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Influence of positioning error on time delay based localization

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B.Sc. Thesis
31/01/2022

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Abstract

This paper investigates the effect of a position error in anchor node position in time of arrival based systems. A positioning error, in the form of a random variable, is added to the anchor node coordinates. Then the localisation accuracy is compared to to the performance of distance error of similar value. The paper shows the results for the following estimation methods: the least squares, the weighted least squares, and the two step least squares. These are compared against the maximum likelihood estimator. This is done in 2D and 3D scenarios.

1 Introduction

Time-delay based localisation methods are becoming more and more common with with the increase of wireless sensor networks. In these networks there are multiple nodes, to which an distinction can be made. Anchor nodes are nodes which position is known. Target nodes are the nodes which position is unknown. An important aspect of time-delay based localisation is the position of anchor sensor nodes. When there are a large number of nodes, it is sometimes impractical to determine the position of each node individually. It is better to calculate the position of most of the nodes using a local positioning system such as time of arrival based localisation. [3] By measuring the time it takes for signals to travel from transmitter to receiver, the position can be determined if enough distance measurements are available. There are a few ways to measure this. An approach this paper uses is using the time of arrival approach (TOA), which will be explained in more detail in a later section. It is desirable to get the most precise estimation possible. Therefore it is useful to investigate sources of measurement uncertainty, which will result in an error in the determination of the target location. This paper mainly investigates the influence of noise in the anchor node position. And how this affects the localisation performance.

1.1 Research question

The main research questions that this paper tries to answer are the following. What is the effect of a positioning error of anchor nodes on localisation accuracy? How can its effect be minimized? To answer these questions, the influence of position error in the anchor nodes will be compared to the influence of another error in TOA localisation, the distance measurement error. These errors are compared in different scenarios. The main scenarios that will be looked at are a symmetric scenario, and an asymmetric scenario. There is also a 3D scenario. The organisation of this paper is as follows. First, the background theory of time based localisation will be explained. Then, in the second section, the different localisation methods are explained. After this, it is explained how the simulation is set up, and how the results are obtained. Finally the conclusions are drawn.

1.2 Time of Arrival

TOA uses the time it takes for a signal to travel from an anchor node to the target node. The signal path can also be a round-trip, in case of , for example, a passive system that is based on a reflected signal such as a radar. However for simplicity, a cooperative scenario is considered where the TOA distances can be directly measured. In order to determine the time, the target also needs to know the time at which the signal was sent, to determine the distance between the nodes. For a 2D scenario, TOA based approaches need at least

3 measurements to be able to determine the desired location. [5] For the 2D scenario, the target position can be calculated as follows.

$$d_{TOA} = \frac{t1 - t0}{c} \tag{1}$$

$$d_{TOA} = \sqrt{(m_{ix} - x)^2 + (m_{iy} - y)^2} \tag{2}$$

In equation 1, $t0$ is the time at which the signal is sent, t1 is the time at which the signal is received. d_{TOA} is the measured distance between the target and node, x and y are the coordinates of the target, and m_{ix} and m_{iy} are the coordinates of the position of anchor node i, which is the number of the corresponding anchor node.

At least 2 equations, thus two TOA measurements, are required to calculate the position of the target. Three equations are needed to guarantee a single solution. Before the location of a target node can be determined, a few things need to be known. The nodes need to be synchronized such that the signals can be sent at the same time. The location of the anchor nodes need to be known. These are all variables in which noise can be present. For example, in the synchronisation, a small time delay can be present. It is important to know how the noise behaves in order to determine the accuracy of the final determined position. This paper will look into the effect of adding noise to the node position.

In the scenario above, there is no noise present. The position noise will be added to the position of the node as n_{ix} and n_{iy}. The distance noise will be n_{id}. The variance of the position noise is σ_p^2, and the variance of the distance noise σ_d^2. This results in the following TOA model.

$$d_{TOA} = \sqrt{(m_{ix} - x + n_{ix})^2 + (m_{iy} - y + n_{iy})^2} + n_d \tag{3}$$

2 Methods used

2.1 Maximum likelihood estimator

To compare the different methods the maximum likelihood estimator is used (MLE). The MLE is a baseline for how accurate an estimation can be[1]. It is expected that the MLE is the best estimation compared to the other methods. The MLE is calculated by calculating the maximum likelihood cost function, and minimizing the cost using optimization algorithms. In this paper, it is calculated using fminunc() function in Matlabs optimization library. A grid based approach is used to make sure that the function does not fall into a local minimum.

2.2 Least squares

One of the most common estimation methods is the least squares method. This method is chosen due to it's computational simplicity, and it is an widely used algorithm. The ordinary least squares algorithm is an algorithm that geometrically finds the point which is closest to all TOA measurements, by minimizing the sum of the squares. [4]

To calculate the linear least squares for the TOA approach, the distance measurement needs to be squared first. The calculations are taken from [3].

$$d_{TOA}^2 = (m_x - x + n_x)^2 + (m_y - y + n_y)^2 + n_d^2 + 2n_d\sqrt{(m_x - x + n_x)^2 + (m_y - y + n_y)^2} \quad (4)$$

Expanding this equation results in the following equation 5

$$\begin{aligned} d_{TOA}^2 = m_x^2 + y^2 + n_x^2 - 2m_x x + m_x n_x - x n_x + m_y^2 + y^2 + n_y^2 \\ -2m_y y + m_y n_y - y n_y + n_d^2 + 2n_d\sqrt{(m_x - x + n_x)^2 + (m_y - y + n_y)^2} \end{aligned} \quad (5)$$

