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Triangular patterns in groups of moving individuals

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Preface

First and foremost, I would like to thank my supervisor, Richard Boucherie for this interesting topic and his insightful comments and our discussions about this problem. I would also like to thank the people around the Abacus room for listening to my ramblings about the topic and trying to help me. Finally, I would like to thank everyone who has proofread my article.

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Abstract

In this paper we discuss an exercise in systemic therapy. In a group of people, each person chooses two others to form an equilateral triangle with. Then, everyone starts moving to reach this goal. A stable state is defined as a state in which no persons are moving anymore. The research question is "Given initial conditions, does the system converge to a stable state?". We find that systems with a specific structure are stable and we find that out of sixty possible configurations with four people, fifty-five are stable.

Keywords: ordinary differential equations, linear difference equations, eigenvalues

1 Introduction

This article will analyze an exercise in systemic therapy. Systemic therapy is a form of psychotherapy in which people are analyzed based on their relationships and life choices, instead of on an individual level [1]. In this exercise, a group of people is put in a room and each person chooses two other people they consider as close friends. Once everybody has chosen, everyone tries to form an equilateral triangle with their two friends. A stable state is defined as a state in which no persons are moving anymore. We will call the entire system stable if it converges to a stable state. We are interested in the patterns that emerge, and specifically whether they are stable. The research question hence is "Given initial conditions, does the system converge to a stable state?".

In section 2, the model will be described mathematically, after which in section 3 the stability of the system is described. Then, in section 4, we investigate systems with a certain structure and their stability properties. Following on, we investigate all possible systems with four persons in section 5. Finally, we draw up our conclusions and make recommendations for future research.

2 Model

The model consists of n persons, represented as points and denoted by P_i, $i = 1, \ldots, n$. Each person has an (x, y) coordinate in the plane. We will denote the column vector of the coordinates of person i at time t by $P_{i,t}$. All persons then also have two friends, with whom they want to form an equilateral triangle. These friends of person i are denoted by $F_{i,1}$ and $F_{i,2}$ and their locations at time t by $F_{i,1,t}$ and $F_{i,2,t}$, respectively. The column vector of all locations at time t is also denoted by $\mathbf{P}_t$. The column vector of the final vertex of the triangle, also known as the goal, of person i at time t is denoted by $G_{i,t}$. The

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coordinates of the goal are calculated by rotating a vector between the other two vertices of the triangle by 60 degrees or $\frac{\pi}{3}$ radians in either direction. This is easily done using rotation matrices. Below, R_+ describes a counterclockwise rotation of 60 degrees, while R_- describes a clockwise rotation of 60 degrees.

$$R_+ = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \qquad R_- = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

There are two possible goal locations. The actual goal will be the one which is closer to the person at $t = 0$. In order to still be able to describe a general system, we will also use $R_\pm$ and $R_\mp$:

$$R_\pm = \begin{bmatrix} \frac{1}{2} & \mp\frac{\sqrt{3}}{2} \\ \pm\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \qquad R_\mp = \begin{bmatrix} \frac{1}{2} & \pm\frac{\sqrt{3}}{2} \\ \mp\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}.$$

We can now calculate the next goal location for each person:

$$G_{i,t+1} = F_{i,1,t} + R_\pm(F_{i,2,t} - F_{i,1,t}), \tag{1}$$

Let I_k denote the identity matrix of dimension k. Since $I_2 - R_\pm = R_\mp$, equation 1 can be rewritten as follows:

$$G_{i,t+1} = R_\pm F_{i,2,t} + R_\mp F_{i,1,t}.$$

Then the step that each person will take is calculated:

$$\Delta P_{i,t+1} = h(G_{i,t+1} - P_{i,t}). \tag{2}$$

Here h determines the step size, and, as we will see soon, should be small if we want the system to converge to a stable state.

After the steps have been calculated for each person, they all move at the same time:

$$\begin{aligned} P_{i,t+1} &= \Delta P_{i,t+1} + P_{i,t} \\ &= h(G_{i,t+1} - P_{i,t}) + P_{i,t} \\ &= h(R_\pm F_{i,2,t} + R_\mp F_{i,1,t} - P_{i,t}) + P_{i,t} \qquad (3) \\ &= h(R_\pm F_{i,2,t} + R_\mp F_{i,1,t}) + (1-h)P_{i,t}. \qquad (4) \end{aligned}$$

Based on equation 3, we can write the complete model using matrices and the previously introduced $\mathbf{P}_t$:

$$\mathbf{P}_{t+1} = h\,A'\,\mathbf{P}_t + \mathbf{P}_t \tag{5}$$

$$A'_{i,j} = \begin{cases} -I_2 & \text{if } i = j \\ R_\pm & \text{if } F_{i,2} = P_j \\ R_\mp & \text{if } F_{i,1} = P_j \\ O_2 & \text{otherwise,} \end{cases}$$

where I_2 denotes the 2 by 2 identity matrix and O_2 denotes the 2 by 2 zero matrix. Each block-row of the matrix consists of one $-I_2$, one R_+ and one R_-, while the remainder is filled with O_2-matrices. The $-I_2$ is always on the diagonal ($i = j$), while the R_+ and R_- are in the columns corresponding to the friends of P_i. Note that each $A'_{i,j}$ as described above is a 2 by 2 matrix, which makes A' a $2n$ by $2n$ matrix.

