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# On equilibria in facility location games in random graphs


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# Preface

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**Abstract**

We consider facility location games where both clients and facilities compete against each other. Facilities compete by selecting a location and clients compete by distributing their spending power strategically. A client minimizes the average facility load, weighted by how much spending power the client spends at the facilities. We show that there always exist an equilibrium for the clients, and that the price of anarchy is generally lower on sparse graphs. We also analyzed the time complexity of our algorithm, which we used to find client and facility equilibria.

## Contents



## 1 Introduction

We have a market with clients, wanting to buy a service from a range of facilities. However, when a facility is busy, their service takes longer. Clients want to minimize that time, by distributing their need across several facilities. On the other hand, facilities want to maximize the number of clients coming to them, and will choose a suitable place to attract as many clients, as they do not compete in price or quality. We want to know if there is a stable market, in which the facilities cannot change their strategy and gain more clients. In addition, if such a stable markets exist we will investigate their price of anarchy. At last we will look at the time complexity of this problem.

# 2 Related work

## 2.1 Facility location games

The simplest facility location game was brought up by Harold Hotelling [7]. He proposed an interval $[0, 1]$ on which customers are uniformly distributed, and $N$ facilities want to locate on that interval. Each customer goes to the facility closest to them. Hotelling then showed that a facility location equilibrium exists for $N = 2$. Later it has been shown that for all $N \neq 3$ equilibria exist [4]. This model has been extended by Downs to non-uniform customer distributions [2], to reflect people's opinion on the political spectrum.

A downside of this model is that discontinuities in the profit function can occur, by the nature of clients always favouring the closest facility. Thus, Kohlberg [8] modified the model such that clients also factor in how busy a certain facility is. In this model the profit functions of facilities are always continuous, however no equilibria exist for any $N > 2$.

Dürr and Thang [3] moved the problem from an interval to a graph with discrete customers on each vertex and looked for Nash equilibria when customers patronize the closest facility. They showed that determining if a Nash-equilibrium exists is a NP-hard problem. Later Krogmann et al. changed the client behaviour such that they want to minimize the load of the facilities they visit [9]. Here it is shown that in that case, a facility equilibrium always exists.

We are interested in a very similar problem as Krogmann, bar the client behaviour. We want clients to minimize their cost and then analyze verify wether facilities can find equilibria under these conditions. We will do this by simulating the game on random graphs.

## 2.2 Random graphs

There are a few random graphs models, the most used one is the Erdös-Renyi graph, of which there are two variants of the model. We will use the following. In his $G(n, p)$ model generates a graph with $n$ vertices and each edge is included with probability $0 \leq p \leq 1$. This model also has been studied by Erdös and Renyi, and gave results for different values of $p$ [5]. One notable observation is that this graph class is not connected necessarily.

Another model is the Barabási-Albert model [1]. This model starts with a connected network with $m_0$ vertices. A vertex is added one at a time, and connected with $m \leq m_0$ vertices with a probability proportional to the number of links the existing vertices already have. That is, the chance that a new vertex is connected to vertex $v_i$ is $p_i = \frac{k_i}{\sum_j k_j}$. Where $k_i$ is the degree of vertex $v_i$ and we sum over all existing vertices. This results in a few vertices with a high degree, and a lot of vertices with only a few links. In contrast to the Erdös-Renyi model, this generates a bidirectional connected graph.

### 2.3 Atomic splittable congestion games

Our client subgame happens to be equivalent to an atomic splittable congestion game. Rosenthal [10] introduced congestion games, where in general, players compete for resources, and the price of a resource is a function of how often a resource is used. He also showed that such games always have at least one equilibrium. A variation on this is that players can choose to take a fraction of a resource, a so called atomic splittable congestion game. Harks and Timmermans [6] showed that there always exist an unique equilibrium in such games, given some conditions. In particular Theorem 4.2 in said paper:

**Theorem 1** ([6])**.** *If for a polymatroid congestion game, the strategy space for every player is the base polytope of a bidirectional flow polymatroid, then the equilibria of this game are unique.*

## 3 Model

We have a directed graph $H = (V, E, w)$, with $V = \{v_1, v_2, \ldots, v_n\}$ and $w : V \to \mathbb{N}$ is the vertex weight. Every vertex $v_i \in V$ is a client with weight $w(v_i)$ corresponding to an amount of spending power at the $i_{\text{th}}$ vertex. Furthermore there are facilities $\mathcal{F} = \{f_1, \ldots, f_k\}$ which can locate at any of the vertices.

