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Modelling vehicular flow on single-lane roundabouts using a discrete time Markov chain

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Abstract

The behavior of traffic on roundabouts is a complex phenomenon and it has been heavily researched in the past decades. In that tendency, a discrete time approach will be used to gain insights on the departure process of a roundabout given Bernoulli arrivals at the entrances. We formulate a Discrete Time Markov Chain (DTMC) to form an analytical framework containing the interarrival times on the roundabout and at the exit of the roundabout. Simulations testthis analytical framework and subsequently the introduction of numerically acquired heuristics complement the analytical framework of these departure processes. Two case studies concerning chains of roundabouts with Bernoulli arrivals show that the departure process at the exit of a chain approximates the departure process at the exit of a single roundabout; additionally one case study demonstrates that the design of a roundabout plays a role in the probability distribution of the interarrival times. A simulation study regarding the classical four-way roundabout show that the theoretical framework along with the numerical heuristics establish a sufficient approximation of the departure process at the exit of the roundabout, but more theory on this subject regarding a discrete time approach is yet to be established.

Contents

1 Introduction

1.1 Motivation

In the last decades, traffic management has grown enormously, and therefore it has become a vast area of research. In particular the vehicular flow on roundabouts has always been a point of interest. For this purpose two of the most popular types of models are considered: microscopic and macroscopic models. Microscopic models show the interaction between the different vehicles. In these models, variables like speed, accelaration and deceleration, and the behavior of vehicles with regard to their predecessors. Macroscopic models are not based on individual vehicle behavior but rather characterizes traffic through global terms like flows, density, and mean speed. This report focuses on the microscopic models, and it opts to give insights on how roundabouts can fit in other traffic disciplines, like traffic sign intersections and priority intersections. New theory on fixed-cycle traffic-light control by Oblakova [1] motivates to scrutinize the behavior of vehicles on roundabouts.

1.2 The Cellular Automata model

For modelling the movement of vehicles on a roundabout, we look at one of the most used car following models: the stochastic Cellular Automata model (CA). The fundamentals of this model were introduced in 1992 by Kai Nagel and Michael Schreckenberg [2]. The CA model was further specifically exploited on roundabouts by M.E. Fouladvand et al. in 2004 [3], and further research was done by N.P. Belz et al. on priority abstaining and priority taking on roundabouts in 2016 [4]. A joint cooperation between the UvA and VU of Amsterdam by P.J. Storm (2019/2020) did research using the CA model to investigate the importance of on- and off-ramps, as well as looking at the entry behavior of vehicles entering a roundabout [5]. Most importantly, C.E.M. Pearce founded a model using a probabilistic discrete time Markov Chain (DTMC), using similar techniques for finding waiting times in the queue for threeway roundabouts [6]. However, most of earlier research is focused on optimizing freeflow on roundabouts and minimizing queueing times, whereas the purpose of this research concentrates on the departure times at roundabouts and the integration of roundabouts in various types of traffic junctions. This makes it so that no earlier research has been conducted on the subject of the departure times in the context of the CA model.

1.3 A modified CA method

The concept of the CA model is to divide the freeway roundabout and its arrival and departure road into cells that equal the average length of a vehicle plus the average distance between two vehicles in free flow speed. These cells would have a length of 7m according to the Storm's research [5].

An alternative for choosing cells with length equal as described above, is to choose smaller cells so that multiple cells can be occupied by the same vehicle. The advantage of splitting the cells is that vehicles can be attributed different lengths, whereas the structure of the model is preserved. The disadvantage is that the modelling becomes more complex and computationally costly as more conditions need to be created for vehicles to pass through multiple cells.

The CA model keeps track for all cells at each time step whether it is occupied by a vehicle or not. After one time step, a vehicle will move from position i to position $i + j$, depending on the speed of the vehicle. Furthermore, a vehicle can be given certain attributes, namely the following:

- The velocity of the vehicle (slow, average or fast driver)
- The position of the vehicle
- Length of the vehicle
- If the driver is a priority taker, priority abstainer, or neither one of them
- The behavior of the vehicle when there is another vehicle in front of him

Priority taking means that vehicles coming from an entrances take priority over vehicles on the roundabouts so that the flow on the roundabout is blocked, whereas priority abstaining causes essentially the same, only the vehicle on the roundabout voluntarily grants vehicles from the entrance of a roundabout priority. The versatility in the attributes that can be given to vehicles shows that the CA model is very suitable as a simulatory model.

1.4 Objective

Whereas most research towards roundabouts focus on rush-hours or traffic jams, this report tries to link the roundabout in the bigger picture of traffic behavior. A principal approach for this concept is to grab a hold of the departure process of roundabouts in its most basic form. Arrivals on roundabouts will therefore be regarded as 'random', which in this case means that arrivals occur according to a Bernoulli trial. Furthermore, when such a representation of the departure process is found, it will be tested against simulations, for which we use SUMO (Simulation of Urban MObility). Keeping in mind the bigger picture of traffic behavior, the overloaded systems of roundabouts do not necessarily concern this paper. This leads to the following research questions:

1. Given the arrival processes at the roundabout, what do the departure processes at the roundabout look like?
2. What influence does a roundabout have on the departure processes of other roundabouts?

2 The Markov Cellular Automata model

The introduction shows that the CA model is very well suited for dissecting the behavior of vehicles. With the help of simulations, a numerical framework for roundabouts can be created. On the other hand, to derive analytical expressions for the arrival and departure processes for entering and exiting a roundabout would allow for a general framework for the behavior of vehicles on roundabouts. The idea of the CA model gives us the motivation of the Markov Cellular Automata (MCA) model. The overall idea of the MCA model is that a Markov chain is created using cells on the roundabout as states, and queues in front of the roundabout as states to describe the most important aspects of the behavior on the roundabout. In this section, the most basic model description is given for the MCA model, along with the transition behavior of its states.

2.1 MCA model description

Consider a threeway roundabout. We formulate the most basic DTMC: the road segments of a roundabout are split into m parts, each with equal length. When a vehicle arrives at a cell on the entrance of the roundabout at time t and it is at the front of the queue (if there is one), it will move to the adjacent road segment of the roundabout at time $t+1$. Subsequently it moves to the next road segment of the roundabout at time $t+2$, and so on until it leaves the roundabout. When there is no queue at the entrance of the roundabout, the vehicle will also move to the adjacent road segment on the roundabout at time $t+1$ if possible. We define the following Discrete Time Markov Chain,

$$\begin{aligned} M(t) &= (\mathbf{S}(t), \mathbf{Q}(t)), \text{ where} \\ \mathbf{S}(t) &= \{S_1(t), S_2(t), ..., S_m(t)\}, \\ \mathbf{Q}(t) &= \{Q_1(t), Q_2(t), Q_3(t)\}, \end{aligned}$$

where:

- $M(t)$ is the complete state space at time t for $t = 0, 1, 2, ...$
- The subspace $\mathbf{Q} = \{(Q_1, Q_2, Q_3)\}$ defines the queue length at the entrances of the roundabout, such that $Q_i \in \mathbb{Z}$ for $i \in \{1, 2, 3\}$.
- The subspace $\mathbf{S} = \{(S_1, ..., S_m)\}$ defines the state of the road segments on the roundabout, with $S_i \in \{0, 1, 2\}$ for $i = 1, ..., m$.