For convenience the terms that contain noise are substituted using n_{toa}

$$n_{toa} = m_x n_x - x n_x + m_y n_y - y n_y + n_d^2 + 2n_d\sqrt{(m_x - x + n_x)^2 + (m_y - y + n_y)^2} \quad (6)$$

$$-2m_x x - 2m_y y + x^2 + y^2 + n_{toa} = d_{toa}^2 - m_x^2 - m_y^2 \quad (7)$$

This equation is just for a single TOA measurement. If there are M anchor nodes. Then the complete set of equations can be expressed in the following matrix form in equation 8

$$\boldsymbol{A\phi} + \boldsymbol{q} = \boldsymbol{b} \quad (8)$$

$$\boldsymbol{A} = \begin{bmatrix} -2m_{1x} & -2m_{1y} & 1 \\ -2m_{2x} & -2m_{2y} & 1 \\ ... & ... & ... \\ -2m_{xi} & -2m_{Mi} & 1 \end{bmatrix} \quad (9)$$

$$\boldsymbol{\phi} = \begin{bmatrix} x & y & (x^2 + y^2) \end{bmatrix} \quad (10)$$

$$\boldsymbol{q} = [n_{toa1}, n_{toa2}, ..., n_{toai}]^T \quad (11)$$

$$\boldsymbol{b} = \begin{bmatrix} (d_{TOA1}^2 - m_{x1}^2 - m_{y1}^2) \\ (d_{TOA2}^2 - m_{x2}^2 - m_{y2}^2) \\ ... \\ (d_{TOAi}^2 - m_{xi}^2 - m_{yi}^2) \end{bmatrix} \quad (12)$$

Then, the least squares solution can be calculated using

$$\boldsymbol{\phi}' = (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T\boldsymbol{b} \quad (13)$$

2.3 Weighted least squares

The least squares method is quite simple. However it can be improved by using the weighted least squares. The weighted least squares (WLS), uses extra information of the known variance of the error. The weighing matrix is calculated as follows. [3]

$$\boldsymbol{W} = E\{\boldsymbol{q}\boldsymbol{q}^T\}^{-1} \tag{14}$$

$$\boldsymbol{W} \approx 4 * diag(4\sigma_1^2 d_1^2, 4\sigma_2^2 d_2^2, ..., 4\sigma_i^2 d_i^2) \tag{15}$$

Here d_i are the actual distances between the target node and anchor node i. This can be approximated with the TOA measurement.

$$\boldsymbol{W} \approx 1/4 * diag(\frac{1}{\sigma_1^2 d_{toa1}^2}, \frac{1}{\sigma_2^2 d_{toa2}^2}, ..., \frac{1}{\sigma_i^2 d_{toai}^2}) \tag{16}$$

Then the final result of the weighted least squares is given by:

$$\boldsymbol{\phi}' = (\boldsymbol{A}^T \boldsymbol{W} \boldsymbol{A})^{-1} \boldsymbol{A}^T \boldsymbol{W} \boldsymbol{b} \tag{17}$$

2.4 Two step least squares

An additional step to the WLS can be added to improve the accuracy at the cost of some calculation time. By using the 3rd component of the result of the WLS. Which is the resulting ϕ' from the WLS. This contains a small error which can again be minimized using another weighted least squares approximation. The weighing matrix for the 2SWLS can be calculated from the weighing matrix of the WLS [3].

$$\boldsymbol{W_2} = ([\boldsymbol{H}(\boldsymbol{A}^T \boldsymbol{W} \boldsymbol{A})^{-1} \boldsymbol{H}^T])^{-1} \tag{18}$$

With $\boldsymbol{H}$ being the following:

$$\boldsymbol{H} = \begin{bmatrix} 2x & 0 & 0 \\ 0 & 2y & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{19}$$

The entries of the WLS are added in $\boldsymbol{b_2}$:

$$\boldsymbol{b_2} = \begin{bmatrix} \phi(1)' & \phi(2)' & \phi(3)' \end{bmatrix} \tag{20}$$

$$\boldsymbol{G} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \tag{21}$$

Together with $\boldsymbol{G}$, the final result can be calculated with equation 22

$$\boldsymbol{\phi}_{\boldsymbol{2}}' = (\boldsymbol{G}^T \boldsymbol{W}_2 \boldsymbol{G})^{-1} \boldsymbol{G}^T \boldsymbol{W}_2 \boldsymbol{b}_{\boldsymbol{2}} \tag{22}$$

The obtain the x and y values of the 2SWLS method, it is required to take the root of $\boldsymbol{\phi}_{\boldsymbol{2}}$. To retain the sign information, the signum function is used in equation 23 and 24

$$x' = signum(\boldsymbol{\phi}(1)')\sqrt{\boldsymbol{\phi}_{\boldsymbol{2}}(1)'} \tag{23}$$

$$y' = signum(\boldsymbol{\phi}(1))'\sqrt{\boldsymbol{\phi}_{\boldsymbol{2}}(2)'} \tag{24}$$

3 Simulation

In the simulations, 4 nodes are used. This results in a total of 4 TOA measurements. The nodes are positioned in a square with sides of 1000 meters. The target position for the symmetric scenario is located at the center of the square at position [500m, 500m]. For the asymmetric situation the target is chosen to be at [200m,300m]. An example is shown in the figure below.