Each facility $f_i \in \mathcal{F}$ selects a single location $s_i \in V$ to set up their facility. Denote $\mathbf{s} = (s_1, \ldots, s_k)$ to be the facility placement profile, and $\mathcal{S} = V^k$ the set of all possible profiles. We will also use the notation $\mathbf{s} = (s_j, s_{-j})$, where $s_{-j}$ denotes the strategies of all facilities, except the one of $f_j$. Given $\mathbf{s}$, we then define the attraction range for a facility $f_i$ on location $s_i \in V$ as $A_{\mathbf{s}}(f_j) = \{s_j\} \cup \{v_i \mid (v_i, s_j) \in E\}$. In general for any set of facilities $F \subseteq \mathcal{F}$ we have $A_{\mathbf{s}}(F) = \{s_j \mid f_j \in \mathcal{F}\} \cup \{v_i \mid (v_i, s_j) \in E, f_j \in \mathcal{F}\}$. Moreover, let $w_{\mathbf{s}}(\mathcal{F}) = \sum_{v_i \in A_{\mathbf{s}}(\mathcal{F})} w(v_i)$ be the total load attracted by facilities.

The range of a client $v_i \in V$ is its direct neighbourhood, denoted as $N(v_i) = \{v_i\} \cup \{z \mid (v_i, z) \in E\}$. Each client wants to distribute its spending power across the facilities in its range. That is, across the facilities in $N_{\mathbf{s}}(v_i) = \{f_j \mid s_j \in N(v_i)\}$. Moreover, let $w(X) = \sum_{v_i \in X} w(v_i)$ for any $X \subseteq V$ be the total spending capacity of client subset $X$.

Let $\sigma : \mathcal{F} \times V \to \mathbb{R}_+^k$ be the client weight distribution function where $\sigma(\mathbf{s}, v_i)$ is the weight distributed by $v_i$ and $\sigma(\mathbf{s}, v_i)_j$ the weight distributed by $v_i$ to facility $f_j$. We say that $\sigma$ is feasible for $\mathbf{s}$ if all clients having at least one facility in their range distribute all their weight to the respective facilities, and all other clients distribute nothing.

The load of a facility $f_k$ can then be notated as $\ell_k(\mathbf{s}, \sigma) = \sum_{i=1}^{n} \sigma(\mathbf{s}, v_i)_k$.
With the load of each facility in mind, each client $v_i$ wants to minimize the weighted average of the facility load, that is, minimize

$$L_i(\mathbf{s}, \sigma) = \sum_{f_j \in \mathcal{F}} \frac{\ell_j(\mathbf{s}, \sigma)\sigma(\mathbf{s}, v_i)_j}{w(v_i)} \tag{1}$$

for all feasible weight distributions $\sigma(\mathbf{s}, v_i)$.

The stable states of this game are the subgame perfect equilibria, since we have a two stage game. First facilities locate somewhere on the graph where they attract the most load, and then clients distribute their spending power across facilities in its range. A state $(\mathbf{s}, \sigma)$ is in a subgame perfect equilibrium if

$$\forall f_j \in \mathcal{F}, \forall s'_j \in V : \ell_j(\mathbf{s}, \sigma) \geq \ell_j((s'_j, s_{-j}), \sigma) \tag{2}$$

and

$$\forall \mathbf{s} \in \mathcal{S}, \forall v_i \in V : L_i(\mathbf{s}, \sigma) \leq L_i(s, \left(\sigma'_i, \sigma_{-i}\right)) \tag{3}$$

for all feasible weight distributions $\sigma'_i$.

We want to know under what conditions these equilibria exists, and if they do, determine the price of anarchy. For this purpose we define $C(\mathbf{s}) = \{v_i \mid v_i \in V, N_{\mathbf{s}}(v_i) \neq \emptyset\}$, that is the set of all clients which are covered by facilities under placement strategy $\mathbf{s}$. We will then compare two states of this problem by measuring the social welfare $W(\mathbf{s}) = \sum_{v_i \in C(\mathbf{s})} w(v_i)$, which is the total spending capacity of all covered clients.
Now we can talk about a social optimum for a host graph $H$ with $k$ facilities. Let $OPT(H, k)$ denote the facility placement profile that maximizes social welfare. And let worstSPE$(H, k)$ be the lowest social welfare across all equilibria on graph $H$ with $k$ facilities. Then we define the price of anarchy to be

$$PoA(H, k) = \frac{W(OPT(H, k))}{W(\text{worstSPE}(H, k))}. \tag{4}$$

In the same way we can define the price of stability, which is the ratio of the social optimum facility placement, and that of the best equilibrium, bestSPE$(H, k)$ in terms of social welfare.

$$PoS(H, k) = \frac{W(OPT(H, k))}{W(\text{bestSPE}(H, k))}. \tag{5}$$

We will attempt to find in what kind of graphs facility location equilibria exists, en determine the price of anarchy if they exist.