The dimension of the subspace $\mathbf{S}$, defines how many road segments the roundabout is being split into. For the purpose of maintaining a small state space we choose $m = 6$. Consequently this means that

$$M(t) = (S_1, S_2, S_3, S_4, S_5, S_6, Q_1, Q_2, Q_3) \tag{1}$$

Q_i describes the number of vehicles in the queue before entrance i of the roundabout. Therefore, $Q_i = \{0, 1, 2, 3, ..., c_i\}$, where the given value of Q_i is the amount of vehicles in the queue, and c_i is the maximum queue length for a given queue i. S_i describes the presence of a vehicle on a road segment.

For a three-way roundabout, $S_i \in \{0, 1, 2\}$. If $S_i(t) = 0$, no vehicle is currently occupying road segment S_i at time t. If $S_i(t) = 1$, a vehicle is occupying road segment $S_i(t)$ it will take the first turn to the right, and at time $t+1$ it 'leaves' the roundabout. If $S_i(t) = 2$, a vehicle is occupying road segment S_i and it will take the second turn to the right, such that at time $t+1$ it is still on the roundabout, but at time $t+2$ it leaves the roundabout. As all states S_i can attain at most 3 different states and Q_i can at most attain $c_i + 1$ different states, the size of the state space will be equal to $3^6 \cdot \prod_{i=1}^{3}(c_i + 1)$. For convenience, all three queues will have equal queue capacity such that $c_1 = c_2 = c_3$. In all further calculations, the queue capacity will be infinite, such that the state space is also infinite.

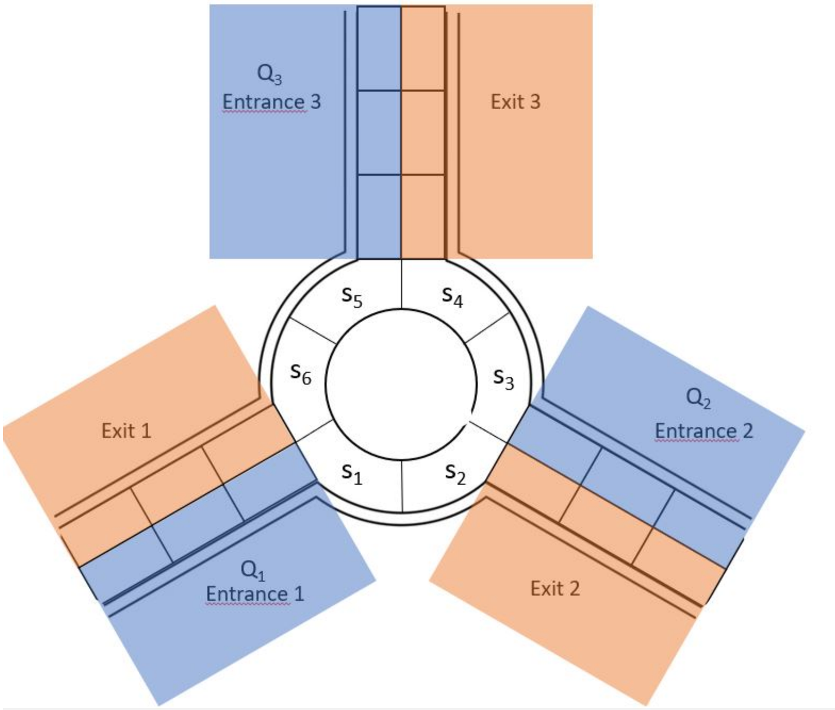

Figure 1: A visualisation of the MCA model. A road segment S_i lies on the roundabout. A queue Q_i is represented at the entry of the roundabout.

A visualisation of the MCA model can be seen in Figure 1. We will sometimes refer to the last road segment state S_6 as S_0, as for the notation of the transition behavior it is shown to be more convenient in some cases.

2.2 Transition behavior MCA model

Now that the discrete time queues and road segments have been defined, it is important to see how the states relate to each other. The Markov chain $M(t)$ jumps to another state as $t = 0, 1, 2, \ldots$. At time t, a vehicle can join some queue with probability q_j. Furthermore, if $Q_j(t) > 0$, a vehicle will leave the queue if this is possible. This will only be possible if the road segment before queue j will not block the queue on the next timestep $t+1$. Notice that for queue j, the road segment before the queue can be expressed as $S_{2(j-1)}(t)$. If $S_{2(j-1)}(t) = 2$, queue j will be blocked at time $t+1$ and no vehicle can leave the queue between time t and time $t+1$..

When a vehicle does leave queue j, it joins road segment $2j - 1$. With probability p_j the vehicle takes a first turn on the roundabout, and with probability $1 - p_j$ the vehicle takes a second turn on the roundabout. So if $Q_j(t) > 0$ and no vehicles are coming from road segment S_j, then with probability p_j, state $S_{2j-1}(t+1) = 1$, and with probability $1 - p_j$, state $S_{2j-1}(t+1) = 2$.

The behavior of the road segments $S_i(t)$ is fairly simple. We differentiate by 'odd' states, so states $S_i(t)$ where i is odd, and 'even' states, states $S_i(t)$ where i is even. If any odd state $S_i(t)$ equals to $0, 1$ or 2, the subsequent even states $S_{i+1}(t+1)$ will take on the exact same value. If an even state $S_i(t) = 2$, inferring the vehicle on this road segment will take a second turn, the subsequent road segment will be $S_{i+1}(t+1) = 1$ with probability 1. If an even state $S_i(t) = 0$ or $S_i(t) = 1$, the subsequent state $S_{i+1}(t+1) = 0$, unless there is at least one vehicle in the queue at the entry of this road segment. The probabilities for these transitions are given: if $Q_j(t) \geq 1$, $S_{2j-1}(t+1) = 1$ with probability p_j, and $S_{2j-1}(t+1) = 2$ with probability $1 - p_j$.

A next state of the Markov chain $M(t)$ will be determined in two steps. First, the new states of the road segements $S_i(t+1)$ will be calculated as shown earlier and $Q_j(t+1)$ will be decreased by 1 if it is non-empty and not blocked. Then a queue $Q_j(t+1)$ will increase by 1 with probability q_i after a vehicle has possibly left the queue. When $Q_j(t)$ has maximum queue capacity c_i and is also being blocked, the queue length will stay the same with probability 1 at time $t+1$ since no one can join, and no one can leave. So, for example, if

$$M(t) = (2, 2, 2, 2, 2, 2, c_1, c_2, c_3),$$

the next state will have no probabilistic attributes, as no new vehicle can enter the roundabout, neither can one enter one of the queues as all have reached maximum capacity c_j. Therefore,

$$M(t+1) = (1, 2, 1, 2, 1, 2, c_1, c_2, c_3).$$

Formally the transition probabilities are defined as follows. For $i = 1, 3, 5$ and $j = \frac{i+1}{2}$:

$$S_i(t+1) = \begin{cases} 1, & \text{if } S_{i-1}(t) = 2 & \text{with probability } 1, \\ 1, & \text{if } S_{i-1}(t) \neq 2 \text{ and } Q_j(t) > 0 & \text{with probability } p_j, \\ 2, & \text{if } S_{i-1}(t) \neq 2 \text{ and } Q_j(t) > 0 & \text{with probability } 1 - p_j, \\ 0, & \text{if } S_{i-1}(t) \neq 2 \text{ and } Q_j(t) = 0 & \text{with probability } 1 - q_j. \\ 1, & \text{if } S_{i-1}(t) \neq 2 \text{ and } Q_j(t) = 0 & \text{with probability } q_j \cdot p_j. \\ 2, & \text{if } S_{i-1}(t) \neq 2 \text{ and } Q_j(t) = 0 & \text{with probability } q_j \cdot (1 - p_j). \end{cases}$$

For 'even' i, that is $i = 2, 4, 6$, we find the expression $S_i(t+1) = S_{i-1}(t)$ with probability 1.