However, we can also write the complete model based on equation 4:

$$\mathbf{P}_{t+1} = A\mathbf{P}_t \tag{6}$$
$$A_{i,j} = \begin{cases} (1-h)I_2 & \text{if } i = j \\ hR_{\pm} & \text{if } F_{i,2} = P_j \\ hR_{\mp} & \text{if } F_{i,1} = P_j \\ O_2 & \text{otherwise.} \end{cases}$$

Once again, A is a $2n$ by $2n$ matrix. The block matrices are distributed the same as before in equation 5.

Remark. With A and A' defined as above, $A = hA' + I_{2n}$.

3 Stability

Now that we have set up the model we can consider its stability. Recall that the model is stable if it converges to a stable state.

Theorem 1. *Let the initial locations $\mathbf{P}_0$ and the friend relations be given. If the corresponding matrix A is diagonalizable and if for all eigenvalues λ of A, we have $|\lambda| \leq 1$, then the system is stable.*

Proof. So let the initial locations $\mathbf{P}_0$ and the friend relations be given. Furthermore, let A be diagonalizable with eigenvalues $|\lambda| \leq 1$. Then we can find matrices Q and D such that D is a diagonal matrix and $A = QDQ^{-1}$. This diagonal matrix D will then have the eigenvalues of A on its diagonal. Given the initial locations $\mathbf{P}_0$ and the friendship relations, we can rewrite equation 6 into

$$\mathbf{P}_t = A^t\mathbf{P}_0.$$

Then

$$\begin{aligned} \mathbf{P}_t &= A^t\mathbf{P}_0 \\ &= QD^tQ^{-1}\mathbf{P}_0 \\ &\xrightarrow{t\to\infty} QO_{2n}Q^{-1}\mathbf{P}_0 \\ &= \vec{0}, \end{aligned}$$

where O_{2n} is the $2n$ by $2n$ zero matrix. Note that the third step follows from the diagonality of D and the fact that $|\lambda| \leq 1$ for all eigenvalues λ. Since the system converges to a stable state, the system is stable. □

We will now show that the eigenvalues of A and A' are related. However, we will first show that if A is diagonalizable, then A' is as well.

Lemma 2. *Let A and A' be $2n$ by $2n$ matrices such that $A = hA' + I_{2n}$. If A is diagonalizable, then A' is also diagonalizable.*

Proof. Let A be diagonalizable and let $A = hA' + I_{2n}$. Since A is diagonalizable, we have $A = QDQ^{-1}$, where D is a diagonal matrix. Then

$$\begin{aligned} hA' &= A - I_{2n} \\ &= QDQ^{-1} - I_{2n} \\ &= Q(D - I_{2n})Q^{-1} \\ \Rightarrow A' &= Q\left(\frac{1}{h}(D - I_{2n})\right)Q^{-1}. \end{aligned}$$

Since D is diagonal, so is $D' = \frac{1}{h}(D - I_{2n})$. Hence A' is diagonalizable with diagonal matrix D'. □

We can now proof the following theorem.

Theorem 3. *With A and A' defined as in lemma 2, their respective eigenvalues λ_j and λ'_j can be expressed in terms of each other as follows:*

$$\lambda_j = h\lambda'_j + 1 \qquad \forall j = 1, \ldots, 2n.$$

Proof. Since the eigenvalues of a diagonalizable matrix are on the diagonal of the corresponding diagonal matrix, we can use the relation between D and D' to find the relation between λ_j and λ'_j.

$$\begin{aligned} &D' = \frac{1}{h}(D - I_{2n}) \iff \\ &D = hD' + I_{2n} \iff \\ &\lambda_j = h\lambda'_j + 1 \quad \forall j = 1, \ldots, 2n. \end{aligned}$$

□

We already have a condition for stability based on the eigenvalues λ of A from theorem 1. However we would like a condition for stability based on the eigenvalues λ' of A', since this matrix is easier to construct. Since $\lambda' \in \mathbb{C}$, we let $\lambda' = a + b\mathrm{i}$. Using theorem 3, we deduce the following:

$$\begin{aligned} 0 \le |\lambda| \le 1 &\iff \text{(Theorem 1)} \\ 0 \le |\lambda|^2 \le 1 &\iff \\ 0 \le |h\lambda' + 1|^2 \le 1 &\iff \text{(Theorem 3)} \\ 0 \le |ah + 1 + bh\mathrm{i}|^2 \le 1 &\iff \\ 0 \le (ah + 1)^2 - (bh)^2 \le 1 &\iff \\ 0 \le (ah)^2 + 2ah + 1 - (bh)^2 \le 1 &\iff \\ -1 \le (ah)^2 + 2ah - (bh)^2 \le 0 &\iff \\ -1 \le h^2|\lambda'|^2 + 2ah \le 0 &\iff \\ -\frac{1}{2h} \le h|\lambda'|^2 + a \le 0. & \end{aligned} \tag{7}$$

Since $h|\lambda'|^2 \ge 0$, we need to have $a \le 0$ in order to satisfy the right side of equation 7. This may not be sufficient, but by choosing h small enough equation 7 will be satisfied.

