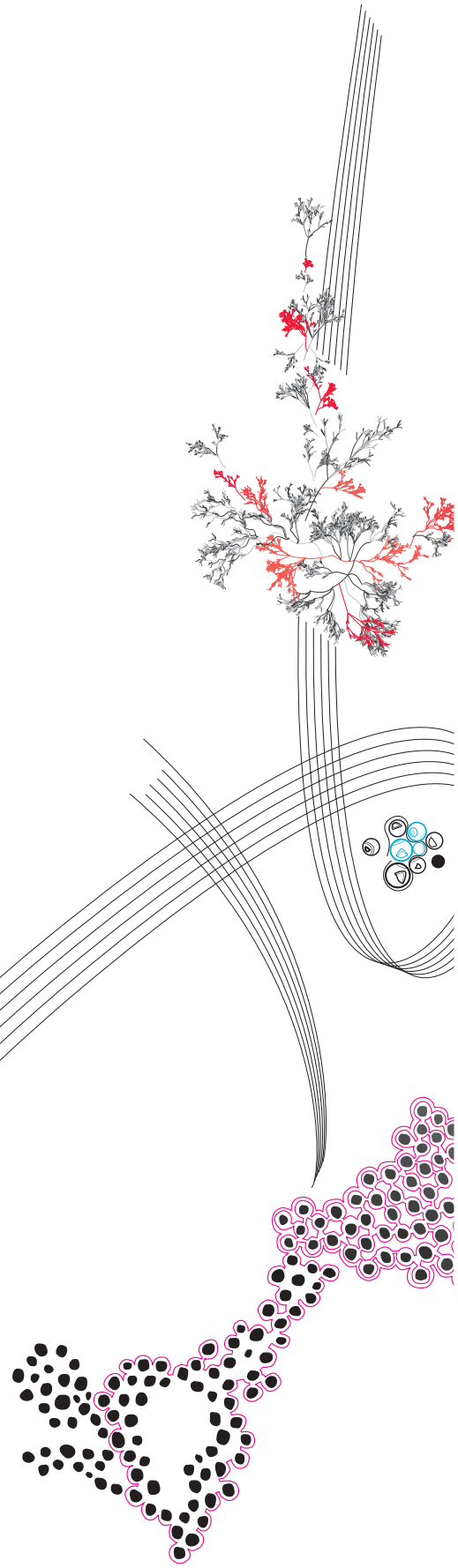

BSc Thesis Applied Mathematics

Exploration on the numerical range and spectral constants of the Volterra operator

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Abstract

The numerical range encodes useful information about a matrix, especially in the case of non-normal matrices, where the eigenvalues provide less information. The Volterra operator is an integrating operator that acts on an infinite-dimensional vector space, but is compact, hence can be finitely approximated. This operator approximation can be done with a Fourier series. Once the numerical range is found, one can do a numerical investigation on its spectral constants in order to then form a conjecture about it. While the numerical results seem to differ from those found analytically, something can still be said about the tendencies of the roots of the polynomials that lead to the best constants.
Keywords: Numerical range, spectral constants, Volterra operator

1 Introduction

When exploring matrices, there are several important features that can be used to describe it. One of the more well-known features of a matrix is its eigenvalues and eigenvectors, which are especially relevant for self-adjoint matrices; The spectral theorem states that if a matrix A is Hermitian (self-adjoint), then there exists an orthonormal basis of V consisting of eigenvectors of A and that each eigenvalue is real. A particular field in which this can be used is in numerical analysis, with large systems of linear equations of matrices with increasing dimension ($Ax = b$). The matrices represent linear mappings on finite-dimensional spaces with standard bases. In the case of self-adjoint matrices, the eigenvalues provide the necessary information for the convergence rates of approximation algorithms. However, when the matrices are far from self-adjoint then the eigenvalues are not sufficient [1]. In those cases, the numerical range (also known as the field of values) becomes a more important feature. As stated in the article by M. Benzi, "knowledge of certain properties of the field of values may be sufficient to prove certain useful bounds and even to obtain optimality results for a class of preconditioners for a given problem". The preconditioners mentioned there refer to preconditioners of a Krylov subspace GMRES method, which is an iterative numerical method that solves those aforementioned large systems of linear equations. The Krylov subspace method projects these large problems into a space of a much lower dimension, and the GMRES method then solves the problem iteratively. The numerical range is more robust than eigenvalues when in the presence of perturbations in the data, which is inevitable when using finitely-precise calculations, thus being better suited to these purposes. Additionally, the use of spectral theorem

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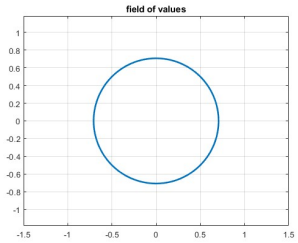
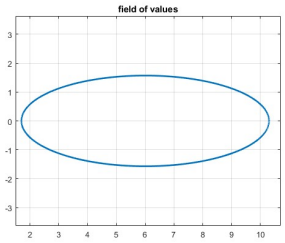
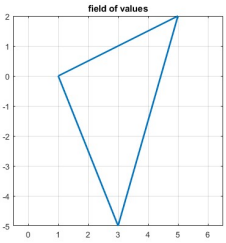


(A) Jordan unitary matrix



(B) Upper diagonal matrix



(C) Normal matrix

FIGURE 1: Examples of three fields of values of matrices

and eigenvalues assumes diagonalizability, while the same is not true of the usage of the numerical range. The important uses mentioned in the quote by M. Benzi also refer to the fact that the numerical range is used in bounding the error of the Krylov subspace method (since the projection into a lower dimensional subspace causes errors), and that there are some cases in which fields of values can be used to draw an equivalence between a problem and a family of preconditioners, hence improving the solutions. So, while the field of values of a matrix is not necessarily the first feature of a matrix that should be looked at, it certainly has importance in particular situations.