For $j = 1, 2, 3$:

$$Q_j(t+1) = \begin{cases} Q_j(t) - 1, & \text{if } S_{2j-2}(t) \neq 2 \text{ and } Q_j(t) \geq 0 & \text{with probability } 1 - q_j, \\ Q_j(t), & \text{if } S_{2j-2}(t) \neq 2 \text{ and } Q_j(t) \geq 0 & \text{with probability } q_j, \\ Q_j(t), & \text{if } S_{2j-2}(t) = 2 \text{ and } Q_j(t) < c_j & \text{with probability } 1 - q_j, \\ Q_j(t) + 1, & \text{if } S_{2j-2}(t) = 2 \text{ and } Q_j(t) < c_j & \text{with probability } q_j, \\ Q_j(t), & \text{if } S_{2j-2}(t) = 2 \text{ and } Q_j(t) = c_j & \text{with probability } 1. \end{cases}$$

3 Interarrival times on the roundabout

3.1 Departure process of isolated flows

Recall Figure 1. In this subsection the merging process of the two flows from S_2 and Q_2 to S_3 is analyzed. Consider two Bernoulli flows on entrance 1 and entrance 2. When entering the roundabout, the Bernoulli arrivals entering at entrance 1 follow the route $S_1 - S_2 - S_3 - S_4$, and then leave the roundabout at exit 3. The Bernoulli arrivals at entrance 2 follow the route $S_3 - S_4$ and then also leave the roundabout at exit 3. No arrivals enter at entrance 3. Because entrance 1 will never be blocked by another flow, there will never be a queue at entrance 1. Therefore the Bernoulli process will also exist on cell S_1 and cell S_2. More specifically, the probability that an occupation occurs on S_2 is equal to p_1, and the probability that a vehicle joins Q_2 is equal to p_2. Therefore the interarrival times for both arrival processes at S_2 and Q_2 are geometrically distributed with parameters p_1 and p_2.

It is observed that the departure process from S_2 proceeding to S_3 on its own follows a geometric distribution, as this flow of vehicles takes precedence over the vehicles that are waiting at the entrance at Q_2. It can be shown however that the departure process from Q_2 proceeding to S_3 also follows a geometric distribution. Firstly it is important to observe that the process at Q_2 can be modelled as a Geo/Geo/1 queue following Kendall's notation, as the vehicles from S_2 that are blocking Q_2 have geometrically distributed interarrival times. This means that with probability $1 - p_1$, vehicles from Q_2 can recieve service. Therefore, the queue that is aimed for will be the Geo(p_2)/Geo($1 - p_1$)/1 queue.

To prove that the departure process of Q_2 follows a geometric distribution, the z-transforms of both the arrival and service times for a general Geo/Geo/1 queue will be used.

Theorem 3.1 *Let $\frac{\lambda}{\mu} = \rho < 1$. Then the departure process of a Geo(λ)/Geo(μ)/1-queue is geometrically distributed with parameter λ.*

Let $A(z)$ be the z-transform of the arrival times, and $S(z)$ the z-transform of the service times. Additionally let $\rho = \frac{\lambda}{\mu}$ be the utility of the system, which adheres to the condition that $\rho < 1$. Furthermore expanding the z-transforms gives

$$A(z) = \frac{\lambda z}{1 - z(1-\lambda)}, \qquad \text{and} \qquad S(z) = \frac{\mu z}{1 - z(1-\mu)}.$$

It is important to see that the z-transform for the departures of a Geo/Geo/1 queue is equal to the probability that the queue is not empty times the z-transform of the service times, added by the probability that the queue is empty multiplied by the z-transform of the arrival times and the service times. Let $G_{dep}(z)$ be the z-transform of the departure process of a Geo/Geo/1 queue. This gives:

$$G_{dep}(z) = \rho \cdot S(z) + (1-\rho) \cdot A(z) \cdot S(z). \tag{2}$$

Evaluating expression (2), we find:

$$\begin{aligned} G_{dep}(z) &= \rho \cdot S(z) + (1-\rho) \cdot A(z) \cdot S(z) \\ &= \rho \cdot \frac{\mu z}{1 - z(1-\mu)} + (1-\rho) \cdot \frac{\lambda z}{1 - z(1-\lambda)} \cdot \frac{\mu z}{1 - z(1-\mu)} \\ &= \frac{\mu z}{1 - z(1-\mu)} \cdot \Big(\frac{\lambda}{\mu} + (\frac{\mu}{\mu} - \frac{\lambda}{\mu}) \cdot \frac{\lambda z}{1 - z(1-\lambda)}\Big) \end{aligned}$$

$$
\begin{aligned}
&= \frac{\mu z}{1 - z(1-\mu)} \cdot \Big(\frac{\lambda(1 - z(1-\lambda)) + (\mu - \lambda)\lambda z}{\mu(1 - z(1-\lambda))}\Big) \\
&= \frac{\mu z}{1 - z(1-\mu)} \cdot \Big(\frac{\lambda - \lambda z + \lambda^2 z + \mu\lambda z - \lambda^2 z}{\mu(1 - z(1-\lambda))}\Big) \\
&= \frac{\mu z}{1 - z(1-\mu)} \cdot \Big(\frac{\lambda(1 - z(1-\mu))}{\mu(1 - z(1-\lambda))}\Big) \\
&= \frac{\lambda z}{1 - z(1-\lambda)} = A(z).
\end{aligned}
$$

Therefore, the departure times are distributed exactly the same as the arrival times for all Geo/Geo/1 queues, as the z-transforms of the two are equal to each other. □

Consequently, the departure process of Q_2 on S_3 is geometrically distributed with parameter p_2.

3.2 Departure process of mixed flows

Now that the isolated departure processes of the two flows meeting at the intersection are found, the following step is to find the joint departure process at road segment S_3, given the geometrically distributed arrivals at S_2 and Q_2. Let the arrivals from S_2 that proceed to S_3 be called X, and let the arrivals from the entrance 2 be called Y. Alternatively, the arrivals of X are oftentimes referred to as 'type 1 vehicles', and the arrivals of Y are referred to as 'type 2 vehicles'. So an arrival of type 1 occurs with probability p_1, and an arrival of type 2 occurs with probability p_2. Let us remember that the independent process X and the independent process Y before merging are each geometrically distributed. A visualisation of the joint process on S_3 can be seen in Figure 2.

x	y	0	0	0	y	x	0	y	0	x	0	0	x	0	y	x	0	0	0	x	Process X+Y
x	0	0	0	0	0	x	0	0	0	x	0	0	x	0	0	x	0	0	0	x	X is geometrically distributed
0	y	0	0	0	y	0	0	y	0	0	0	0	0	0	y	0	0	0	0	0	Y is geometrically distributed

Figure 2: A visualisation of the process $X + Y$, the process X and the process Y, where the last two processes are geometrically distributed. A 0 means no arrival.