Hence, in order to have a stable system, we need all eigenvalues λ' to have $\mathrm{Re}(\lambda') \le 0$ and h sufficiently small.

4 One by one

We can simplify this problem by starting with three people who have all chosen each other and then letting the others join one by one. If we then let the others only choose people who joined before them, we will always have a stable solution. In this and the next section, we will assume that the step-size h is constant and close to zero, i.e., $0 < h \ll 1$.

Theorem 4. *If we have a system consisting of at least three persons where the first three persons all chose each other as friends and if we let everyone after the first three persons only choose people who joined before them, then the system will be stable for all initial locations.*

Proof. Let A_n be the matrix such that $\mathbf{P}_t = A_n\mathbf{P}_{t-1}$ for the system with n persons. For stability, we need the A_n to be diagonalizable and all the eigenvalues of A_n need to be less than or equal to 1 in absolute value. We will show this by induction.

We will first have the base case, $n = 3$. The matrix A_3 for this system is then as follows:

$$A_3 = \begin{bmatrix} 1-h & 0 & \frac{h}{2} & -h\frac{\sqrt{3}}{2} & \frac{h}{2} & h\frac{\sqrt{3}}{2} \\ 0 & 1-h & h\frac{\sqrt{3}}{2} & \frac{h}{2} & -h\frac{\sqrt{3}}{2} & \frac{h}{2} \\ \frac{h}{2} & h\frac{\sqrt{3}}{2} & 1-h & 0 & \frac{h}{2} & -h\frac{\sqrt{3}}{2} \\ -h\frac{\sqrt{3}}{2} & \frac{h}{2} & 0 & 1-h & h\frac{\sqrt{3}}{2} & \frac{h}{2} \\ \frac{h}{2} & -h\frac{\sqrt{3}}{2} & \frac{h}{2} & h\frac{\sqrt{3}}{2} & 1-h & 0 \\ h\frac{\sqrt{3}}{2} & \frac{h}{2} & -h\frac{\sqrt{3}}{2} & \frac{h}{2} & 0 & 1-h \end{bmatrix}$$

We can now calculate the characteristic polynomial

$$\chi_3(\lambda) = |A_3 - \lambda I_6| = (\lambda - 1)^4(\lambda - (1 - 3h))^2$$

and find the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 1 - 3h$. We can now also find the eigenvectors corresponding to λ_1,

$$\begin{bmatrix} \sqrt{3} \\ 1 \\ 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -\sqrt{3} \\ 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -\sqrt{3} \\ 1 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ \sqrt{3} \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and the eigenvectors corresponding to λ_2,

$$\begin{bmatrix} -\sqrt{3} \\ -1 \\ \sqrt{3} \\ 1 \\ 0 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ \sqrt{3} \\ -1 \\ -\sqrt{3} \\ 2 \\ 0 \end{bmatrix}.$$

Since we have six linearly independent eigenvectors, the matrix A_3 is diagonalizable. Furthermore, since $0 < h \ll 1$, we have $h < \frac{1}{3}$, hence all $|\lambda| \leq 1$.

Now for the induction step, we assume for $n = k > 3$ that the eigenvalues of A_k are less than or equal to 1 and we want to show that this then also holds for the eigenvalues of A_{k+1}.

Therefore, let A_k be a matrix describing the change in location of a system of k people, where everyone after the first three persons can only choose their friends from the group of persons who are already there. By the induction step, A_k will be diagonalizable with all eigenvalues less than or equal to 1. We can then describe A_{k+1} as follows:

$$A_{k+1} = \begin{bmatrix} A_k & O \\ B & (1-h)I_2 \end{bmatrix},$$

where O is a $2k$ by 2 zero-matrix and B is a 2 by $2k$ matrix consisting of one R_+-matrix, one R_--matrix and 2 by 2 zero matrices elsewhere, with the exact locations of the R matrices determined by who the new person chooses as their friends. We will now find the eigenvalues of A_{k+1} using the characteristic polynomial

$$\begin{aligned} \chi_{k+1}(\lambda) &= |A_{k+1} - \lambda I_{2k+2}| \\ &= \begin{vmatrix} A_k - \lambda I_{2k} & O \\ B & (1-h-\lambda)I_2 \end{vmatrix} \\ &= |A_k - \lambda I_{2k}| \cdot |(1-h-\lambda)I_2| \\ &= \chi_k(\lambda)\ (\lambda - (1-h))^2. \end{aligned}$$

Here the third equality follows by expanding the determinant about the final two columns, which have $(2n-1)$ zero elements and one non-zero element [2]. Hence we find that the eigenvalues of A_{k+1} are the eigenvalues of A_k and $\lambda = 1-h$. For the new eigenvalue λ we can find the following eigenvectors:

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

We can extend each of the eigenvectors of A_k with two entries to create eigenvectors for A_{k+1}. Even if these entries are non-zero, the eigenvectors will still all be linearly independent. Together with the 2 new eigenvectors, we now have $2k+2$ eigenvectors, so A_{k+1} is diagonalizable. Furthermore, since the eigenvalues of A_k are less than or equal to 1 by the induction hypothesis and $|1-h| \leq 1$, we have that all eigenvalues are less than or equal to 1 and hence completing the proof by induction.