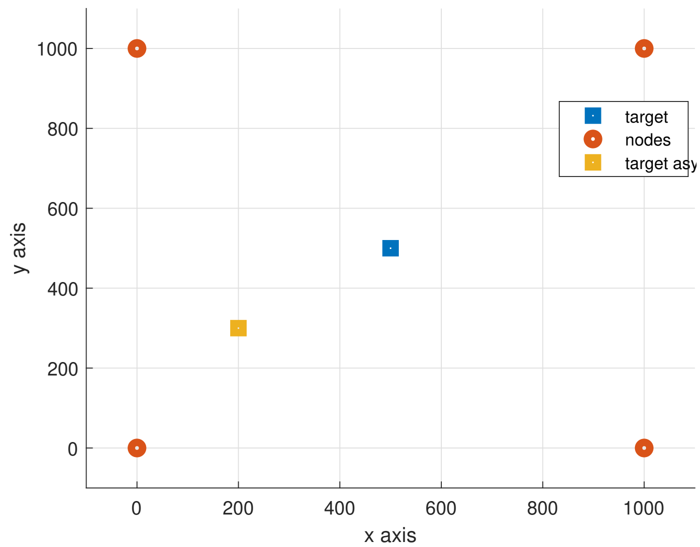


Figure 1: Simulation Layout

For the variance of the simulations there are two main noise parameters that are varied: the variance of the position error, and the distance error. Where both of these errors are modelled as zero-mean independent Gaussian variables. Most simulations are done for 1000 runs for each 1000 values of noise. The result is evaluated with the root mean squared position error (RMSE), which is calculated between the resulting target estimation and the real target value.

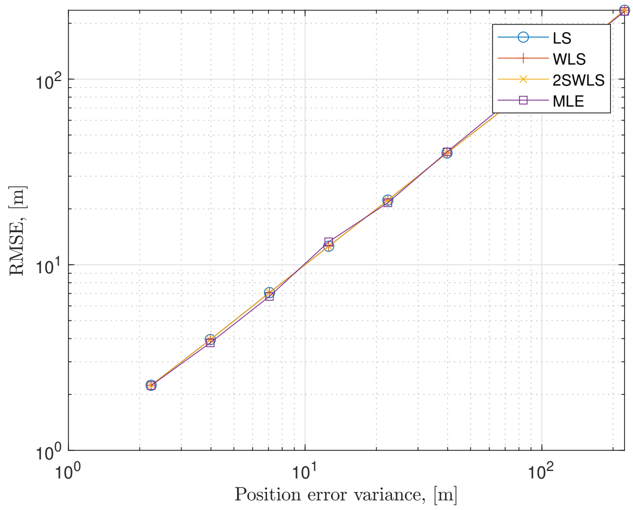
4 Results

4.1 2D Scenario

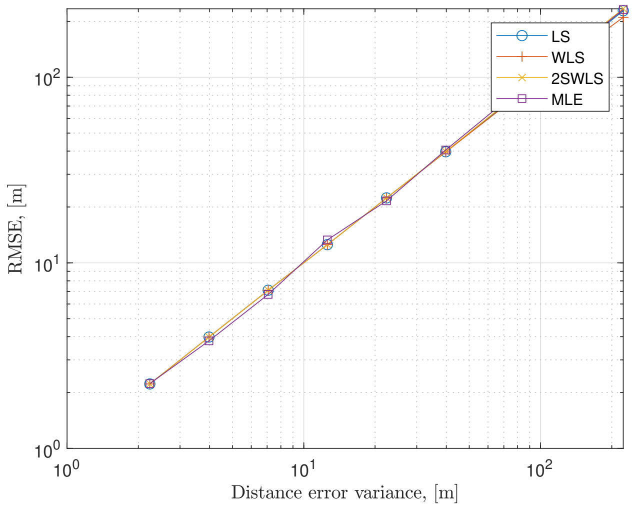
In figure 2 and figure 3, different types of noise are separately plotted for the 2D scenario. Figure 2 shows the symmetric scenario and figure 3 shows the asymmetric scenario. For these figure 10000 runs are simulated. The MLE is calculated using 100 runs. In these figures, the result of the methods and the maximum likelihood are shown. Sub figure 2a there is an error in the position of the anchor nodes. Sub figure 2b shows the results when a distance error is added to the distance of the TOA measurement. The variance of these errors are plotted on the x-axis. The resulting mean squared position error is plotted on the y-axis. The WLS and the 2SWLS are weighted on distance error variance. This is why only the asymmetric scenario shows an improved estimation for the WLS, and 2SWLS calculation.

In figure 4 the methods are plotted against distance error on the x-axis, all methods are plotted two times, one time with with no position error added, and one time with a position error with variance $5m^2$. The distance error is expressed in SNR, such that the variance is proportional to the distance. For each SNR values, a 1000 values of position errors are computed, for which again are a 1000 values of distance errors are computed. This results in a RMSE calculated from a million different values. Interestingly, with the position error added, the value converges to about 5m, which is the same as the position error variance. The maximum likelihood estimator is also plotted. The result of the MLE is sometimes worse than the estimated value. This is likely because the MLE is calculated using only a 100 runs, compared to the 10000 runs of the other methods. Also for the symmetric scenario, the error is as large as the MLE, while the asymmetric scenario is worse compared to the MLE.