To find the probability mass function, we must look at the interarrival times of the joint process $\{X \text{ or } Y\}$, where X has priority over Y when the two arrive at the same time, and Y is being put in a queue when this happens. We define the random variable I_{X+Y} as the interarrival time of the joint process of any occupation on S_3 of type 1 or 2. So $P(I_{X+Y} = 0)$ is the probability that two consecutive occupations at S_3 of type 1 or 2 takes place. In this subsection, we will construct the proof of the first lemma:

Lemma 3.2 *Consider the discrete random variables $X(t)$ and $Y(t)$ to be geometrically distributed, and the merging process of the two to be described as above, where type 1 vehicles have priority over type 2 vehicles. Then the interarrival times of the merging process takes on the following probability mass function:*

$$
P(I_{X+Y} = k) = \begin{cases} 1 - \alpha + \frac{p_1 p_2}{p_1 + p_2}, & k = 0 \\ \alpha^{k-1}(1-\alpha) \cdot (1 - P(I_{X+Y} = 0)), & k \in \mathbb{Z}_+ \end{cases}
$$

where $\alpha = (1 - p_1)(1 - p_2)$.

Let the random variable $A(t)$ be an occupation at S_3 at time t of a specific type of vehicle: $A(t) = 0$ means that no vehicle occupies S_3 at time t, $A(t) = 1$ means that a type 1 vehicle occupies S_3 at time t, and $A(t) = 2$ means that a type 2 vehicle occupies S_3 at time t. It should be noted that an *arrival* of type 2 at time t does not necessarily mean an *occupation* of type 2, as a type 1 vehicle can arrive at time t as well at S_3, and type 1 vehicles have priority over type 2 vehicles. A consecutive occupation at S_3 is defined as $P(I_{X+Y} = 0)$. Therefore $P(I_{X+Y} = 0) = P\Big(A(t+1) \in \{1,2\} \mid A(t) \in \{1,2\}\Big)$.

By the definition of conditional law of probability,

$$P(I_{X+Y} = 0) = \frac{P\Big(A(t) \in \{1,2\}, A(t+1) \in \{1,2\}\Big)}{P\Big(A(t) \in \{1,2\}\Big)}. \tag{3}$$

Now as $P\Big(A(t) \in \{1,2\}\Big)$ is interpreted as the probability that any occupation occurs, it will be equal to $p_1 + p_2$. This leaves $P\Big(A(t) \in \{1,2\}, A(t+1) \in \{1,2\}\Big)$, the numerator of equation (3), to be calculated.

Additionally the short notation of the random variable $Q(t) = Q_2(t)$ is introduced, which describes the length of the queue for type 2 vehicles at time t. A notable detail in the definition of $Q(t)$ is that *early arrivals* are considered, such that $Q(t+1) = 0$ if there are no vehicles in the queue between time t and $t+1$, and a vehicle can always move onto the intersection if no vehicle of type 1 is coming, so that the vehicle does not have to wait in the queue. As the queue is a Geo/Geo/1-queue, the queue length distribution follows a geometric distribution, and it is found that $P(Q(t) = k) = (1 - \frac{p_1 \cdot p_2}{\alpha})(\frac{p_1 \cdot p_2}{\alpha})^k$, where it should be noted that $\rho = \frac{p_1 \cdot p_2}{\alpha}$. The derivation of the queue length distribution can be found by solving the balance equation for the transition diagram of Figure 3, where $\lambda = p_2$ and $\mu = 1 - p_1$.

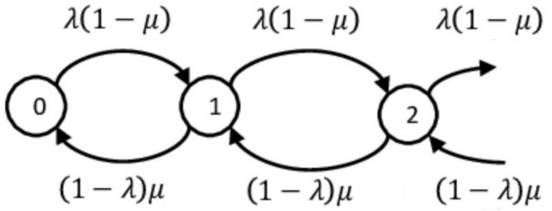

Figure 3: Transition diagram for a discrete Geo/Geo/1 queue.

An expression will be found for $P\Big(A(t) \in \{1,2\}, A(t+1) \in \{1,2\}\Big)$, where it will be split into two parts:

$$P\Big(A(t) \in \{1,2\}, Q(t+1) = 0\Big) \cdot (1 - \alpha) + P\Big(A(t) \in \{1,2\}, Q(t+1) > 0\Big) \tag{4}$$

The left hand side of equation (4) can be interpreted as follows: the probability that at time t an occupation takes place and the queue has no vehicles at time $t+1$, times the probability $(1 - \alpha)$, meaning that at least one arrival takes place at time $t+1$, so that $A(t+1) \in \{1,2\}$. The right hand side of equation (4) means: the probability that at time t an occupation takes place, which is $A(t) \in \{1,2\}$, and the queue being non-empty at time $t+1$, so that at time $t+1$

an occupation takes place regardless of whether an arrival occurs. Elaborating on both parts of equation (4) yields for the left hand side:

$$\begin{aligned}
&P\Big(A(t) \in \{1,2\}, Q(t+1) = 0\Big) \cdot (1-\alpha) \\
&= (1-\alpha) \cdot \Bigg[P(Q(t)=1) \cdot \alpha + P(Q(t)=0) \cdot \Big(p_1(1-p_2) + p_2(1-p_1)\Big) \Bigg] \\
&= (1-\alpha) \cdot \Bigg[(1 - \frac{p_1 \cdot p_2}{\alpha}) \cdot \frac{p_1 \cdot p_2}{\alpha} \cdot \alpha + (1 - \frac{p_1 \cdot p_2}{\alpha}) \cdot \Big(p_1(1-p_2) + p_2(1-p_1)\Big) \Bigg] \\
&= (1-\alpha) \cdot \Bigg[p_1p_2 - \frac{(p_1p_2)^2}{\alpha} + p_1 + p_2 - 2p_1p_2 - \frac{p_1p_2}{\alpha}(p_1 + p_2 - 2p_1p_2) \Bigg] \\
&= (1-\alpha) \Bigg[p_1 + p_2 - p_1p_2 + \frac{p_1p_2}{\alpha}(p_1 + p_2 - p_1p_2) \Bigg] \\
&= (1-\alpha)(1 - \alpha - \frac{p_1p_2}{\alpha}(1-\alpha)) \\
&= (1-\alpha)^2 \cdot (1 - \frac{p_1p_2}{\alpha}).
\end{aligned}$$

For the right hand side of equation (4) we find:

$$\begin{aligned}
&P\Big(A(t) \in \{1,2\}, Q(t+1) > 0\Big) \\
&= P(Q(t)=0) \cdot p_1p_2 + P(Q(t)=1) \cdot (1-\alpha) + P(Q(t) \geq 2) \\
&= (1 - \frac{p_1p_2}{\alpha})p_1p_2 + (1 - \frac{p_1p_2}{\alpha})\frac{p_1p_2}{\alpha}(1-\alpha) + \Big(\frac{p_1p_2}{\alpha}\Big)^2 \\
&= p_1p_2 - \frac{(p_1p_2)^2}{\alpha} + \Big(\frac{p_1p_2}{\alpha} - (\frac{p_1p_2}{\alpha})^2\Big)(1-\alpha) + \Big(\frac{p_1p_2}{\alpha}\Big)^2 \\
&= p_1p_2 - \frac{(p_1p_2)^2}{\alpha} + \frac{p_1p_2}{\alpha} - \Big(\frac{p_1p_2}{\alpha}\Big)^2 - p_1p_2 + \frac{(p_1p_2)^2}{\alpha} + \Big(\frac{p_1p_2}{\alpha}\Big)^2 \\
&= \frac{p_1p_2}{\alpha}.
\end{aligned}$$

Hence,

$$\begin{aligned}
P\Big(A(t) \in \{1,2\}, A(t+1) \in \{1,2\}\Big) &= (1-\alpha)^2 \cdot (1 - \frac{p_1p_2}{\alpha}) + \frac{p_1p_2}{\alpha}. \\
&= (1-\alpha)^2 - \frac{p_1p_2}{\alpha} + 2p_1p_2 - \alpha p_1p_2 + \frac{p_1p_2}{\alpha} \\
&= (1-\alpha)^2 + 2p_1p_2 - \alpha p_1p_2 \\
&= (1-\alpha)(p_1 + p_2 - p_1p_2) + 2p_1p_2 - \alpha p_1p_2 \\
&= (1-\alpha)(p_1 + p_2) + p_1p_2.
\end{aligned}$$