The definition of this numerical range is:
For a bounded linear operator A – a linear mapping from a vector space to the same vector space – on a Hilbert space H, the numerical range $W(A)$ is a set defined as:

$$W(A) = \{\langle Ax, x\rangle : ||x||_2 = 1, \} \tag{1}$$

where x is some vector in H, and $\langle \cdot, \cdot \rangle$ is the inner product in H. Since every complex square matrix represents a linear mapping on a Hilbert space (with respect to some basis), the numerical range of such matrices can be studied. This set clearly includes, but is not limited to the operator's eigenvalues. When plotted on a graph in the complex plane, a convex set can be seen (as proven in section 2). The shape of the set is also related to some of the matrix or the operator's properties: Jordan unitary matrices have the form of circles (figure 1a), upper triagonal matrices are elliptical (figure 1b), and normal matrices are just the convex hull of their eigenvalues (figure 1c).

Related to the numerical range is the spectral constant: Let A be a bounded linear operator. Then $(\psi > 0) \in \mathbb{C}$ is called a spectral constant for A if for every function f holomorphic on an open set containing $\overline{W(A)}$, with $||\cdot||$ being the operator norm, the following inequality holds:

$$||f(A)|| \leq \psi \sup_{z \in W(A)} |f(z)|. \tag{2}$$

It is known that this spectral constant ψ cannot be greater than $(1 + \sqrt{2})$ [2] [3] and conjectured $\psi \leq 2$ [4] for any given matrix A (see more in section 5). For a specific matrix A, however, ψ could be even further reduced.

In particular, the exploration on the numerical range and spectral constants being done is specifically about the Volterra operator. That is a bounded linear operator on the space of complex-valued square-integrable functions on the interval of $[0, 1]$, which can also represent indefinite integration. The operator is also not self-adjoint, hence its numerical range is more relevant, but there are difficulties arising from the fact that the underlying vector space is infinite-dimensional. It can be described more formally as the

linear transformation: $V : \mathcal{L}^2[0,1] \to \mathcal{L}^2[0,1]$ defined by:

$$(Vf)(x) = \int_0^x f(s)ds, \tag{3}$$

where $\mathcal{L}^2$ refers to the Lebesgue space of square-integrable functions.

As such, the research question being addressed is: *What can be said about the spectral constant (and by extension, numerical range) of the Volterra operator?*

Additionally, as mentioned previously, the usual property mentioned when discussing matrices is its eigenvalues and eigenvectors. Thus, the following subquestions may also be addressed: *What are some potential advantages (and disadvantages) of observing its numerical range instead? What polynomials result in the highest spectral constants?*

2 Basic Definitions

In this chapter, we list both the obvious and less obvious definitions that will be used. Due to the role of inner products in the definition of the numerical range, a brief reminder of its definition is stated: An inner product space is a vector space V over a field F with a inner product $\langle \cdot, \cdot \rangle : V \times V \to F$ which satisfies the following properties for vectors $x, y, z \in V$ and scalars $\alpha, \beta \in F$:

- $\langle x, y \rangle = \overline{\langle y, x \rangle}$
- $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
- $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$
- for $x \neq 0$, $\langle x, x \rangle > 0$.

In the complex space $\mathcal{L}^2[0,1]$ space which the Volterra operator takes as input and output, the inner product is defined as:

$$\langle f(x), g(x) \rangle = \int_0^1 f(x)\overline{g(x)}dx \tag{4}$$

for $f(x), g(x) \in \mathcal{L}^2[0,1]$. The integral considered here is the Lebesgue integral, which encompasses more spaces than the Riemann integral.

The Volterra operator is also a compact, bounded linear operator.

Definition 1 (Bounded Linear Operator [5])**.** *Let $(E, ||\cdot||_E)$ and $(F, ||\cdot||_F)$ be normed spaces. A linear operator $T : E \to F$ is called bounded if there is a scalar $0 \leq c$ such that*

$$||Tf||_F \leq c||f||_E \qquad \text{for all } f \in E.$$

Any bounded linear operator from E to F is now denoted by $\mathcal{L}(E; F)$

From that same book, we have

Lemma 1. *A linear operator $T : E \to F$ is bounded if and only if its operator norm*

$$||T|| := ||T||_{\mathcal{L}(E;F)} := \sup_{||f||_E \leq 1} ||Tf||_F$$

is a finite number. If T is bounded, then

$$||Tf||_F \leq ||T||\ ||f||_E \qquad \text{for all} \quad f \in E. \tag{5}$$

It is a known result that the operator norm of the Volterra operator is $\frac{2}{\pi}$ [6], which implies that it is bounded. Now, the most important result that allow us to explore the operator as a finite-dimensional approximation:

Theorem 1. *Let H be a Hilbert space, E a Banach space, and $A : E \to H$ a linear operator. Then A is finitely approximable if and only if it is compact. [5]*

The Volterra operator can be shown to be compact using the Arzeli-Ascoli [7][8][9] theorem. We can also show that it is finitely-approximable.