The same result has been plotted for the asymmetric scenario in figure 5. It again shows a slightly higher RMSE compared to the symmetric scenerio. This is because the distance variance is based on the distance between the target and anchor node. The sum of all these distances is larger in the asymmetric scenario resulting in a slightly higher total variance. It is also interesting that once the position error is way smaller than the distance error, the WLS starts to outperform the 2SWLS. This could be because the calculation of the weighing matrix does not hold anymore for the 2SWLS.

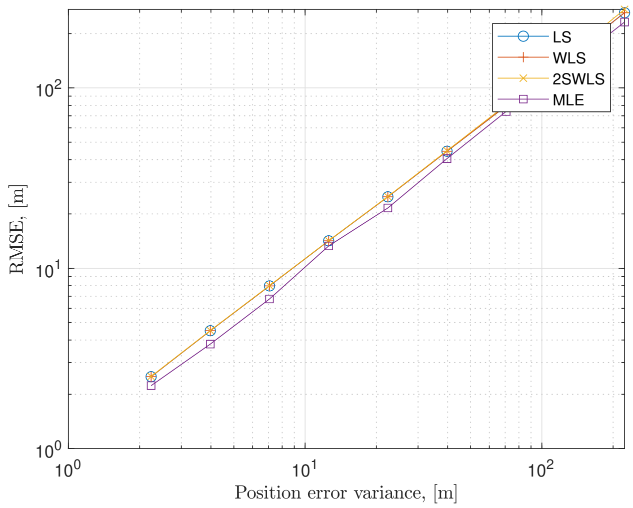


(a) Varying the position error

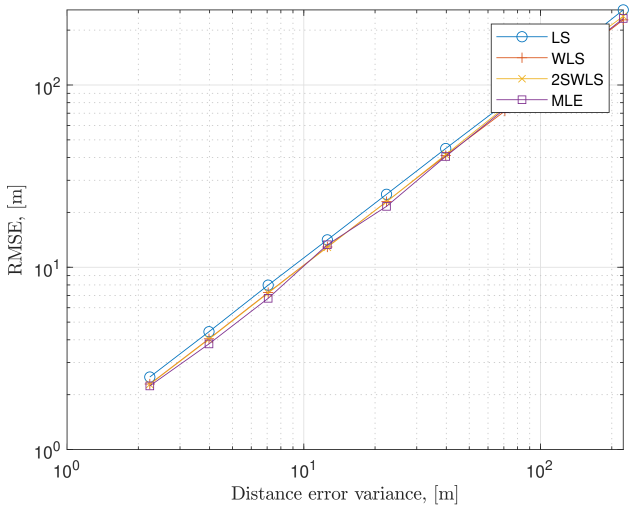