And to conclude, by using equation (3),

$$\begin{aligned}
P\big(I_{X+Y} = 0\big) &= \frac{P\Big(A(t) \in \{1,2\}, A(t+1) \in \{1,2\}\Big)}{P\Big(A(t) \in \{1,2\}\Big)} \\
&= \frac{(1-\alpha)(p_1 + p_2) + p_1p_2}{p_1 + p_2},
\end{aligned}$$

so

$$P\big(I_{X+Y} = 0\big) = 1 - \alpha + \frac{p_1 p_2}{p_1 + p_2}. \tag{5}$$

Thus the interarrival time for two consecutive occupations are found. For $I_{X+Y} > 0$, we observe that the queue length must be equal to zero when an occupation occurs. So the queue is always empty for $I_{X+Y} > 0$, and therefore the probability that $P(I_{X+Y} = 1)$ for example, will be equal to the probability that any occupation occurs at time $t + 2$, times the probability that there has not been a consecutive occupation at time $t + 1$. This argument can be given because there can not be a queue at time $t+1$ because no vehicle occupies S_3 at this time. The probability that any occupation occurs at time $t+2$ is equal to $1-\alpha$. So $P(I_{X+Y} = 1) = \Big(1-P(I_{X+Y} = 0)\Big)\cdot(1-\alpha)$. For the probability that $I_{X+Y} = 2$, so an occupation at time t, two empty occupations at $t+1$ and $t+2$ and then again an occupation at $t+3$, with the same argument will be equal to the probability that no consecutive occupation has occured at time $t+1$, times the probability that no new vehicle arrives from S_2 or entrance 2 at time $t+2$, times the probability that at time $t+3$ a vehicle *does* arrive from S_2 or entrance 2. Therefore $P(I_{X+Y} = 2) = \Big(1 - P(I_{X+Y} = 0)\Big) \cdot \alpha \cdot (1 - \alpha)$.

It follows that the probability $P(I_{X+Y} = k)$, given that $k \in \mathbb{Z}_+$, follows a geometric distribution with parameter α. Even more, $P(I_{X+Y} = k)$ for $k \in \mathbb{Z}_+$ can be formally expressed as $P(I_{X+Y} = k|I_{X+Y} > 0)P(I_{X+Y} > 0)$. Since $P(I_{X+Y} > 0) = 1 - P(I_{X+Y} = 0)$, we find that $P(I_{X+Y} = k) = P(I_{X+Y} = k|k > 0)\Big(1 - P(I_{X+Y} = 0)\Big)$. This leads to the probability mass function for the interarrival times of the process $X + Y$:

$$P(I_{X+Y} = k) = \begin{cases} 1 - \alpha + \frac{p_1 p_2}{p_1+p_2}, & k = 0 \\ \alpha^{k-1}(1-\alpha)\cdot(1 - P(I_{X+Y} = 0)), & k \in \mathbb{Z}_+, \end{cases} \tag{6}$$

which concludes the proof of Lemma 3.2. □

One can describe the interarrival times as delayed geometric, as for $I_{X+Y} > 0$ the interarrival times indeed follow a geometric distribution. The interpretation for $k = 0$ in equation (6) can be the following: the probability $1 - \alpha$ represents the probability that at least one of the type 1 or type 2 vehicles arrives at the entrance of the roundabout at time $t + 1$, given that at time t there has been an arrival. There is some other factor however, clearly influenced by the queue formed by the type 2 vehicles, that accounts for the element $\frac{p_1 p_2}{p_1+p_2}$ in equation (6). We argue that this element fully accounts for the probability that a queue has formed at Q_2, as this is the only other influence on this merging process. Moreover, this element is equal to the probability that no arrival takes place, which is α, times the probability that the queue is not empty, $P(Q(t + 1) > 0) = \frac{p_1 p_2}{\alpha}$, divided by the total utility of the process, which is equal to $p_1 + p_2$, such that $\alpha \cdot \frac{p_1 p_2}{\alpha} \cdot \frac{1}{p_1+p_2} = \frac{p_1 p_2}{p_1+p_2}$.

4 Interarrival times at the exit of the roundabout

As the interarrival times on the roundabout are found, a following step must be made to find the interarrival times at the exit of a roundabout. The difference between the process on the roundabout and the process right after exiting the roundabout is that vehicles of type Y, as illustrated in Figure 2, have the possibility not to exit the roundabout on the first turn, but to stay on the roundabout and leave on the second turn. Therefore we introduce a new random variable Z, otherwise referred to as 'type 3 vehicles'. Type 3 vehicles do arrive at Q_2 and they exit the roundabout at the second turn. Type 2 vehicles, also referring to an arrival at Q_2, still leave at the first possible turn. Note that both type 2 and type 3 vehicles arrive at Q_2, but they can never arrive at the same time: either a type 2 vehicle arrives with probability p_2, or a type 3 vehicle arrives with probability p_3, or no vehicle arrives with probability $1 - p_2 - p_3$.

It is also important to observe that with this altered definition of type 2 arrivals the distribution for the interarrival times, equation (6), changes, namely each p_2 is altered to $p_2 + p_3$. Hence the distribution for the interarrival times $I_{X,Y,Z}$ of the process $X + Y + Z$ becomes

$$P(I_{X,Y,Z} = k) = \begin{cases} 1 - \beta + \frac{p_1(p_2+p_3)}{p_1+p_2+p_3}, & k = 0 \\ (1-\beta) \cdot \beta^{k-1} \cdot (1 - P(I_{X,Y,Z} = 0)), & k \in \mathbb{Z}_+, \end{cases} \tag{E.1}$$

where $\beta = (1 - p_1)(1 - p_2 - p_3)$. Also the ρ that is used in the queue length distribution having only type 1 and 2 vehicles, will from now on be defined as $\rho = \frac{p_1(p_2+p_3)}{\beta}$ when accounting for type 3 vehicles as well. It can be verified filling in $p_3 = 0$ that equation (E.1) is equal to equation (6).

4.1 Important relationships between queue and occupations

Let us call I_{exit} the interarrival times of the process $X + Y$ at the exit of a roundabout, to make a distinction between the earlier analysed interarrival times I_{X+Y} for the interarrivals on a roundabout. Just like with the distribution of the interarrival times on the roundabout, it is convenient to start with finding $P(I_{exit} = 0)$, the probability that there are no vehicles between to arrivals of type 1 or 2. There are multiple methods to find the distribution of I_{exit}. The straightforward one is to use the same method as used before in section 3. However, a simpler method is established by looking at the conditional probabilities of the exit process. Therefore we need to find some important relationshipis between the occupation slot and the queue length.

