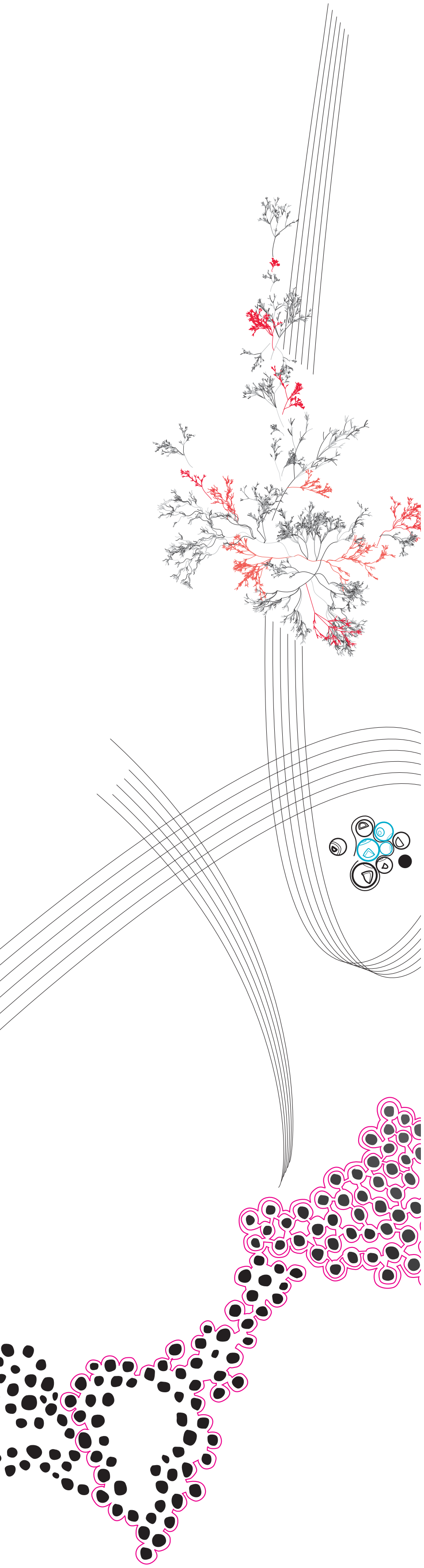
MSc Systems & Control
Thesis



PID tuning for electromechanical systems based on dilated LMI optimisation

Thijs David Rakels

Examination committee:
Dr. ir. W.B.J. Hakvoort
Dr. H. Köroğlu
Dr. ir. W. Roozing

July, 2023

Chair of Precision Engineering
Department of Mechanics of Solids,
Surfaces and Systems (MS3)
Faculty of Engineering Technology
University of Twente.

UNIVERSITY OF TWENTE.

Preface

I am delighted to present my thesis, titled *PID tuning for electromechanical systems based on dilated LMI optimisation* as the culmination of my studies in Systems & Control at the University of Twente. In this work, I attempt to determine PID tuning parameters using linear matrix inequality optimisation. Using linear matrix inequalities, which are new to me, to optimise the well-established PID controller is something that intrigued me, and I am honoured to share my journey.

First and foremost, I would like to express my heartfelt gratitude to my supervisor, Hakan Köroğlu. Our weekly sessions started off with very informative private lectures but quickly shifted to interesting discussions on the topic. Secondly, I want to thank Wouter Hakvoort, the chair of my graduating committee. During our biweekly discussions, you were always a voice of reason that linked the sometimes dry material to practical implementation and interpretation. Dear Hakan and Wouter, I am very grateful for your guidance on my journey over the last seven months. Thank you!

This thesis aims to find PID tuning parameters for electromechanical systems using linear matrix inequality optimisation. By formulating the problem as a static output feedback synthesis, the controller parameters are optimised. Using static output feedback synthesis comes with its limitations, resulting in a suboptimal controller.

The thesis is structured as follows: Chapter 1 introduces the work, Chapter 2 provides an introduction to the theoretical foundation of linear matrix inequalities up until the point of static output feedback. Chapter 3 presents a stand-alone paper titled: *PID tuning for mechanical systems based on dilated LMI optimisation.* Chapter 4 shows additional results that could not be added to the paper, among them PID optimisation for an electromechanical plant. Chapter 5 offers the concluding remarks on the work.

As a final remark, I would like to thank my family, friends, and colleagues who believed in me, helped me, and sometimes challenged me during my studies. In one way or another, you are all responsible for the work presented here.

Thijs Rakels
July, 2023
Enschede

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Chapter 1

Introduction

Proportional Integral Differential (PID) control is often used in the control of systems and is widely praised because of its simplicity and effectiveness [ÅH01]. A PID controller has three tuning parameters, the effects of which are well described in the literature and with this knowledge, numerous methods have been developed to assist in parameter tuning [Bor+21]. Although these methods are an effective way to find a controller, they do not give the best-performing controller for a specific objective. Finding such controllers can be done with optimal control, where a controller is optimised to minimise a performance objective, such as worst-case energy gain and energy to peak square gain, called the $\mathscr{H}_\infty$ and generalised-$\mathscr{H}_2$ norms, respectively. Because of the popularity of PID control, hardware has been designed that has an integrated PID controller that cannot be changed. Therefore, the possibility of using optimal control with a fixed PID structure is desirable.

Finding optimal PID parameters can be done by formulating the PID optimisation as a Static Output Feedback (SOF) synthesis, which has been done before in the literature [ZWL02; BHÅ16; Sae06]. Linear Matrix Inequalities (LMIs) are very popular for solving optimal control problems because they can be solved using powerful linear optimisation techniques. LMIs also have appealing features for multi-objective synthesis, robust synthesis for uncertain systems, and gain scheduling. Unfortunately, SOF synthesis cannot be optimised using LMIs directly as the LMI becomes bilinear in the design variables, making it a Bilinear Matrix Inequality (BMI). Because of this, SOF remains an unsolved problem and might not be solvable at all [BT97].

The BMIs can be reduced to LMIs using different techniques and manipulations. However, these tricks rely on assumptions and restrictions and lead to a potentially conservative solution. The controller obtained by SOF synthesis is therefore suboptimal. Different approaches to tackling the SOF synthesis have been published in the literature, from which three main trends can be distinguished. The first approach uses iterative linearization to find a controller [CLS98; Syr+97; Mer18], the second approach shapes the behaviour in the frequency domain for $\mathscr{H}_\infty$ performance [Sae06; SOW10; BHÅ16], and the last approach finds an LMI formulation with the help of LMI dilation [KF14]. In Section 2.5 each of these methods is explained in more detail.

In this thesis, a SOF synthesis is formulated to optimise a PID controller with a low-pass filter, known as a tame PID. Tame PID is often used in practise but not researched when considering PID optimisation. In this work, electromechanical systems are considered, for which a controller is optimised. Chapter 3 presents a paper where PID controllers are optimised for mechanical systems. Chapter 4 provides additional results that could not be added to the paper. Section 4.1 and 4.2 investigate the system presented in Chapter 3 in more detail. Section 4.3 presents the optimisation of an electromechanical system with an additional tuning parameter and presents two approaches to optimising this system. Chapter 5 concludes the results and gives recommendations for future work.

Chapter 2

Basic LMI Theory for Optimal Control

2.1 What is an LMI

The story of LMIs starts with the introduction of the binary relations $\prec$, $\preceq$, $\succ$ and $\succeq$ on the set of symmetric and Hermitian matrices. A complex-valued matrix A is Hermitian if it is square and $A = A^*$, where $\cdot^*$ denotes the complex conjugate transpose and the set is represented by $\mathbb{H}^n$. If A is real, this amounts to $A = A^T$ and A is said to be symmetric and in $\mathbb{S}^n$. A Hermitian or symmetric matrix A is negative definite if $x^*Ax < 0$ for all non-zero complex vectors x. Similarly, A is said to be negative semi-definite if $x^*Ax \leq 0$ holds. The matrix A is positive definite and positive semi-definite if $-A$ is negative definite and negative semi-definite respectively. The symbols $\prec$, $\preceq$, $\succ$ and $\succeq$ define binary relations on both $\mathbb{H}^n$ and $\mathbb{S}^n$ as follows:

$A \prec B$	if $A - B$ is negative definite
$A \preceq B$	if $A - B$ is negative semi-definite
$A \succ B$	if $A - B$ is positive definite
$A \succeq B$	if $A - B$ is positive semi-definite

Hermitian and symmetric matrices have real eigenvalues. A positive definite matrix $A \prec 0$ has only negative eigenvalues, and a similar $A \preceq$ has only non-positive eigenvalues.

Definition 2.1.1 (Convex sets)**.** A set $\mathscr{I}$ in a linear vector space is said to be convex if

$$\{x_1, x_2 \in \mathscr{I}\} \Rightarrow \{x := \alpha x_1 + (1-\alpha)x_2 \in \mathscr{I} \text{ for all} \alpha \in (0,1)\}.$$

In geometric therms, this states that a convex set is characterised by the property that the line segment connecting any two points in the set belongs to the set.

Definition 2.1.2 (Convex functions)**.** A function $F : \mathscr{I} \to \mathbb{R}$ is convex if $\mathscr{I}$ is a non-empty convex set and if for all x_1, $x_2 \in \mathscr{I}$ and $\alpha \in (0,1)$ holds that

$$F\left(\alpha x_1 + (1-\alpha)\, x_2\right) \leq \alpha\left(x_1\right) + (1-\alpha)\, F\left(x_2\right). \tag{2.1}$$

The function F is called strictly convex if the inequality with $\leq$ replaced by $<$ holds for all x_1, $x_2 \in \mathscr{I}$, $x_1 \neq x_2$ and all $\alpha \in (0,1)$.

The importance of studying convex functions becomes apparent when studying local and global minima. Suppose that $F : \mathscr{I} \to \mathbb{R}$ is convex. Every local optimal solution of F is a global optimal solution. Moreover, if F is strictly convex, then the global optimal solution is unique.
A linear matrix inequality is an expression of the form

$$F(x) := F_0 + x_1F_1 + ... + x_nF_n \prec 0, \tag{2.2}$$

where $x = (x_1, ..., x_n)$ is vector of n real numbers called the decision variables and $F_0, ..., F_n$ are Hermitian matrices i.e. $F_j = F_j^* \in \mathbb{H}$, for $j = 0, ..., n$.

Definition 2.1.3 (Linear Matrix Inequality)**.** A linear matrix inequality (LMI) is an inequality

$$F(x) \prec 0, \tag{2.3}$$

where F is an affine function mapping a finite dimensional space $\mathscr{K}$ to either the set $\mathbb{H}$ of Hermitian or the set of $\mathbb{S}$ of symmetric matrices.

The linear matrix inequality $F(x) \prec 0$ defines a convex constraint on x. In other words, the set

$$\mathscr{I} := \{x | F(x) \prec 0\}, \tag{2.4}$$

of solutions of the LMI $F(x) \prec 0$ is convex.
If there exist multiple LMI constraints on x, $F_1(x) \prec 0$ $F_2(x) \prec 0$ ... $F_n(x) \prec 0$, these LMIs can be combined into a single LMI without losing any generality. This LMI will take the form of

$$F(x) := \begin{bmatrix} F_1(x) & 0 & \cdots & 0 \\ 0 & F_2(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & F_n(x) \end{bmatrix} \prec 0 \tag{2.5}$$

Another important property of LMI's is obtained from algebraic observation. If M is a square matrix and T is non-singular, then the product T^*MT is called a congruence transformation of M. For Hermitian and symmetric matrices M, the number of positive and negative eigenvalues does not change after the congruence transformation. From this property, an important relation can be obtained called the Schur Complement.

Definition 2.1.4 (Schur Complement)**.**

$$\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} \succ 0 \Longleftrightarrow \begin{bmatrix} X & 0 \\ 0 & Z - Y^T X^{-1} Y \end{bmatrix} \succ 0 \Longleftrightarrow \begin{bmatrix} X - YZ^{-1}Y^T & 0 \\ 0 & Z \end{bmatrix} \succ 0 \tag{2.6}$$

Many optimisation problems in control can be formulated or reformulated as linear matrix inequalities. Since the linear matrix inequality $F(x) \prec 0$ defines a convex constraint on the variable x, optimisation problems involving the minimization (or maximisation) of the performance function $f : \mathscr{I} \to \mathbb{R}$ with $\mathscr{I} := \{x|F(x) \prec 0\}$ belong to the class of convex optimisation problems. Therefore, the LMI problems can be solved in an effective and efficient manner. Two generic problems related to linear matrix inequalities:

- **Feasibility:** The question of whether there exists an $x \in \mathscr{K}$ such that $F(x) \prec 0$ is called the feasibility problem.
- **Optimisation:** Consider an objective function $f : \mathscr{I} \to \mathbb{R}$ where $\mathscr{I} = \{x|F(x) \prec 0\}$. The problem to determine

  $$f_{opt} = \inf_{x \in \mathscr{I}} f(x)$$

  is called an optimisation problem with an LMI constraint. The problem involves the determination of f_{opt} and the calculation of the optimal solutions $x_{opt}(x_{opt} \in \mathscr{I}$ such that $f_{opt} = f(x_{opt}))$.

2.2 Lyapunov Stability

Consider the problem of investigating the internal stability of the LTI system

$$\dot{x} = Ax, \tag{2.7}$$

where $A \in \mathbb{R}^{n \times n}$. From the Lyapunov stability theorem, it is known that the system is exponentially stable if and only of there exists an $X \in \mathbb{S}^n$ such that $X \succ 0$ and $A^T X + XA \prec 0$. These two inequalities are LMI constraints on the problem and can be treated as a system of LMIs resulting in the LMI constraint.

$$\begin{bmatrix} -X & 0 \\ 0 & A^T X + XA \end{bmatrix} \prec 0, \tag{2.8}$$

which can be solved as a feasibility problem.

2.3 LMIs for $\mathscr{H}_\infty$ and $\mathscr{H}_2$ performance

Consider the plant P(s) as

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t),\\ y(t) &= Cx(t) + Du(t),\end{aligned} \tag{2.9}$$

where $x(t) \in \mathbb{R}^k$, $u(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^m$, at any $t \in \mathbb{R}$. Suppose the system matrices A, B, C and D of $P(s)(A, B, C, D)$ are known. Then the following are equivalent:

1. $||P||_{H_\infty} < \gamma$
2. There exists a $X \succ 0$ such that

$$\begin{bmatrix} A^TX + XA & XB \\ B^TX & -\gamma I \end{bmatrix} + \frac{1}{\gamma}\begin{bmatrix} C^T \\ D^T \end{bmatrix}\begin{bmatrix} C & D \end{bmatrix} \prec 0. \tag{2.10}$$

This is known as the KYP or Bounded Real Lemma. The KYP lemma shown is nonlinear in γ. With the use of the Schur complement, the following LMI can be obtained:

$$\begin{bmatrix} A^TX + XA & XB & C^T \\ B^TX & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} \prec 0. \tag{2.11}$$

The $\mathscr{H}_\infty$ norm of a system can be obtained by minimising γ subjected to the LMI constraint in (2.11). Assuming that $P(s) = C(sI - A)^{-1}B$, this means that the following are equivalent:

1. A is Hurwitz and $||P||_{H_2} < \gamma$
2. $\begin{cases} \text{trace } (CQC^T) & < \gamma^2 \\ AQ + QA^T + BB^T & \prec 0 \\ Q & \succ 0 \end{cases}$

This can be reformulated as an LMI optimisation problem in the form of

$$\begin{bmatrix} A^TX + XA & XB \\ B^TX & -\gamma I \end{bmatrix} \prec 0; \qquad \begin{bmatrix} X & C^T \\ C & W \end{bmatrix} \succ 0; \qquad \text{trace}(W) < \gamma. \tag{2.12}$$

Similarly, the LMI for Generalised $\mathscr{H}_2$ performance is given by

$$\begin{bmatrix} A^TX + XA & XB \\ B^TX & -\gamma I \end{bmatrix} \prec 0; \qquad \begin{bmatrix} X & C^T \\ C & \gamma I \end{bmatrix} \succ 0. \tag{2.13}$$

2.4 LMIs for controller design

The last section of LMI basics is concerned with finding an optimal controller for the standard control problem, as seen in Fig. 2.1. In this problem formulation, the objective is to find a controller K that optimises the transfer function $H_{z/w}$ which goes from w to z. In the following section, LMIs for $\mathscr{H}_\infty$-optimal control with full state feedback and static output feedback are presented. The LMIs for $\mathscr{H}_2$ and generalised -$\mathscr{H}_2$ can be determined from the information provided.