(b) Varying the distance error

Figure 2: symmetric scenario

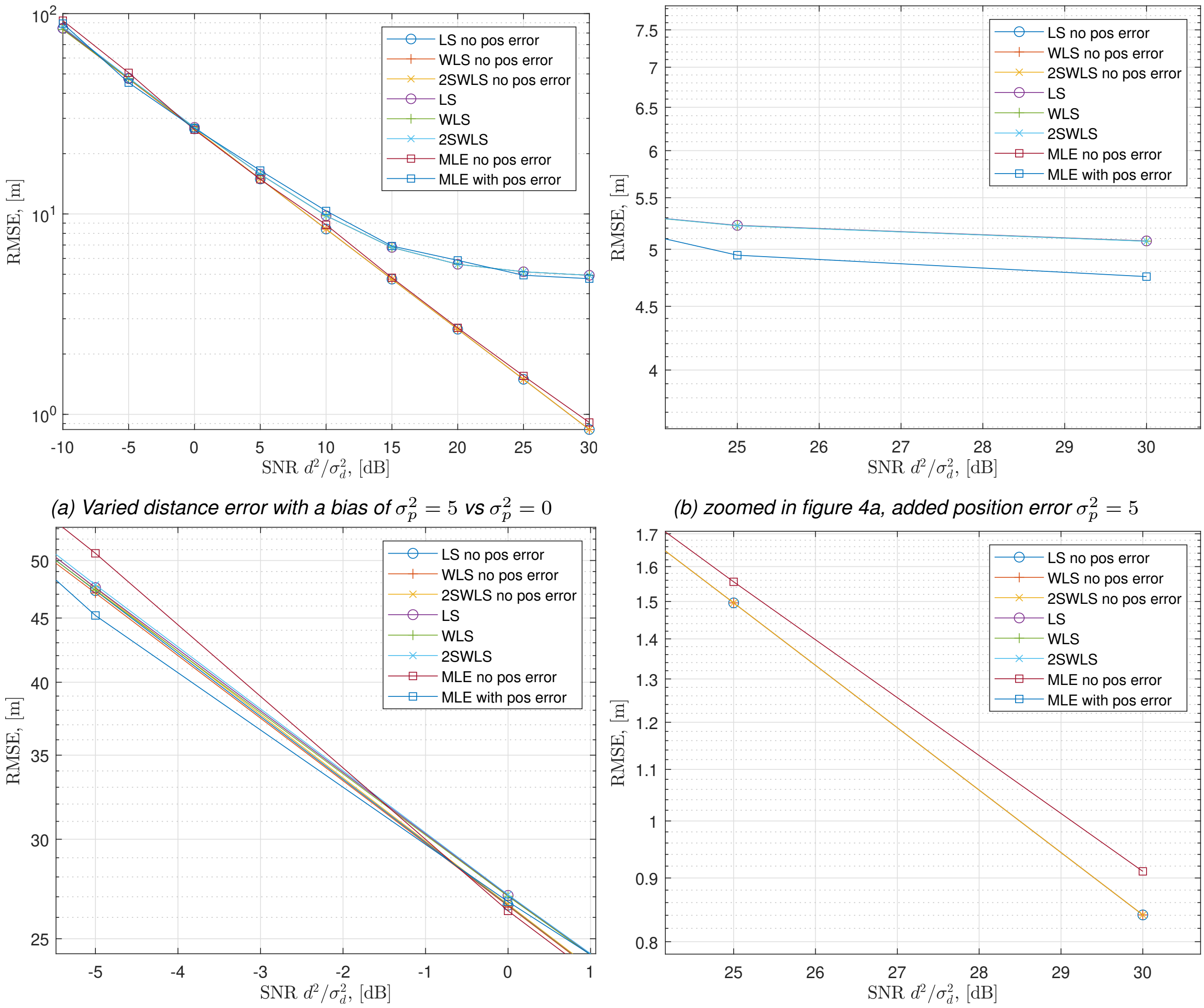


(a) Varying the position error



(b) Varying the distance error

Figure 3: Asymmetric scenario



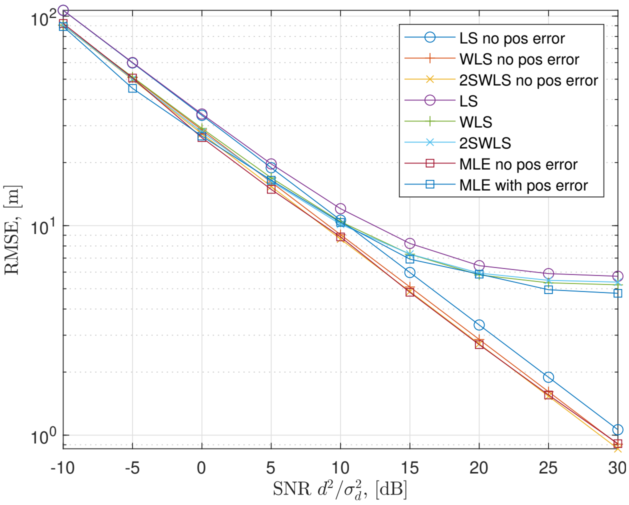
(a) Varied distance error with a bias of $\sigma_p^2 = 5$ vs $\sigma_p^2 = 0$

(b) zoomed in figure 4a, added position error $\sigma_p^2 = 5$

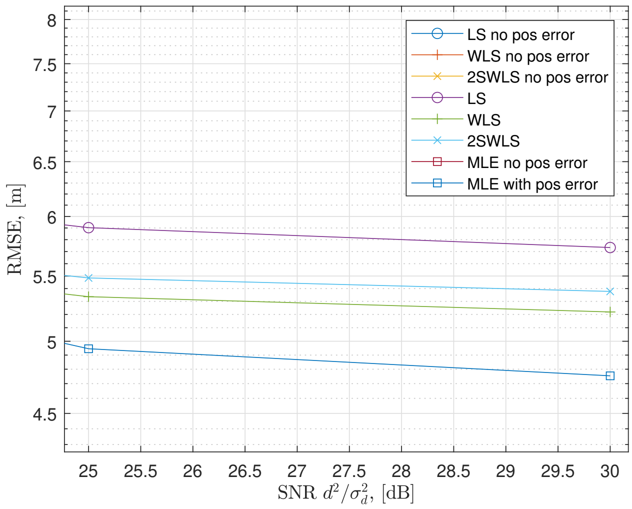
(c) zoomed in figure 4a, low SNR

(d) zoomed in figure 4a, no added position error $\sigma_p^2 = 0$

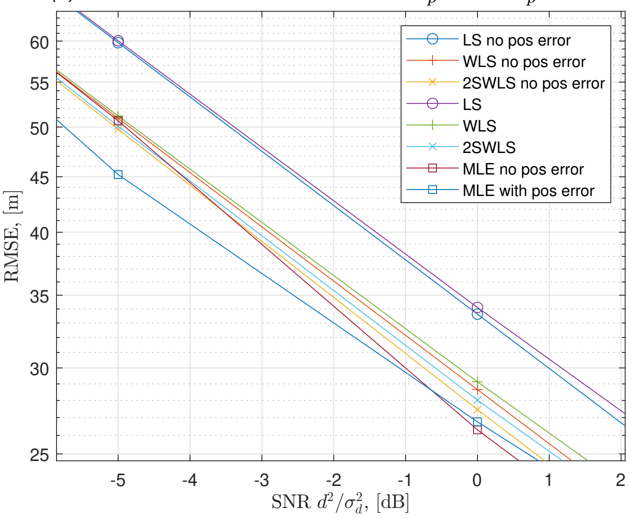
Figure 4: Symmetric scenario



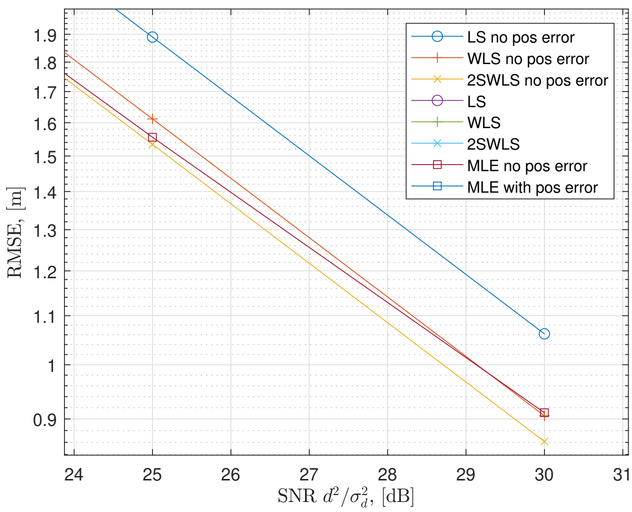
(a) Varied distance error with a bias of $\sigma_p^2 = 5$ vs $\sigma_p^2 = 0$