For this purpose, type 3 vehicles, are added in the definition of the arrival process $A(t)$: if an arrival of type 3 occurs on the road segment at time t, we say that $A(t) = 3$. Secondly, the new notation $Q_{i(j)}(t)$ is introduced, in which i stands for the type vehicle at the front of the queue if the queue is non-empty, and j describes the amount of type i vehicles at the front of the queue. Subsequently, we find for example that $P(Q_{2(1)}(t) > 0) = \frac{p_2}{p_2+p_3}\rho = \frac{p_1 p_2}{(1-p_1)(1-p_2-p_3)}$. More generally we find that $P(Q_{3(j)}(t) = i)$ with $j < i$ is equal to $(\frac{p_3}{p_2+p_3})^j \cdot (1 - \rho)\rho^i = \frac{1-p_1-p_2-p_3}{(1-p_1)(1-p_2-p_3)} \frac{p_1^i p_3^j (p_2+p_3)^{i-j}}{(1-p_1)^i(1-p_2-p_3)^i}$. We illustrate the difference between the two departure processes $X + Y$ and $X + Y + Z$ where X, Y and Z stand for the processes of occupation of type 1, type 2 and type 3 vehicles in Figure 4.

Formally, $\{I_{exit} = k\} = \Big\{A(t) \in \{1, 2\}, A(t + k + 1) \in \{1, 2\}, A(t + i) \in \{0, 3\}$ for $i = 1, 2, ..., k\Big\}$, for $k > 0$. Additionally for consecutive occupations, $\{I_{exit} = 0\} = \Big\{A(t) \in \{1, 2\}, A(t + 1) \in \{1, 2\}\Big\}$

x	y	z	0	0	y	x	0	y	z	x	z	0	x	0	y	x	0	0	z	x	X+Y+Z is distributed according to (E.1)
x	y	0	0	0	y	x	0	y	0	x	0	0	x	0	y	x	0	0	0	x	Joint process X+Y
x	0	0	0	0	0	x	0	0	0	x	0	0	x	0	0	x	0	0	0	x	X is geometrically distributed
0	y	0	0	0	y	0	0	y	0	0	0	0	0	0	y	0	0	0	0	0	Y is geometrically distributed
0	0	z	0	0	0	0	0	0	z	0	z	0	0	0	0	0	0	0	z	0	Z is geometrically distributed

Figure 4: A visualisation of the processes $X+Y$ and $X+Y+Z$, the processes X, Y and Z, where the last three processes are geometrically distributed. A 0 means no arrival.

We now introduce a very important property of the probability that a queue has length i after time t, given that some at the same time t the occupation of the slot is known. This probability can be defined by $P(Q(t+1)=i|A(t)=j)$.

Lemma 4.1 *For $j=2$ and $j=3$,*

$$P\Big(Q(t+1)=i|A(t)=j)\Big)=P(Q(t)=i). \tag{7}$$

We use the law of conditional probability to determine that $P(Q(t+1)=i|A(t)=2)=\frac{P(Q(t+1)=i,A(t)=2)}{P(A(t)=2)}$. The probability that both events $Q(t+1)=i$ and $A(t)=2$ occur, is explained as follows: if there is a type 2 vehicle being accepted at time t, then it must be true that at time t no type 1 vehicle has arrived, which occurs with probability $(1-p_1)$. The queue length at time $t+1$ can therefore be composed in two ways: by an arrival of type 2 or 3 if the queue has i vehicles at time t, or no arrival of type 2 or 3 if the queue has $i+1$ vehicles at time t. This means that for a non-empty queue, so $i>0$:

$$\begin{aligned}
P(Q(t+1)=i,A(t)=2) &= (1-p_1)\Big((p_2+p_3)P(Q_2(t)=i)+(1-p_2-p_3)P(Q_2(t)=i+1)\Big) \\
&= (1-p_1)(p_2+p_3)\frac{p_2}{p_2+p_3}\frac{(1-p_1-p_2-p_3)p_1^i(p_2+p_3)^i}{(1-p_1)^{i+1}(1-p_2-p_3)^{i+1}} \\
&+ (1-p_1)(1-p_2-p_3)\frac{p_2}{p_2+p_3}\frac{(1-p_1-p_2-p_3)p_1^{i+1}(p_2+p_3)^{i+1}}{(1-p_1)^{i+2}(1-p_2-p_3)^{i+2}} \\
&= p_2\frac{(1-p_1-p_2-p_3)p_1^i(p_2+p_3)^i}{(1-p_1)^{i+1}(1-p_2-p_3)^{i+1}}(1-p_1+p_1) \\
&= p_2\cdot P(Q(t)=i).
\end{aligned}$$

So we know that $P(Q(t+1)=i,A(t)=2)=p_2\cdot P(Q(t)=i)$, and as $P(A(t)=2)=p_2$, it becomes clear that

$$\begin{aligned}
P(Q(t+1)=i|A(t)=2) &= \frac{P(Q(t+1)=i,A(t)=2)}{P(A(t)=2)} \\
&= \frac{p_2\cdot P(Q(t)=i)}{p_2} \\
&= P(Q(t)=i).
\end{aligned}$$

Therefore Lemma 4.1 holds for $i>0$. For an empty queue, that is $i=0$, we find using the same logic:

$$P(Q(t+1)=0|A(t)=2)=\frac{P(Q(t+1)=0,A(t)=2)}{P(A(t)=2)}$$

$$
\begin{aligned}
&= \frac{(1-p_1)p_2 P(Q(t)=0) + (1-p_1)(1-p_2-p_3)P(Q_2(t)=1)}{p_2} \\
&= \frac{(1-p_1)(1-p_1-p_2-p_3) + p_1(1-p_1-p_2-p_3)}{(1-p_1)(1-p_2-p_3)} \\
&= \frac{1-p_1-p_2-p_3}{(1-p_1)(1-p_2-p_3)} \\
&= P(Q(t)=0).
\end{aligned}
$$

So $P(Q(t+1)=i|A(t)=2) = P(Q(t)=i)$ for any integer $i \in \mathbb{Z}$. Since the process of type 2 vehicles and type 3 vehicles is shared and their behaviour is analogous, it follows that $P(Q(t+1)=i|A(t)=3) = P(Q(t)=i)$ as well. This concluldes Lemma 4.1. □

It can also be proven that $P(Q_{2(i)}(t+1)=i|A(t)=2) = P(Q_{2(i)}(t)=i)$, using the exact same technique of the previous proof. Lemma 4.1 invites to look at the case where $j=1$:

Lemma 4.2 *For $i > 0$,*

$$
P(Q(t+1)=i|A(t)=1) = \frac{1-p_2-p_3}{p_1} \cdot P(Q(t)=i),
$$

and for the special case that $i=0$,

$$
P(Q(t+1)=0|A(t)=1) = (1-p_2-p_3) \cdot P(Q(t)=0),
$$

For $j=1$ in equation (7), we find that

$$
\begin{aligned}
P(Q(t+1)=i|A(t)=1) &= \frac{P(Q(t+1)=i, A(t)=1)}{P(A(t)=1)} \\
&= \frac{p_1(p_2+p_3)P(Q(t)=i-1) + p_1(1-p_2-p_3)P(Q(t)=i)}{p_1} \\
&= \frac{(1-p_1-p_2-p_3)p_1^{i-1}(p_2+p_3)^i}{(1-p_1)^i(1-p_2-p_3)^i} + \frac{(1-p_1-p_2-p_3)p_1^i(p_2+p_3)^i}{(1-p_1)^{i+1}(1-p_2-p_3)^i} \\
&= (1-p_1-p_2-p_3)\frac{p_1^{i-1}(p_2+p_3)^i}{(1-p_1)^{i+1}(1-p_2-p_3)^i}(1-p_1+p_1) \\
&= (1-p_1-p_2-p_3)\frac{p_1^i(p_2+p_3)^i}{(1-p_1)^{i+1}(1-p_2-p_3)^{i+1}} \cdot \frac{1-p_2-p_3}{p_1} \\
&= \frac{1-p_2-p_3}{p_1} \cdot P(Q(t)=i).
\end{aligned}
$$

