University of Twente

Applied Mathematics

Optimal Transport and the Structure of Cities

Bachelor's Assignment

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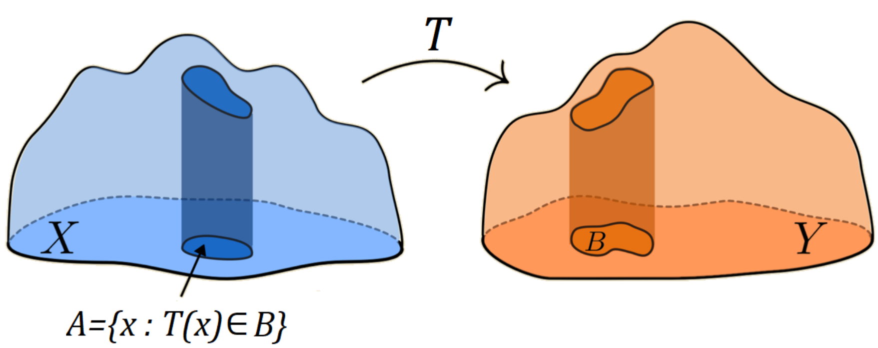

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1 Introduction

Cities are complex systems that display patterns influenced by various factors. For understanding these patterns, we need some kind of model we can use that can simulate the possible uses of the city. The first model explaining the structure of cities was given by Lucas and Rossi-Hansberg [1]. This model was given in a pathbreaking paper and explains the internal structure of a city as an equilibrium solution between different possible uses of land. This classic paper was remarkable since Lucas and Rossi-Hanserg were the first ones that worked out an equilibrium model for this. However, the original model had limitations, such as the condition of rotational symmetry of the city, which needed to be addressed.

In recent years, advancements in the field have been made with as a result better models. One of the more recent models, described in 'Optimal transportation and the structure of cities' [2], seems more promising. This model does not need the condition of rotational symmetry of the city and also can be interpreted through the lenses of Optimal Transport. We can start by defining x to be the place where individuals are living and y to be the place where they work at. You can see x as the starting destination, and y as the end destination. Moving from x to y will cost $c(x, y)$. For the most simple models, $c(x, y) = ||x-y||_2$, the distance between those points, but the cost function $c(x, y)$ can be changed to something more desirable when changing the model. By modeling inhabitants and firms as probability distributions we can use Optimal Transport theory to compare them and find the most efficient way to transform one into another.

Optimal transport has recently become popular in various research areas since it offers an efficient framework to compare probability distributions. Signal processing [3], image processing [4], inverse problems [5], and many other fields have benefited from the usage of Optimal Transport method. It is no surprise that such tools have proven useful also for studying the structure of cities. In this thesis, we will demonstrate this point.

In this thesis we will analyze the model described in 'Optimal transportation and the structure of cities' [2]. After explaining the model of Lucas and Rossi-Hansberg [2] we will follow [6] to explain how to use Optimal Transport to interpret the model. Lastly, we will improve and extend the model, by studying generalization and adding new features to it. We will have the following research questions, to set a clear goal

1. How can the structure of cities be modeled using Optimal Transport theory?
2. How can the model be improved such that it can be used in real-world scenarios?

2 Basic model

2.1 Table of variables

Let us define all variables of the model in a table to easily keep track of them

x	living place for individuals
y	working place for individuals
$c(x, y)$	transportation costs
ψ	wage of each individual
z	productivity, reflects employment at neighboring locations
Q	rent for an inhabitant
φ or $V(Q)$	the available revenue of inhabitants/workers
$\overline{u}$	utility level of workers
$c_l(\varphi)$	consumption level
$S(\varphi)$	residential space, units of land rented and used for living
$N(\varphi)$	the relative density of residents
$n(z, \psi)$	relative density of jobs, number of jobs per unit of land used for production
$\Pi(z, \psi)$	the rent for business use
$\theta(\Pi, Q)$	land allocated to business, $1 - \theta(\Pi, Q)$ is land allocated to residential use
$\tilde{\mu}$	overall density of residents
$\tilde{\nu}$	overall density of jobs

Table 1: Table of variables

2.2 Descripton

Before describing the basic model, it is important to highlight two important properties. First of all, all individuals contribute one unit to the labor market. This means that the total number of workers in the city is equal to the total supply of labor. This property is very important since it is used for finding the solution. Second of all, each district must be mixed, meaning there cannot be a district fully consisting of working places or fully consisting of living places. This is to evade mathematical complexities.

The city consists of three types: inhabitants, firms, and land owners. These all fit inside Ω, an open bounded connected subset in $\mathbb{R}^2$. All inhabitants are identical, and all items of production will be used for consumption or sale. All inhabitants use the same utility function $(c_l, S) \mapsto U(c_l, S)$, where c_l is the consumption level and S is the residential space rented. The utility function represents the satisfaction or pleasure the inhabitants receive for their respective consumption level c_l and residential place S.