(b) zoomed in figure 5a, added position error $\sigma_p^2 = 5$

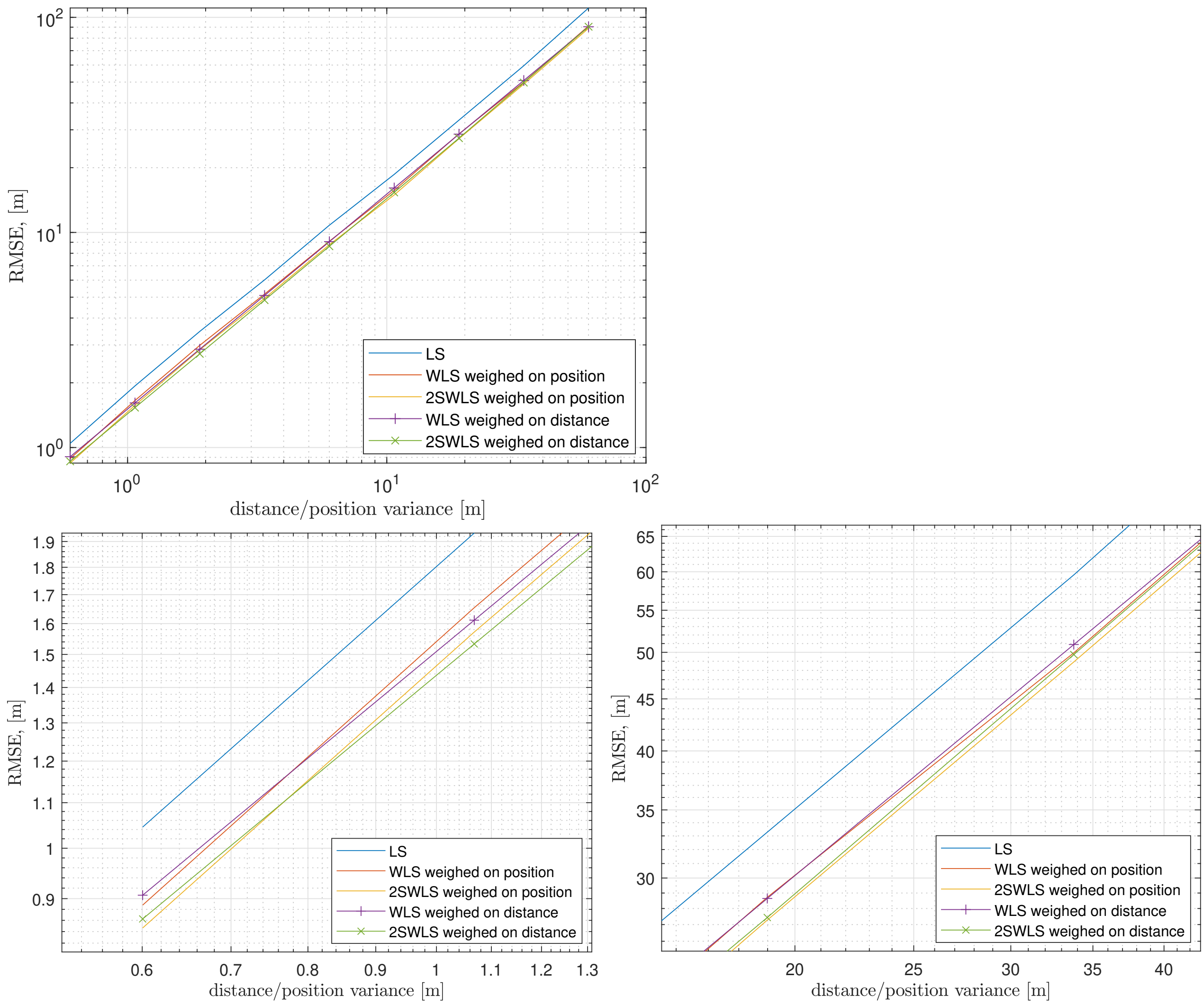


(c) zoomed in figure 5a, low SNR



(d) zoomed in figure 5a, no added position error $\sigma_p^2 = 0$

Figure 5: Asymmetric scenario



(a) zoomed figure, low variance

(b) zoomed figure, high variance

Figure 6: Caption for this figure with two images

Weighing on position vs distance

In the next figures the difference between weighing based on position error, and based on distance error is explored. Figure 6 shows the WLS and 2SWLS weighed on position and on distance in the same figure. The x-axis shows the variance of the distance error in case of the methods weighing on distance, or the variance of position error based on when the methods are weighed on position. In the second figure there is an additional noise added. In case of weighing on position error, a distance error is added with variance 5. And in case of weighing on the distance error, an additional position error is added, again with a value of 5. Again this is simulated with a 1000 different distance errors, for each 1000 different position errors. It does not look like either one outperforms the other. It again shows that the WLS outperforms the 2SWLS in figure 6a.