This concludes the first part of Lemma 4.2. For $i=0$, we find that

$$
\begin{aligned}
P(Q(t+1)=0|A(t)=1) &= \frac{P(Q(t+1)=0, A(t)=1)}{P(A(t)=1)} \\
&= \frac{p_1(1-p_2-p_3)P(Q(t)=0)}{p_1} \\
&= (1-p_2-p_3)P(Q(t)=0)
\end{aligned}
$$

This concludes the proof of Lemma 4.2. □

Analogously, $P(Q_{2(i)}(t+1) = i|A(t) = 1) = \frac{1-p_2-p_3}{p_1} \cdot P(Q_{2(i)}(t) = i)$ and $P(Q_{3(i)}(t+1) = i|A(t) = 1) = \frac{1-p_2-p_3}{p_1} \cdot P(Q_{3(i)}(t) = i)$ using the same concept of proof again. Additionally, it must be observed that for $j = 2$ and $j = 3$:

$$P(Q(t+1) > i|A(t) = j) = P(Q(t) > i), \tag{8}$$

arguing that $P(Q(t+1) > i|A(t) = 2) = \sum_{i+1}^{\infty} P(Q(t+1) = i|A(t) = 2)$, and then using Lemma 4.1 to find that $\sum_{i+1}^{\infty} P(Q(t+1) = i|A(t) = 2) = \sum_{i+1}^{\infty} P(Q(t) = i) = P(Q(t) > i)$, for $i \in \mathbb{Z}$. Once more it should be noted that for $j = 1$, we find that

$$P(Q(t+1) > i|A(t) = 1) = \frac{1-p_2-p_3}{p_1} P(Q(t) > i). \tag{9}$$

Lemma 4.2, Lemma 4.1 and Equations (8) and (9) are sufficient to calculate the interarrival times for two consecutive occupations of type 1 and type 2 vehicles.

Theorem 4.3 *The probability that two consecutive occupations of type 1 or 2 take place, defined by $I_{exit} = 0$, is equal to*

$$P(I_{exit} = 0) = 1 - \alpha + \frac{p_1 p_2}{p_1 + p_2}(1 - p_3), \tag{10}$$

where $\alpha = (1-p_1)(1-p_2)$.

Proof:
Let us recall that

$$P(I_{exit} = 0) = \frac{P\Big(A(t) \in \{1,2\}, A(t+1) \in \{1,2\}\Big)}{P\Big(A(t) \in \{1,2\}\Big)}$$

It pertains that the denominator $P\Big(A(t) \in \{1,2\}\Big) = P(A(t) = 1) + P(A(t) = 2) = p_1 + p_2$, as this is the occupation rate of the process $X+Y$. We find that the numerator of can be rewritten:

$$\begin{aligned} &P\Big(A(t) \in \{1,2\}, A(t+1) \in \{1,2\}\Big) \\ &= P\Big(A(t) = 1, A(t+1) = 1\Big) + P\Big(A(t) = 1, A(t+1) = 2\Big) \\ &+ P\Big(A(t) = 2, A(t+1) = 1\Big) + P\Big(A(t) = 2, A(t+1) = 2\Big). \end{aligned}$$

The law of conditional probability can be used again to find that

$$P\Big(A(t) = i, A(t+1) = j\Big) = P\Big(A(t+1) = j|A(t) = i\Big) \cdot P\Big(A(t) = i\Big). \tag{11}$$

We can recall that $P\Big(A(t) = 1\Big) = p_1$ and $P\Big(A(t) = 2\Big) = p_2$. The next step is to calculate the probability $P\Big(A(t+1) = j|A(t) = i\Big)$ for $i = 1,2$ and $j = 1,2$, leaving four probabilities to calculate. The probability that a type 1 vehicle arrives after any kind of event is equal to p_1, as type 1 vehicles are never blocked by some other vehicle. So $P\Big(A(t+1) = 1|A(t) = 1\Big) = p_1$ and $P\Big(A(t+1) = 1|A(t) = 2\Big) = p_1$.

The probability that a type 2 vehicle occupies at time slot $t+1$, given that its predecessor is also a type 2 vehicle, will be equal to the probability that no type 1 vehicle arrives, times the probability that a type 2 vehicle arrives from the queue, or either arrives while the queue is empty. So, using Lemma 4.1 and equation (8) we find that

$$
\begin{aligned}
P(A(t+1)=2|A(t)=2) &= (1-p_1)\cdot\Big[p_2\cdot P(Q(t)=0|A(t)=2)+P(Q_2(t)>0|A(t)=2)\Big]\\
&= (1-p_1)\Big[(p_2(1-\frac{p_1(p_2+p_3)}{(1-p_1)(1-p_2-p_3)})+\frac{p_1p_2}{(1-p_1)(1-p_2-p_3)}\Big]\\
&= (1-p_1)\Big[\frac{p_2(1-p_1)}{(1-p_1)}+\frac{p_1p_2(1-p_2+p_3)}{(1-p_1)(1-p_2-p_3)}\Big]\\
&= (1-p_1)\Big[\frac{p_2(1-p_1)+p_1p_2}{1-p_1}\Big]\\
&= (1-p_1)\cdot\frac{p_2}{1-p_1}\\
&= p_2.
\end{aligned}
$$

Lastly, the conditional probability $P\Big(A(t+1)=2|A(t)=1\Big)$ can be found as well, arguing that a type 2 occupation at time $t+1$ is the sum of two probabilities: the first is the probability that no type 1 vehicle arrives, times the probability that a type 2 arrives while the queue is empty. The second is the probability that no type 1 vehicle arrives, times the probability that a type 2 vehicle arrives from the queue, given that the queue is not empty. This yields the following expression:

$$
\begin{aligned}
&P(A(t+1)=2|A(t)=1)\\
&=(1-p_1)\cdot p_2\cdot P(Q(t+1)=0|A(t)=1)+(1-p_1)P(Q_2(t+1)>0|A(t)=1)
\end{aligned}
$$

Expanding on this equality, using Lemma 4.2 and equation (9) yields:

$$
\begin{aligned}
P(A(t)=2|A(t)=1) &= (1-p_1)\Big[(p_2(1-p_2-p_3)P(Q(t)=0)+p_2P(Q(t)=0)+P(Q_2(t)>0)\Big]\\
&= (1-p_1)p_2\Big[(1-p_2-p_3)(1-\frac{p_1(p_2+p_3)}{\beta})+(1-\frac{p_1(p_2+p_3)}{\beta})+\frac{p_1}{\beta}\Big]\\
&= (1-p_1)p_2\Big[(1-p_2-p_3)\frac{1-p_1-p_2-p_3}{(1-p_1)(1-p_2-p_3)}+\frac{1-p_2-p_3}{(1-p_1)(1-p_2-p_3)}\Big]\\
&= (1-p_1)p_2\Big[\frac{1-p_1-p_2-p_3}{1-p_1}+\frac{1}{1-p_1}\Big]\\
&= p_2(2-p_1-p_2-p_3).
\end{aligned}
$$

Using equation (11), the following expressions are found:

- $P\Big(A(t)=1|A(t)=1\Big)\cdot P\Big(A(t+1)=1\Big)=p_1\cdot p_1$
- $P\Big(A(t)=1|A(t)=2\Big)\cdot P\Big(A(t+1)=2\Big)=p_1\cdot p_2$
- $P\Big(A(t)=2|A(t)=1\Big)\cdot P\Big(A(t+1)=1\Big)=p_2(2-p_1-p_2-p_3)\cdot p_1$
- $P\Big(A(t)=2|A(t)=2\Big)\cdot P\Big(A(t+1)=2\Big)=p_2\cdot p_2$