Definition 2 (Numerical Range)**.** *Let H be a complex Hilbert space and let A be a bounded linear operator on H. $D(A)$ is the domain of the linear operator A. The numerical range $W(A)$ of A is defined by*

$$W(A) := \{\langle Ax, x\rangle : x \in D(A), ||x|| = 1\}.$$

3 Properties of the numerical range

From the definition of the inner product, we can then show that the numerical range has the property of linearity :

Proposition 1 (Linearity of Numerical Range)**.** *For the numerical range $W(A)$ on a bounded linear operator A on a Hilbert space H, it holds that*

$$W(\alpha I + \beta A) = \alpha + \beta W(A) \tag{6}$$

for $\alpha, \beta \in \mathbb{C}$.

Proof. Take any $k \in W(A)$. There exists an $x_k \in D(A)$ such that $\langle Ax_k, x_k\rangle = k$ and $||x_k|| = 1$.

Now, if we consider $\langle(\alpha I + \beta A)x_k, x_k\rangle$, we get:

$$\begin{aligned}\langle(\alpha I + \beta A)x_k, x_k\rangle &= \langle \alpha I x_k, x_k\rangle + \langle \beta A x_k, x_k\rangle \\ &= \alpha\langle I x_k, x_k\rangle + \beta\langle A x_k, x_k\rangle \\ &= \alpha||x_k|| + \beta k \\ &= \alpha + \beta k\end{aligned}$$

Since that extends to any $k \in W(A)$, it can then be concluded that $W(\alpha I + \beta A) = \alpha + \beta W(A)$

□

Additionally, the field of values is a convex set [10] [11]. The following proof uses Karl Gustafson's proof as reference [12]:

Proposition 2. *[Toeplitz-Hausdor Theorem] The numerical range $W(A)$ of an arbitrary linear operator A in a Hilbert space is convex.*

Proof. In order for $W(A)$ to be convex, the following must be shown:

if $k_1, k_2 \in W(A)$ and $\lambda \in (0, 1)$,
then $\lambda k_1 + (1 - \lambda)k_2 \in W(A)$.

By 1, it is known that $W(\mu A + \gamma) = \mu W(A) + \gamma$ for $\mu, \gamma \in \mathbb{C}$. Thus, we can limit ourselves to observing the case $k_1 = 0$ and $k_2 = 1$.

Suppose that $0, 1 \in W(A)$. Then, there exists some $x_1 \in D(A) : \langle Ax_1, x_1\rangle = 0$ and some $x_2 \in D(A) : \langle Ax_2, x_2\rangle = 1$, where $||x_1||$ and $||x_2|| = 1$.

Let $x = \alpha x_1 + \beta x_2$ for $\alpha, \beta \in \mathbb{C}$, where

$$\begin{aligned} ||x||^2 = 1 &= ||\alpha x_1 + \beta x_2|| \\ &= \alpha^2 + \beta^2 + 2\alpha\beta \text{Re}\langle x_1, x_2\rangle \end{aligned} \tag{7}$$

Now, we want to show that for any $\lambda \in (0, 1)$

$$\begin{aligned} \langle Ax, x\rangle &= \langle A(\alpha x_1 + \beta x_2), \alpha x_1 + \beta x_2\rangle = \lambda \\ &= \beta^2 + \alpha\beta[\langle Ax_1, x_2\rangle + \langle Ax_2, x_1\rangle] = \lambda, \end{aligned} \tag{8}$$

since that will fulfill the convexity definition.

If $\langle Ax_1, x_2\rangle + \langle Ax_2, x_1\rangle$ is real, then equation (7) describes an ellipse (on the axes β and α) with points $(0, 1), (0, -1), (-1, 0), (1, 0)$. This is due to the Cauchy-Schwarz inequality, which implies that $|\text{Re}\langle x_1, x_2\rangle| < 1$. Equation (8) describes a hyperbola which has points at $(\beta, \alpha) = (\sqrt{\lambda}, 0)$ and $(-\sqrt{\lambda}, 0)$. Since $|\lambda| < 1$, the system (8) – (7) always has solutions. It is inevitable that the ellipse (from (7)) would intercept with the hyperbola (from (8)), because the hyperbola has points within the ellipse. Hence, an α and β can always be found such that both of the conditions (set by the equations) are met.