All inhabitants bring home a revenue $\psi - c(x, y)$, this is the wage ψ they get minus the commuting cost $c(x, y)$. The production function per unit of land at a given location is $(z, n) \mapsto f(z, n)$, where z is the productivity and n is the relative density of jobs. This function defines a relation between the productivity z combined with the relative density of jobs n and the output of work at that location. The productivity z can be expressed in the following way

$$z(x) = Z_\nu(x) := \chi\left(\int_\Omega \rho(x, y)v(y)dy\right), \quad x \in \Omega$$

where $\rho(\cdots)$ is a non-negative continuous weighting kernel, χ is a continuous increasing bounded function and ν is the overall density of jobs at each location x. The function χ can be used for the positive externality of labor, meaning that a third party gains positive effects from a party that decides to consume or produce a product or service. In the model, we choose χ to be $\chi(x) = x$, resulting in

$$z(x) = Z_\nu(x) := \int_\Omega \rho(x, y)v(y)dy. \tag{1}$$

This is assumed for simplicity. Luckily the reader can also modify χ, displaying positive externality by for instance choosing χ to be quadratic, $\chi(x) = x^2$, resulting in

$$z(x) = Z_\nu(x) := \left(\int_\Omega \rho(x, y)v(y)dy\right)^2.$$

2.3 Behaviour of various quantities

2.3.1 Workers

An inhabitant of the city living at x spends his revenue on consuming c_l and renting land S at respective prices 1 and Q, where Q is the rent per unit of the residential area. Based on this argumentation, we can calculate the available revenue φ or $V(Q)$ for the inhabitant using the following equation

$$V(Q) := \min\{c_l + QS \mid U(c_l, S) \geq \overline{u}\}. \tag{2}$$

Here c_l and S must satisfy $U(c_l, S) \geq \overline{u}$, since we want every inhabitant to have at least a satisfaction $\overline{u}$. This equation states that the firms want to give the workers as little as possible but still want them to be satisfied.

Also, the relative density of residents $N(\varphi)$ can be calculated by

$$N = \frac{1}{S(\varphi)} =: N(\varphi). \tag{3}$$

Here we assume the relative density of residents to be the inverse of the residential land S. This can be modified by the reader.

2.3.2 Firms

Assume that at location y, the wage ψ and the productivity z are decided. Then the firms can decide the level of employment at y, by solving

$$\Pi(z, \psi) := \max\{f(z, n) - \psi \cdot n \mid n \geq 0\}. \tag{4}$$

Here $n(z, \psi)$ is the corresponding optimal level of employment, ψ is the wage, $f(z, n)$ is the production function per unit of land, and the function $\Pi(z, \psi)$ is the rent for business use. The reasoning of Equation (4) is that the landlords want to have as much money for rent by maximizing the production output while minimizing the money they give to their workers.

Note that the optimal level of employment $n(z, \psi)$ is equivalent to the relative density of jobs(the number of jobs per unit of land used for production).

2.3.3 Landlords

The residential area is owned by landlords. These landlords extract all surplus from workers and firms. Landlords at location x offer two rents: $\Pi = \Pi(z, \psi)$ for business use and $Q = Q(\varphi)$ for residential use. These rents were determined in the previous subsections. Landlords decide how the land is split, here the fraction $\theta(\Pi, Q)$ of land is allocated to businesses so that $1 - \theta(\Pi, Q)$ is the fraction of land for residential use. θ is bounded away from 0 and 1, since we want all districts to be mixed.

2.3.4 Free mobility of labor

All workers want to bring as much money home as they can, meaning they want to maximize their revenue $\psi(y) - c(x, y)$. So,

$$\varphi(x) = \sup\{\psi(y) - c(x, y) \mid y \in \Omega\}. \tag{5}$$

This condition implies

$$\psi(y) \leq \inf\{\varphi(x) + c(x, y) \mid x \in \Omega\}. \tag{6}$$

Moreover, the wage ψ is regulated by the following formula

$$\psi(y) = \inf\{\varphi(x) + c(x, y) \mid x \in \Omega\}. \tag{7}$$

This equation can be interpreted that firms at location y want to attract inhabitants that work for the lowest wages. Equations (5) and (7) ensure that the wage ψ and the revenue φ are conjugated functions.
We can also define a commuting network over the city using the wage ψ and the revenue φ

$$s(x) = \text{arg max}\{\psi(y) - c(x, y) \mid y \in \Omega\} \tag{8}$$

$$t(y) = \text{arg min}\{\varphi(x) + c(x, y) \mid x \in \Omega\}. \tag{9}$$

Here the maps s and t are inverse to each other. So given x, you can find where they work using s, and given y, you can find where they live using t.