Figure 8 shows the same figure however, here there is an position error added with

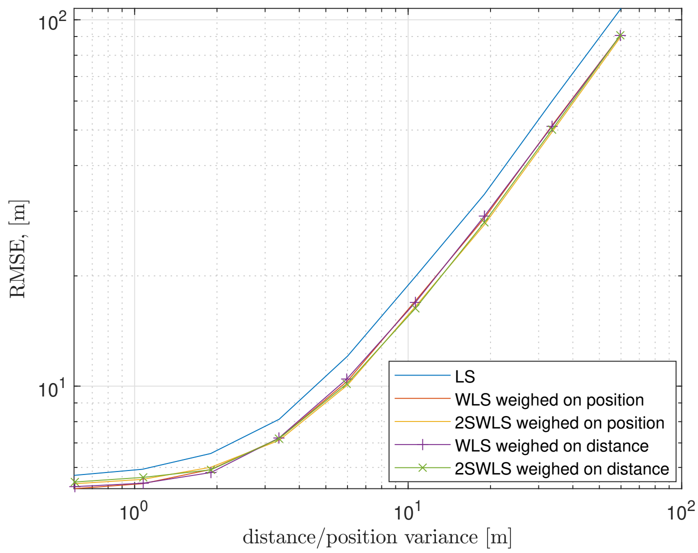
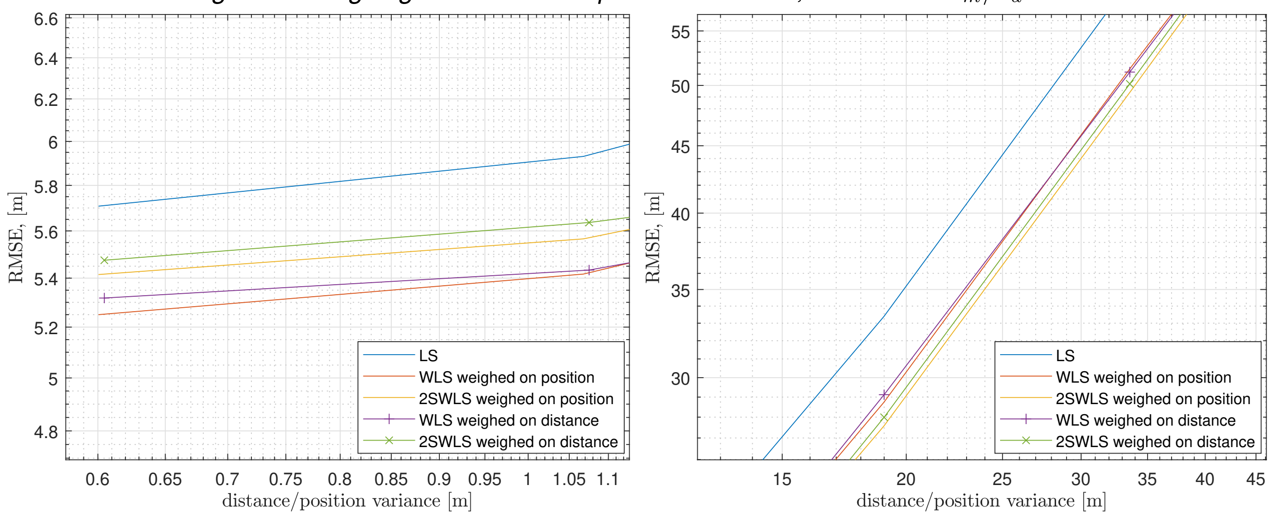


Figure 7: Weighing on distance vs position variance, with added $\sigma_m/\sigma_d = 5$



(a) zoomed figure, low variance

(b) zoomed figure, high variance

Figure 8: Caption for this figure with two images

again a variance of 5m. Again both types of weighing perform quite similar. However weighing on position does seem to give a slight performance increase when the distance error is higher than the position error.