# 4 Uniqueness of client equilibrium

In order to prove that clients have an unique equilibrium we reduce the client game to a similar game, which has been proven to have unique equilibria. In particular, the reduction will be to an *Atomic splittable singleton congestion game (CG)* which is a tuple $(N, E, S_i, d_i)$ with

- $N$: Set of players
- $E$: Set of resources
- $S_i$: a collection of allowable subsets of $E$
- $d_i$: demand of player $i \in N$

Each player $i \in N$ has a strategy $x_i = (x_{i,1}, \ldots, x_{i,e})$ which is a distribution of $d_i$ over $S_i$. For some strategy profile $x = x_1, \ldots, x_n$ we define the load of a resource as $\ell_e(x) = \sum_{i \in N} x_{i,e}$. The cost of a resource $e \in E$ in $x$ for player $i$ is defined as $\Pi_i(x, e) = x_{i,e} \cdot \ell_e(x)$

### 4.1 Reduction

We have our game on the graph $H = (V, E, w, \mathbf{s})$ as described in Section 3 and will reduce the client subgame to an atomic splittable singleton congestion game. Map the clients $i \in V$ to the players in $N$ such that $V = N$, and the set of facilities $\mathcal{F}$ is mapped one to one to resources $E$. The set of facilities in the neighbourhood of a client $i \in V$ then map to a set $S_i$. And finally the demand of a client is mapped to the demand of a player, $d_i = w_i$. Now we have a similar game, we can easily find a bijective function $F$ from the strategies from the FLG to the CG, namely $F(\sigma(\mathbf{s}, v_i)_j) = x_{i,e}$ for all $\sigma(\mathbf{s}, v_i)_j \in \sigma$

**Theorem 2.** *$F(\sigma)$ is in equilibrium in CG if and only if $\sigma$ is in equilibrium in FLG.*

*Proof.* We have the same cost function in both games, and since $F$ is a bijection, we have that the costs of a player $i$ in the FLG with strategy $\sigma(\mathbf{s}, v_i)$ is the same as a player $i$ in the CG with strategy $x_i$. Furthermore the strategy space of both games is It follows that an equilibrium in one game implies an equilibrium in the other game.

□

**Lemma 3.** *There exists a unique client equilibrium in FLG.*

*Proof.* Since each strategy consists of singleton sets we have a polymatroid congestion game. This follows directly from Theorem 1 in the paper of Harks and Timmermans [6]. □

## 5 Methods

In this section we will discuss how we determined whether a game as equilibria or not. The first step is finding the client's spending distribution, given the locations of the facilities. Then we can determine where facility locations are in an equilibrium, and find the associated Price of Anarchy.

All computation done by us is done with Python 3.10.5.

### 5.1 Client equilibrium

This method to find the equilibrium in the client spending power distribution consists of two steps. We loop over the set of all clients and minimize their cost function consecutively. After each individual minimization we compare the spending power distribution with the previous spending power distribution, and if the difference is within a certain range, we say that the strategy has converged and thus is in equilibrium.

Minimizing a client's cost function is trivial when there are 0 or 1 neighbouring facilities, and has a nice closed form solution when there are 2 facilities in the neighbourhood.

The cost for a client $v_i$ neighbouring two facilities with loads $F_1$ and $F_2$ is

$$C_i(x) = x(F_1 + x) + (w_i - x)(F_2 + w_i - x)$$

Here $x$ is the spending power distributed to $f_1$. To minimize this function we obtain the derivative and set it to zero.

$$C_i'(x) = 4x + F_1 - F_2 - 2w_i = 0$$

Solving for $x$ gives us

$$x = \frac{2w_i + F_2 - F_1}{4}.$$

Which is how much spending power is spend at facility 1, and $w_i - x$ at facility 2. Note that in our case $w_i = 1$.

In the case where there are more than two facilities in the neighbourhood, minimizing is a least square problem, as seen in (6).