Let $B = \langle Ax_1, x_2\rangle + \langle Ax_2, x_1\rangle$. Now, we will show that B can always be assumed to be real, given an appropriate scalar multiple of x_1.
Let $x_1' = \gamma x_1$ and $B(x_1') = \langle Ax_1', x_2\rangle + \langle Ax_2, x_1'\rangle$. Here, $\gamma = a + bi$ where $a, b \in \mathbb{R}$ and satisfies

$$|\gamma|^2 = a^2 + b^2 = 1. \tag{9}$$

In order to prove that B can always be chosen to be real (with an appropriate scalar multiple), it must hold that:

$$\text{Im } B(x_1') = 0,$$

for $B(x_1') = \langle Ax_1', x_2\rangle + \langle Ax_2, x_1'\rangle$.
Therefore, we rewrite

$$\begin{aligned} B(x_1') &= \langle Ax_1', x_2\rangle + \langle Ax_2, x_1'\rangle = \langle A(\gamma x_1), x_2\rangle + \langle Ax_2, \gamma x_1\rangle \\ &= \langle A(a + bi)x_1, x_2\rangle + \langle Ax_2, (a + bi)x_1\rangle. \end{aligned}$$

Now, the imaginary part of $B(x_1')$ is considered:

$$\begin{aligned} \text{Im } B(x_1') &= \text{Im } [a\langle Ax_1, x_2\rangle + ib\langle Ax_1, x_2\rangle + a\langle Ax_2, x_1\rangle - ib\langle Ax_2, x_1\rangle] \\ &= \text{Im } [a\langle Ax_1, x_2\rangle + a\langle Ax_2, x_1\rangle] + \text{Re } [b\langle Ax_1, x_2\rangle - b\langle Ax_2, x_1\rangle] \\ &= a\text{Im } B + b\text{Re } [\langle Ax_1, x_2\rangle - \langle Ax_2, x_1\rangle] \end{aligned}$$

Therefore, the equation that needs to be considered in the system of equations is

$$\text{Im } B(x_1') = a\text{Im } B + b\text{Re } [\langle Ax_1, x_2\rangle - \langle Ax_2, x_1\rangle] = 0. \tag{10}$$

As we will now explain, the system (9) (10) has at least two solutions (a, b), so B can always be chosen real with an appropriate scalar multiple of x_1. The reason these solutions

exist in this system are the same as in system (8) - (7): there is a hyperbola and an ellipse, and there are points of the hyperbola that exist within the ellipse, so there must be at least two coordinates (a, b) that are a solution to this system. Hence, the system (9) (10) also allows for a solution for a, b, α, β, which implies that $W(A)$ is convex.

□

The numerical range is also bounded if the operator is bounded by the operator norm

Proposition 3 (Boundedness of Numerical Range)**.**

$$k \leq ||A|| \qquad \textit{for all } k \in W(A).$$

Proof. First, we let $k \in W(A)$ as described in 2. By the definition of the numerical range, $k = \langle Ax_k, x_k \rangle$ for some x_k such that $||x_k|| = 1$. Using one of the properties of the numerical range, we know that $\langle z, y \rangle \leq ||z||\ ||y||$ for some vector $z, y \in H$ [13]. Thus,

$$\langle Ax_k, x_k \rangle \leq ||Ax_k||\ ||x_k|| = ||Ax_k||$$

A is a bounded operator, so by 5, we state that

$$||Ax|| \leq ||A||\ ||x|| \qquad \text{for all } x \in D(A).$$

Thus,

$$\langle Ax_k, x_k \rangle \leq ||Ax_k||\ ||x_k|| = ||Ax_k|| \leq ||A||\ ||x_k|| = ||A||.$$

□

4 The Numerical Range of the Volterra operator

The numerical range of the Volterra operator can be found in literature [14], where this set was found without approximating the operator into a finite-dimensional matrix. In this case, however, such an approximation was indeed used, namely using the Fourier Series. Once such a matrix is constructed, a MATLAB script can be run on it such that the boundary points of the numerical range are found. The numerical range seems to be quite clearly shown through this method, even when the dimension of the matrix is relatively low.

4.1 Finite-dimensional approximation - Fourier series

Due to the fact that the Volterra operator is a compact operator, it can be approximated finitely. The well-known Fourier Series is a viable option for a finite-dimensional approximation of infinite-dimensional operators. The infinite Fourier series for a function f is:

$$f(x) = \sum_{k=-\infty}^{\infty} c_k \cdot e^{i2\pi kx}, \tag{11}$$

for c_k being the Fourier coefficients defined as

$$c_k = \langle f, e^{2\pi ik} \rangle = \int_0^1 f(t) \cdot \overline{e^{-2\pi kit}} dt. \tag{12}$$

The desired form of this approximation is an operator matrix A: the dimension of this matrix depends on how far the Fourier series will go. If (11) is taken to some $k = N, N \in \mathbb{N}$ instead of ∞, the dimension of the matrix A is then $2N + 1$. The approximation turns the linear transformation $V : \mathcal{L}^2[0,1] \to \mathcal{L}^2[0,1]$ into the 'alternative' transformation $P_n V J_n : \text{span}\{e^{-2i\pi(n)} : n \in \{-N, \ldots, N\}\} \to \text{span}\{e^{-2i\pi(n)} : n \in \{-N, \ldots, N\}\}$. Here, P_n is a projection from the infinite dimensional space into the finite-dimensional one, and J_n is the embedding from the finite-dimensional space to $\mathcal{L}^2$.