Since all the eigenvalues are less than or equal to 1 for this type of system with at least three persons, we conclude that each of these systems must also be stable. □

As we will see in more detail in section 5.1, we can switch the numbering of the persons to create new matrices with the same behaviour. For example, take the matrices in equation 8. The matrix in equation 8a is in the one-by-one form, while the matrix in equation 8b is not. Therefore, by theorem 4, the matrix in equation 8a obviously corresponds to a stable system. However, we can swap the 2$^{\text{nd}}$ and 5$^{\text{th}}$ persons in equation 8a to get equation 8b. Since changing the numbering does not affect the behaviour of a system, the system corresponding to the matrix in equation 8b must also be stable.

$$\begin{bmatrix} H & R_+ & R_- & O_2 & O_2 \\ R_- & H & R_+ & O_2 & O_2 \\ R_+ & R_- & H & O_2 & O_2 \\ R_- & R_+ & O_2 & H & O_2 \\ R_+ & O_2 & O_2 & R_- & H \end{bmatrix} \quad (8a) \qquad \begin{bmatrix} H & O_2 & R_+ & O_2 & R_- \\ R_- & H & O_2 & R_+ & O_2 \\ R_- & O_2 & H & O_2 & R_+ \\ R_+ & O_2 & O_2 & H & R_- \\ R_+ & O_2 & R_- & O_2 & H \end{bmatrix} \quad (8b)$$

5 The case of $n = 4$

In order to describe the behaviour of the system with four persons, we will find all possible matrices that describe such a system. In general, the diagonal is fixed and each row has one R_+, one R_- and one O_2 matrix. This yields six possibilities per row. With four rows, we then have $6^4 = 1296$ possible matrices. However, we will first only look at the different locations we could place the O_2 matrices, and after that look at the locations of the R_+ and R_- matrices. For the O_2 matrix, there are three options per row, so this yields $3^4 = 81$ different matrices.

5.1 The locations of the O_2 matrices

Of these 81 matrices, some will describe the same behaviour. This occurs since the ordering of the people does not matter. For example, take the matrix in equation 9a. If we swap the numbers of persons one and three, we get the matrix in equation 9b. We achieved this by switching the elements at $(1, 3)$ and $(3, 1)$ and then swapping the remainder of the first and third rows with each other and swapping the remainder of the first and third columns with each other.

$$\begin{bmatrix} H & O_2 & R & R \\ O_2 & H & R & R \\ R & R & H & O_2 \\ R & R & O_2 & H \end{bmatrix} \quad (9a) \qquad \begin{bmatrix} H & R & R & O_2 \\ R & H & O_2 & R \\ R & O_2 & H & R \\ O_2 & R & R & H \end{bmatrix} \quad (9b)$$

In order to find the number of distinct matrices, we will transform the matrix to a simpler form. Since we currently only care about the placement of the O_2 matrix, we can describe each matrix as a vector with four entries. Each entry will correspond to a row, and will denote in which column the O_2 matrix of that row is located. For example, the matrices in equation 9a and equation 9b can be represented as follows:

$$(2, 1, 4, 3) \quad (10a) \qquad (4, 3, 2, 1) \quad (10b)$$

Not all vectors with four entries correspond to a matrix. The first requirement is that all entries are integers between one and four, such that they correspond to a column of the matrix. The second requirement comes from the fact that the diagonal entries of the matrix are already taken. Therefore, for a vector (b_1, b_2, b_3, b_4), we must have $b_j \neq j \quad \forall j \in \{1, 2, 3, 4\}$.

We are now also able to do the permuting of rows and columns in this form. For the switching two rows, we switch the corresponding entries, while for switching two columns we change all the numbers corresponding to the one column to the number of the other column and vice versa. For example, in equation 10a and equation 10b we swapped the first with the third person. Starting with equation 10a, we first swap the first and third entries to get $(4, 1, 2, 3)$ (row swap) and then change all the 1's for 3's and vice versa to get equation 10b (column swap). Note that $(4, 1, 2, 3)$ is just an intermediary step and not necessarily a permutation of equation 10a and equation 10b. We can then create all other permutations from this, since every permutation can be written as the product of 2-cycles [4].

We can now start with a matrix, calculate all of its permutations and see how many there are. This will yield at most $4! = 24$ (vector representations of) matrices. However, due to symmetry, there may be fewer.