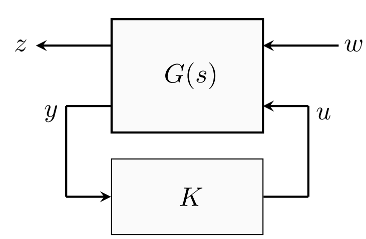

Figure 2.1: The standard control problem with a static gain matrix

LMI for Full State Feedback with $\mathscr{H}_\infty$ performance

Consider a plant model $G(s)$ in Fig. 2.1 as

$$G(s): \begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ I & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix}, \tag{2.14}$$

where $x(t) \in \mathbb{R}^k$, $u(t) \in \mathbb{R}^n$ and $y = x$. With the control law $u = Kx$, the closed loop dynamics are described by

$$\begin{bmatrix} \dot{x} \\ z \end{bmatrix} = \begin{bmatrix} A + B_2 K & B_1 \\ C_1 + D_{12} K & D_{11} \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}. \tag{2.15}$$

Computing the closed-loop $\mathscr{H}_\infty$ norm with the LMI provided earlier results in

$$\begin{bmatrix} \mathrm{He}\left\{X\left(A + B_2 K\right)\right\} & X B_1 & * \\ * & -\gamma I & * \\ C_1 + D_{12} K & D_{11} & -\gamma I \end{bmatrix} \prec 0, \tag{2.16}$$

where $X \in \mathbb{S}_+^k$ and $K \in \mathbb{R}^{n \times k}$ are the design variables and $\gamma \in \mathbb{R}$ is the guaranteed performance level that is optimised. The function He{} is defined as $\mathrm{He}\left\{X\right\} = X + X^T$ and the $*$ represents entries that can be determined by symmetry. It can be seen that this LMI is bilinear in the design variables, which makes the problem non-convex. By pre- and post-multiplying the LMI with the block diagonal matrix $\mathrm{diag}\begin{pmatrix} Y, & 0, & 0 \end{pmatrix}$, where $Y = X^{-1}$. The dual condition can be obtained by

$$\begin{bmatrix} \mathrm{He}\{AY + B_2 Z\} & B_1 & * \\ * & -\gamma I & * \\ C_1 Y + D_{12} Z & D_{11} & -\gamma I \end{bmatrix} \prec 0, \tag{2.17}$$

where $K = ZY^{-1}$. This optimisation problem is an LMI and convex in the design variables.

LMI for static output feedback with $\mathscr{H}_\infty$ performance

The last example discussed in this section is the design of a static output feedback controller with $\mathscr{H}_\infty$ performance. Consider the standard control problem in Fig. 2.1 with the generalised plant defined as

$$G(s): \begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix} \tag{2.18}$$

where $x(t) \in \mathbb{R}^k$, $u(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^m$. With the control law $u = Ky$ the closed loop can be determined as

$$\begin{bmatrix} \dot{x} \\ z \end{bmatrix} = \begin{bmatrix} A + B_2 K C_2 & B_1 + B_2 K D_{21} \\ C_1 + D_{12} K C_2 & D_{11} + D_{12} K D_{21} \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}. \tag{2.19}$$

With the closed-loop and the LMI for $\mathscr{H}_\infty$ the LMI for SOF can be determined as

$$\begin{bmatrix} \mathrm{He}\left\{X(A + B_2 K C_2)\right\} & X\left(B_1 + B_2 K D_{21}\right) & * \\ * & -\gamma I & * \\ (C_1 + D_{12} K C_2) & (D_{11} + D_{12} K D_{21}) & -\gamma I \end{bmatrix} \prec 0. \tag{2.20}$$

The LMI for $\mathscr{H}_2$ and generalised-$\mathscr{H}_2$ can be determined from context if the equality constraint $D_{11} + D_{12} K D_{21} = 0$ holds, which complicates things. This complication can be avoided by assuming $D_{11} = 0$ and either $D_{12} = 0$ or $D_{21} = 0$. The LMIs for SOF are bilinear in the design variables X and K; it is still unknown how to reformulate this as an LMI problem for a generic plant without introducing any conservatism. It is also unclear whether this is possible at all for a generic plant model.

2.5 Tackling SOF controller design

The previous sections presented the problem of static output feedback syntheses for a generic plant model using LMIs. Although it is not possible to use the Bilinear Matrix Inequality (BMI) to find an optimal controller directly, it is an ongoing topic in research to find optimal SOF controllers. One approach is to reduce the BMIs to LMIs using a variety of tricks and conditions, resulting in a solution that is often conservative and therefore a suboptimal controller. Literature suggested a vast array of strategies and conditions for reducing BMIs to LMIs; however, three primary approaches were distinguished. In the next sections, the central concept of each approach is presented.

Iterative LMI

The first approach originates from the work of Cao et al. [CLS98], where an Iterative LMI (ILMI) approach was proposed to find a stabilising controller for SOF. The BMI for SOF is transformed to a Quadratic Matrix Inequality (QMI) using some conservative conditions in the BMI. Next, an iterative procedure is applied, which starts with an initial Lyapunov matrix that is found using an algebraic Ricatti equation. The iterative optimisation attempts to find a solution with the lowest system poles. The optimisation is terminated as all closed loop poles are on the Left Half Plane (LHP), resulting in a stabilising controller. Continuations of the ILMI approach are focused on the convexivation of the problem, which would result in a controller that is closer to the actual optimum [Syr+97; SP16].

Frequency shaping

A different approach is presented by Boyd et al. [BHÅ16], which was inspired by the work of Saeki et al [Sae06; SOW10]. Boyd et al. found that new QMI conditions could be formulated by using the closed-loop system properties at sampled frequencies. With these QMIs, closed-loop performance could be enforced at each of these frequencies. Using iterative convex-concave optimisation, the QMIs are solved to find a sub-optimal controller.

LMI dilation

The final approach is based on LMI dilation. By introducing an additional design matrix in the controller, extra design freedom is obtained. Coutinho et al. combine controller dilation with the S-procedure [Cou+05] to obtain an LMI formulation that can be solved. Inspired by this work, Köroğlu and Falcone presented a new approach to the multi-objective SOF syntheses problem [KF14].

Chapter 3

Paper: PID tuning for mechanical systems based on dilated LMI optimisation

PID tuning for mechanical systems based on dilated LMI optimisation

Thijs D. Rakels
Chair of Precision Engineering, MS3
University of Twente
Enschede, Netherlands

***Abstract*—PID feedback control is widely used in control systems due to its effectiveness and simplicity. Although numerous approaches for PID tuning exist, optimising the variables remains an open question. In this work, PID parameter tuning for two mechanical systems is formulated as a static output feedback synthesis. The static output feedback synthesis is optimised using a dilated LMI formulation and successive iterative linearizations. It is shown that the approach presented in this work is successful in obtaining PID parameters with $\mathscr{H}_\infty$ and generalised-$\mathscr{H}_2$ performance objectives. The presented synthesis seems appropriate for use with robust uncertainty modelling and stable weighting filters.**

***Index Terms*—Feedback control, PID, $\mathscr{H}_\infty$, $\mathscr{H}_2$, Static output feedback (SOF), Linear matrix inequality (LMI)**

Proportional Integral Derivative (PID) feedback control is widely known and used in the automation industry. It is estimated that 90%–95% of control loops used in industrial applications are in the PID form [1]. Over the past century, PID control has been used in applications such as motion control, process control, power electronics, and more [2]. The PID controller is praised for its simplicity; it has three tuning parameters (gains), and the effects on closed-loop performance are well understood. Numerous methods have been developed for tuning the parameters to achieve desirable performance [3]–[5]. These tuning procedures are developed for single-input, single-output (SISO) systems and focus on finding a robust and stabilising controller. Most of these tuning methods are based on loop shaping, where desired open-loop characteristics are defined. However, despite the effectiveness of the tuning methods, the resulting controller is not necessarily the best one that can be obtained.

A different approach to controller design is the optimal control method. In this approach, a controller is constructed to minimise a closed-loop system norm. Common performance objectives in control are the $\mathscr{H}_\infty$ and generalised-$\mathscr{H}_2$ norms, which represent worst-case energy gain and minimise energy to peak square gain, respectively. An ongoing topic of research is to formulate fixed-structure optimal controllers. A form of fixed-structure controller that is often considered is multiple-input multiple-output (MIMO) PID designs, as the problem is far more complex than the SISO case. Linear matrix inequalities (LMIs) are often used in the design of optimal controllers as they are convex and can therefore efficiently be solved [6]–[11]. More appealing features of LMIs are shown when optimising systems with multi-objective synthesis, robust synthesis for uncertain systems and scheduled synthesis [12].

Even though a large number of problems in the field of control theory, like calculating system norms and state feedback control, can be formulated as LMIs [13], many others are non-convex and cannot be solved using LMIs. An approach to optimal PID controller design is to formulate the problem as a static output feedback (SOF) problem [6], [14]. The norm optimisation of an SOF is non-convex in the controller variables and leads to a bilinear matrix inequality (BMI). In order to find approximate solutions, it is necessary to make an effort to transform the problem into an LMI structure. It hasn't yet been possible to derive an LMI formulation of SOF that is not potentially conservative. Two well-known methods to transform BMIs into LMIs are convex-concave decomposition and the linearization method [15]. Solving the SOF optimisation problem remains an open question in control theory and is stated to be NP-hard [16]. Although the problem is hard to solve in general, there may be workable solutions for particular situations. Therefore, research into PID controller synthesis for plants with a generic structure is worth investigating.

The first notable work on PID optimisation with the LMI approach was shown in [6]. In this work, the PID control problem is formulated as a SOF problem and solved with the help of a previously designed iterative LMI (ILMI) approach [17]. Lin et al. [7] improved on the work of [6] and removed some of the shortcomings. Later work [8], [9] provided even more improvements on the same method with the use of convexification of the non-convex problem. These improvements allowed the addition of weighting filters in the generalised plant formulation. The developments in linearising the BMI problem are summarised in two surveys [10], [11], where both initial and more recent work are reviewed. A different approach to the PID optimisation problem can be found in [18]–[20]. Here, the non-convexity of the SOF $\mathscr{H}_\infty$ design problem is overcome by reformulating the problem in the frequency domain. A different approach is taken by Köroğlu and Falcone [21]. With the use of a dilation inspired by [22]. LMIs are formulated that depend on a scalar parameter which needs to be fixed beforehand. This approach allows for the reduction of some of the restrictive assumptions about the plant that are necessary with some of the other methods.

The PID-specific research previously mentioned focuses on the perfect PID controller formulation. In industry, the PID controller is often used in combination with a low-pass filter, also known as a tame PID. In this paper, a SOF synthesis of the tame PID controller is presented for mechanical systems. This synthesis is optimised with the SOF optimisation in Köroğlu and Falcone [21] and extended with a successive iterative linearization.

Sections I and II are used to define the SOF formulation and present an improved optimisation method to find a sub optimal. Next, SOF synthesis is formulated for two different mechanical systems in Section III. The optimisation results are presented and discussed in Sections IV before the concluding remarks.

I. Static Output Feedback formulation

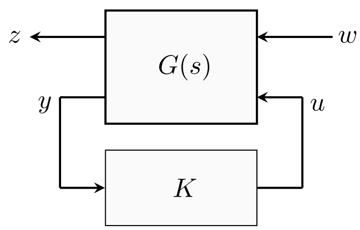

Fig. 1. The standard control problem for static output feedback

Consider the standard control problem as depicted in Fig. 1, where $G(s)$ is a known Linear time-invariant (LTI) generalised plant and K is a to-be-designed controller. The plant model can be given as

$$G(s) : \left[\begin{array}{c} \dot{x}(t) \\ \hline z(t) \\ y(t) \end{array}\right] = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] \left[\begin{array}{c} x(t) \\ \hline w(t) \\ u(t) \end{array}\right], x(0) = 0, \tag{1}$$

where $x(t) \in \mathbb{R}^k$, $u(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}^m$ represent the state vector, control input, and measured output vectors, respectively. Furthermore, $w(t) \in \mathbb{R}^q$ denotes the generalised disturbance, and $z(t) \in \mathbb{R}^p$ gives the performance output.
In this paper, we consider the problem of static output feedback (SOF) synthesis, for which the control input is constructed as

$$u(t) = Ky(t), \tag{2}$$

where $K \in \mathbb{R}^{n\times m}$ is a static gain matrix. The closed-loop system dynamics can be determined by imposing (2) onto (1), which results in

$$\left[\begin{array}{c} \dot{x} \\ \hline z \end{array}\right] = \left[\begin{array}{c|c} A + B_2KC_2 & B_1 + B_2KD_{21} \\ \hline C_1 + D_{12}KC_2 & D_{11} + D_{12}KD_{21} \end{array}\right] \left[\begin{array}{c} x \\ \hline w \end{array}\right], \tag{3}$$

where the time dependencies have been suppressed for notational simplicity. In this paper, we consider static output feedback synthesis with $\mathscr{H}_2$ and $\mathscr{H}_\infty$ performance objectives. Recall the definitions of $\mathscr{L}_2$ and $\mathscr{L}_\infty$ as

$$||x||_2 := \left(\int_0^\infty ||x(t)||^2 dt\right)^{1/2}, \tag{4}$$

$$||x||_\infty := \sup_{t\geq 0} ||x(t)||, \tag{5}$$

where $||a|| := \sqrt{a^T a}$. The objective is to find a matrix $K \in \mathbb{R}^{n\times m}$ for which the closed-loop is stable (i.e. $A + B_2KC_2$ is Hurwitz) and minimises a certain performance objective.

II. Proposed method

In this section, a solution to the SOF problem is proposed. This solution consists of a method based on the work of Köroğlu and Falcone [21]. In their work, sufficient LMI conditions are presented for H_∞ and H_2 performance. In this paper, an ensuing iterative linearization process is added to reduce the resulting conservatism.

A. LMIs for SOF by Köroğlu and Falcone [21]

Köroğlu and Falcone [21] proposed a dilated LMI formulation for the static output feedback problem, inspired by the work of Coutinho et al. [22]. The idea of dilation is to define the feedback gain matrix K as

$$K = NW^{-1}, \tag{6}$$

with $N \in \mathbb{R}^{n\times m}$ and an extra Lyapunov matrix $Y \in \mathbb{S}_+^k$ which is different from $W \in \mathbb{R}^{m\times m}$. By defining a transformed state vector and a new signal, the closed-loop dynamics can be written in a way that avoids the bilinear dependencies on the design variables when deriving the LMIs. From the new closed-loop dynamics, the LMIs for $\mathscr{H}_\infty$ and $\mathscr{H}_2$ performance are formed.