To build the matrix, each of the basis vectors has the Volterra operator applied to it (integrated) and that result is represented in terms of the bases. Because the operation being performed is an integration and the bases vectors are in the form of $e^{-2\pi ikx}$, the matrix is quite sparce since those values can be easily represented for $k \neq 0$. Namely,

$$\int_0^x e^{-2\pi ikt} dt = \frac{-1}{2\pi ik}[e^{-2\pi ikx} - 1].$$

Both of those terms are directly representable by bases vectors, since they are visibly just multiples of the bases vectors. Hence, for each column of A except for the $(N+1)^{\text{th}}$, the values are known. The column m has all zeros other than in row m and in row $(N+1)$. In those rows, the matrix will have the value $\frac{-1}{2\pi im}$ and $\frac{1}{2\pi im}$ respectively. As for the final

$$\begin{bmatrix} \frac{-1}{2\pi i(-2)} & 0 & a_{1,3} & 0 & 0 \\ 0 & \frac{-1}{2\pi i(-1)} & a_{2,3} & 0 & 0 \\ \frac{1}{2\pi i(-2)} & \frac{1}{2\pi i(-1)} & a_{3,3} & \frac{1}{2\pi i(1)} & \frac{1}{2\pi i(2)} \\ 0 & 0 & a_{4,3} & \frac{-1}{2\pi i(1)} & 0 \\ 0 & 0 & a_{5,3} & 0 & \frac{-1}{2\pi i(2)} \end{bmatrix}$$

FIGURE 2: Example of Volterra approximation matrix for $N = 2$. The missing $N + 1^{\text{th}}$ column is yet to be defined.

missing column: the result of integrating that basis vector is not as easily described.

$$\begin{aligned} \int_0^x e^{-2\pi i(0)t} dt &= \int_0^x (1) dt \\ &= x. \end{aligned}$$

In order to describe the function $f = x$ in terms of the basis vectors, we use (12). Hence, for each $k \in \{-N, \ldots, N\}$, we find

$$c_k = \langle x, e^{2\pi ik} \rangle = \int_0^1 x \cdot \overline{e^{-2\pi kit}} dt. \tag{13}$$

The complex conjugate of $e^{-2\pi kit}$ can be found if Euler's formula is used: $e^{it} = \cos t + i \sin t$, hence

$$e^{-2\pi kit} = \cos(-2\pi kit) + i \sin(-2\pi kit)$$

and

$$\overline{e^{-2\pi kit}} = \cos(-2\pi kit) - i \sin(-2\pi kit).$$

With that, the computation of (13) becomes:

$$\begin{aligned}\int_0^1 x \cdot \overline{e^{-2\pi kit}} &= \int_0^1 x \cdot [\cos(-2\pi kit) - i\sin(-2\pi kit)]dt \\ &= \int_0^1 x \cdot \cos(-2\pi kit)dt - i\int_0^1 x \cdot \sin(-2\pi kit)dt.\end{aligned} \tag{14}$$

Those two separate integrals are solvable using partial integration. We solve the first term as

$$\begin{aligned}\int_0^1 x \cdot \cos(-2\pi kit)dt &= (x \cdot \frac{-1}{2\pi k}\sin(-2\pi kx))\Big|_0^1 - \int_0^1 \frac{-1}{2\pi k}\sin(-2\pi kx)dx \\ &= 0 + \frac{1}{2\pi k}(-\cos(-2\pi kx) \cdot \frac{-1}{2\pi k})\Big|_0^1 \\ &= 0\end{aligned}$$

and the second term as

$$\begin{aligned}-i\int_0^1 x \cdot \sin(-2\pi kit)dt &= -i(x \cdot \frac{1}{2\pi k}\cos(-2\pi kx))\Big|_0^1 + i\int_0^1 \frac{1}{2\pi k}\cos(-2\pi kx)dx \\ &= \frac{-i}{2\pi k} - \frac{i}{2\pi k}(\sin(-2\pi kx) \cdot \frac{-1}{2\pi k})\Big|_0^1 \\ &= \frac{-i}{2\pi k} - 0 \\ &= \frac{-i}{2\pi k}.\end{aligned}$$

Thus, the $(N+1)^{\text{th}}$ column of the matrix has the value $\frac{-i}{2\pi k}$ on each row $k \in \{-N.\ldots,N\}$. Incorporating this into the previous example describing a Volterra matrix approximation (figure 2) of dimension 5 leads to:

$$\begin{bmatrix} \frac{-1}{2\pi i(-2)} & 0 & \frac{-i}{2\pi i(-2)} & 0 & 0 \\ 0 & \frac{-1}{2\pi i(-1)} & \frac{-i}{2\pi i(-1)} & 0 & 0 \\ \frac{1}{2\pi i(-2)} & \frac{1}{2\pi i(-1)} & a_{3,3} & \frac{1}{2\pi i(1)} & \frac{1}{2\pi i(2)} \\ 0 & 0 & a_{4,3} & \frac{-1}{2\pi i(1)} & 0 \\ 0 & 0 & a_{5,3} & 0 & \frac{-1}{2\pi i(2)} \end{bmatrix}$$

FIGURE 3: Volterra approximation matrix with dimension 5. As compared to the previous matrix 2, the 3rd column has now been defined.