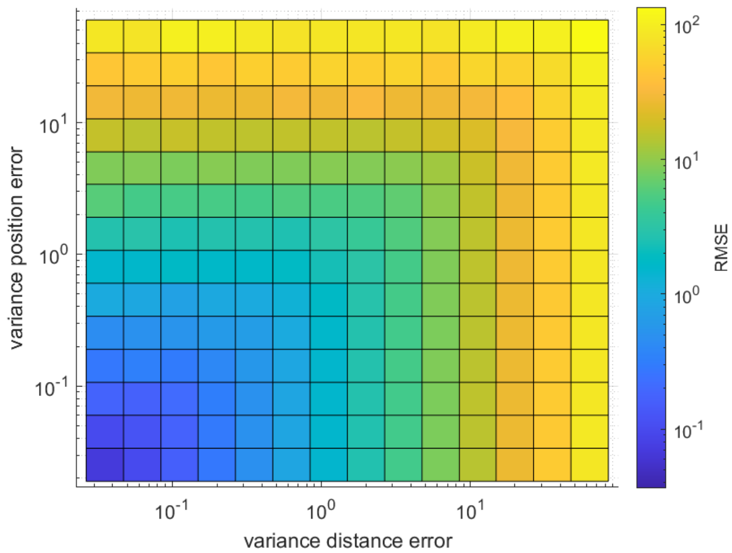


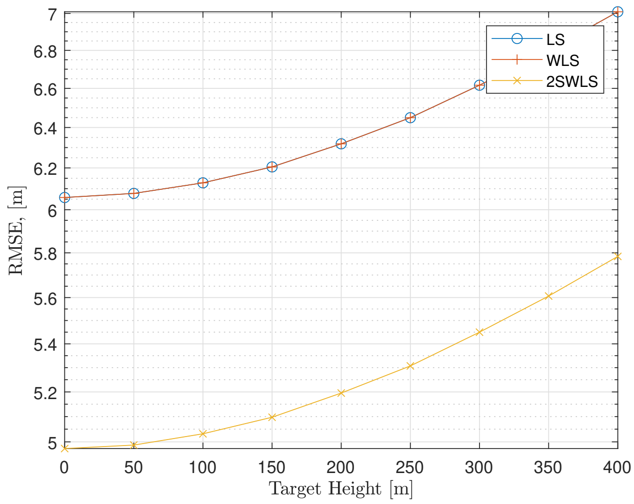
Figure 9: Heatmap of the least squares result

Heatmap

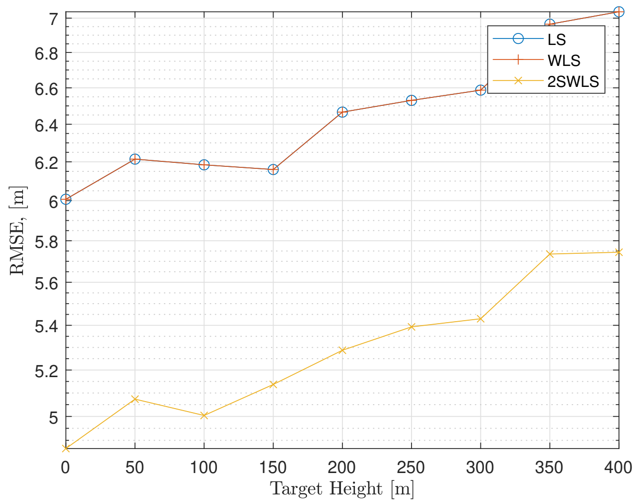
A heatmap is shown in figure 9 for the least squares method. On the x axis the position error is varied and on the y axis the distance error is varied. The color intensity shows the value of the RMSE. It shows symmetric and linear behaviour for both axes. Increasing the distance error does introduce more noise in the results than increasing the position error.

4.2 3D scenario

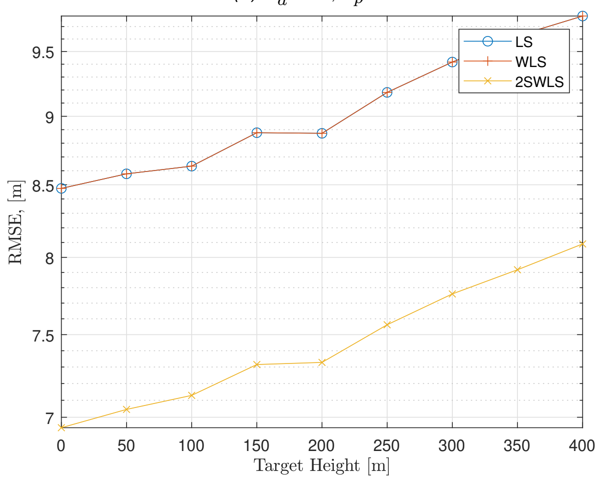
For this section, the calculations are extended to a 3D scenario[2]. In this case the horizontal positions of the nodes and target are identical to the 2D scenario. However the height of the target is varied starting at 0, while the variances of the errors stayed the same. Three different error combinations are shown are shown in figure 10. What becomes immediately clear is that the error increases with the height, even though the total error is kept the same. Also the 2SWLS provides a improvement over the WLS even though the situation is still completely symmetric. Figure 10b shows that the position error give an more noisy result. This means that the variance of the RMSE is higher when adding position noise. Adding both types of noises such as in figure 10c results in a higher RMSE, with the added noise from the position error.



(a) $\sigma_d^2 = 5, \sigma_p^2 = 0$



(b) $\sigma_p^2 = 5, \sigma_d^2 = 0$



(c) Varying height with $\sigma_p = 5^2$, $\sigma_d^2 = 5$

Figure 10: varying height in a 3D scenario

5 Conclusions

The goal of this report was to study the effect of adding a position error to the anchor nodes. This was done by comparing localisation methods such as the least squares, weighted least squares, and the two step least squares to the maximum likelihood estimator in different scenarios. Independently the position error and the distance error perform very similar. Independently, they result in comparable RMSE. It appears that the errors behave additive and their effect is uncorrelated. The RMSE of the MLE seems to scale exactly with the position error variance: position noise causes a higher variance in results than distance error. However it does still converge to the same RMSE. This could be because for the position error, for both the x axis, and y-axis an error is added for each node, compared to just an error based on the distance. Comparing the result of the LS to the results of WLS and 2SWLS, it is clear that in symmetric scenarios all three methods give nearly the same accuracy for both position variance and distance variances. However when introducing asymmetry, the methods start to perform worse than the asymmetric scenario, with the least squares having consistently the lowest accuracy, then the WLS, then the 2SWLS in most cases. The 2SWLS needs to be weighed correctly Comparing the 2D scenario to the 3D scenario, it becomes clear that the geometry matters. Just like introducing asymmetry, increasing the height decreases the accuracy of all the methods. For future research, it would be interesting to see if that also were to happen to the MLE.

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