2.4 Equilibrium

After a certain time period, our model will reach a state of balance or stability. When this state is reached, we can say that the equilibrium is reached and therefore the variables in the city will not change. The equilibrium can be regarded as the solution to the model. Note that in the real world, there is no such thing as an equilibrium since there are too many random variables to keep in mind.

Before going into the definition of the equilibrium, we still need to define the overall density of residents $\tilde{\mu}$ and the overall density of jobs $\tilde{\nu}$.

$$\tilde{\mu}(z, \psi, \varphi) := (1 - \theta(z, \psi, \varphi))N(\varphi) \tag{10}$$
$$\tilde{\nu}(z, \psi, \varphi) := \theta(z, \psi, \varphi)n(z, \psi). \tag{11}$$

These seem very logical since the density of residents $\tilde{\mu}$ is indeed the land for residential use times the relative density of residents. Also, the overall density of jobs $\tilde{\nu}$ is the land for business use times the relative density of jobs.

Definition 2.1 (Equilibrium)**.** An equilibrium is a collection $(\mu, \nu, z, \psi, \varphi)$ of positive continuous functions on Ω and a pair of measurable maps $(s, t) : \Omega \to \Omega$ satisfying

- $\int_\Omega \mu = \int_\Omega \nu$
- $z = Z_\nu$
- The pair (ψ, φ) satisfies

$$\varphi(x) = \sup\{\psi(y) - c(x, y) \mid y \in \Omega\}$$
$$\psi(y) = \inf\{\varphi(x) + c(x, y) \mid x \in \Omega\}$$

- For all $x \in \Omega$: $\mu(x) = \tilde{\mu}(z(x), \psi(x))$ and $\nu(x) = \tilde{\nu}(z(x), \psi(x), \varphi(x))$
- $s(\mu) = \nu$ and $t(\nu) = \mu$
- $s(t(y)) = y$ and $t(s(x)) = x$
- The pair (s, t) satisfies

$$s(x) = \arg\max\{\psi(y) - c(x, y) \mid y \in \Omega\}$$
$$t(y) = \arg\min\{\varphi(x) + c(x, y) \mid x \in \Omega\}$$

3 Optimal Transport

After defining a model in Section 2, we also need a framework to interpret this. This can be done using the theory of Optimal Transport. In this section, we are going to discuss which concepts of Optimal Transport we need for solving the model. The definition of these concepts was gotten out of 'Computational Optimal Transport' [6].

3.1 Optimal Transport, simply explained

Optimal transport provides the framework for transforming one given distribution into another, which in our case is transporting the density of inhabitants to the density of firms. For creating a better understanding we can have an example. Let us have our inhabitants located at a, b, and c, and the firms they need to get to at A, B, and C as in Figure 1.

Using Optimal Transport, we can determine the optimal pairing of locations with destinations. Every couple has an underlying cost, which in our case would be the total distance between points. Using Optimal transport we can minimize this cost, and therefore minimize the transformation needed. In our example, it can be seen that it was the most efficient that the sources from a were transported to B, the sources of b to C, and the sources of c to A.

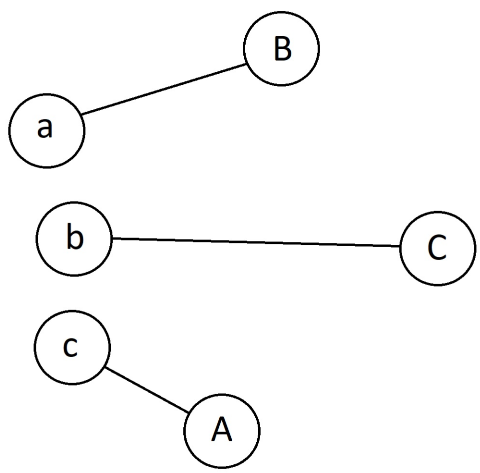

Figure 1: Basic example of a transportation problem

3.2 Basic concepts

Before diving into the theory of Optimal Transport, we need to consider a few basic concepts.

Definition 3.1 (Dirac delta)**.** Let point $x \in X$, with the probability measure

$$\delta_x(A) = \begin{cases} 0 & \text{if } x \notin A \\ 1 & \text{if } x \in A \end{cases} \tag{12}$$

This probability measure is referred to as the Dirac delta.

Empirical measures are linear combinations of Dirac deltas at different points, defined by $v = \sum_{i=1}^{N} c_i \delta_{x_i}$, where $c_i > 0$, $N \in \mathbf{N}$ and $x_i \in X$. Here v is said to be a probability measure if $\sum_i c_i = 1$. Also, v is said to have uniform weights if $v = \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i}$.