5 Spectral Constant

Once the numerical range has been defined, the spectral constant can then be addressed. As mentioned previously: Given a bounded linear operator A, $(\psi > 0) \in \mathbb{C}$ is called a spectral constant for A if for every function f holomorphic on an open set containing $\overline{W(A)}$, with $||\cdot||$ being the operator norm, the following inequality holds:

$$||f(A)|| \leq \psi \sup_{z \in W(A)} |f(z)|, \tag{15}$$

where the numerical range of A is $W(A)$. Notably, such a constant should take into consideration any possible function f, which is a daunting task. When A is a general matrix, it has been famously conjectured by Michel Crouzeix that $\psi = 2$ [4]. There have been proofs that have shown that $\psi \leq (1 + \sqrt{2})$ [15], improving on the previous result of 11.08 [16]. In a way, we can consider the question: Does the Crouzeix conjecture hold for the Volterra operator? As we consider this question, it should be mentioned that it is unlikely that the conjecture wouldn't hold, and if it does not hold, then the methods should be thoroughly examined to guarantee that it is a valid answer.

6 Method of Numerical Exploration

In the exploration of spectral constants, the work of Michael Overton was quite heavily used [17]. While his work generally attempts to find the spectral constant for any matrix [18] using numerical methods, the methods used can still be applied to this problem (albeit with some steps removed due to its simpler nature: we have a fixed matrix - that of the Volterra operator).

In Overton's work, the Crouzeix ratio for a given polynomial p is: $f(p, A) = \frac{||p||_{W(A)}}{||p(A)||_2}$. Here, $W(A)$ is the numerical range of an operator matrix A in a Hilbert space, $||p(A)||_2$ is the matrix 2-norm of the evaluation of polynomial p in a matrix sense [19], and $||p||_{W(A)} = \sup_{z \in W(A)} |p(z)|$. The Crouzeix ratio $f(p, A)$ is the inverse of the constant being searched in this paper, so while his method for finding this ratio will be used, the spectral constant will be $\frac{1}{f(p,A)}$.

This process to finding the Crouzeix ratio with a specified polynomial can be implemented quite straightforwardly in MATLAB: There is a `polyvalm` function that evaluates a polynomial in a matrix sense ($p(A)$, and from then on, one can simply use the `norm(·,2)` to find $||p(A)||_2$. As for the numerator: it is known that the $\sup_{z \in W(A)} |p(z)|$ is the same as $\sup_{z \in \delta W(A)} |p(z)|$, where $\delta W(A)$ is the boundary of the numerical range. That boundary can be found using the Chebfun `fov` function. By the nature of numerical methods, there are a limited amount of boundary points taken into consideration: that quantity can be defined (to some upper limit defined by the Chebfun function itself). All of those boundary points can be evaluated on the polynomial using `polyval`, and its highest value is then found using `norm(·, inf`. Now, both the denominator and numerator have been found for a specified matrix and polynomial.

Now that the ratio can be computed for a given polynomial p, we must write a program that goes through multiple options of p, selecting the highest ratio in all of those options and storing it. If it were possible to go through every possible polynomial, we would then have a certain answer. However, there are infinitely many such polynomials. While there are less 'oblivious' ways to go about this numerical exploration, the method used here is to create polynomials of a selected degree with randomized coefficients. Once the degree of polynomials used is chosen, the program will generate randomized coefficients for the polynomial using a uniform distribution (whose parameters are also chosen). The Crouzeix ratio of the Volterra operator approximation matrix will then be found for that particular polynomial. Then, the program creates another set of randomized coefficients within the same uniform distribution parameters and compares that new Crouzeix ratio to the previous one and stores the lowest ratio (hence highest spectral constant). That process then iterates several times (as defined beforehand in the program), and the final result of the program is the lowest ratio across the total amount of iterations, as well as the polynomial that resulted in that constant.

6.1 Results

The disadvantage of such a method is its discrepancies due to this finite-dimensional restriction. Additionally, the MATLAB method will provide the points on the boundary of the numerical range, but not the equations of the boundaries themselves. However, increasing the dimension does not significantly change the shape of the numerical range. Below, one can see the results of plotting out the numerical range of an approximation of dimension 5 and 501:

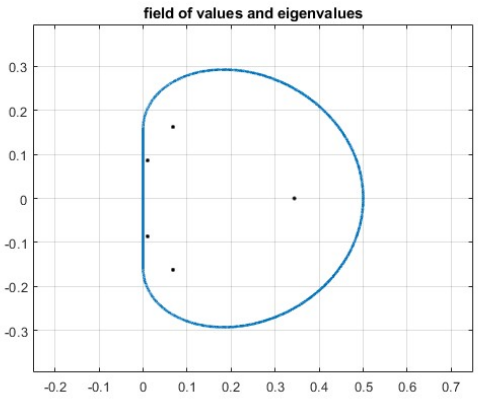


FIGURE 4: Numerical range of the Volterra operator approximated in a matrix of dimension 5.