This least square problem is solved by using the SciPy optimize library of python. In particular, their Sequential Least Squares Programming method is used to minimize the client cost, with an absolute tolerance of 0,001.

Here $\mathcal{L}(v_i) = [\ell_1(\mathbf{s}, \sigma), \ldots, \ell_k(\mathbf{s}, \sigma)]$, with $\sigma(\mathbf{s}, v_i) = \mathbf{0}$ is the load vector of all the facilities.

$$\begin{array}{lll} \text{minimize} & \sigma(\mathbf{s}, v_i) \cdot \sigma(\mathbf{s}, v_i) + \mathcal{L}(v_i) \cdot \sigma(\mathbf{s}, v_i) & \\ \text{subject to} & \sum_{j \in N_{\mathbf{s}}(v_i)} \sigma(\mathbf{s}, v_i)_j = w(v_i) & \\ & 0 \leq \sigma(\mathbf{s}, v_i)_j \leq w(v_i) & \forall f_j \in N_{\mathbf{s}}(v_i) \end{array} \tag{6}$$

## 5.2 Facility equilibrium

To find the facility equilibria we calculate the load of all facilities, over all possible combinations, that is, all $\binom{|V|+|\mathcal{F}|-1}{|\mathcal{F}|}$ possible locations. Then for each permutation it is easily checked whether it is in equilibrium or not. This way we find all equilibria.

In addition to that it is easy to find the Price of Anarchy, as we can easily find the maximum cumulative facility load, the social optimum and compare that to the best and worst equilibria, in a social sense.

## 5.3 Generating instances

In order to generate we did the following: In every case we gave each client a spending power equal to one.

We generate the Erdös-Renyi graph by creating $n$ clients, and then for each pair $(v_i, v_j)$, for $v_i, v_j \in V$ and $i \neq j$ we add an edge with probability $p$. Note that this is a directed graph.

For the Barabási-Albert model we start with $m_0$ clients, all connected with each other. Then for each new client, we select one existing client, with the probability distribution $p = \frac{\text{degree of client}}{\text{sum of all degrees}}$. If the randomly chosen client already has an edge with the new client, we repeat this until a non-connected client is chosen. This is repeated $m_0$ times, such that the new client has exactly $m_0$ edges. Once we have $n$ clients we are done.

# 6 Results

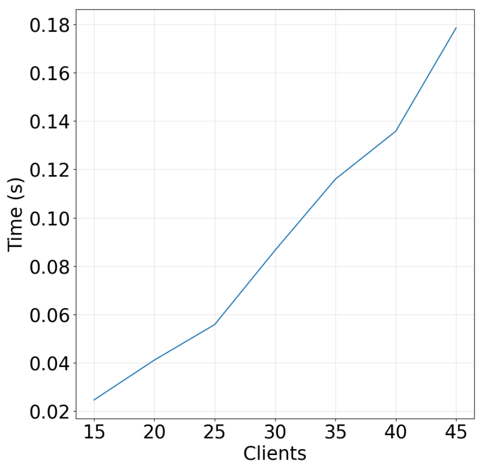
## 6.1 Time complexity

For the time complexity we only take a look at the times of games which use the Erdös-Renyi random graph.

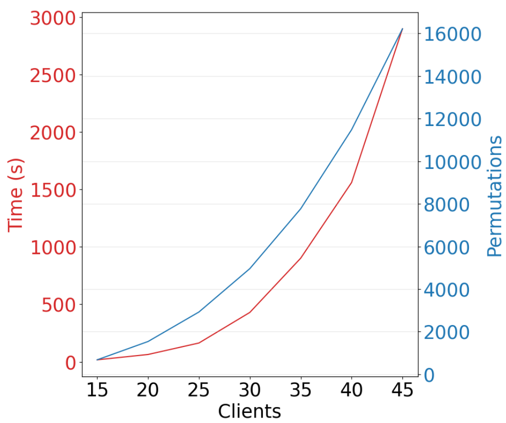
In Figure 1a we see the computation time for the client equilibrium. The number of facilities are fixed at three, and we use the $G(n, 0.4)$ graph, only varying the number of clients. We see that when the graph size increases, so does the time to compute the client equilibrium. Note that increasing the number of clients, also increases the number of edges each client has, on average. This is due to the nature of Erdös-Renyi graphs.

In Figure 1b we look the computation time of the facilities equilibria. This thus includes finding every client equilibrium. There is, as expected, a correlation between the amount of time it takes to solve one game, and the number of permutations the facilities can be in. The same holds true when you increase the number of facilities, see Figure 2a. One can see that due to the computation time we did not look at the case with 6 facilities.