3.3 Push-forward

Push-forward is another concept that needs to understand for understanding Optimal Transport. The push-forward will get one probability measure to another probability measure. For this, we have the following definition

Definition 3.2 (Push-forward)**.** Let $\mu \in M(X)$ and $T : X \to Y$. Then the push-forward of μ with respect to f is $T(\mu) \in M(Y)$ defined for every $B \subset Y$ as

$$T(\mu)(B) = \mu(T^{-1}(B)). \tag{13}$$

Here $T^{-1}(B)$ is also defined when T is not invertible.

The push-forward is also defined for empirical measures and measures with densities. For an empirical measure f the push-forward is defined to be $T(\mu) = \sum_{i=1}^{N} c_i \delta_{T(x_i)}$. For a measure of density f of μ, we get

$$T(\mu(A)) = \int_A f(T(x))|JT|(x) d\mathcal{L}^n(x).$$

Here JT is the Jacobian of T.

3.4 Monge formulation

The basic definition of Optimal Transport was defined by Monge and is therefore called the Monge formulation.

Definition 3.3 (Monge transportation problem)**.** Consider $\mu \in P(X)$ and $v \in P(Y)$. Define a cost function c: $X \times Y \to \mathbb{R}_+$ that is continuous in both variables. The Monge transportation problem amounts to solve

$$\min \Big\{ \int_X c(x, T(x)) d\mu : T : X \to Y,\ s.t.\ T(\mu) = v \Big\}. \tag{14}$$

In this equation, T is said to be the transport map or Monge map.

3.5 Example

To enhance the clarity of the Optimal Transport theory, let us illustrate it with an example. Let us take for our example a one-dimensional city that will be divided according to a mass distribution. In this city, the population mass distribution and the work mass distribution are not the same. Meaning that the places where everybody lives and everybody has to go to work are not equal. This is intuitive since your home is not your working place. In Figure 2, you can see the two probability mass distributions.

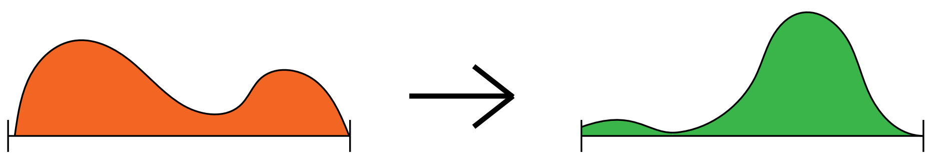

Figure 2: Conversion of probability mass distribution

Here the x-axis is the location in the city, and the y-axis is the number of people living or working at that location. The goal of optimal transport using the Monge formulation is to minimize the cost of moving from one probability mass distribution to another. In this example, it means that you want to move the people at home to the work location while minimizing the distance of traveling. This can be done by first turning the probability mass into Dirac delta's, like in Figure 3(Note that this Dirac delta approximation is done by hand and therefore is not completely accurate), which is called sampling.

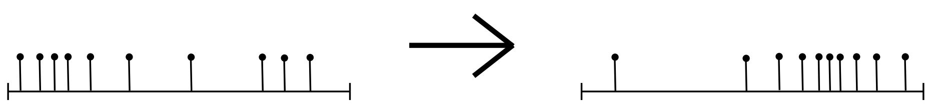

Figure 3: Conversion of Dirac delta's using sampling

Here we divided the probability mass into 10 Dirac deltas with uniform weight. Now we can define the empirical measures μ and ν to be

$$\mu = \frac{1}{10}\sum_{i=1}^{10}\delta_{x_i} \in L$$

$$\nu = \frac{1}{10}\sum_{j=1}^{10}\delta_{y_j} \in L.$$

Optimal transport can now transport the measure μ to measure ν using Monge's formulation

$$\min\Big\{\int_X c(x, T(x))d\mu : T : X \to Y,\ s.t.\ T(\mu) = \nu\Big\}.$$

Here we define the cost $c(x, y)$ to be equal to the distance between the points $||x - y||$, meaning $c(x, y) = ||x - y|| = |x - y|$. It is needed to calculate all the 10! possibilities for T being $T_1, T_2, \ldots, T_{3628800}$ where each tests a different hypothesis of combinations of x_i and y_j. The combination having the least cost is the solution to the transportation problem. Fortuitously, for calculating the solution, Optimal Transport offers optimization tools that are able to solve the problem more efficiently.
Note if we take a two-dimensional city that the probability mass distribution is depicted in three dimensions.

3.6 Dual formulation

As you may have observed, the Monge formulation solves a minimization problem by minimizing the costs. However, you can also formulate the Optimal Transport problem in a way that maximizes the profit, which is called the dual formulation.