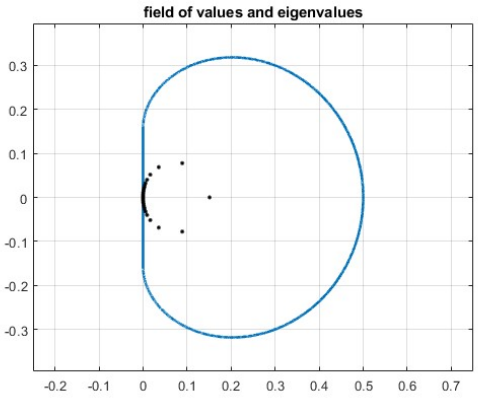


FIGURE 5: Numerical range of the Volterra operator approximated in a matrix of dimension 501.

However, as we come to see, the results found using this method are less than ideal.

The first attempt at finding the spectral constant was simply with polynomials with random coefficients of increasing degree. Below, one can find the table of some results for the spectral constants of degrees from 0 to 2. These were the results found after 500,000 iterations, with 2000 boundary points used on a matrix of dimension 21. The column 'Coefficients' describes what distribution was used during the randomization of the polynomial: $U(-500, 500)$, for example, means that the coefficients were uniformly distributed between -500 and 500. The real and imaginary coefficients were sometimes also generated with slightly different parameters; as such, the parameters for the real and imaginary coefficients are described separately.

Using random polynomials of degree 0 lead to a resulting constant of 1, regardless of the coefficient. The results for polynomials of degree 1 are shown in table 1 while the

results for polynomials of degree 2 are in table 2.

Coefficients	Polynomial used	Constant
Re : U(-50, 50) Im : U(-50, 50)	$(11.09 + 25.26i)x + (35.68 + 24.56i)$	1.7453
Re : U(-500, 500) Im : U(-500, 500)	$10^2 \cdot [(-3.8913 + 7.5263i)x + (-1.4322 + 7.4555i)]$	1.2587
Re : U(-500, 500) Im : U(0, 1000)	$10^2 \cdot [(4.3316 + 3.3386i)x + (-2.4874 + 6.7445i)]$	1.6402

TABLE 1: Resulting best spectral constant after 500,000 iterations on a matrix of dimension 21 and the polynomial of degree 1 that gave the result.

Coefficients	Polynomial used	Constant
Re : U(-50, 50) Im : U(-50, 50)	$(-6.11 - 9.86i)x^2 + (-20.33 + 1.90i)x + (4.93 - 0.77i)$	1.7469
Re : U(-500, 500) Im : U(-500, 500)	$10^2 \cdot [(-0.36 + 6.89i)x^2 + (-0.67 + 3.34i)x + (-7.49 + 6.74i)]$	1.2395
Re : U(-500, 500) Im : U(0, 1000)	$10^2 \cdot [(0.07 + 5.56i)x^2 + (0.84 + 5.04i)x + (2.35 + 4.01i)]$	1.5848

TABLE 2: Resulting best spectral constant after 500,000 iterations on a matrix of dimension 21 and the polynomial of degree 2 that gave the result.

This, however, yielded a mixed bag of results, with much higher spectral constants resulting from a seemingly arbitrary difference (imaginary coefficients distributed from 0 to 1000 instead of -500 to 500). Additionally, not much could be said about pattern and tendencies in the coefficients of the functions that yielded the best results. Further results were found with higher dimensional matrices (dimension 51), but the spectral constants resulting from that numerical exploration tended to be in a ± 0.1 range of the lower-dimensional results.

Therefore, it was decided that the roots of the polynomials should be noted, rather than the coefficients (as per a suggestion). Additionally, in order to provide insight about the tendencies of the results as the matrices increased in dimension (hence hopefully giving an idea of what the infinite-dimensional result might be), it was decided that the coefficient distribution remain the same (uniformly between -50 and 50 for both the real and imaginary coefficients), while the dimension of the approximation matrix increases. This was done for linear polynomials, shown in table 3, and for quadratic polynomials, shown in table 4.

Dimension	Roots	Constant
5	0.2010-0.0003i	1.7361
11	0.1897+0.0001i	1.7386
51	0.1956+0.0012i	1.7338
101	0.1891+0.0016i	1.7264
501	0.2078 + 0.0006i	1.7288

TABLE 3: Resulting best spectral constant after 100,000 iterations for polynomials of degree 1 with coefficients uniformly distributed from -50 to 50.

Dimension	Roots	Constant
5	(-1.5050+0.3008i), (0.2338+0.0009i)	1.7052
11	(-1.5395+0.1793i), (0.2246-0.0009i)	1.7176
51	(-0.4376+1.4986i), (0.2310-0.0550i)	1.7313
101	(-1.3613-1.0990i), (0.2363+0.0289i)	1.7301
501	(-3.0518 +1.0893i), (0.2141-0.0108)	1.7343

TABLE 4: Resulting best spectral constant after 100,000 iterations for polynomials of degree 2 with coefficients uniformly distributed from -50 to 50.