After doing this until we've seen all 81 matrices, we find that there are six groups of matrices, which can not be permuted to each other. From each group one matrix is shown in equation 11, together with its vector representation. All 81 vector representations can be found in appendix A.

$$\begin{bmatrix} H & R_+ & R_- & O_2 \\ R_+ & H & O_2 & R_- \\ R_+ & O_2 & H & R_- \\ O_2 & R_+ & R_- & H \end{bmatrix} \quad (4,3,2,1) \tag{11a}$$

$$\begin{bmatrix} H & R_+ & R_- & O_2 \\ O_2 & H & R_+ & R_- \\ R_+ & O_2 & H & R_- \\ R_+ & R_- & O_2 & H \end{bmatrix} \quad (4,1,2,3) \tag{11b}$$

$$\begin{bmatrix} H & O_2 & R_+ & R_- \\ R_+ & H & R_- & O_2 \\ R_+ & R_- & H & O_2 \\ R_+ & O_2 & R_- & H \end{bmatrix} \quad (2,4,4,2) \tag{11c}$$

$$\begin{bmatrix} H & R_+ & O_2 & R_- \\ R_+ & H & R_- & O_2 \\ R_+ & R_- & H & O_2 \\ R_+ & O_2 & R_- & H \end{bmatrix} \quad (3,4,4,2) \tag{11d}$$

$$\begin{bmatrix} H & R_+ & R_- & O_2 \\ R_+ & H & R_- & O_2 \\ R_+ & R_- & H & O_2 \\ R_+ & R_- & O_2 & H \end{bmatrix} \quad (4,4,4,3) \tag{11e}$$

$$\begin{bmatrix} H & R_+ & R_- & O_2 \\ R_+ & H & R_- & O_2 \\ R_+ & O_2 & H & R_- \\ R_+ & R_- & O_2 & H \end{bmatrix} \quad (4,4,2,3) \tag{11f}$$

5.2 Burnside's Lemma

We can check that there are indeed six groups of matrices using Burnside's Lemma. Before that, we need the definition of the fix of a permutation ϕ. It is defined as follows: [3]

$$\text{fix}(\phi) = \{i \in S \mid \phi(i) = i\}$$

This means that $\text{fix}(\phi)$ is the set of all elements $i \in S$ that are not altered by ϕ. Burnside's Lemma then states the following: [3]

Lemma 5 (Burnside's Lemma)**.** *If G is a finite group of permutations on a set S, then the number of orbits of elements of S under G is*

$$\frac{1}{|G|} \sum_{\phi \in G} |\text{fix}(\phi)|.$$

In our case, G is the set of all 24 permutations of numberings, while S is the set of all 81 (vertex representations of) matrices. We can split the permutations up into five types, each of which fixes the same amount of elements in S. For each type we can then count the amount of elements it fixes, multiply them and then add them all up. For example, there are 6 permutations that swap two persons. Consider the permutation ϕ_1 which swaps the first two persons. The matrices in fix(ϕ_1) must then look like equation 12, since $\phi_1(B) = B$. Each C_j, $j = 1, \ldots, 7$, in equation 12 is either an R-matrix or an O_2-matrix.

$$B = \begin{bmatrix} H & C_1 & C_2 & C_3 \\ C_1 & H & C_2 & C_3 \\ C_4 & C_4 & H & C_5 \\ C_6 & C_6 & C_7 & H \end{bmatrix} \tag{12}$$

Recall that each row must have one O_2 and two R-matrices. We can choose one of C_1, C_2 and C_3 to be the O_2, and then the other two must both be R. Hence there are three choices for these three matrices. However, the remainder is forced. In order to satisfy the condition of having two R-matrices in the third and fourth row, we need $C_4 = C_6 = R$, which then forces $C_5 = C_7 = O_2$. Since each permutation that swaps two persons fixes matrices which have a similar structure to B, each permutation that swaps two persons fixes three elements. This results of this example are collected in the second row of table 1, with the other rows calculated similarly. For each type of permutation an example is given using cycle notation [5].

TABLE 1: Applying Burnside's Lemma (not distinguishing between R_+ and R_-)

Type of permutation	#Permutations	#Elements fixed	Multiplied
Identity [(1) (2) (3) (4)]	1	$3^4 = 81$	81
Swapping two persons [(1 2) (3) (4)]	6	3	18
Swapping two pairs of persons [(1 2) (3 4)]	3	$3^2 = 9$	27
Swapping three persons [(1 2 3) (4)]	8	0	0
Swapping all four persons [(1 2 3 4)]	6	3	18
Total	$\lvert G \rvert = 24$		$\sum_{\phi \in G} \lvert \text{fix}(\phi) \rvert = 144$

This indeed yields that the number of orbits is $144/24 = 6$, when we don't distinguish between R_+ and R_-.

We can also apply Burnside's Lemma when we do distinguish between R_+ and R_-. The procedure is the same, just the numbers are different. The results are in table 2.