Theorem II.1 (LMI for $\mathscr{H}_\infty$ [21])**.** The closed-loop system (3) is stable and satisfies the $\mathscr{H}_\infty$ performance objective

$$||z||_2 < \sigma||w||_2, \tag{7}$$

with a gain matrix as in (6), if there exist $Y \in \mathbb{S}_+^k$, $W \in \mathbb{R}^{m\times m}$ and $N \in \mathbb{R}^{n\times m}$ that satisfy

$$\begin{bmatrix} -\phi \text{He}\{W\} & \phi(C_2Y - WC_2) + N^TB_2^T & \phi D_{21} & * \\ * & \text{He}\{AY + B_2NC_2\} & B_1 & * \\ * & * & -\sigma I & * \\ D_{12}N & C_1Y + D_{12}NC_2 & D_{11} & -\sigma I \end{bmatrix} \prec 0, \tag{8}$$

where $\phi \in \mathbb{R}_+$ is an arbitrary positive scalar.

The function He() is defined as $\text{He}(Q) = Q + Q^T$, I represents the identity matrix with dimensions that can be determined from context, and $*$ denote entries that are identifiable from symmetry. In the work of Köroğlu and Falcone [21] LMI conditions for generalised $\mathscr{H}_2$ performance are presented as:

Theorem II.2 (LMI for generalised-$\mathscr{H}_2$ [21])**.** Under the condition that $D_{11} = D_{12} = 0$, the closed-loop system (3) is stable and satisfies

$$||z||_\infty < \gamma||w||_2, \tag{9}$$

with a gain matrix as in (6), if there exist $Y \in \mathbb{S}_+^k$, $W \in \mathbb{R}^{m\times m}$ and $N \in \mathbb{R}^{n\times m}$ that satisfy

$$\begin{bmatrix} -\psi \text{He}\{W\} & \psi(C_2Y - WC_2) + N^T B_2^T & \psi D_{21} \\ * & \text{He}\{AY + B_2NC_2\} & B_1 \\ * & * & -\gamma I \end{bmatrix} \prec 0, \tag{10}$$

$$\begin{bmatrix} Y & * \\ C_1Y & \gamma I \end{bmatrix} \succ 0, \tag{11}$$

where $\psi \in \mathbb{R}_+$ is an arbitrary positive scalar.

The positive scalars ϕ and ψ are introduced to reduce the introduced conservatism of the dilated LMIs. By performing a line search over the scalar variable, the least conservative controller can be found.

B. Iterative Linearization

Starting from the BMIs of the SOF problem for $\mathscr{H}_\infty$ and $\mathscr{H}_2$ performance, an iterative process can be formulated to reduce conservatism. The closed-loop system (3) is stable and satisfies the $\mathscr{H}_\infty$ performance measure in (7) if there exists a gain matrix $K \in \mathbb{R}^{n\times m}$ and $X \in \mathbb{S}_+^k$ that satisfy

$$\begin{bmatrix} \text{He}\{X(A + B_2KC_2)\} & X(B_1 + B_2KD_{21}) & * \\ * & -\sigma I & * \\ C_1 + D_{12}KC_2 & D_{11} + D_{12}KD_{21} & -\sigma I \end{bmatrix} \prec 0, \tag{12}$$

Similarly, we can define the LMI for the generalised $\mathscr{H}_2$ performance measure. The closed-loop system (3) is stable and satisfies (9) if there exists a gain matrix $K \in \mathbb{R}^{n\times m}$ and $X \in \mathbb{S}_+^k$ that satisfy

$$\begin{bmatrix} \text{He}\{X(A + B_2KC_2)\} & X(B_1 + B_2KD_{21}) \\ * & -\gamma I \end{bmatrix} \prec 0, \tag{13}$$

$$\begin{bmatrix} X & * \\ (C_1 + D_{12}KC_2) & \gamma I \end{bmatrix} \succ 0. \tag{14}$$

From the equations (12) and (13) it can be seen that there exists a bilinear dependency on the design variables K and X, resulting in the problem being a BMI problem, which is nonlinear and non-convex.

By fixing one of the design variables, the BMI reduces to an LMI, which is a convex optimisation problem. This linearization allows for local optimisation of the problem for one of the design variables when the other variable is known. As a new value for the design variable is found, this value can be fixed in order to linearize the problem around this new optimum, allowing the previously known variable to be optimised. This linearization can be performed iteratively in order to improve upon the initially known solution.

The controller obtained by each step of the iterative optimisation is guaranteed to have a closed-loop system norm that cannot be higher than the initial one. The LMI constraints in each iteration are convex when one of the variables is fixed, which guarantees that the controller found in the optimisation is the global optimum with a combination of $K \in \mathbb{R}^{n\times m}$ and $X \in \mathbb{S}_+^k$. Fixing one of these variables and optimising the other keeps the previous solution in the viable set of solutions. As a result, the newly found optimum is either the previously found solution or has an improved system norm. In theory, this would guarantee that there is no negative effect of the iterative optimisation. In practise, care needs to be taken to avoid numerical issues.

The stabilising controller found by the method of Köroğlu and Falcone [21] can be used as a starting point for the iterative process. The conservatism that might exist as a result of the method may be reduced or at least stay the same. The iterative process should be terminated when no significant improvements are made. Given that the BMI problem is nonlinear, this stagnation of improvement is not proof of a local or global optimum.

C. Method Implementation

This section provides some additional information about the practical implementation of the aforementioned method. The pseudo-code in Algorithm 1 provides a crude outline of the implementation with an $\mathscr{H}_2$ performance measure. Reformulation for $\mathscr{H}_\infty$ performance can be found in the content of this paper. The implementation consists of two phases: The initialization phase, where an initial stabilising controller is constructed, and the iteration phase, consisting of ”Opt 1” and ”Opt 2”, where the iterative process is used to further optimise.

Each LMI optimisation is written down in the following way. The **Constraints** section gives the LMI constraints that represent the performance measure, and the **Objective** section shows which parameter is optimised and the design variable. The black parameters represent constants in the optimisation procedure that are known prior to the optimisation step. The red parameters represent the variables that can be optimised during the LMI optimisation. The green variables represent the variables used in the line search. The variables that are optimised in the LMI optimisation gain the subscript $\bullet_{opt}$.

A line search needs to be performed to find a value for ψ, as the optimisation problem is non-convex in this variable. The set $\Psi \in \mathbb{R}_+$ is defined and represents all the numbers in the line search. For each of the values of $\psi \in \Psi$, the optimum K_{opt} and γ_{opt} are computed. The best performing controller will be stored as K^* which will be used in the next optimisation.

The iterative optimisation is performed until no further improvement is made. This is done by comparing the previously known γ^* with the newly found γ_{opt}. If the new norm is less than ρ times the previous optimum, the optimum is terminated, and the last known controller is said to be the optimum controller. In practise, the performance measures γ and σ do not always go down with each iteration.

III. PID CONTROLLER DESIGN AS SOF SYNTHESIS

This section will provide the SOF synthesis for a tame PID controller design. The controller that is optimised is a tame

PID controller in the form of

$$C(s) = \frac{c_1 s^2 + c_2 s + c_3}{s^2 + c_4 s}, \tag{15}$$

Algorithm 1: Proposed SOF optimisation method with H_2 performance measure. The black parameters represent constants, red parameters represent the variables that are optimised, and green parameters represent the variables of the line search.

Initialization:

forall $\psi \in \{\psi_{min} = \psi_1, \psi_2, ..., \psi_{N-1}, \psi_N = \psi_{max}\}$ **do**

Constraints:

$$\begin{bmatrix} -\psi \text{He}\{W\} & \psi(C_2 Y - W C_2) + N^T B_2^T & \psi D_{21} \\ * & \text{He}\{AY + B_2 N C_2\} & B_1 \\ * & * & -\gamma I \end{bmatrix} \prec 0$$

$$\begin{bmatrix} Y & * \\ C_1 Y & \gamma I \end{bmatrix} \succ 0$$

Objective:

$$N_{opt}, W_{opt} \leftarrow \arg\min_{N,W} \gamma$$

$$\gamma_{opt} \leftarrow \min \gamma$$

$K_{opt} \leftarrow N_{opt} {W_{opt}}^{-1}$

if $\gamma_{opt} < \gamma^*$ **then**

$K^* \leftarrow K_{opt}$

$\gamma^* \leftarrow \gamma_{opt}$

end

end

Iterative Optimization:

while $\gamma^* \geq \rho \gamma_{opt}$ **do**

Opt 1:

Constraints:

$$\begin{bmatrix} \text{He}\{X(A + B_2 K^* C_2)\} & X(B_1 + B_2 K^* D_{21}) \\ * & -\gamma I \end{bmatrix} \prec 0$$

$$\begin{bmatrix} X & * \\ (C_1 + D_{12} K^* C_2) & \gamma I \end{bmatrix} \succ 0$$

Objective:

$$\arg\min_{X} \gamma$$

$\gamma^* \leftarrow \gamma_{opt}$

$X^* \leftarrow X_{opt}$

Opt 2:

Constraints:

$$\begin{bmatrix} \text{He}\{X^*(A + B_2 K C_2)\} & X^*(B_1 + B_2 K D_{21}) \\ * & -\gamma I \end{bmatrix} \prec 0$$

$$\begin{bmatrix} X^* & * \\ (C_1 + D_{12} K C_2) & \gamma I \end{bmatrix} \succ 0$$

Objective:

$$\arg\min_{K} \gamma$$

$\gamma^* \leftarrow \gamma_{opt}$

$K^* \leftarrow K_{opt}$

end

where the constants c_1, c_2, c_3 and $c_4 \in \mathbb{R}$ represent the controller variables. Consider the feedback system in Fig. 2, where $P(s)$ is the plant model, y is the plant output, r and n represent the reference signal and input disturbance, respectively. Measurement noise is not considered in this work. The performance outputs are z_1 and z_2, and constant $\lambda \in \mathbb{R}_+$ is a scaling factor for the performance output of the control input, which has a regularising purpose. In this paper, PID optimisation for mechanical systems is considered. As a start, a mass spring damper plant is considered, and next, a controller is optimised for a double mass spring damper plant.

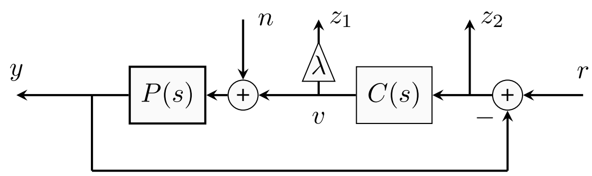

Fig. 2. Feedback System

Theorem III.1 (Mass Spring Damper)**.**

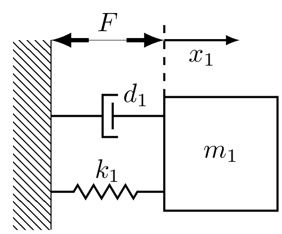

Fig. 3. Mass spring damper system

The feedback system in Fig. 2 with a plant model $P(s)$ in Fig. 3 with $F = v$ and controller $C(s)$ as

$$P(s) = \frac{1}{ms^2 + ds + k}, \tag{16}$$

$$C(s) = \frac{c_1 s^2 + c_2 s + c_3}{s^2 + c_4 s}, \tag{17}$$

is formulated as a SOF synthesis by

$$G(s): \begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \left[\begin{array}{cccc|c|c} -\frac{d}{m} & -\frac{k}{m} & 0 & -\frac{1}{m} & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right] \begin{bmatrix} x \\ w \\ u \end{bmatrix}, \tag{18}$$

with gain matrix $K = \begin{bmatrix} c_1 & c_2 & c_3 & -c_4 \end{bmatrix}$ and generalised input $w = \ddot{r} + \frac{d}{m}\dot{r} + \frac{k}{m} r - \frac{1}{m} n$.

Proof. Consider the plant in Fig. 2 where the plant and controller are defined as

$$P(s) = \frac{1}{ms^2 + ds + k}, \tag{19}$$

$$C(s) = \frac{c_1 s^2 + c_2 s + c_3}{s^2 + c_4 s}. \tag{20}$$

Define the error states as

$$e = r - y, \tag{21}$$

$$\dot{e} = \dot{r} - \dot{y}, \tag{22}$$

$$\ddot{e} = \ddot{r} - \ddot{y}. \tag{23}$$

The controller dynamics are given by

$$\ddot{v} + c_4 \dot{v} = c_1 \ddot{e} + c_2 \dot{e} + c_3 e, \tag{24}$$

and can be integrated once to

$$\dot{v} = c_1 \dot{e} + c_2 e + c_3 e' - c_4 v, \tag{25}$$

where
$codt'$ represents the integrated state. By defining $w = \left(\ddot{r} + \frac{d}{m}\dot{r} + \frac{k}{m}r - \frac{1}{m}n\right)$, the dynamics of the plant can be computed as

$$\ddot{y} = -\frac{d}{m}\dot{y} - \frac{k}{m}y + \frac{1}{m}\left(v + d\right), \tag{26}$$

$$\ddot{r} - \ddot{e} = -\frac{b}{m}(\dot{r} - \dot{e}) - \frac{k}{m}(r - e) + \frac{1}{m}\left(v + d\right), \tag{27}$$

$$\ddot{e} = \left(\ddot{r} + \frac{d}{m}\dot{r} + r - n\right) - \frac{d}{m}\dot{e} - \frac{k}{m}e - \frac{1}{m}v, \tag{28}$$

$$\ddot{e} = w - \frac{d}{m}\dot{e} - \frac{k}{m}e - \frac{1}{m}v. \tag{29}$$

At this point the states can be defined as

$$x = \begin{bmatrix}\dot{e} & e & e' & v\end{bmatrix}^T, \tag{30}$$

and the input $u = \dot{v}$. Looking now at (24) it can clearly be seen that full state feedback is obtained for $u = \begin{bmatrix}c_1 & c_2 & c_3 & -c_4\end{bmatrix} x$. The performance channels z can be constructed in a variety of different ways depending on the objectives. Here the performance channels are chosen to be $z_1 = e$ and $z_2 = \lambda v$, from which the following generalized plant is obtained:

$$G(s): \begin{bmatrix}\ddot{e} \\ \dot{e} \\ e \\ \dot{v} \\ \hline z \\ y\end{bmatrix} = \left[\begin{array}{cccc|c|c} -\frac{d}{m} & -\frac{k}{m} & 0 & -\frac{1}{m} & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right] \begin{bmatrix}\dot{e} \\ e \\ e' \\ v \\ \hline w \\ u\end{bmatrix}. \tag{31}$$

□

Theorem III.1 shows that the generalised plant for PID controller design of a mass spring damper plant model results in full state feedback synthesis. This is not a surprising result since both the controller and plant are second-order systems. However, the result is convenient as the controller variables appear directly. The synthesis can easily be extended to a robust synthesis for uncertain systems and multi-objective synthesis. A common practise is to shape the synthesis with elaborate weighting filters. Including a stable weighting filter would turn the problem into a SOF synthesis. Since state feedback formulation will take care of stabilisation (even for an unstable second-order plant), there will be hopes of getting good synthesis results from the static output synthesis problem, as obtained with the addition of the weighting filter.