Finally we varied the $p$-value to see how this effects the computation time in Figure 2b. A higher $p$-value means the graph is more connected. Since the amount of permutations remains constant in this graph, we can conclude that the computation time of the client equilibrium is slower the more dense the graph is. This makes sense, as the more neighbours clients have, the higher the chance is that it has three facilities, so we have to use the least square method to find the minimum cost as opposed to the closed form solution.

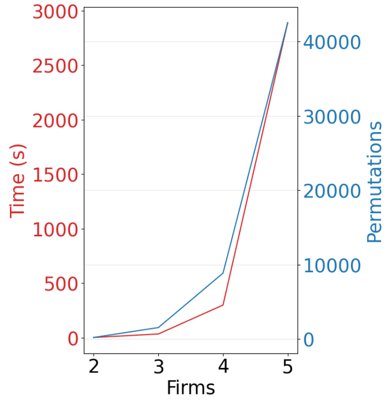



(A) mean computation time for the client equilibrium, with 3 facilities and an Erdös-Renyi graph with p = 0.4

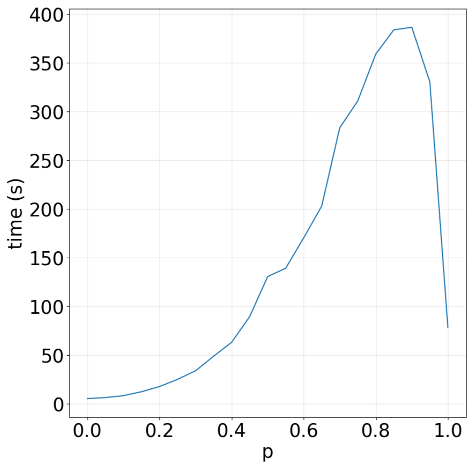



(B) The mean computation time for facilities equilibria compared to the number of clients. 3 facilities and $p = 0.4$

Figure 1




(A) The mean computation time for facilities equilibria compared to the number of facilities. 20 clients and $p = 0.3$




(B) The mean computation time for facilities equilibria compared to the $p$-value. 20 clients, 3 facilities

Figure 2

## 6.2 Price of anarchy

For investigating the price of anarchy we varied some parameters and looked at the mean price of anarchy, and the variance of it. We analyze the two different random graphs and their parameters.

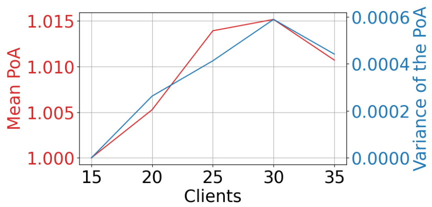
First the barabasi graph class. We looked at two parameters, the number of clients and the $m_0$. When increasing the number of clients we notice the mean PoA increasing from 1 to 1.015 before decreasing again at 35 clients. The decline can be explained since there are more highly connected clients, meaning facilities can each pick such client, and not interfere too much with each other.

When increasing $m_0$ we notice a steep drop approaching 1. This is probably for the same reason why the PoA drops when increasing clients, more clustered vertices and thus less interfering facilities.

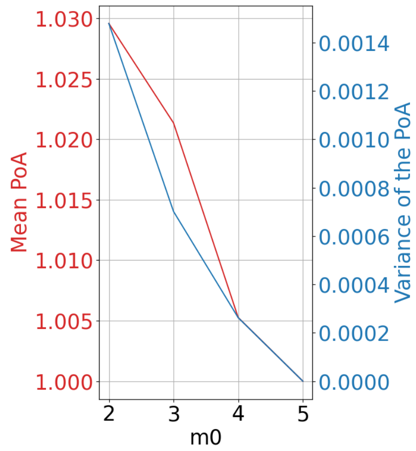
We looked at similar parameters in the Erdös graphs, clients and $p$. The PoA increases as the number of clients increases, as can be seen in Figure 4.

When looking at values for $p \in [0, 1]$, in Figure 5, the first observation is the steep drop after $p = 0.35$. At this point the graph is probably too connected, each client is connected to roughly half of clients, thus placing two facilities could capture the demand of all clients. This is then also the most profitable, apparently. Also after $p = 0.45$ the sample size is only 5 for each value of p, as opposed to 40 before that point, due to time constraints. This will limit the amount of times the PoA can be higher than 1.