TABLE 2: Applying Burnside's Lemma (distinguishing between R_+ and R_-)

Type of permutation	#Permutations	#Elements fixed	Multiplied
Identity [(1) (2) (3) (4)]	1	$6^4 = 1296$	1296
Swapping two persons [(1 2) (3) (4)]	6	0	0
Swapping two pairs of persons [(1 2) (3 4)]	3	$6^2 = 36$	108
Swapping three persons [(1 2 3) (4)]	8	0	0
Swapping all four persons [(1 2 3 4)]	6	$6^1 = 6$	36
Total	$\lvert G \rvert = 24$		$\sum_{\phi \in G} \lvert \text{fix}(\phi) \rvert = 1440$

This gives that there are $1440/24 = 60$ different orbits, when we distinguish between R_+ and R_-. Note that in these sixty different orbits, the initial locations are taken into account, since the order of the R_+ and R_- in each row has already been determined. In order to find all 60 different orbits we can use a similar procedure as before. Choose any matrix, calculate all 24 permutations and keep going until we've got all 1296 matrices. Using a computer, this does not take much time. In order to write them down succinctly, we extend the vector representation with a plus or minus in each element. A plus denotes that in the corresponding row the R_+ matrix is to the left of R_-, while a minus denotes that the R_- is to the left of the R_+ in the corresponding row. For example, $(4+, 3+, 2+, 1+)$ would be the new representation of the matrix in equation 11a. In fact, each of the matrices in equation 11 would get four pluses added to the existing representation. In appendix B one matrix from each group can be found.

5.3 Calculating the eigenvalues

For each of the 60 cases we will now calculate the eigenvalues. To start, take for example the matrix in equation 11a, but with a different arrangement of pluses and minuses:

$$\begin{bmatrix} H & R_- & R_+ & O_2 \\ R_+ & H & O_2 & R_- \\ R_- & O_2 & H & R_+ \\ O_2 & R_+ & R_- & H \end{bmatrix} \tag{13}$$
$$(4-, 3+, 2-, 1+)$$

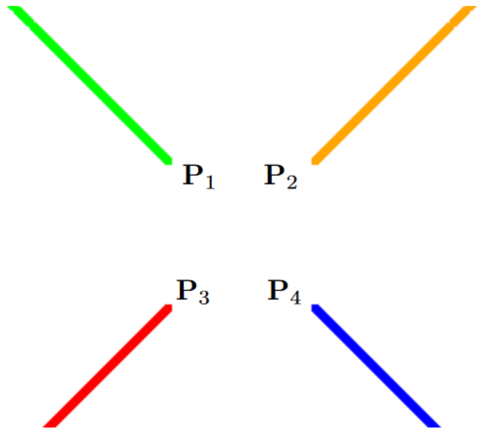

FIGURE 1: A simulation of four persons standing in a square, as described by equation 13

This matrix describes four people standing in a square, each having their neighbours as a friend. They will all move backwards indefinitely, see figure 1. When we calculate the eigenvalues for this matrix, we find 2 positive eigenvalues, 4 negative eigenvalues and 2 zero eigenvalues. This can also be seen in the final row of table 3. The first four columns of tables 3 through 8 denote the plus/minus part of the extended vector representation. In the headers of the following columns, the first sign (+, − or 0) in each of the columns denotes the sign of the real part, while the second sign (± or 0) denotes the sign of the imaginary part, where ± denotes complex conjugate pairs. The final column denotes whether the system is stable. Since there is at least one eigenvalue with positive real part for the matrix of equation 13 (in the final row of table 3), this system is not stable. There are of course $2^4 = 16$ possibilities of writing down the pluses and minuses in the first row to create new systems, however the left-out combinations of pluses and minuses can be reached by permutation from combinations that are present.

Table 3: The number of occurrences of signs of the eigenvalues of the matrix in equation 11a

Extended vector	+0	+±	−0	−±	00	0±	Stable?
+ + + −			6		2		yes
+ + − −			2	4	2		yes
+ − − +			6		2		yes
− + − +	2		4		2		no

We see that out of these four configurations of the matrix in equation 11a, three are stable and one is not.

Table 4: The number of occurrences of signs of the eigenvalues of the matrix in equation 11b

Extended vector	+0	+±	−0	−±	00	0±	Stable?
+ + − +				6	2		yes
+ − − +				6	2		yes
+ − − −			2	4	2		yes
− + − −			6		2		yes
− − + −				6	2		yes
− − − −				6	2		yes

We see that out of these six configurations of the matrix in equation 11b, all six are stable.

Table 5: The number of occurrences of signs of the eigenvalues of the matrix in equation 11c

Extended vector	+0	+±	−0	−±	00	0±	Stable?
+ + + +			2	4	2		yes
+ + + −				6	2		yes
+ − + +				4	4		yes
+ − − +				6	2		yes
− + + +			2	4	2		yes
− + + −				6	2		yes
− + − −				4	4		yes
− − + −			4		4		yes
− − − +				6	2		yes
− − − −			2	4	2		yes

We see that out of these ten configurations of the matrix in equation 11c, all ten are stable.