Definition 3.4 (Dual problem formulation)**.** Let $\mathcal{L}_c(\mu, \nu)$ be the Monge formulation. The Dual formulation of optimal transport between μ and ν can be computed as

$$\mathcal{L}_c(\mu, \nu) = \sup_{(f,g)\in\mathcal{R}(c)} \int_X f(x)d\mu(x) + \int_Y g(y)d\nu(y) \tag{15}$$

where the set of admissible dual potentials is

$$\mathcal{R}(c) = \{(f, g) \in \mathcal{C}(X) \times \mathcal{C}(Y) : \forall\ (x, y), f(x) + g(y) \leq c(x, y)\}. \tag{16}$$

Here, (f, g) is a pair of continuous functions.

4 Connection between the model and Optimal Transport

The goal of this section is to explain how the equilibrium model presented in Section 2 can be interpreted using the Optimal Transport framework described in Section 3. First, the distributions of inhabitants and firms are the probability measures $\mu \in P(X)$ and $\nu \in P(Y)$ for which we want to compute the optimal transport.

Then, the commuting cost $c(x, y)$ that is used in the model to measure how costly is for an inhabitant that lives in x to work in y, will be the cost used in the Monge Transportation model (see Definition 3.3). Instead of minimizing the distance described in the Monge Transportation model, we can also maximize the profit $f(x) + g(x)$ as described in Subsection 3.6 with the dual formulation.

The profit we want to maximize is the wage of the inhabitants ψ minus the available revenue of inhabitants φ, so in our case $\psi(y)-\varphi(x)$. After this observation, we can define the set of admissible dual potentials as

$$\begin{aligned}\mathcal{R}(c) &= \{(f,g) \in \mathcal{C}(X) \times \mathcal{C}(Y) : \forall\ (x,y), f(x)+g(y) \leq c(x,y)\} \\ &= \{(\psi,\varphi) \mid \psi(y)-\varphi(x) \leq c(x,y), (x,y) \in \Omega \times \Omega\}.\end{aligned} \tag{17}$$

Note that, from the previous relation that for all $(x,y) \in \Omega \times \Omega$ it holds that

$$\varphi(x) \geq \sup_{y\in\Omega} \psi(y) - c(x,y), \qquad \psi(y) \leq \inf_{x\in\Omega} \varphi(x) + c(x,y).$$

This suggests that if the supremum is attained at (ψ,φ) then ψ and φ are conjugate to each other, meaning it will satisfy condition 3 of the Equilibrium. Now we can define the dual formulation to be

$$\begin{aligned}\mathcal{L}_c(\mu,\nu) &= \sup_{(f,g)\in\mathcal{R}(c)} \int_X f(x)d\mu(x) + \int_Y g(y)d\nu(y) \\ &= \sup_{(\psi,\varphi)\in\Omega} \left\{ \int_\Omega \psi\nu dy - \int_\Omega \varphi\mu dx \mid \psi(y)-\varphi(x) \leq c(x,y), (x,y) \in \Omega^2 \right\}.\end{aligned} \tag{18}$$

The commuting network s, is there to convert the masses from the density of inhabitants μ into the density of jobs ν, meaning $s(\mu)=\nu$. Note there is also a map $t(\nu)=\mu$, which is an inverse map to s. These maps were defined in Definition 2.1 as

$$s(x) = \arg\max\{\psi(y) - c(x,y) \mid y \in \Omega\} \tag{19}$$

$$t(y) = \arg\min\{\varphi(x) + c(x,y) \mid x \in \Omega\} \tag{20}$$

These maps can be derived from the dual formulation described in Equation (18). The map s as defined in (19) is the Optimal Transport map in the Monge formulation. It can be also shown that [2, Lemma 2]

$$t(y) = y + \nabla C^*(-\nabla\psi(y)) \tag{21}$$

where C^* denotes the Frenchel Transform of C. Therefore thanks to Brenier theorem [7, Theorem 1.17], we obtain that t is the Optimal Transport map in the Monge formulation.

5 Changes to the model

In this section, we are going to discuss some changes we can make to the model. In each subsection, we are going to discuss how we can implement this in the model. First, let us start by stating the implementations we can make to the model

- Obstacles in the city. We can define an object that cannot be moved through or an object that can be slowly moved through.
- Areas with restrictions. We can make another restriction that in some areas, we cannot have firms if it is a non-industrial space.
- Preferation for local workers. We can determine which inhabitants are preferred working for a firm.
- The division of land. We can look at the motives of the land owner in dividing the land.

5.1 Obstacles in the city

In every city, there exist various obstacles that require effort to pass. Some obstacles, such as traffic jams, can be slowly traversed with patience. On the other hand, there are obstacles that cannot be directly overcome and need an alternative route, like hills, forests, and other similar barriers. As these obstacles are not accounted for in the basic model, we can include them in this section of the report so that we have a more realistic model.