The PoA being 1 when $p$ is 0 and 1, is expected since we then have an empty and complete graph respectively. Thus all locations are independent from each other.




(A) The mean and variance of the PoA. With three facilities and $m_0$ of




(B) The mean and variance of the PoA. With 20 clients and 3 facilities

FIGURE 3

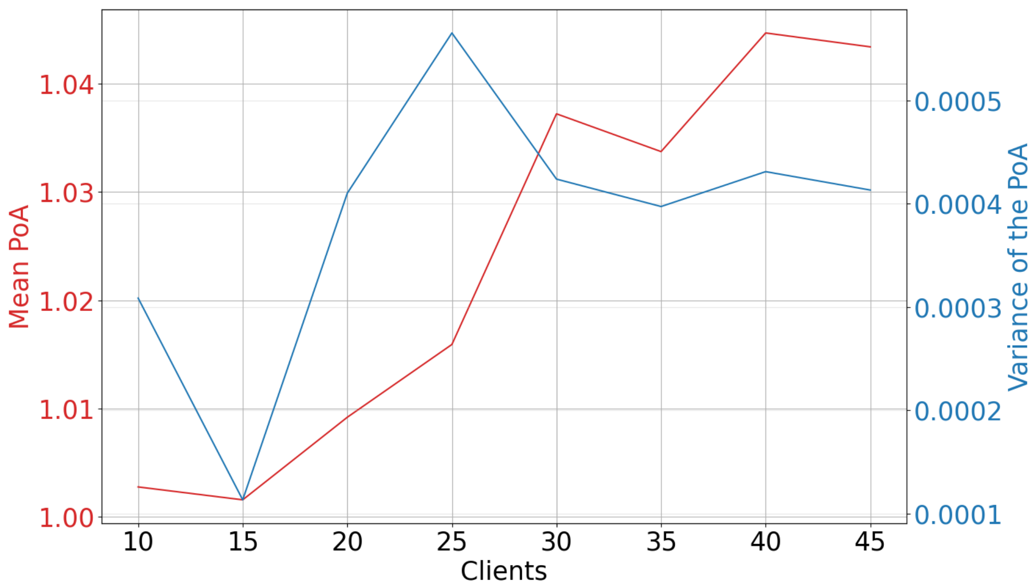



FIGURE 4: The mean and variance of the PoA. On an erdös graph with three facilities and $p = 0.4$.

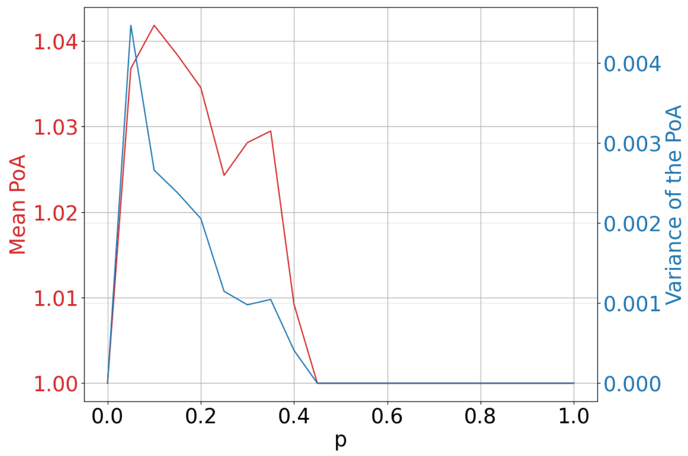



FIGURE 5: The mean and variance of the PoA. With 20 clients and three facilities

# 7 Conclusion and discussion

We conclude that equilibria can exist for facilities, but are not guaranteed. Also the price of anarchy seems to depend on the graph density.

A major improvement would be a faster algorithm to find the client spending distribution. This is the only computation done by our algorithm so speeding this up helps a lot.

Additionally look for ways to limit the search space for facility equilibria. One way is keeping into account that the price of anarchy is always less than 2 [9]. Thus any facility location vector must be in that range. Computing the social optimum can be computed rather easily, since it does not require computing the client equilibrium.

In this game, more things which can be investigated is the behaviour in smaller games. We only looked into games with more than 15 players, but games with 5 to 14 players maybe show different results. Furthermore in the case of Erdös graphs, it can be interesting to pair the clients and $p$ value such that the expected degree of a client is fixed.

Moreover looking more into the two facility case can be interesting since this is the most basic case, which is not trivial. Also the spending power per client was one in every case investigated in this paper.

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