In most systems, and especially precision systems, feed-forward is often incorporated into the control scheme. An argument can be made that reference tracking is obsolete, as feed-forward is more effective at minimising the error, and the feedback controller needs to minimise the effect of noise, small disturbances, and model uncertainties [23]. From Theorem III.1 we obtain the generalised input w, which can be separated into w_m(measurable), and w_u (unmeasurable) as

$$w_m = \ddot{r} + \frac{d}{m}\dot{r} + \frac{1}{m}r, \tag{32}$$

$$w_u = -\frac{1}{m}n. \tag{33}$$

The input w_m can be compensated with feed-forward when $\ddot{r}$ and $\dot{r}$ are known in addition to r. This also relies on exact knowledge of the system parameters. The LMI-based synthesis can be generalised if system uncertainty is assumed, which would result in an additional term in w_u. For this paper, exact system knowledge is assumed, and therefore the input will be reduced to the unknown portion w_u. In practise, this is done by assuming set-point tracking by $r = 0$.

Consider the feedback system in Fig. 2 with a plant model as in Fig. 4. Where F is the input of the system and x_2 is the output. This plant model is higher order then the plant in Theorem III.1 which results in a SOF synthesis as can be seen in Theorem III.2

Theorem III.2 (Resonance Mode)**.**
The feedback system in Fig. 2 with a plant model $P(s)$ as in

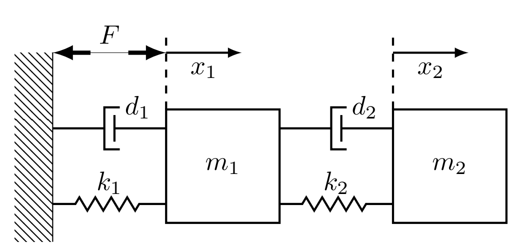

Fig. 4. Double mass spring damper system.

Fig. 4 with $F = v$ and controller $C(s)$

$$P(s) : \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \dot{x}_1 \\ \dot{x}_2 \\ \hline y \end{bmatrix} = \left[\begin{array}{cccc|c} -\frac{(d_1+d_2)}{m_1} & \frac{d_2}{m_1} & -\frac{(k_1+k_2)}{m_1} & \frac{k_2}{m_1} & \frac{1}{m_1} \\ \frac{d_2}{m_2} & -\frac{d_2}{m_2} & \frac{k_1}{m_2} & -\frac{k_2}{m_2} & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \end{array}\right] \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ x_1 \\ x_2 \\ \hline F \end{bmatrix}, \tag{34}$$

$$C(s) = \frac{c_1 s^2 + c_2 s + c_3}{s^2 + c_4 s}, \tag{35}$$

is formulated as a SOF synthesis for standstill optimisation by $G(s)$ as

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{e} \\ \dot{x}_1 \\ \dot{e} \\ e \\ \dot{v} \\ \hline z_1 \\ z_2 \\ \hline y \end{bmatrix} = \left[\begin{array}{cccccc|c|c} -\frac{(d_1+d_2)}{m_1} & -\frac{d_2}{m_1} & -\frac{(k_1+k_2)}{m_1} & -\frac{k_2}{m_1} & 0 & \frac{1}{m_1} & 1 & 0 \\ -\frac{d_2}{m_2} & -\frac{d_2}{m_2} & -\frac{k_1}{m_2} & -\frac{k_2}{m_2} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right] \begin{bmatrix} \dot{x}_1 \\ \dot{e} \\ x_1 \\ e \\ e' \\ v \\ \hline w \\ \hline u \end{bmatrix}, \tag{36}$$

with gain matrix $K = \begin{bmatrix} c_1 & c_2 & c_3 & -c_4 \end{bmatrix}$ and generalised input $w = \frac{1}{m} n$

Proof. Starting from the Equations of Motion (EoM) of the plant model in Fig. 4

$$\begin{aligned} m_1 x_1 &= -(d_1 + d_2)\dot{x}_1 + d_2\dot{x}_2 - (k_1 + k_2)x_1 + k_2 x_2 + F, \\ m_2 x_2 &= d_2\dot{x}_1 - d_2\dot{x}_2 + k_1 x_1 - k_2 x_2. \end{aligned} \tag{37}$$

Define the error states with $r = 0$ as

$$e = -x_2, \tag{38}$$

$$\dot{e} = -\dot{x}_2, \tag{39}$$

$$\ddot{e} = -\ddot{x}_2. \tag{40}$$

Substituting the error states and $F = n + v$ in the EoM

$$\begin{aligned} m_1\ddot{x}_1 &= -(d_1 + d_2)\dot{x}_1 - d_2\dot{e} - (k_1 + k_2)x_1 - k_2 e + n + v, \\ m_2\ddot{e} &= -d_2\dot{x}_1 - d_2\dot{e} - k_1 x_1 - k_2 e. \end{aligned} \tag{41}$$

At this point the states can be defined as

$$x = \begin{bmatrix} \dot{x}_1 & \dot{e} & x_1 & e & e' & v \end{bmatrix}^T, \tag{42}$$

and the input $u = \dot{v}$. The performance channels z can be constructed in a variety of different ways depending on the objectives. Here the performance channels are chosen to be $z_1 = e$ and $z_2 = \lambda v$, from which the following generalized plant is obtained:

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{e} \\ \dot{x}_1 \\ \dot{e} \\ e \\ \dot{v} \\ \hline z_1 \\ z_2 \\ \hline y \end{bmatrix} = \left[\begin{array}{cccccc|c|c} -\frac{(d_1+d_2)}{m_1} & -\frac{d_2}{m_1} & -\frac{(k_1+k_2)}{m_1} & -\frac{k_2}{m_1} & 0 & \frac{1}{m_1} & 1 & 0 \\ -\frac{d_2}{m_2} & -\frac{d_2}{m_2} & -\frac{k_1}{m_2} & -\frac{k_2}{m_2} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right] \begin{bmatrix} \dot{x}_1 \\ \dot{e} \\ x_1 \\ e \\ e' \\ v \\ \hline w \\ \hline u \end{bmatrix}, \tag{43}$$

with gain matrix $K = \begin{bmatrix} c_1 & c_2 & c_3 & -c_4 \end{bmatrix}$ and generalised input $w = \frac{1}{m} n$ □

From Theorem III.2 it can be seen that a SOF synthesis for PID parameter optimisation can be obtained from the EoM without a lot of trouble.

IV. Results

Section II provided an approach to the SOF optimisation problem, and Section III formulated a SOF synthesis for PID parameter tuning. This chapter will provide the optimisation results of both Theorem III.1 and Theorem III.2.

A. *Results of optimisation of the system in Theorem III.1*

Theorem III.1 shows that the optimisation of a PID controller for a mass spring damper system can be formulated as a FSF synthesis. Optimisation of the FSF synthesis can be done directly by LMI optimisation, which would result in the global optimum solution. In theory, the dilated LMI approach presented by Köroğlu and Falcone [21] is not conservative for FSF synthesis optimisation, as shown in Appendix A. As there might be limitations in the implementation, the PID variables are obtained using the LMIs for FSF and the approach presented in Section II. The mass spring damper system used in the optimisations is inspired by the nominal system described by Vogel [24], as $m = 0.2$, $d = 0.74$ and $k = 11.46$. The regularisation parameter is set as $\lambda = 0.1$.

Fig. 5 and Fig. 6 display the line search of the dilated LMIs. It can be seen that the performance is optimal for large ϕ and ψ in the line search. The system norms obtained by the optimisation are shown in Table II, Phase 1 is the performance after the line search of the dilated LMIs, Phase 2 is after the iterative optimisation, and FSF is the norm obtained from using the FSF LMIs directly. The results clearly show that the system norms after Phase 1 are equal to the FSF optimisation. This indicates that the theory in Appendix A holds in practise. Because Phase 1 obtained an optimal result, Phase 2 is not able to improve the performance of the FSF optimisation.

TABLE I
OVERVIEW OF THE SYSTEM NORMS, PHASE 1 INDICATES THE LINE SEARCH, PHASE 2 IS THE ITERATIVE OPTIMISATION, AND FSF IS OPTIMISED WITH FSF LMIS DIRECTLY.

	Phase 1	Phase 2	FSF
$\mathscr{H}_\infty$	0.1000	0.1000	0.1000
$\mathscr{H}_2$	0.1455	0.1455	0.1455

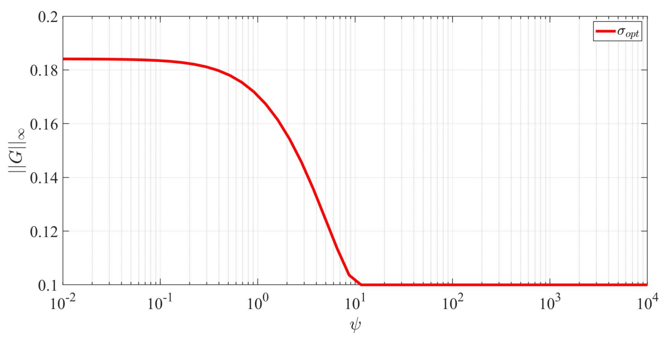

Fig. 5. Found norms by the line-search of the dilated LMIs for a mass spring damper with $\mathscr{H}_\infty$ performance.

Next, the controllers found in Table II are evaluated. The open-loop system $P(s)\cdot C(s)$ with stability margins is evaluated, and the closed-loop system is shown.

1) $\mathscr{H}_\infty$: The PID controllers obtained for $\mathscr{H}_\infty$ optimisation are shown in (44) and (45). The open-loop and closed-loop behaviours can be seen in Fig. 7 and Fig. 8 respectively. It can be seen that the behaviour is different except for the 10 rad/s region, which is the resonance mode of the plant. This is not surprising, as both controllers have the same $\mathscr{H}_\infty$ norm, which is highest around this region.

$$C_{SOF}(s) = \frac{256.1s^2 + 500.5s + 21.23}{s^2 + 219.2s} \tag{44}$$

$$C_{FSF}(s) = \frac{32.14s^2 + 46.95s + 0.5036}{s^2 + 27.37s} \tag{45}$$

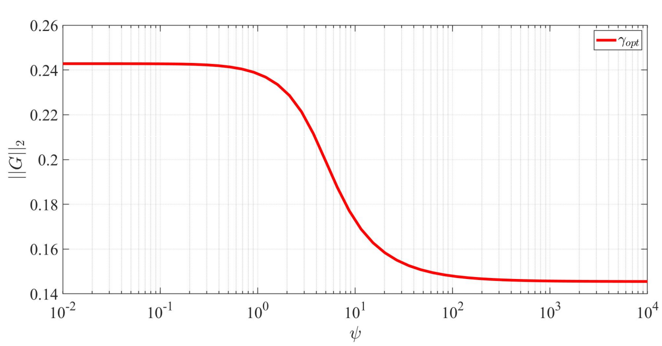

Fig. 6. Found norms by the line-search of the dilated LMIs for a mass spring damper with $\mathscr{H}_2$ performance.

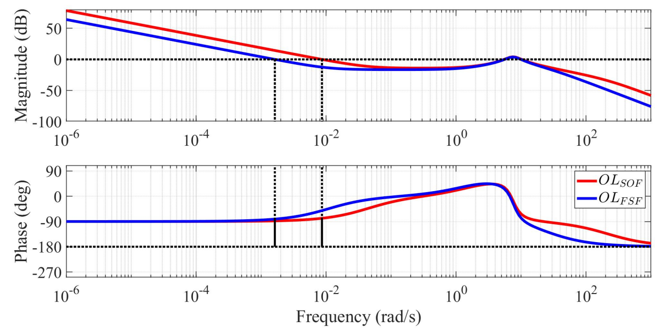

Fig. 7. Open-loop Bode plots of the mass spring damper system with optimised controllers for $\mathscr{H}_\infty$ performance.

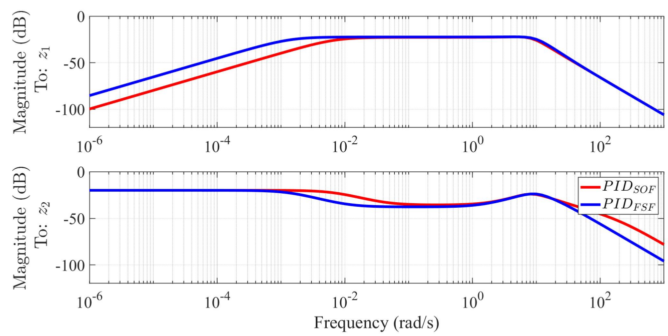

Fig. 8. Closed-loop Bode plots of the mass spring damper system with optimised controllers for $\mathscr{H}_\infty$ performance.

2) $\mathscr{H}_2$: The controllers found for $\mathscr{H}_2$ performance are given in (46) and (47). It can clearly be noticed that the controllers are different, as the SOF optimised controller has a negative value for c_2. This is unusual for conventional PID designs, where all controller variables are positive. The effect can be seen in the open-loop frequency response in Fig. 9. As a result of the phase shift, a gain margin is introduced for the SOF-optimised controller around 0.01 rad/s. It can be seen that the integrator is pushed to very low frequencies and the differentiator is pushed to very high frequencies. Fig. 10 shows the frequency response of the closed-loop system; here it can be seen that the behaviour does not differ a lot as a result of the negative c_2 value.

$$C_{SOF}(s) = \frac{1.399\cdot 10^4 s^2 - 75.55s + 8.824}{s^2 + 1.059\cdot 10^4 s} \tag{46}$$

$$C_{FSF}(s) = \frac{5.102\cdot 10^7 s^2 + 311s + 1.206\cdot 10^4}{s^2 + 3.862\cdot 10^7 s} \tag{47}$$

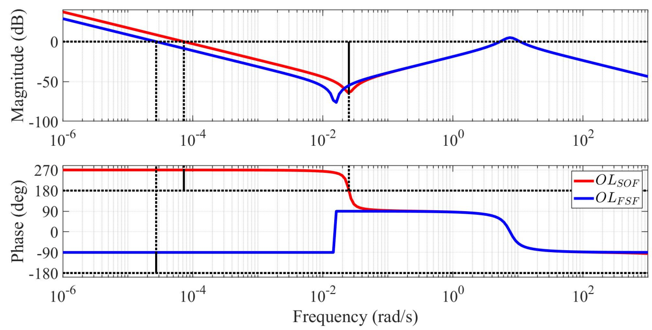

Fig. 9. Open-loop Bode plots of the mass spring damper system with optimised controllers for $\mathscr{H}_2$ performance.

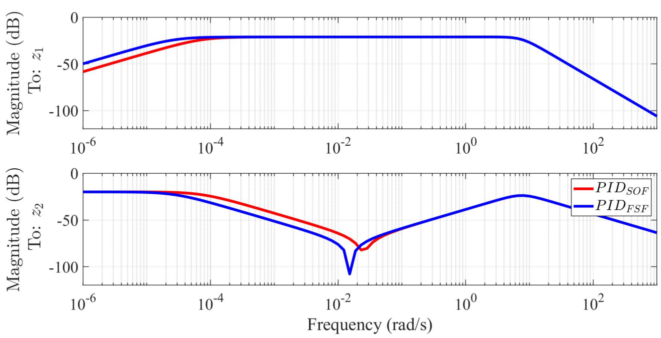

Fig. 10. Closed-loop Bode plots of the mass spring damper system with optimised controllers for $\mathscr{H}_2$ performance.

3) Remarks on the results: In the previous section, it was shown that the optimal PID controllers found are not unique and can vary depending on the optimisation approach. The absence of an unique controller might be because a unique optimal controller does not exist. Another reason can be because of the numerical optimisation performed in MATLAB. However, it can be seen that the behaviour of equal-performing controllers can be different, with properties that can be desirable. This emphasises the necessity of formulating a PID optimisation synthesis that incorporates all desired properties, which can be done by characterising input and output signals with weighting filters and choosing the correct performance measures.