These new parameters seem more promising than the previous, both due to their implications about the results as the approximation of the Volterra operator and the Volterra operator itself converge, as well as the roots seemingly having a better pattern than the polynomial coefficients.

7 Discussion

The results found in tables 1 and 2, as well as the other omitted but similarly-structured tables with similar values had some unexpected results. The search for the best spectral constant when cycling through coefficients between -50 and 50 yielded better results than those going through coefficients between -500 and 500 (as seen in the first two rows of the aforementioned tables). This seems rather counterintuitive because the $U(-50, 50)$ distribution is a subset of $U(-500, 500)$, and therefore the 'best polynomial coefficients (and thus spectral constants)' that appeared in the first row's search should also have appeared in the second. One explanation might be that these 'good' results are rare, and thus 500,000 iterations is not enough to cycle through enough coefficients (in the given distribution) to get a meaningful results. It seems to suggest that the coefficients do not need to be very high to result in 'good' spectral constants.

As previously mentioned, the column labeled 'polynomial used' did not have that much value: there was not much of a discernable pattern in the results. As such, for the next sequence of numerical tests, the roots of the polynomials were used instead, to great effect. For linear polynomials, the root was generally close to (0.2+0i). Quadratic polynomials also shared that same (approximated), but the second root of the quadratic polynomials is a bit more unclear. The real part was always negative, but the magnitude of the real parts had quite some variety. The imaginary part was mostly positive, but that doesn't seem to give much insight. The constants seemed to decrease as the dimension of the approximation grew in the linear polynomial, but seemed to increase in the case of the quadratic polynomial. The quadratic polynomial had an increase of 0.3 as the dimension grew, while the linear polynomial decreased by around 0.1. Both, however, pointed towards a spectral constant of around 1.73

After much of this numerical investigation had already been done, a new paper by Thomas Ransford and Nathan Walsh was published, titled Norms of Polynomials of the Volterra Operator". This paper computes the operator norm of linear and real-quadratic polynomials of the Volterra operator, and later use those values to test the Crouzeix conjecture. Within that article, the spectral constants of the Volterra operator are also discussed in the case of real-quadratic polynomials. The results presented in that paper suggest that the spectral constant of the Volterra operator when observing real-quadratic polynomials is 1.5278 [20]. Another significant insight presented in paper is that the operator norm of the Volterra operator quickly tends to 0 in the higher powers ($||V^n|| \to 0$ quickly as $n \to \infty$),

so the lower power coefficients will dominate the resulting spectral constant. That value of 1.5278 does not match with the results shown from our numerical investigation. Even when limiting ourselves to real-quadratic polynomials, the results from our numerical methods give spectral constants around 1.76 ± 0.2 (shown in table 5), which is quite significantly different. Obviously there is the matter of the finite-dimensional approximation, which definitely accounts for some error compared to the actual value, but the difference between 1.5278 and 1.76 seems to be quite high. The polynomial that Ransford and Walsh found with the highest spectral constant is $z^2 + 0.685z - 0.167$, and if that exact polynomial is used in our numerical algorithm, the resulting spectral constant seems to match, as shown in table 6. Those differences could be a result of a finite dimensional approximation: if that case, then the numbers should converge as the dimension of the approximation matrix increases. However, it is still unclear if there is something fundamentally wrong with the numerical method used in our paper or if the values would converge if taken at some limit. One obvious improvement that could be done is the selection of the polynomials: instead of using random coefficients, some particular selection algorithm could be implemented, such as the BFGS (Broyden, Fletcher, Goldfarb and Shannon) algorithm, as suggested by Michael Overton.

Dimension	Roots	Constant
5	(-2.3006), (0.2332)	1.7626
11	(-3.404), (0.2155)	1.7715
51	(-3.9396), (0.2131)	1.7516
101	(-3.7145), (0.2161)	1.7491
201	(-4.3908), (0.2143)	1.7480

TABLE 5: Resulting best spectral constant after 100,000 iterations for polynomials of degree 2 with only real coefficients uniformly distributed from -50 to 50.

Dimension	Constant
5	1.4383
11	1.4877
51	1.5193
101	1.5235
201	1.5257

TABLE 6: Resulting best spectral constant for the Ransford-Walsh polynomial as dimension of approximation matrix increases.

8 Conclusion

We have identified features of the numerical range of a matrix and identified situations in which it is more useful than eigenvalues, that is, when solving linear systems of equations with non-normal matrices, specifically when using Krylov subspace methods with GMRES. Furthermore, during the numerical investigation of the spectral constants of the Volterra operator, we can find some convergence pattern as the dimensions of the approximation matrices increase as well as similarities between the roots of the polynomials that result in these matrices. Those similarities in roots are limited, however. The quadratic polynomials have only one root in common with each other (and with the linear polynomial); the

second root has positive real coefficients, but their magnitudes differ (and the imaginary coefficients don't have an obvious pattern). The spectral constants found seem to be around 1.73 when considering general results. However, there is a significant discrepancy when comparing the results of the spectral constants to Ransford and Walsh, and while their investigation did not have finite-dimensional approximations (so some differences can be explained), we can still identify that the numerical methodology used in our paper can be improved.

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