We can model an obstacle in the city by setting an additional cost to that obstacle when crossing it. For the most simple case, we first assume that the obstacle divides the city into multiple parts. For illustrating this we will have Figure 4 regarding the division of the city.

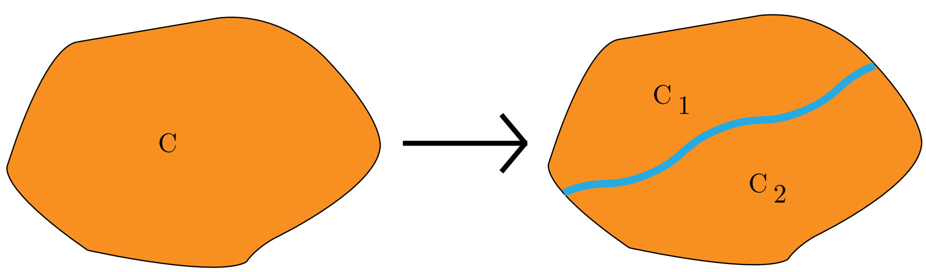

Figure 4: Division of city by obstacle

In our example we take the river to be our obstacle O. As you can see the river splits the city C into components C_1 and C_2 With this knowledge we can easily set a cost for passing this river by going with a boat and therefore changing components. Now, let us define the cost function

$$C(x,y) = \begin{cases} ||x-y|| & \text{for } (x,y) \in C_1 \text{ or } (x,y) \in C_2 \\ ||x-y|| + C_O & \text{for } x \in C_1 \text{ and } y \in C_2 \text{ or } x \in C_2 \text{ and } y \in C_1. \end{cases}$$

Here C_O is the cost of passing the obstacle O. You can divide the city into even more components, and define more cases to your function with each having a different cost C_{O_n} for passing river n. We can also define passing an obstacle by checking if the route taken intersects with the obstacle, more about this later in this section.

Now we can talk about the case if we can not go through the obstacle but we can go around it as illustrated in Figure 5. As you can see we assume that the obstacle is square to simplify the problem.

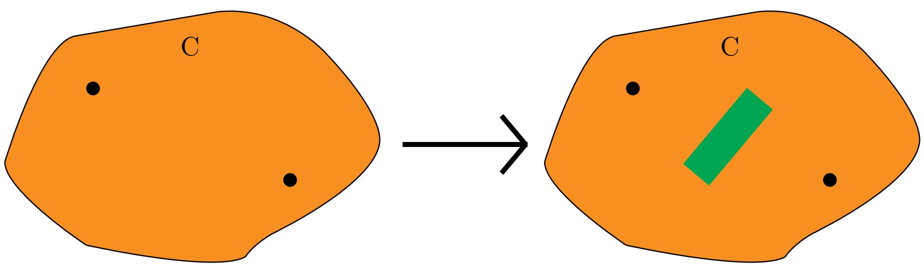

Figure 5: Obstacle in city

Obviously, the path from one point to the other is not direct, since there is an object in between. Now we need to look at which path is the shortest without going through the obstacle. In Figure 6 we took a look a look at the possibilities.

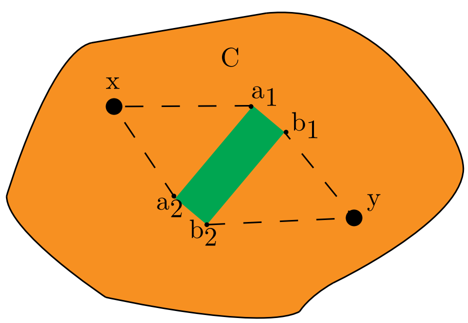

Figure 6: Obstacle in city with points

Here you notice that we can either pass the top or the bottom side of the obstacle. To look at which route is the best, we take the minimum distance out of all the possible routes. This leads to the following function

$$C(x,y) = \min_{r \in R}\{||x - a_r|| + ||a_r - b_r|| + ||b_r - y||\}.$$

Here we simply look at which route r is the shortest out of all possible routes R, where in this case $R = \{1, 2\}$. This definition is only useful for this specific example, so we need to make a more general equation

$$C(x,y) = \inf\left\{L(\gamma) = \int_0^1 |\dot{\gamma}(t)|\,dt : \gamma \in \Gamma \text{ and } \gamma(0) = x, \gamma(1) = y\right\}$$

where we have

$$\Gamma = \{\gamma : [0,1] \to D \text{ differentiable}\}.$$

Here we define Γ to be the set of all differentiable curves γ on domain D. This domain D is defined to be the city C without the obstacle(s) O_n. This results in an equation that consists of all the curves from x to y that go around the obstacle. For the cost, we will grab the infimum out of the length of all curves. This results in getting the shortest path from point x to y that goes around the obstacle.