B. Results of optimisation of the system in Theorem III.2

PID optimisation for the plant in Fig. 4 cannot be done with FSF and has to be optimised as a SOF synthesis instead. In this section, the PID parameters are optimised using the SOF optimisation method presented in Section II. For SOF synthesis, the dilated LMIs presented in Köroğlu and Falcone [21] are expected to be conservative. At first, the effectiveness of the optimisation method is analysed, and then the controller results are shown. The plant parameters used are $m_1 = m_2 = 0.2$, $d_1 = d_2 = 0.74$, $k_1 = k_2 = 11.46$ and $\lambda = 0.1$.

1) Optimisation: In this section, the results of the optimisation process are presented. The first step of the optimisation, Phase 1, is the line search over the dilated LMIs as presented in Köroğlu and Falcone [21]. The results can be seen in Fig. 11 and Fig. 12 for $\mathscr{H}_\infty$ and $\mathscr{H}_2$ performance, respectively. For both line searches, it can be seen that a clear optimum exists. The controller variables of this optimum are used as a starting point for the iterative optimisation, the result of which can be seen in Fig. 13 and Fig. 14 for $\mathscr{H}_\infty$ and $\mathscr{H}_2$ performance, respectively. The norm shown are the results after the optimisation of K, defined as Opt 2 in the Algorithm 1.

TABLE II
OVERVIEW OF THE SYSTEM NORMS, PHASE 1 IS AFTER THE LINE SEARCH AND PHASE 2 IS AFTER THE ITERATIVE OPTIMISATION.

	Phase 1	Phase 2
$\mathscr{H}_\infty$	0.0250	0.0227
$\mathscr{H}_2$	0.0352	0.0320

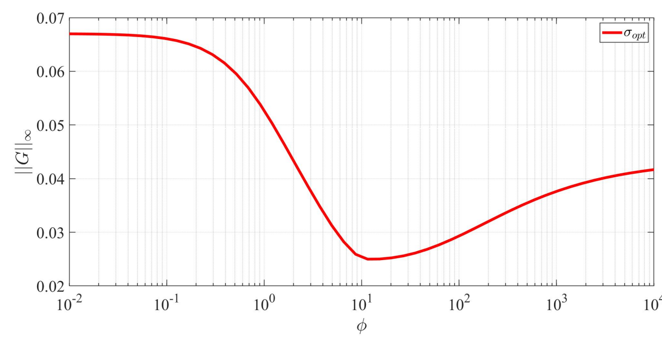

Fig. 11. Found norms by the line-search of the dilated LMIs for a system in Fig. 4 with $\mathscr{H}_\infty$ performance.

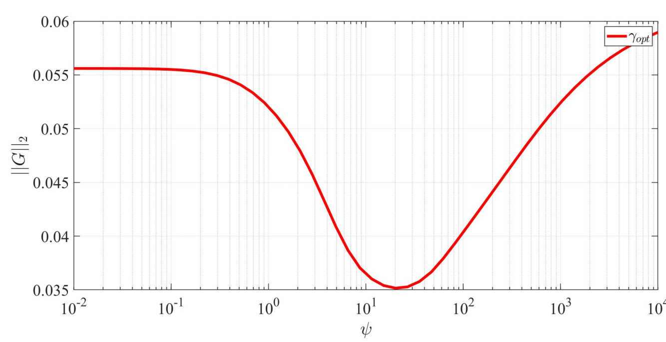

Fig. 12. Found norms by the line-search of the dilated LMIs for a a system in Fig. 4 with $\mathscr{H}_2$ performance.

It can be seen that the iterative optimisation, Phase 2, improves the performance for both $\mathscr{H}_\infty$ and $\mathscr{H}_2$ performance. Performance for $\mathscr{H}_2$ optimises in a single iteration, and $\mathscr{H}_\infty$ requires multiple iterations to find an optimum. Table II shows the norms that are obtained by the optimisations after the different phases. It is seen that both $\mathscr{H}_\infty$ and $\mathscr{H}_2$ improve by approximately 10% as a result of the iterative phase. As mentioned in Section II the performance increase is not guaranteed.

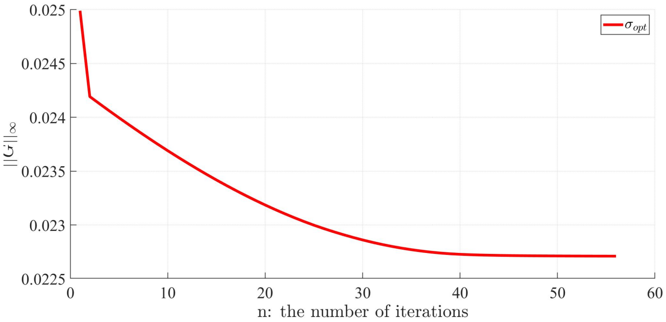

Fig. 13. The system norms as a result of the iterative optimisation after Opt 2, for $\mathscr{H}_\infty$ performance.

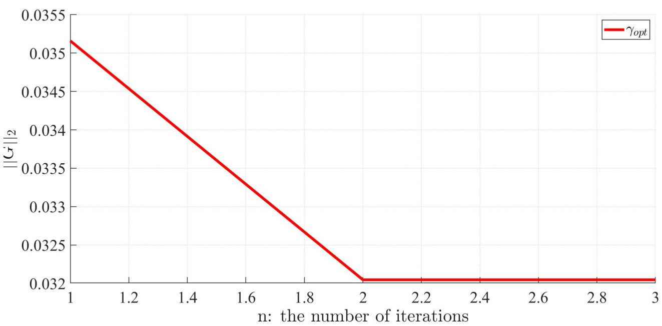

Fig. 14. The system norms as a result of the iterative optimisation after OPT 2, for $\mathscr{H}_2$ performance.

2) PID results: In the previous section, suboptimal tuning parameters for the PID controllers were determined. The obtained controllers are

$$C_{\mathscr{H}_\infty}(s) = \frac{59.1s^2 + 34.08s + 0.02019}{s^2 + 30.79s}, \tag{48}$$

$$C_{\mathscr{H}_2}(s) = \frac{37.4s^2 - 50.56s + 0.0093}{s^2 + 19.13s}. \tag{49}$$

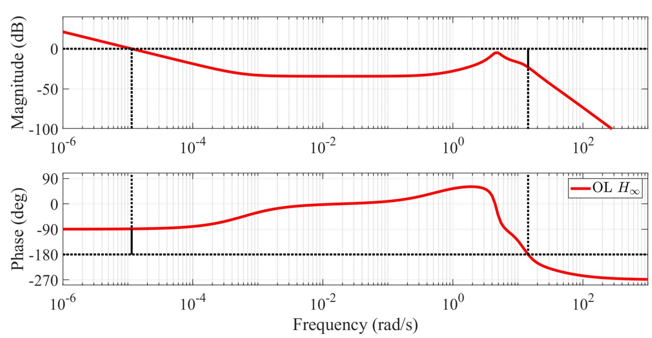

Fig. 15. Open-loop frequency response of PID controller and plant in Fig. 4 optimised for $\mathscr{H}_\infty$ performance.

Again, it can be noted that the c_2 value for $\mathscr{H}_2$ becomes negative, which is different from traditional PID tuning where all parameters are equal. Fig. 15 and Fig. 16 show that both PID controllers have very similar stability margins at similar frequencies. The closed-loop frequency response in Fig. 17 has

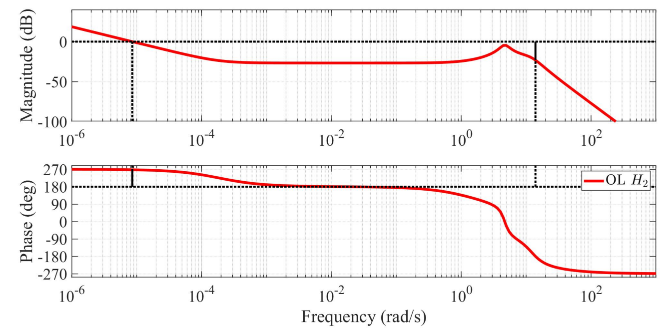

Fig. 16. Open-loop frequency response of PID controller and plant in Fig. 4 optimised for $\mathscr{H}_2$ performance.

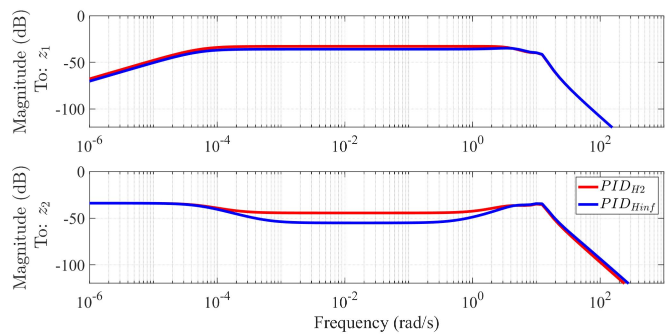

Fig. 17. Closed-loop frequency response of PID controller and plant in Fig. 4.

more similar behaviour between the $\mathscr{H}_\infty$ and $\mathscr{H}_2$ performance objectives than the results for the mass spring damper system in Fig. 8 and Fig. 10.

C. Additional remarks

Optimising the PID tuning parameters for the two plant models was successful, as shown in the previous sections. However, some interesting things were noticed that need some additional remarks. The integrator in all optimised PID controllers was pushed to very low frequencies. This is not surprising as the input n as presented in Fig. 2 has no colorization. As this signal has 0 mean there is no need for the integrator, and it can be pushed away to low frequencies. For real systems, this might result in some problems and emphasise the necessity of correct characterization of the input signals by weighting filters. Using stable weighting filters in the SOF synthesis presented in this work should not impose any problems. The SOF synthesis should also generalise nicely to Multiple-Input Multiple-Output (MIMO) PID optimisation and robust uncertain systems.

V. Conclusion and Recommendations

In this work, PID parameter tuning is formulated as a SOF synthesis problem. For a second-order system, it was shown that the tuning parameters appear directly, which results in a FSF synthesis problem. A higher-order system remains a SOF synthesis problem; however, it shows similarities in structure to the FSF example. It was shown that the PID parameter optimisation was successful for $\mathscr{H}_\infty$ and generalized-$\mathscr{H}_2$ performance objectives. This optimisation is performed with dilated LMI constraints by Köroğlu and Falcone [21] and a successive iterative linearization.

The PID controllers obtained by optimisation are different then controllers that are designed using traditional loop-shaping. This is not surprising, as the plant model is a simplification of a plant that can appear in a real-world example. To find a controller better suited for real-world implementation, it is recommended to more accurately characterise the input signals and performance measures to match an actual system. This can be done by using weighting filters on the input and performance output and uncertainty modelling.

Another recommendation for future work is to compare different LMI optimisation techniques on the SOF synthesis for PID optimisation. Because SOF optimisation is not convex, the solution found is not the optimum. A different approach might be better suited for the PID optimisation synthesis and find a more optimal solution. An overview of approaches is provided in the introduction; however, some notable options are provided in a survey by Sadabadi and Peaucelle [11].

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APPENDIX

A. *Köroğlu and Falcone [21] for FSF*

In this appendix, it is shown that the LMI constraints by Köroğlu and Falcone [21] are not conservative for FSF synthesis. This result is shown for the $\mathscr{H}_\infty$ optimisation LMI. The plant model in (1) is a FSF synthesis when $C_2 = I$ and $D_{21} = 0$. Using the LMI in (8) with $W = Y$ and applying the Schur complement results in

$$\begin{bmatrix} \text{He}\{AY + B_2N\} & B_1 & * \\ * & -\sigma I & * \\ C_1Y + D_{12}N & D_{11} & -\sigma I \end{bmatrix} + \begin{bmatrix} B_2N \\ 0 \\ D_{12}N \end{bmatrix} (\phi\text{He}\{Y\})^{-1} \begin{bmatrix} N^TB_2^T & 0 & N^TD_{12}^T \end{bmatrix} \prec 0, \tag{50}$$

with $Y \in \mathbb{S}$. Because Y is a symmetric matrix and the inverse of a positive definite matrix is also positive definite, the second term can be simplified to

$$\begin{bmatrix} B_2N \\ 0 \\ D_{12}N \end{bmatrix} \frac{1}{2\phi} Y^{-1} \begin{bmatrix} N^TB_2^T & 0 & N^TD_{12}^T \end{bmatrix}. \tag{51}$$

Because Y^{-1} is positive definite the second term of (52) is conservative, except for $\phi = \infty$ for which the second term equals 0. This leaves the LMI

$$\begin{bmatrix} \text{He}\{AY + B_2N\} & B_1 & * \\ * & -\sigma I & * \\ C_1Y + D_{12}N & D_{11} & -\sigma I \end{bmatrix} \prec 0, \tag{52}$$

which is the LMI for FSF with $\mathscr{H}_\infty$ performance and $K = ZY^{-1}$.

Chapter 4

Supplementary Results and Analysis

The paper in Chapter 3 showed that PID tuning parameters could be obtained successfully using the presented SOF synthesis. In this section, some additional results are presented that could not be presented in a coherent way in the paper. In Section 4.1, a comparison is made between the found system norm of the LMI optimisation and the actual system norms. The next section shows the effect of the regularisation variable λ, which was assumed constant in the paper. In the final Section 4.3, PID optimisation for an electromechanical system with tunable first-order actuator dynamics is shown. The optimisation of this new tuning parameter is approached in two different ways.

4.1 System norm comparison

Using LMIs to find an optimal controller is an optimisation problem, which means that the LMI conditions are used while minimising a certain variable. This variable is σ and γ for $\mathscr{H}_\infty$ and $\mathscr{H}_2$ respectively. Recall the inequalities for performance from Section 2.3 as

$$||G||_{H_\infty} < \sigma, \tag{4.1}$$

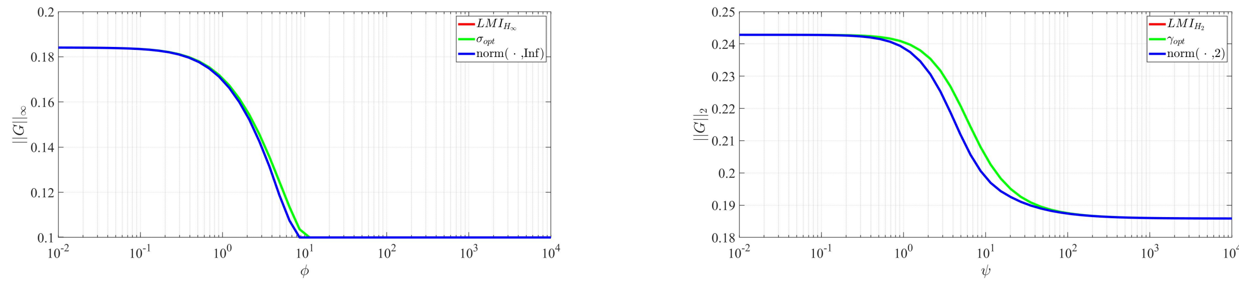
$$||G||_{H_2} < \gamma. \tag{4.2}$$

This clearly shows that the objective variables σ and γ are at most the actual system norm. However, the actual system norm might be lower than the norm found from the LMI optimisation directly. The optimisation procedure presented in Chapter 3 finds the minimum value of the objective variable with a line search and uses this controller as a starting point for the iterative optimisation. However, the true system norm might show that a different controller has better performance.

