$, where the semiring is the one-dimensional tropical semiring, the convex sets behave similarly to tropical polyhedra in $\mathbb{T}^n$.

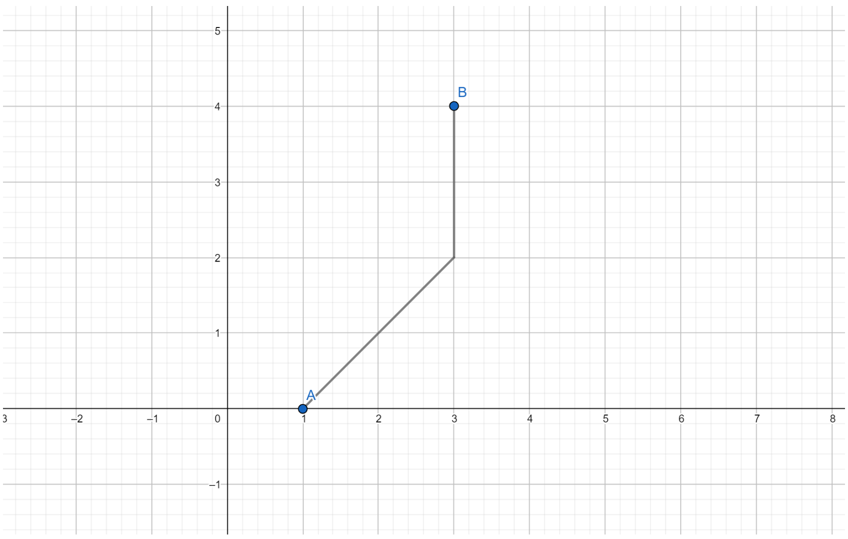

FIGURE 7: The possible locations of the vertex of convex hull of Example 7.1

Example 7.1 (Tropical convex hulls). Before trying to calculate the result, let us interpret the condition of scalars: the convex hull of the scalars should be the zero polyhedron. By Theorem 6.7, one of the scalars should have the vertex at the ordinary zero and no vertices of any scalar polyhedra should be outside the nonnegative orthant. This means that all polyhedra should be contained in the nonnegative orthant and one should be exactly the nonnegative orthant. In case only two vectors are given, one can clearly interpret the scalars: by setting one scalar to $[0, \infty]$ and the other one to $[t, \infty]$ with $t \geq 0$, all cases can clearly be determined. One should note that here we use the word "scaling" as the elementwise Minkowski sum, as that is what scaling means in the context of polyhedron semiring. For the example, assume the two vectors given are:

$$\begin{bmatrix}[0,\infty]\\ [1,\infty]\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}[4,\infty]\\ [3,\infty]\end{bmatrix}.$$

Then scaling the first vector by the zero scalar and the second vector by the $[t,\infty]$ scalar results in the first vector as taking the convex hull elementwise will always result in the first vector. Therefore let us look at the second vector being scaled by the zero scalar and the first vector being scaled by the $[t,\infty]$ scalar.

$$[t,\infty]\odot\begin{bmatrix}[0,\infty]\\ [1,\infty]\end{bmatrix} = \begin{bmatrix}[0+t,\infty]\\ [1+t,\infty]\end{bmatrix}$$

Now by taking the elementwise convex hull of the scaled first vector and the second vector, one can see that there are cases to be considered.

$$\lambda_1\odot\begin{bmatrix}[0,\infty]\\ [1,\infty]\end{bmatrix} \oplus \lambda_2\odot\begin{bmatrix}[4,\infty]\\ [3,\infty]\end{bmatrix} = \begin{cases}\begin{bmatrix}[t,\infty]\\ [1+t,\infty]\end{bmatrix}, & \text{for } 0\le t\le 2\\ \begin{bmatrix}[t,\infty]\\ [3,\infty]\end{bmatrix}, & \text{for } 2< t\le 4\\ \begin{bmatrix}[4,\infty]\\ [3,\infty]\end{bmatrix}, & \text{for } t> 4\end{cases}$$

This can be visualised in Figure 7, where the first dimension in the graph corresponds to the first dimension of the vectors, and similarly for the second dimension. The grey tropical line segment is all the possible locations of the finite vertex of the convex hull.

Clearly for $m=1$, the relation to tropical spaces can be explored, as the tropical line segments and tropical convex hulls have been defined previously. However, this is under the restriction that $m=1$ and can be generalized. For $m\ge 2$, it will not be straightforwardly visualisable unless $n=1$, which resembles the topic covered in the previous sections. For $m,n\ge 2$ the convex hull will be at least 4-dimensional, and the scalars also have less restrictions as they can have more than one vertex.