Table 6: The number of occurrences of signs of the eigenvalues of the matrix in equation 11d

Extended vector				+0	+±	−0	−±	00	0±	Stable?
+	+	+	+			2	4	2		yes
+	+	+	−				6	2		yes
+	+	−	+			2	4	2		yes
+	+	−	−				6	2		yes
+	−	+	+		2		4	2		no
+	−	+	−			2	4	2		yes
+	−	−	+				6	2		yes
+	−	−	−			2	4	2		yes
−	+	+	+			2	4	2		yes
−	+	+	−				6	2		yes
−	+	−	+			2	4	2		yes
−	+	−	−		2		4	2		no
−	−	+	+				6	2		yes
−	−	+	−			2	4	2		yes
−	−	−	+				6	2		yes
−	−	−	−			2	4	2		yes

We see that out of these sixteen configurations of the matrix in equation 11d, fourteen are stable and two are not.

Table 7: The number of occurrences of signs of the eigenvalues of the matrix in equation 11e

Extended vector				+0	+±	−0	−±	00	0±	Stable?
+	+	+	−			2	4	2		yes
+	+	−	−			2	4	2		yes
+	−	+	−				4	4		yes
+	−	−	+			2	4	2		yes
+	−	−	−			2	4	2		yes
−	+	−	−				4	4		yes
−	−	−	+			2	4	2		yes
−	−	−	−			2	4	2		yes

We see that out of these eight configurations of the matrix in equation 11e, all eight are stable. Note that this was expected since this matrix is of the one-by-one form described in section 4.

Table 8: The number of occurrences of signs of the eigenvalues of the matrix in equation 11f

Extended vector				+0	+±	−0	−±	00	0±	Stable?
+	+	+	+			2	4	2		yes
+	+	+	−			2	4	2		yes
+	+	−	+				6	2		yes
+	+	−	−				6	2		yes
+	−	+	+		2		4	2		no
+	−	+	−				6	2		yes
+	−	−	+				6	2		yes
+	−	−	−			2	4	2		yes
−	+	+	+			2	4	2		yes
−	+	+	−				6	2		yes
−	+	−	+				6	2		yes
−	+	−	−		2		4	2		no
−	−	+	+				6	2		yes
−	−	+	−				6	2		yes
−	−	−	+			2	4	2		yes
−	−	−	−			2	4	2		yes

We see that out of these sixteen configurations of the matrix in equation 11f, fourteen are stable and two are not.

This means that in total, out of the sixty distinct configurations with $n = 4$, there are fifty-five stable configurations and five non-stable configurations.

6 Conclusion

We have found that matrices with the one-by-one structure always correspond to stable systems. We have also found that for systems with four persons, there are sixty possible systems and that fifty-five of them are stable, while the other five are not.

6.1 Recommendations

Further research into this topic could look into the step size h. We could make it a function of time and then see if there is a specific function that would make all systems stable.

Another thing to be researched further is the proof for the one-by-one case, since the preconditions may be stricter than what is necessary for the proof. They can likely be relaxed to starting with any system that is stable instead of starting with the system of three people. If this is indeed the case then this may lead to more systems easily proven to be stable.

Finally, using a vector representation that denotes per row in which column the R_+ and R_- are can easily be extended to systems with more than four people, since the locations of the O_2 matrices then automatically follow, regardless of how many people are in the system. However, in order to get the number of stable systems you would also need to rewrite the program or write a new program used for calculating the different groups of matrices, in order for that program to understand the new representation. The total number of groups can then be checked using Burnside's Lemma, as was done in this article.

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A All vector representations of $n = 4$ by type

A.1 Equation 11a – (4, 3, 2, 1)

(4, 3, 2, 1) (2, 1, 4, 3) (3, 4, 1, 2)

A.2 Equation 11b – (4, 1, 2, 3)

(4, 1, 2, 3) (2, 3, 4, 1) (2, 4, 1, 3) (4, 3, 1, 2) (3, 4, 2, 1) (3, 1, 4, 2)

A.3 Equation 11c – (2, 4, 4, 2)

(2, 4, 4, 2)	(3, 4, 4, 3)	(4, 4, 1, 1)	(4, 3, 4, 3)	(4, 1, 4, 1)	(3, 1, 1, 3)
(2, 1, 1, 2)	(2, 1, 2, 1)	(2, 3, 2, 3)	(4, 4, 2, 2)	(3, 3, 2, 2)	(3, 3, 1, 1)

A.4 Equation 11d – (3, 4, 4, 2)