To test if the minimum of the objective variable is actually the best-performing controller, an investigation is necessary. Both plants presented in Chapter 3 are optimised as shown in Phase 1, using the method presented by Koroglu and Falcone [KF14]. For each value of the line search, the objective variable is determined, the system norm is determined using the performance LMIs, and the norm is determined using the MATLAB function norm(). The results are shown in the next sections. The MATLAB function norm() cannot calculate the generalised-$\mathscr{H}_2$ norm therefore, the $\mathscr{H}_2$ norm is used here instead.

Mass spring damper plant

The results from the line search of the dilated LMI optimisation are seen in Fig. 4.1. The first thing to notice is that the LMI norm and the MATLAB function norm() give the same norm result. This is not surprising, as the LMIs for finding the system norms are directly obtained without using tricks that could potentially introduce conservatism. Fig. 4.1a and Fig. 4.1b show that σ_{opt} and γ_{opt} are conservative for some values of the line search, as the actual norm of the same controller shows better performance. However, it can be seen that for the minima of σ_{opt} and γ_{opt} the norm is equal. This shows that it does not matter if the minimum is determined using σ_{opt} and γ_{opt} or the actual system norm, as both would choose the same controller variables.



(a) $\mathscr{H}_\infty$-norm (b) $\mathscr{H}_2$-norm

Figure 4.1: Norm results from the dilated LMI search for the mass spring damper system in Chapter 3.

Double mass spring damper plant

The double-mass spring damper plant from Chapter 3 resulted in a SOF synthesis, which potentially leads to conservatism. In Fig. 4.2 it is shown that the norm obtained by the performance LMI and the MATLAB function norm() remain equal. Obtaining the system norm uses the closed-loop system; therefore, the LMIs for performance are not influenced by the fact that the system is controlled using SOF and no operations are used that might introduce conservatism. Fig. 4.2a and Fig. 4.2b show that σ_{opt} differs from the actual system norms. This shows similarities to the mass spring damper system, apart from the fact that there is a clear difference in the minimum. Using the controller indicated by the minimal values of σ_{opt} and γ_{opt} results in worse performance than a controller that has been been determined based on the minimum value of the real norm.
Algorithm 1 in Chapter 3 includes an iterative optimisation, as conservatism of the SOF optimisation was expected with the dilated LMI method. To test if determining the controller based on σ_{opt} and γ_{opt} has equal performance to a controller determined on the actual system norm, both are used as initial controllers for the iterative optimisation, the results of which can be seen in Table 4.1. Because Opt 1 in Algorithm 1 is the LMI for determining system performance, the results after Phase 2, the iterative optimisation, can be seen as the true system norm and are not conservative.

	Phase 1	Phase 2
σ_{opt}	0.0250	0.0227
$\|\|G\|\|_\infty$	0.0241	0.0218

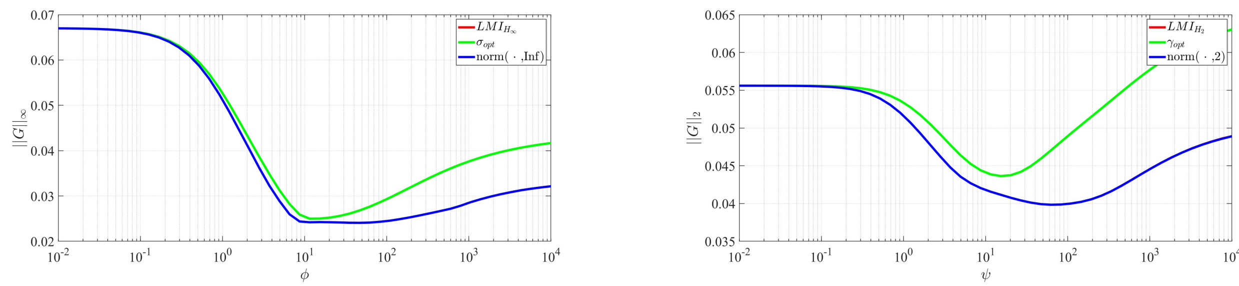
(a) $\mathscr{H}_\infty$

	Phase 1	Phase 2
γ_{opt}	0.0436	0.0376
$\|\|G\|\|_2$	0.0398	0.0351

(b) $\mathscr{H}_2$

Table 4.1: Optimisation results, showing the difference between using σ_{opt} and γ_{opt} and the actual system norm. Phase 1 indicates the norm after the dilated LMI line search, and Phase 2 is the moment after the iterative optimisation. All values represent the actual system norm.

The results in Table 4.1 show a clear pattern: the controller that is determined using the true minimum system norm outperforms the controller that is determined using the σ_{opt} and γ_{opt} values, both in Phase 1 and Phase 2. This result shows to be careful with the norm that is obtained by LMI constraints that are potentially conservative for a line, as the wrong optimum might be chosen. However, this also shows a simple improvement for the Algorithm used in Chapter 3.



(a) $\mathscr{H}_\infty$-norm (b) $\mathscr{H}_2$-norm

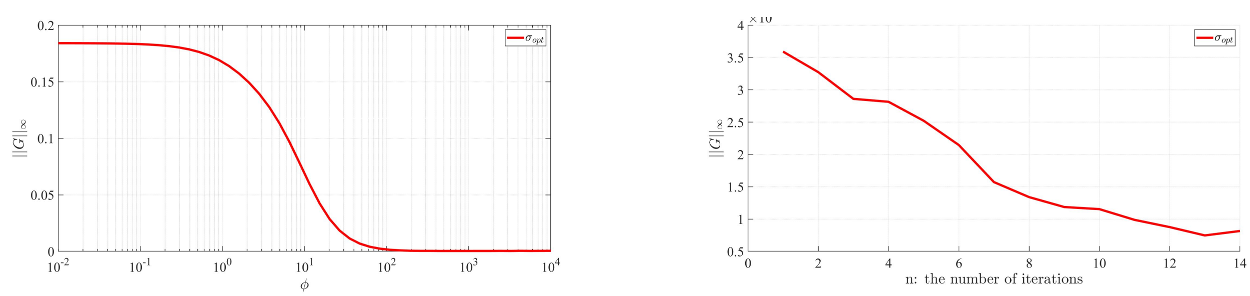
Figure 4.2: Norm results from the dilated LMI search for the double mass spring damper system in Chapter 3.

4.2 Input regularisation

The SOF synthesis presented in Chapter 3 uses two performance outputs, z_1 is the error, and z_2 is a regularisation output on the control input. If z_2 is not used, the control input could go to infinity if the closed loop stability allows for it. For a mass spring damper system, the gain margin lies at infinity. As a result, there might not exist an optimal controller, as the control parameters can be infinitely high. When a resonance mode is introduced in the double mass spring damper system, a gain margin exists, which naturally limits the controller gain, and there might be no need for a regularisation parameter in order to find a solution. However, one might choose to keep this performance output to obtain the desired behaviour.
The effect of the regularisation parameter will be shown in the following two sections: In the first section, the effect of no controller input regularisation is shown for the mass spring damper system and double mass spring damper system. In the second section the effect of different values of the regularisation variable are shown for the mass spring damper system.

The effect of $\lambda = 0$

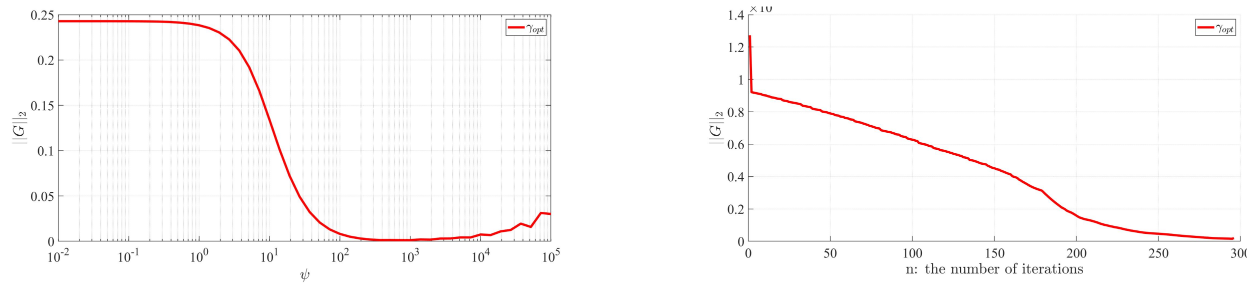
Consider the SOF synthesis presented in Theorem III.1 and Theorem III.2 from Chapter 3 with $\lambda = 0$. Both synthesises are optimised using the optimisation procedure as presented in Algorithm I of the same chapter. First, the results of the mass spring damper system are shown, followed by the results of the double mass spring damper system.



(a) Line search results (b) Iterative optimisation results

Figure 4.3: Results of the PID optimisation for the mass spring damper system with $\mathscr{H}_\infty$ performance.

Fig. 4.3 and Fig. 4.4 show that the line search finds large values of ϕ and ψ to be optimal, similar to the results in Chapter 3. This is as expected, as the mass spring damper synthesis is FSF, as shown in the chapter. However, a difference is that the controller obtained by the line search can be improved upon with iterative optimisation, as shown in Fig. 4.3b and Fig. 4.4b. Because the controller can be improved, this indicates that there is no optimum and that the norm could be improved indefinitely. However, this controller

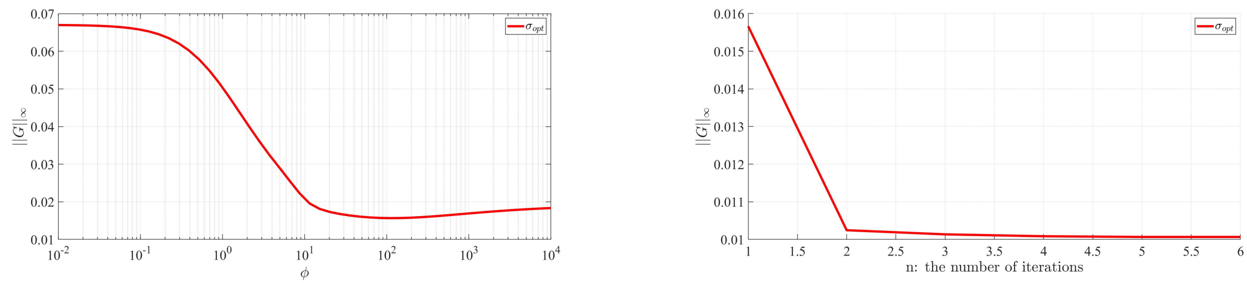


(a) Line search results

(b) Iterative optimisation results

Figure 4.4: Results of the PID optimisation for the mass spring damper system with $\mathscr{H}_2$. performance.

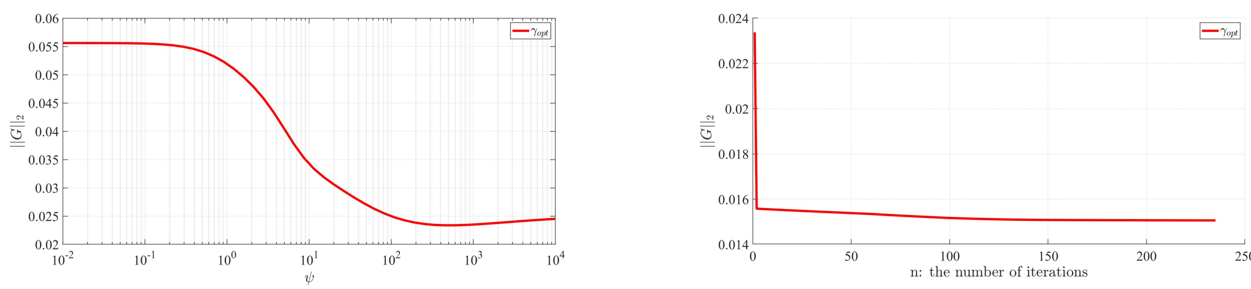
cannot be obtained in practise due to the LMI solvers in MATLAB; as values become very large and very small, the numerical optimisation runs into trouble. A clear indication of these problems can be seen in Fig. 4.4a, for high values ψ the norm increases and fluctuates. In Chapter 3 it was shown that for the mass spring damper, $\psi = \infty$ would result in a solution that is not conservative; therefore, this increasing norm is not in line with the theory. Appendix A discusses this effect in more detail.



(a) Line search results.

(b) Iterative optimisation results

Figure 4.5: Results of the PID optimisation for the double mass spring damper system with $\mathscr{H}_\infty$ performance.



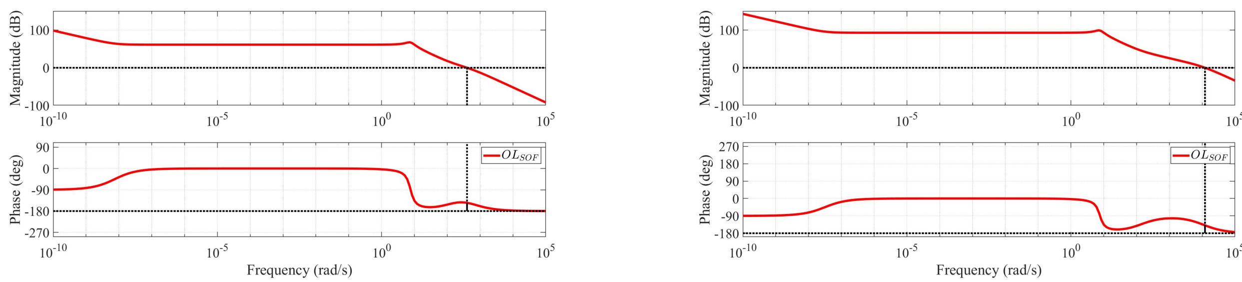
(a) Line search results

(b) Iterative optimisation results

Figure 4.6: Results of the PID optimisation for the double mass spring damper system with $\mathscr{H}_2$ performance.

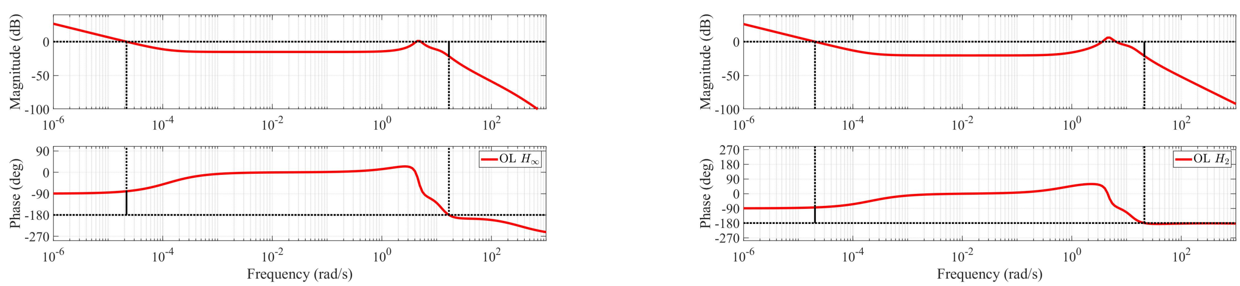
The optimisation results for the double mass spring damper system are shown in Fig. 4.5 and Fig. 4.6. In the results of the line search Fig. 4.5a and 4.6a it can be seen that an optimal controller exists and is not lost as a result of $\lambda = 0$. This indicates that an optimal controller that is not infinitely fast exists.
To confirm the statement that the gains would be pushed very high for the mass spring damper system while remaining similar for the double mass spring damper system, the open-loop characteristics are shown. Fig. 4.7a and Fig. 4.7b show the open-loop characteristics for the mass spring damper system, and Fig. 4.8a

and Fig. 4.8b shows the open-loop characteristics for the double mass spring damper system. Comparing the results with the controllers obtained in Chapter 3, it can be seen that the open-loop gain of the mass spring damper is much higher when $\lambda = 0$, and the double mass spring damper has a similar open-loop characteristic, as expected.