We can also combine these functions for an obstacle that can be passed slowly, and get moved around. We can for example think of a river that can be passed by boat but also can get driven around by using a bridge. This would lead to our final function

$$C(x,y) = \inf\left\{L(\gamma) = \int_0^1 |\dot{\gamma}(t)|\,dt + \sum_O c_O : \gamma \in C \text{ and } \gamma(0) = x, \gamma(1) = y\right\} \tag{22}$$

where

$$c_O = \begin{cases} C_O & \text{for } (\gamma \cap O) \neq \emptyset \\ 0 & \text{for } (\gamma \cap O) = \emptyset. \end{cases} \tag{23}$$

Note that we look at all curves γ in the city C and if the curve goes through an obstacle O it will get an additional cost C_O related to this obstacle O. This cost C_O will be equal to ∞ if the obstacle cannot be passed through.

5.2 Areas with restriction

In the current world, we have areas with only houses and some areas with only working places. This is because some people prefer to only live next to houses instead of working places. The same goes for firms. In the model, we will include this restriction to make it more realistic to the real-world situation.

For explaining how we are going to implement this into the model, we will first illustrate it in Figure 7.

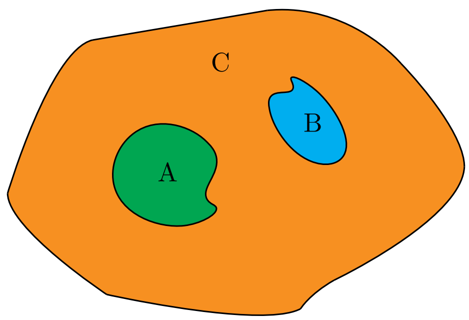

Figure 7: City with different types of areas

Here we say that area A should not contain firms, and area B should not contain homes of inhabitants. Note that $A \subseteq C$ and $B \subseteq C$. Also, it should be considered that landlords cannot have any impact on this. For this, we can have an extra condition to the equilibrium added as described in Definition 2.1, which is

- $\mu(x) = 0 \; \forall \; x \in A$ and $\nu(x) = 0 \; \forall \; y \in B$, where $A \subseteq C$, $B \subseteq C$

Here the areas A and B could be defined however one pleases.

5.3 Workers at the office

In some situations firms will try to have as many people as possible working at the office, avoiding letting people work online. How can we decide which people to choose at the office and which people to let work online?

For this we can consider a situation where the density of people will be bigger than the density of firms, resulting in $\int \mu > \int \nu$. This does not satisfy the equilibrium condition $\int \mu = \int \nu$, so we need to change the density. The only solution for this is to decrease the density of residents μ. This can be done by transforming the continuous density to something discrete using a method like sampling.

For choosing the people that will work at the office, we will need to find the workers that are the closest to the firms. So, let each worker i work at x_i and let each firm j be located at y_j. Say that there are a total of N workers and a total of firms M where $N > M$. For the use of our model, we need N and M to be equal. So we look at all the $\binom{N}{M}$ subsets of N that have M elements.

Afterward, for deciding which subset is the most efficient for transportation, you will calculate the Monge transportation problem for all these subsets. The subset with the least total transportation needed will be chosen as workers. To not leave the $N - M$ inhabitants without any money, we will let them work in another way, like online, which will get paid accordingly. We will end this section with a process describing the selection of office workers.

Process for the selection of office workers
Let N be the number of inhabitants, and M be the number of firms. Now divide the N inhabitants into $\binom{N}{M}$ unique subsets P of M inhabitants. Here each inhabitant is located at a unique x_i, and each firm at y_i. Also, let μ be the density of inhabitants, and ν to be the density of jobs after the division into subsets. Here we consider $\mu \in P(X)$ and $\nu \in P(Y)$. Compute the Monge transportation problem

$$v_P = \min\Big\{ \sum_i c(x_i, T(x_i)) : T : X \to Y,\ s.t.\ T(\mu) = \nu \Big\}$$

for each unique subset P. Afterwards, find $\min_P\{v_P\}$. The subset P that will be found, is the subset P of inhabitants that will get assigned to a firm.

5.4 The division of land

As discussed in Subsection 2.3.3, landlords decide how the land is split. Where the fraction $\theta(\Pi, Q)$ of land is allocated to business so that $1 - \theta(\Pi, Q)$ is the fraction of land for residential use. This fraction $\theta(\Pi, Q)$ is related to the rent for business use Π and the rent for residential use Q, but how are they related?

First off, we can just set $\theta(\Pi, Q)$ to a constant such that we know that the land division is how we want it. However, this is not the case in the real world, since landlords will simply rent their land to their highest bidder. We can implement that the highest bidder will rent their land into the model, as they do in the paper by Lucas and Rossi-Hansberg [1].
In this implementation, the rational behavior of landlords will be different, and therefore we need to only replace Subsection 2.3.3 with Subsection 5.4.1.