For example, for $n=1$, $m=2$ and the polyhedra vectors having single vertices, even restricting one of the scalars to be the cone polyhedron still allows the other polyhedron to have an arbitrary amount of vertices. For example, let $\mathbf{v}_1=[\{(1,0)\}+\mathbb{O}]$ and $\mathbf{v}_2=[\{(3,3)\}+\mathbb{O}]$, see Figure 8. Clearly setting the scalar for $\mathbf{v}_1$ to be the zero scalar will always result in the convex hull being just $\mathbf{v}_1$ itself, so let us look at when the scalar for $\mathbf{v}_2$ is the nonnegative orthant. First assume that the scalar only has one vertex, then the resulting scaled polyhedron $\lambda_1\odot\mathbf{v}_1$ has a vertex inside of $\mathbf{v}_1$ by Lemma 4.6. If the resulting vertex ends in the hatched rectangle, then the convex hull will be determined by this vertex only. Similarly, if the resulting vertex is located inside the polyhedron of $\mathbf{v}_2$, then the convex hull of $\lambda_1\odot\mathbf{v}_1$ and $\mathbf{v}_2$ will be the polyhedron $\mathbf{v}_2$. On the other hand, if the vertex of $\lambda_1\odot\mathbf{v}_1$ is located in the cross-hatched areas, then the convex hull will have two vertices, both the vertex of $\lambda_1\odot\mathbf{v}_1$ and the vertex of $\mathbf{v}_2$. Finally, we can lift the restriction that the scalar must have only one vertex and we can see that the convex hull in this case is already extremely hard to generalize. Some potential parallels can be drawn to Gaubert and Katz [10], where points are used instead of polytopes.

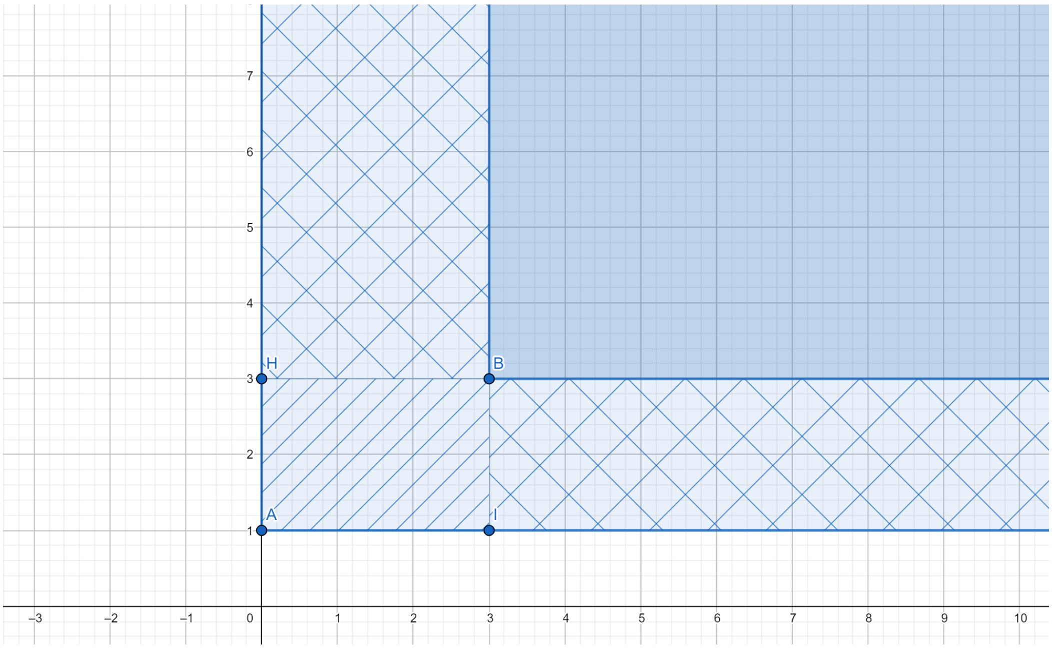

FIGURE 8: Convex hull of two polyhedra with $m = 2$, $n = 1$

8 Conclusion

As said in the introduction, the polyhedron semiring has not been explored before and therefore Sections 3 and 4 lay the essential basics of this semiring. Algorithm 2 we presented in Section 5 is the first ever algorithm provided to approximate an equation for these polyhedra, and Section 6 expands on this even more. Finally, Section 7 includes the first advances for convex sets of vectors of polyhedra.

There are similar problems to $A \odot X = B$ as in Section 6 as well, for example $A \odot X = B \odot Y$. In case of rings or fields, this can be simplified by just subtracting Y from both sides and solving $A + (X - Y) = B$ in the same method as previously, however, subtraction (in our case, tropical division) is not defined for semirings. Clearly by substituting $Y = \mathbb{O}$, since it is the multiplicative unity, we get $A \odot X = B \odot \mathbb{O} = B$ which can be solved using the methods above. There is also a clear translational degree of freedom, as shifting both X and Y by the same arbitrary vertex will not change the shape of the solution. This is left for future research.

The given properties and theorems can mostly be applied to subsemirings from Examples 3.4, 3.5, 3.6 and 3.7. One exception is the convex hull of vectors with elements from the integer polyhedron semiring, as the convex hull is not well defined in the field of integers. However, the properties that do hold, can become even stronger as the extra restrictions give the polyhedra more structure.

Finally, the exploration of vectors of polyhedra as in Section 7 can be done for other properties as well, such as spans or matrix properties. There have been papers about the ranks of tropical matrices [27], and this can potentially be generalized to ranks of polyhedra matrices as well.

The results of this paper can be used to improve understanding of neural networks, as categorizing high-dimensional points can be done using polyhedra. Categorizing requires a considerable amount of computing power, and even slightly optimizing this procedure can have a great impact. Now, this paper does not claim to solve this problem outright. Nevertheless, the basic structure has now been set; just a few more improvements could already have important implications. Similarly, the convex hull and Minkowski addition can be useful for working with measurements, as taking the convex hull of two error polyhedra can approximate the total location of the true value, and Minkowski addition can be used to improve the error bounds. By working with error polyhedra instead of precise points, one takes into account uncertainty and gets a result that gives more information than just a single point. The results of this paper can be used to advance and improve properties of the aforementioned topics.

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