(a) $\mathscr{H}_\infty$ (b) $\mathscr{H}_2$

Figure 4.7: Open-loop characteristic of the optimised PID controller for the mass spring damper system.

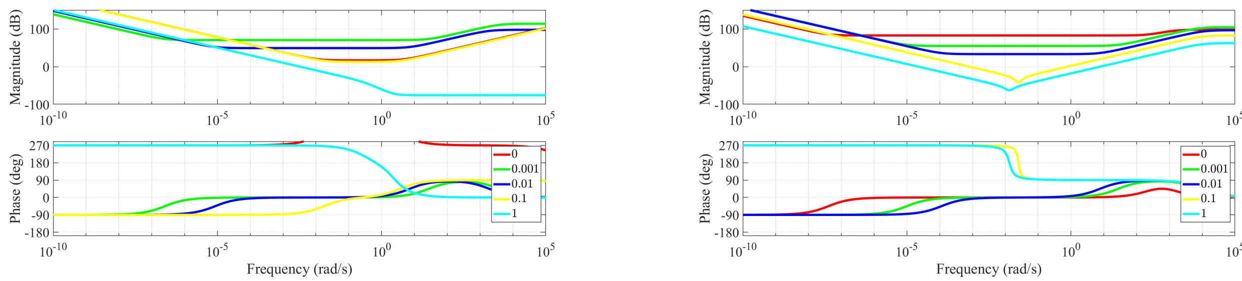


(a) $\mathscr{H}_\infty$ (b) $\mathscr{H}_2$

Figure 4.8: Open-loop characteristic of the optimised PID controller for the double mass spring damper system.

The effect of λ

The regularisation parameter λ has an effect on the controller obtained by the optimisation. The open-loop characteristic of the mass spring damper system for different values of λ is shown in Fig. 4.9. It can be seen that the open-loop characteristic changes a lot when λ becomes larger. It can be seen that the negative value of c_2, as seen in Chapter 3, is a result of a high weighting on the control effort. This clearly shows that a correct choice of λ is necessary if regularisation of the control input is necessary. As shown before, when the plant has a resonance mode, which limits the gain of the controller, input regularisation might not be necessary. In that situation, a good characterization of the input and output is most important, and a natural limitation on the controller gains exists as a result of the stability margins.



(a) $\mathscr{H}_\infty$ (b) $\mathscr{H}_2$

Figure 4.9: Open-loop characteristic of the optimised PID controller for the double mass spring damper system.

4.3 Electromechanical system

The plant models considered in Chapter 3 are two mechanical systems. In this section, the mass spring damper plant is considered with a current controller. The current controller has tunable first-order dynamics and is inspired by the work of Seinhorst [Sei]. As the dynamics of the actuator can be tuned, a new variable is introduced in the feedback system. Tuning this parameter can be done in two different ways: The parameter can be set as a constant over which a line search is performed, or the parameter can be tuned simultaneously with the PID parameters. Both methods are presented in Theorem 4.3.1 and 4.3.2 respectively.

Theorem 4.3.1 (First order actuator SOF)**.**

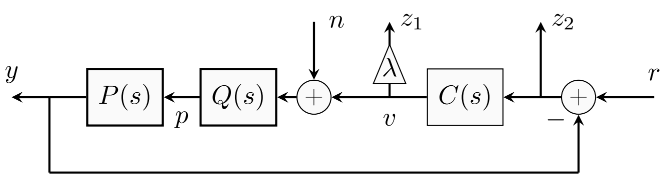


Figure 4.10: Feed-back system with actuator dynamics

The feedback system in Fig. 4.10 with a plant model $P(s)$, controller $C(s)$ and actuator dynamics $Q(s)$

$$P(s) = \frac{1}{ms^2 + ds + k} \tag{4.3}$$

$$C(s) = \frac{c_1 s^2 + c_2 s + c_3}{s^2 + c_4 s} \tag{4.4}$$

$$Q(s) = \frac{1}{s + c_5} \tag{4.5}$$

is formulated as a SOF synthesis for standstill optimisation by

$$G(s): \begin{bmatrix} \dot{x} \\ \hline z_1 \\ z_2 \\ \hdashline y \end{bmatrix} = \left[\begin{array}{ccccc|c:c} -\frac{d}{m} & -\frac{k}{m} & 0 & 0 & -\frac{1}{m} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{1}{c_5} & -\frac{1}{c_5} & \frac{1}{c_5} & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\ \hdashline 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} x \\ \hline w \\ \hdashline u \end{bmatrix}, \tag{4.6}$$

with gain matrix $K = \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix}$, generalised input $w = -\frac{1}{m}n$ and a constant c_5.

Theorem 4.3.2 (First order actuator SSOF)**.** The feedback system in Fig. 4.10 with a plant model $P(s)$, controller $C(s)$ and actuator dynamics $Q(s)$

$$P(s) = \frac{1}{ms^2 + ds + k} \tag{4.7}$$

$$C(s) = \frac{c_1 s^2 + c_2 s + c_3}{s^2 + c_4 s} \tag{4.8}$$

$$Q(s) = \frac{1}{s + c_5} \tag{4.9}$$

is formulated as a SSOF synthesis for standstill optimisation by

$$G(s) : \begin{bmatrix} \dot{x} \\ \hline z_1 \\ z_2 \\ \hdashline y \end{bmatrix} = \left[\begin{array}{ccccc|c:cc} -\frac{d}{m} & -\frac{k}{m} & 0 & 0 & -\frac{1}{m} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\ \hdashline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 \end{array}\right] \begin{bmatrix} x \\ \hline w \\ \hdashline u \end{bmatrix} \tag{4.10}$$

with block-diagonal controller

$$K = \begin{bmatrix} c_1 & c_2 & c_3 & c_4 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{c_5} \end{bmatrix}, \tag{4.11}$$

and generalised input $w = -\frac{1}{m}n$. This is a Structured Static Output Feedback (SSOF) synthesis and can be obtained by enforcing structure Matrices in the LMI optimisation.

Proof. Theorem 4.3.1 and Theorem 4.3.2 are proven simultaneously. Consider the plant in Fig. 4.10 where $r = 0$, and the plant, controller and current controller are defined as

$$P(s) = \frac{1}{ms^2 + ds + k}, \tag{4.12}$$

$$C(s) = \frac{c_1 s^2 + c_2 s + c_3}{s^2 + c_4 s}, \tag{4.13}$$

$$Q(s) = \frac{1}{c_5 s + 1}. \tag{4.14}$$

Now a distinction is made between v which is the output of the PID controller, and p which is the output of the current controller. Define the error states as

$$e = -y, \tag{4.15}$$

$$\dot{e} = -\dot{y}, \tag{4.16}$$

$$\ddot{e} = -\ddot{y}. \tag{4.17}$$

The dynamics of the plant in therms of the error states expressed as

$$\ddot{y} = -\frac{d}{m}\dot{y} - \frac{k}{m}y + \frac{1}{m}p, \tag{4.18}$$

$$-\ddot{e} = \frac{b}{m}\dot{e} + \frac{k}{m}e + \frac{1}{m}p, \tag{4.19}$$

$$\ddot{e} = -\frac{d}{m}\dot{e} - \frac{k}{m}e - \frac{1}{m}p. \tag{4.20}$$

The actuator dynamics are expressed as

$$c_5\dot{p} + p = v + n, \tag{4.21}$$

$$\dot{p} = -\frac{1}{c_5}p + \frac{1}{c_5}v + \frac{1}{c_5}n. \tag{4.22}$$

Then the dynamics of the PID controller are defined as

$$\dot{v} = -c_4 v + c_1\dot{e} + c_2 e + c_3 e'. \tag{4.23}$$

Defining the state vector x as

$$x = \begin{bmatrix} \dot{e} & e & e' & v & p \end{bmatrix}^T. \tag{4.24}$$

The generalized plant can be formulated with the following definitions, $u = \dot{v}$, $z_1 = e$ and $z_2 = \lambda w$. If c_5 is assumed to be constant the generalised plant of Theorem 4.3.1 is given by:

$$G(s) : \begin{bmatrix} \ddot{e} \\ \dot{e} \\ e \\ \dot{v} \\ \dot{p} \\ \hline z_1 \\ z_2 \\ \hdashline y \end{bmatrix} = \left[\begin{array}{ccccc|c:c} -\frac{d}{m} & -\frac{k}{m} & 0 & 0 & -\frac{1}{m} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{1}{c_5} & -\frac{1}{c_5} & \frac{1}{c_5} & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\ \hdashline 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} \dot{e} \\ e \\ e' \\ v \\ p \\ \hline w \\ \hdashline u \end{bmatrix}, \tag{4.25}$$

with controller $K = \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix}$.

When c_5 is assumed a to-be-found controller parameter the generalised plant of Theorem 4.3.2 is found as

$$G(s) : \begin{bmatrix} \ddot{e} \\ \dot{e} \\ e \\ \dot{v} \\ \dot{p} \\ \hline z_1 \\ z_2 \\ \hdashline y \end{bmatrix} = \left[\begin{array}{ccccc|c:cc} -\frac{d}{m} & -\frac{k}{m} & 0 & 0 & -\frac{1}{m} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\ \hdashline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 \end{array}\right] \begin{bmatrix} \dot{e} \\ e \\ e' \\ v \\ p \\ \hline w \\ \hdashline u \end{bmatrix} \tag{4.26}$$

with a block diagonal gain matrix K is defined as

$$K = \begin{bmatrix} c_1 & c_2 & c_3 & c_4 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{c_5} \end{bmatrix} \tag{4.27}$$

□

Theorem 4.3.1 and Theorem 4.3.2 provide two variations to find the PID controller parameters for a mass spring damper system with tunable actuator dynamics. When all parameters are optimised simultaneously, the SOF synthesis changes to a SSOF synthesis. The block diagonal structure of the controller can be imposed by defining structure in the design matrices of the LMIs. For the dilated LMI presented in Chapter 3 the matrices become

$$N = \begin{bmatrix} x & x & x & x & 0 \\ 0 & 0 & 0 & 0 & x \end{bmatrix}, \tag{4.28}$$

$$W = \begin{bmatrix} x & 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 \\ 0 & 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 & x \end{bmatrix}, \tag{4.29}$$

$$Y \in \mathbb{S}^5_+, \tag{4.30}$$

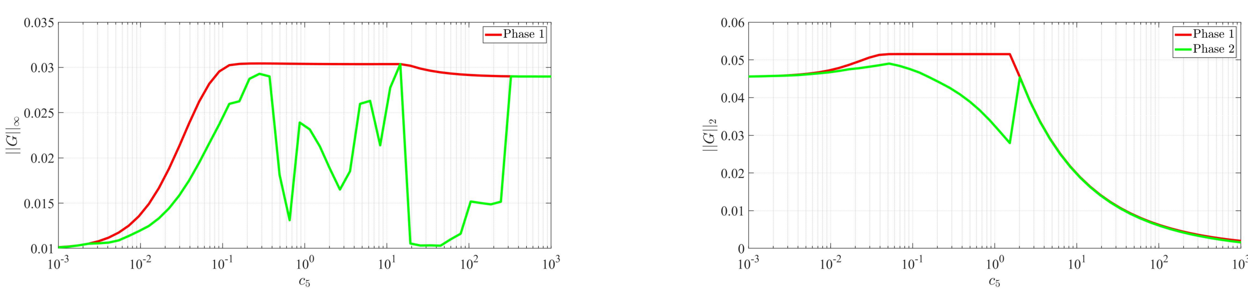
where x represents variables that can be optimised in the LMI optimisation and 0 is fixed. The matrix W is a full matrix for SOF; thus, defining the controller stricter for SSOF comes at the cost of the design freedom of this matrix, which might lead to more conservatism.

Results

The SOF and SSOF synthesis from Theorems 4.3.1 and 4.3.2 are optimised using Algorithm 1, as presented in Chapter 3. The controller will be optimised for $\mathscr{H}_\infty$ and a generalised $\mathscr{H}_2$ performance objective. The plant parameters are set as $m = 0.2$, $d = 0.74$ and $k = 11.46$ inspired by [Vog19]. From Section 4.2 it was shown that using a larger λ resulted in unexpected controllers; therefore, the control input regularisation will be set as $\lambda = 0.01$. This should result in a limitation on the control input while not weighing it too heavily.

Optimisation results of Theorem 4.3.1

This section shows the results of optimising the SOF synthesis in Theorem 4.3.1. The optimal controllers are computed with a line search over the new variable c_5 which characterises the actuator dynamics. The results of the line search can be seen in Fig. 4.11a and Fig. 4.11b. First, the results of $\mathscr{H}_2$ are discussed in detail, and next, the results of $\mathscr{H}_\infty$



(a) $\mathscr{H}_\infty$ (b) $\mathscr{H}_2$

Figure 4.11: line search results over c_5.

The results for $\mathscr{H}_2$ in Fig. 4.11b show three distinct regions in the line search: The variable $c_5 \lesssim 0.01$, $0.01 \lesssim c_5 \lesssim 2$ and $2 \lesssim c_5$. For all values of c_5 except for $0.01 \lesssim c_5 \lesssim 2$, the norms of Phase 1 and Phase 2 are very similar. Each of these regions is investigated in more detail below.

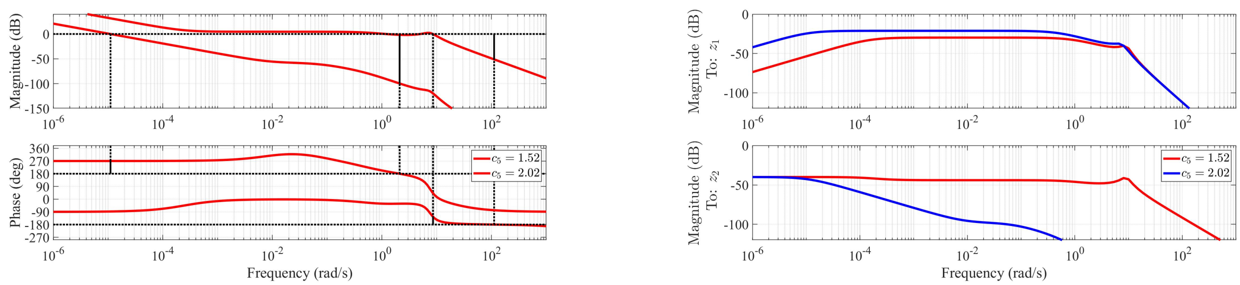
$c_5 \lesssim 0.01$

The actuator dynamics describe a low-pass filter with a cross-over frequency $\omega_c = \frac{1}{c_5}$. As c_5 becomes very small, the cross-over frequency is pushed to very high frequencies, and everything lower than the cross-over frequency behaves as if it were not there. As a result, for very small values of c_5 the plant model behaves as a normal mass spring damper system. The minimal norm found for small values of c_5 is indeed equal to the norm obtained from optimisation of the synthesis in Theorem III.1 in Chapter 3 [1].

[1] note that $\lambda = 0.01$ is not optimised in the Chapter 3

$0.01 \lesssim c_5 \lesssim 2$

In this region, a clear improvement by the iterative optimisation can be seen as c_5 becomes larger. However, at $c_5 \approx 2$ the performance drops and Phase 1 and Phase 2 become equal again. Fig. 4.12a shows the open-loop characteristics of $c = 1.52$ and $c = 2.02$ and Fig. 4.12b shows the closed-loop transfer. It can be seen that the PID controller optimised for $c_5 = 2.02$ gives a very different open-loop characteristic. In the closed-loop, it can be seen that this is largely to reduce the transfer to the z_2 output, which is the control input regularisation.