5.4.1 Landlords (new case)

At location $x \in \Omega$, given the rent for business use $\Pi(z(x), \psi(x))$ and the rent for residential use $Q(\varphi(x))$, landowners will determine a fraction a $\theta(x) \in [0,1]$ which satisfies

$$\Pi(z(x), \psi(x)) > Q(\varphi(x)) \Rightarrow \theta(x) = 1 \tag{24}$$

$$\Pi(z(x), \psi(x)) < Q(\varphi(x)) \Rightarrow \theta(x) = 0 \tag{25}$$

Note that, if $\Pi = Q$, landlords don't have a preferation for allocating this land. The values θ on the set $\{x \mid \Pi(z(x), \psi(x)) = Q(\varphi(x))\}$ will be determined using the equilibrium conditions.
Here the overall density of employment ν is given by

$$\nu(x) = \theta(x) n(z(x), \psi(x)) \tag{26}$$

and the overall density of residents μ by $\mu(x) = (1 - \theta(x))N(\varphi(x))$.

5.4.2 Equilibrium (new case)

The conjugacy relations between wages ψ and revenues φ are defined slightly different, since there is only employment on $\text{supp}(\nu)$ and residents on $\text{supp}(\mu)$. Now we define the relations in the following way

$$\varphi(x) = \sup\{\psi(y) - c(x, y) \mid y \in \text{supp}(\nu)\} \tag{27}$$

$$\psi(y) = \inf\{\varphi(x) + c(x, y) \mid x \in \text{supp}(\mu)\} \tag{28}$$

Here $\text{supp}(\nu)$ and $\text{supp}(\mu)$ denote the support of μ and ν, which is the subset of the function domain containing the elements which are not mapped to zero. From here we can define the equilibrium definition as

Definition 5.1 (New equilibrium)**.** A general equilibrium is a pair of nonnegative Lesbesgue integrable function (μ, ν) on Ω, a collection of functions$(z, \psi, \varphi, \theta)$ on Ω, and a pair of measurable maps $(s, t) : \Omega \to \Omega$ satisfying

- $\int_\Omega \mu = \int_\Omega \nu > 0$
- $z = Z_\mu$
- The pair (ψ, φ) satisfies

$$\varphi(x) = \sup\{\psi(y) - c(x, y) \mid y \in \text{supp}(\nu)\}$$
$$\psi(y) = \inf\{\varphi(x) + c(x, y) \mid x \in \text{supp}(\mu)\}$$

- For almost every $x \in \Omega,\ \theta(x) \in [0,1]$ and:
  $\mu(x) = (1 - \theta(x))N(\varphi(x)),\ \nu(x) = \theta(x)n(z(x), \psi(x))$
- The following conditions will be satisfied

$$\Pi(z(x), \psi(x)) > Q(\varphi(x)) \Rightarrow \theta(x) = 1$$
$$\Pi(z(x), \psi(x)) < Q(\varphi(x)) \Rightarrow \theta(x) = 0$$

- $s(\mu) = \nu$ and $t(\nu) = \mu$
- $s(t(y)) = y$ $\nu-a.e.$ and $t(s(x)) = x$ $\mu-a.e.$
- The pair (s, t) for $\mu-a.e.$ x and $\nu-a.e.$ y satisfies

$$\varphi(x) = \psi(s(x)) - c(x, s(x)) \tag{29}$$

$$\psi(y) = \varphi(t(y)) + c(t(y), y) \tag{30}$$

6 Conclusion

How can the structure of cities be modeled using Optimal Transport theory?
First off, we are using the model described in 'Optimal transportation and the structure of cities' [2]. Within this model, we have identified the various components of a city, the variables that influence these components, and the interrelationships between them. Some of these identifications needed interpretation, so we used the theory of Optimal Transport for this, namely the Dual Formulation.

How can the model be improved such that it can be used in more real-world scenarios?
The model we were describing in Section 2 could not be applied to all situations, so we wanted to make it more applicable. We could now add obstacles in the city that could be passed through with a certain cost related to it or cannot be passed through. We can add areas with restrictions, where we can choose if in the area only firms, only workers or both could be placed. We could describe which office workers would be chosen if there were more inhabitants than firms. Lastly, we looked at how the land can be divided, where we also stated our final equilibrium in Definition 5.1.

7 Discussion

The process of doing research went quite well since the research was done consistently throughout time. The communication went well between the supervisor and the student, there was at least one meeting each week. There was not much that could have gone better within the given time frame. It was only preferred that there was more time given since there were lots of other ideas that could be researched. Overall this research could be considered successful since the theory was understood, and there were some additions to existing concepts made by the researcher.

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