(3, 4, 4, 2)	(3, 4, 2, 2)	(2, 4, 4, 3)	(4, 1, 2, 1)	(2, 1, 4, 1)	(2, 3, 4, 3)
(3, 4, 1, 3)	(4, 3, 4, 1)	(2, 1, 2, 3)	(3, 3, 2, 1)	(3, 3, 1, 2)	(4, 1, 4, 3)
(3, 1, 1, 2)	(4, 3, 2, 3)	(2, 1, 1, 3)	(2, 3, 2, 1)	(2, 4, 1, 2)	(4, 3, 1, 1)
(3, 1, 4, 3)	(2, 1, 4, 2)	(4, 4, 2, 1)	(3, 4, 1, 1)	(4, 3, 2, 2)	(4, 4, 1, 2)

A.5 Equation 11e – (4, 4, 4, 3)

(4, 4, 4, 3)	(2, 1, 2, 2)	(3, 1, 1, 1)	(2, 4, 2, 2)	(3, 3, 4, 3)	(4, 1, 1, 1)
(4, 4, 4, 2)	(3, 3, 1, 3)	(3, 3, 2, 3)	(2, 1, 1, 1)	(2, 3, 2, 2)	(4, 4, 4, 1)

A.6 Equation 11f – (4, 4, 2, 3)

(4, 4, 2, 3)	(4, 1, 1, 2)	(2, 3, 1, 1)	(3, 3, 4, 2)	(3, 1, 2, 1)	(3, 1, 4, 1)
(2, 3, 4, 2)	(2, 3, 1, 3)	(3, 4, 4, 1)	(3, 1, 2, 3)	(3, 3, 4, 1)	(4, 4, 1, 3)
(2, 4, 2, 1)	(2, 4, 4, 1)	(4, 1, 1, 3)	(2, 3, 1, 2)	(3, 4, 2, 3)	(4, 1, 2, 2)
(4, 1, 4, 2)	(3, 1, 2, 2)	(4, 3, 4, 2)	(2, 4, 2, 3)	(2, 4, 1, 1)	(4, 3, 1, 3)

B All distinct extended vector representations of $n = 4$

B.1 Equation 11a – (4, 3, 2, 1)

$(4+, 3+, 2+, 1-)$ $(4+, 3+, 2-, 1-)$ $(4+, 3-, 2-, 1+)$ $(4-, 3+, 2-, 1+)$

B.2 Equation 11b – (4, 1, 2, 3)

$(4+, 1+, 2-, 3+)$ $(4+, 1-, 2-, 3+)$ $(4+, 1-, 2-, 3-)$ $(4-, 1+, 2-, 3-)$
$(4-, 1-, 2+, 3-)$ $(4-, 1-, 2-, 3-)$

B.3 Equation 11c – (2, 4, 4, 2)

$(2+, 4+, 4+, 2+)$ $(2+, 4+, 4+, 2-)$ $(2+, 4-, 4+, 2+)$ $(2+, 4-, 4-, 2+)$
$(2-, 4+, 4+, 2+)$ $(2-, 4+, 4+, 2-)$ $(2-, 4+, 4-, 2-)$ $(2-, 4-, 4+, 2-)$
$(2-, 4-, 4-, 2+)$ $(2-, 4-, 4-, 2-)$

B.4 Equation 11d – (3, 4, 4, 2)

$(3+, 4+, 4+, 2+)$ $(3+, 4+, 4+, 2-)$ $(3+, 4+, 4-, 2+)$ $(3+, 4+, 4-, 2-)$
$(3+, 4-, 4+, 2+)$ $(3+, 4-, 4+, 2-)$ $(3+, 4-, 4-, 2+)$ $(3+, 4-, 4-, 2-)$
$(3-, 4+, 4+, 2+)$ $(3-, 4+, 4+, 2-)$ $(3-, 4+, 4-, 2+)$ $(3-, 4+, 4-, 2-)$
$(3-, 4-, 4+, 2+)$ $(3-, 4-, 4+, 2-)$ $(3-, 4-, 4-, 2+)$ $(3-, 4-, 4-, 2-)$

B.5 Equation 11e – (4, 4, 4, 3)

$(4+, 4+, 4+, 3-)$ $(4+, 4+, 4-, 3-)$ $(4+, 4-, 4+, 3-)$ $(4+, 4-, 4-, 3+)$
$(4+, 4-, 4-, 3-)$ $(4-, 4+, 4-, 3-)$ $(4-, 4-, 4-, 3+)$ $(4-, 4-, 4-, 3-)$

B.6 Equation 11f – (4, 4, 2, 3)

$(4+, 4+, 2+, 3+)$ $(4+, 4+, 2+, 3-)$ $(4+, 4+, 2-, 3+)$ $(4+, 4+, 2-, 3-)$
$(4+, 4-, 2+, 3+)$ $(4+, 4-, 2+, 3-)$ $(4+, 4-, 2-, 3+)$ $(4+, 4-, 2-, 3-)$
$(4-, 4+, 2+, 3+)$ $(4-, 4+, 2+, 3-)$ $(4-, 4+, 2-, 3+)$ $(4-, 4+, 2-, 3-)$
$(4-, 4-, 2+, 3+)$ $(4-, 4-, 2+, 3-)$ $(4-, 4-, 2-, 3+)$ $(4-, 4-, 2-, 3-)$