(a) Open-loop

(b) closed-loop

Figure 4.12: frequency response of the electromechanical system for two different values of c_5.

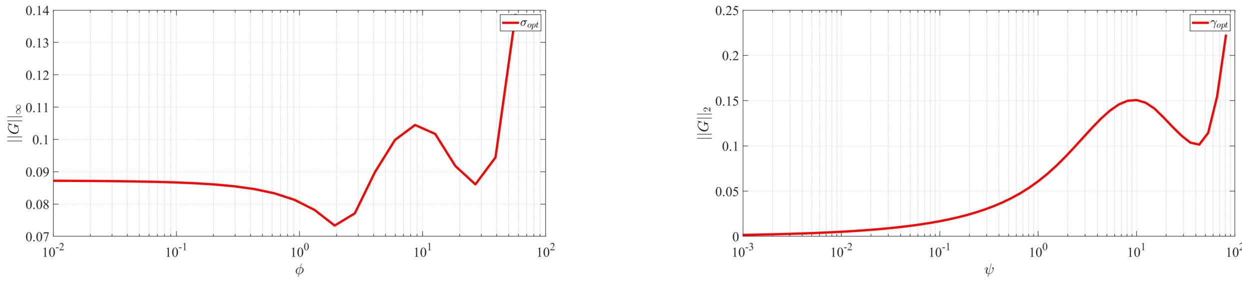
$2 \lesssim c_5$

As the value of c_5 becomes very large, the cross-over frequency $\omega_c = \frac{1}{c_5}$ of the actuator goes to very low frequencies. As a result, the gain of the open loop is suppressed at all frequencies. As c_5 becomes larger, the effect seen in Fig. 4.12a becomes more extreme.

Next, the line search for $\mathscr{H}_\infty$ performance is reviewed. Although the results of Phase 1 look like a mirror image of the results for $\mathscr{H}_2$, the results should be interpreted in a different way. On the left, where c_5 is very small, the system behaves as the mass spring damper system presented in Theorem III.1 in Chapter 3. As c_5 becomes larger, the effect of the low-pass filter is influencing the open-loop characteristic. However, from the results of Phase 1, it can be seen that better performance cannot be obtained as c_5 increases. The results of Phase 2 are very unpredictable and appear to be very noisy. Appendix A goes into more detail on the issues with the iterative optimisation. The optimal controller obtained in the optimisation is shown in the section: Comparison of the obtained controllers.

Optimisation results of Theorem 4.3.2

Theorem 4.3.2 shows the possibility of simultaneously obtaining all the design variables using the SSOF synthesis. The optimisation results can be seen in Fig. 4.13a and 4.13b for the $\mathscr{H}_\infty$ and $\mathscr{H}_2$ performance objectives, respectively. With the introduction of fixed structure in the controller, it can be seen that the line search, as presented in Koroglu and Falcone [KF14], has local minima and maxima, which were not seen before. The globally optimal controller will be shown in the next section.



(a) $\mathscr{H}_\infty$ (b) $\mathscr{H}_2$

Figure 4.13: Line search for optimal SSOF controllers.

Comparison of the obtained controllers

In this section, the controllers obtained by the different optimisation approaches are presented. The PID controllers and current controller for $\mathscr{H}_\infty$ are

$$PID_{SOF}(s) = \frac{6.08 \cdot 10^5 s^2 + 3.52 \cdot 10^6 + 5.596}{s^2 + 2.57 \cdot 10^4 s} \qquad Q_{SOF}(s) = \frac{1}{1 \cdot 10^{-3}s + 1}, \tag{4.31}$$

$$PID_{SSOF}(s) = \frac{7.961s^2 + 27.21s + 1.284 \cdot 10^{-4}}{s^2 + 2.315s} \qquad Q_{SSOF}(s) = \frac{1}{0.8032s + 1}. \tag{4.32}$$

The PID controllers and current controller for $\mathscr{H}_2$ are

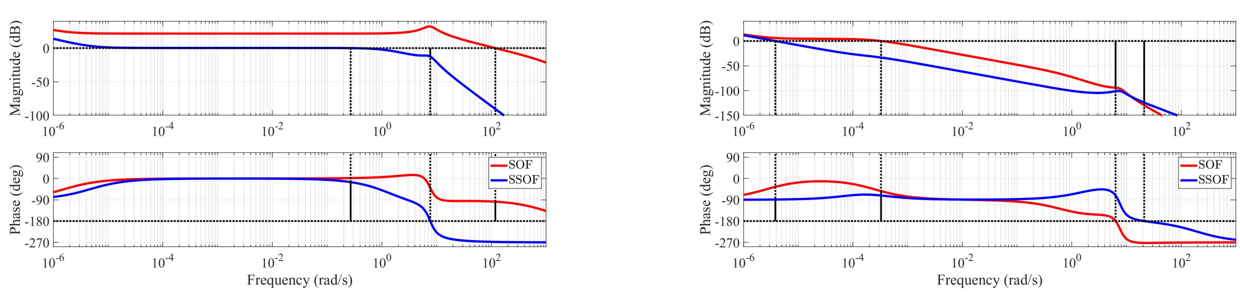
$$PID_{SOF}(s) = \frac{2.21s^2 + 12.96s + 3.386 \cdot 10^{-5}}{s^2 + 0.6834s} \qquad Q_{SOF}(s) = \frac{1}{4044s + 1}, \tag{4.33}$$

$$PID_{SSOF}(s) = \frac{39.05s^2 + 75.32s + 0.008475}{s^2 + 193.9s} \qquad Q_{SSOF}(s) = \frac{1}{4044s + 1}. \tag{4.34}$$

In Fig. 4.11b it was seen that performance should improve as c_5 becomes larger. Both optimisation approaches can find a better $\mathscr{H}_2$-norm by increasing their respective line search ranges. Therefore, c_5 in the SOF optimisation is set to the same value as found in the SSOF optimisation. The final norm of the optimised systems can be seen in Table 4.2. It can be seen that controllers found by SOF outperform the SSOF, which is as expected. Fig. 4.14 and Fig. 4.15 show the open-loop characteristics and closed-loop responses of the found controllers.

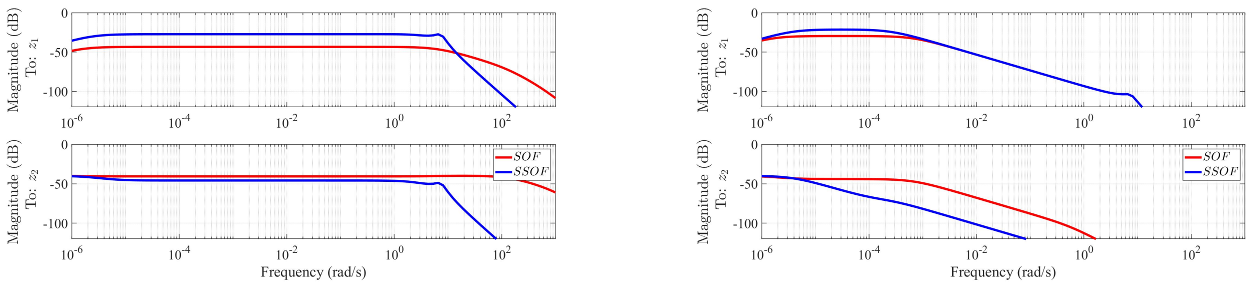
	SOF	SSOF
$\mathscr{H}_\infty$	0.0111	0.0434
$\mathscr{H}_2$	$6.0887 \cdot 10^{-4}$	$9.55 \cdot 10^{-4}$

Table 4.2: System norms for the electromechanical system.



(a) $\mathscr{H}_\infty$ (b) $\mathscr{H}_2$

Figure 4.14: Open-loop characteristic of the optimised controllers.



(a) $\mathscr{H}_\infty$ (b) $\mathscr{H}_2$

Figure 4.15: Closed-loop response of the optimised controller.

The controller results show, similar to the results in Chapter 3, that the integrator is pushed to the very low frequency region. The results show that the choice of the performance objective results in a very different controller. The line search with SOF finds the best controller at the cost of performing multiple LMI optimisations.

Chapter 5

Conclusions and Recommendations

In this thesis, PID controller parameters are successfully optimised for two mechanical systems in Chapter 4.3 and an electromechanical system in Section 4.3. This chapter provides concluding remarks and recommendations for future work.

5.1 Conclusions

The SOF formulation for a second order system in Theorem III.1 has a convenient structure as the design parameters appear directly. This synthesis is well suited to add features that make LMIs interesting, such as multi-objective synthesis, robust synthesis for uncertain systems, and gain scheduling. The presented SOF synthesis is also appropriate to be extended with input and output characterization in the form of weighting filters. In the results, it is shown that a system with a resonance mode or an added low-pass filter is optimised successfully.
Section 4.3 presented results on the optimisation of an additional tunable variable. Two approaches to optimising the system are presented: the first was using a line search with SOF, and the second was using a structured controller to optimise the additional variables simultaneously. Both methods were successful in obtaining controller parameters, although the controller optimised with SSOF synthesis was outperformed by the controller obtained with SOF synthesis.
Section 4.2 showed the effect of input regularisation. For systems where there is no limitation on the controller gains, input regularisation is necessary. However, it is also shown that the controller behaviour can be influenced by the input regularisation until the point where the behaviour is very different. Using a plant model where the controller gains are limited, for example, the system in Theorem III.2 of Chapter 3, input regularisation is not necessary to find an optimum.
Though the results looked promising, Section 4.1 and Appendix A also show the pitfalls of using LMIs for SOF optimisation. Both the conservatism that is imposed by reformulating the BMIs for SOF into LMIs and the issues that were encountered by the MATLAB implementation have pitfalls that can hinder the optimisation. A partial solution for conservatism is presented in Section 4.1; however, the optimisation problems in MATLAB remain unsolved.

5.2 Recommendations

The recommendations for future work consist of two main topics, the first of which is to improve the SOF synthesis. The plant models used in this work are a starting point for future work, but they are not yet ready to optimise controllers that can be implemented in an experimental setup. The next step would be to define an experimental set-up and define the SOF synthesis to accurately depict the set-up. Using weighting filters to characterise the input and output signals and modelling at least the lowest-order mode should lead to a synthesis that can obtain an implementable optimal controller. Further extensions would include robust synthesis and multi-objective synthesis.

The second topic for future research would be improving the optimisation of SOF synthesis. In this research, the approach presented by Köroğlu and Falcone [KF14] is used. However, as SOF synthesis leads to (sub-)optimal controllers, a different method might be more effective for the presented synthesis. Inspiration for different optimisation approaches can be obtained from Sadabadi and Peaucelle [SP16]. Another point of interest would be to pinpoint where and why the MATLAB implementation fails sometimes. This could help find a solution for these issues or a way to circumvent them.

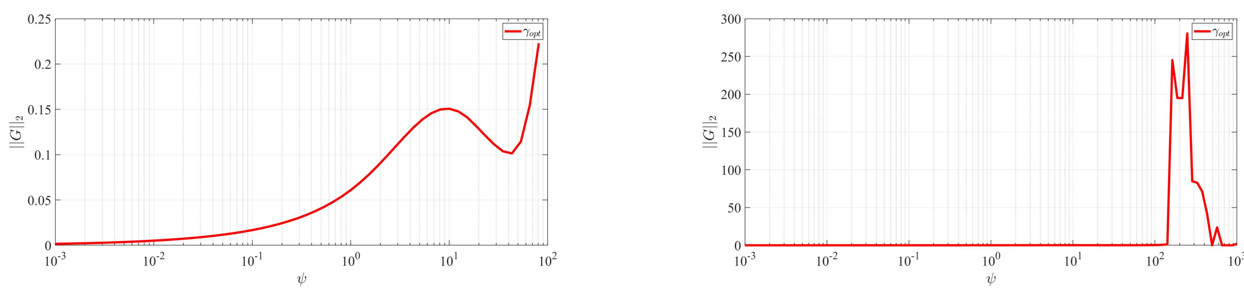
Appendices

A Notions On Implementation

During the implementation of Algorithm 1 presented in Chapter 3 some difficulties were encountered. In this section, these problems are discussed, the implemented solution is presented, and if applicable, recommendations for further improvements are provided.

A.1 Line Search

The optimal value of ψ cannot be predicted prior to the line search, so defining the set Ψ is therefore not trivial. To find an initial value, a wide range on the logarithmic scale can be defined, followed by a finer search around the initially determined optimum. However, for large values of ψ the optimisation runs into numerical issues with the solver, resulting in unrealistic values of the found system norm. The definition of large is ambiguous and depends on the generalised plant definition. Using a safety margin is not possible either, as the optimal value can be close to the point where no solution is found. An example is given in Fig. 1b.



(a) With Stop after large change

(b) Without Stop after large change

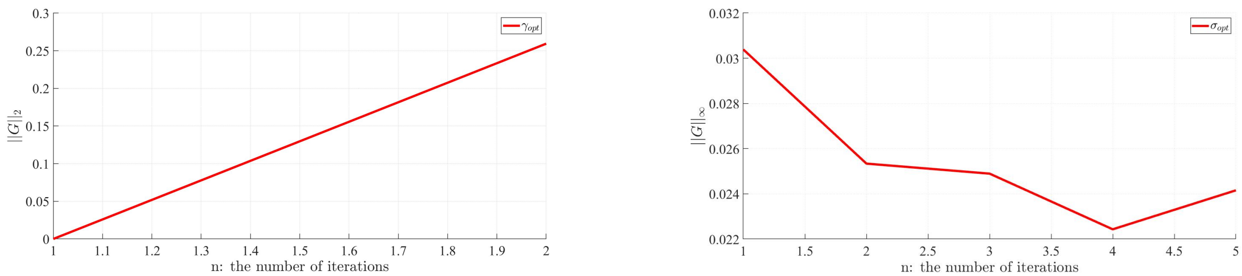
Figure 1: Line search example with numerical problems.

Due to the numerical issues, the choice of the controller with minimal norm cannot be done by simply taking the minimum for all values of $\psi \in \Psi$. The easiest method is to plot the norm against the values of ψ and identify the minimum by visual inspection. However, an automated process might be desired. It was discovered that multiple local minima might exist, so the line search cannot simply be terminated when the norm increases (when ψ goes from low to high). However, it can be noted that the norm underwent rapid change prior to the issues. As a result, the implementation in this paper is to terminate the line-search when the system norm changes more than for example 30% in a single step of ψ, the value should depend on the choice of line search. Fig. 1 shows the difference in line search with and without the stop.

The implemented method is a very rough solution to the problem. The solution works sufficiently for the systems that are analysed in this paper but might run into trouble in other cases. It is recommended to find a better solution or to compare the results to a visual inspection of the graphs.

A.2 Increasing γ and σ

In Chapter 3 it was stated that in theory, the system norm cannot increase as a result of the iterative optimisation. With each linearization step, the previous solution exists in the feasible set, and it is either the best solution or a better one exists. However, in practise, this optimisation runs into trouble as, at some iterations, the norm increases. The increasing norm cannot be predicted, as sometimes the norm increases with the first iteration and sometimes it takes multiple iterations to increase. It was seen that the choice of LMI solver has a large influence, and *sedumi* was found to work best.



(a) Example

(b) Example

Figure 2: Examples of iterative optimisation where the norm increases.

Fig. 2 shows two examples where the system norm increases as a result of the iterative optimisation. The first step $n = 1$ is the controller obtained from the dilated LMI line search. The workaround implemented for this thesis is: If the norm increases at n as a result of the iterative optimisation, the controller at $n-1$ is used instead.

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