The sum of these probabilities divided by $P\Big(A(t) \in \{1,2\}\Big)$ equals the interarrival times for 0 interarrivals:

$$\begin{aligned} P(I_{exit}=0) &= \frac{P\Big(A(t)\in\{1,2\}, A(t+1)\in\{1,2\}\Big)}{P\Big(A(t)\in\{1,2\}\Big)} \\ &= \frac{p_1^2 + p_1p_2 + p_1p_2(2-p_1-p_2-p_3)+p_2^2}{p_1+p_2} \\ &= \frac{(1-\alpha)(p_1+p_2)+p_1p_2\cdot(1-p_3)}{p_1+p_2}. \end{aligned}$$

Simplifying concludes the proof:

$$P(I_{exit}=0) = 1-\alpha+\frac{p_1p_2}{p_1+p_2}(1-p_3). \qquad \square$$

It is important to notice the saliant detail between the properties of equations (6) and (10): the only difference between the two probabilities $P(I_{exit}=0)$ and $P(I_{X+Y}=0)$ is the factor $(1-p_3)$ before the component $\frac{p_1p_2}{p_1+p_2}$, where $(1-p_3)$ is the probability that no type 3 vehicle arrives at Q_2.

4.2 Interarrival times for k $\in \mathbb{Z}_+$

Like with the joint arrival process $X+Y+Z$ on the roundabout, it is not allowed to assume that for $k \in \mathbb{Z}_+$ the interarrival times of $X+Y$ is distributed geometrically.The process $X+Y+Z$ is a merger between the process X and the process $Y+Z$, whereas the process $X+Y$ is a bifurcation of the processes $X+Y+Z$ and Z. Intuitively and informally, the process $X+Y$ is the splitting between the delayed geometric distribution of $X+Y+Z$ and the geometric distribution of Z. The main challenge lies within the fact that type 3 vehicles can interpose between type 2 vehicles in the queue, and therefore interfere in the splitted process $X+Y$.

4.2.1 Concept of proof

Let us introduce a new notation:
$P(\overset{k+1}{(\mathrm{x_0x_1\ldots x_k})}|A(t)=x_{k+1}) = P(A(t+k)=x_0, A(t+k-1)=x_1, ..., A(t+1)=x_k|A(t)=x_{k+1})$. This notation is specifically introduced to find the interarrival times for k time steps between two arrivals of type x_{k+1} and x_0. In the following subsections, we will learn that the interarrival times of $k \in \mathbb{Z}_+$ are equal to

$$P(I_{exit}=k) = \alpha^k(1-\alpha+\phi) - \phi(1-p_3)\sum_{n=0}^{k-1}\alpha^{k-1-n}p_3^n, \quad k \in \mathbb{Z}_+ \tag{12}$$

The concept of the proof is similar to that of finding the interarrivals for $k=0$: all the possible combinations of probabilities $P(\overset{k+1}{(\mathrm{x_0x_1\ldots x_k})}|x_{k+1})$ must be found, where $x_0, x_{k+1} \in \{1,2\}$, and where all $x_1, x_2, ..., x_k \in \{0,3\}$, where a type 0 vehicle means no occupation at all. Then using equation (11), the conditional law of probability will give us the interarrival times for $k>0$.

Since the proof is rather long and requires multiple steps, the structure of the proof will be as follows:

1. Find the four probabilities $P((\mathrm{x}_0\overset{\mathrm{k+1}}{33\ldots3})\,|x_{k+1})$ where $x_0, x_{k+1} \in \{1,2\}$.
2. Find the two probabilities $P((0\overset{\mathrm{k+1}}{33\ldots3})\,|x_{k+1})$ where $x_{k+1} \in \{1,2\}$.
3. Find the probability $P((\mathrm{x}_0\overset{\mathrm{k+1}}{\mathrm{x}_1\ldots\mathrm{x}_\mathrm{k}})\,|0)$ where $x_0 \in \{1,2\}$ and $x_1, x_2, ..., x_k \in \{0,3\}$.
4. Combine all previous steps and evaluate the expression.

4.2.2 Preliminaries

Before the first step of the concept of proof can be treated, some additional properties of the system should be looked at. The following Lemma is an extension to Lemma 4.1:

Lemma 4.4 *Consider the probability $P(Q_{3(i)}(t+k+1) = i, (\overset{k}{33\ldots3})\,|A(t) = 1)$, which indicates the probability that given an occupation of type 1 at time t, another k consecutive occupations occur on time $t+1, t+2, ..., t+k$, and the queue at time $t+k+1$ is equal to i with all the vehicles in the queue being of type 3. Then for $i \geq 1, k \geq 1$:*

$$(1-p_1)P\Big(Q_{3(i)}(t+k+1) = i, (\overset{k}{33\ldots3})\,|A(t) = 1\Big) = p_3^{k+i}p_1^{i-1}\frac{(1-p_1-p_2-p_3)}{(1-p_1)^i(1-p_2-p_3)^i} \tag{13}$$

The proof will be carried out using mathematical induction. The basis step is to prove that equation (13) holds for $k = 1$. Let us call the probability for the basis step $P_{BS} = (1-p_1)P\Big(Q_{3(i)}(t+2) = i, A(t+1) = 3|A(t) = 1\Big)$. Using the concept that the probability $P(Q_{3(i)}(t+1) = i, A(t) = 3)$ can be splitted and expressed as $(1-p_1)p_3P(Q_{3(i)}(t) = i) + (1-p_1)(1-p_2-p_3)P(Q_{3(i+1)}(t) = i+1)$, it becomes clear that

$$\begin{aligned}
P_{BS} &= (1-p_1)P\Big(Q_{3(i)}(t+2) = i, A(t+1) = 3|A(t) = 1\Big) \\
&= (1-p_1)^2p_3P\Big(Q_{3(i)}(t+1) = i|A(t) = 1\Big) \\
&\quad + (1-p_1)^2(1-p_2-p_3)P\Big(Q_{3(i+1)}(t+1) = i+1|A(t) = 1\Big)
\end{aligned}$$

.

Using Lemma 4.2, we find that

$$\begin{aligned}
P_{BS} &= (1-p_1)^2\Big[p_3\frac{1-p_2-p_3}{p_1}P(Q_{3(i)}(t) = i) + \frac{(1-p_2-p_3)^2}{p_1}P(Q_{3(i+1)}(t) = i+1)\Big] \\
&= (1-p_1)^2p_3\frac{1-p2-p3}{p_1}\frac{p_1^ip_3^i(1-p_1-p_2-p_3)}{(1-p_1)^{i+1}(1-p_2-p_3)^{i+1}} \\
&\quad + (1-p_1)^2\frac{(1-p_2-p_3)^2}{p_1}\frac{p_1^{i+1}p_3^{i+1}(1-p_1-p_2-p_3)}{(1-p_1)^{i+2}(1-p_2-p_3)^{i+2}} \\
&= p_3^{i+1}p_1^{i-1}\frac{1-p_1-p_2-p_3}{(1-p_1)^i(1-p_2-p_3)^i}\Big((1-p_1)+p_1\Big) \\
&= p_3^{i+1}p_1^{i+1}\frac{1-p_1-p_2-p_3}{(1-p_1)^i(1-p_2-p_3)^i},
\end{aligned}$$

concluding the basis step, as for $k = 1$ this last expression is equal to equation (13). Now suppose for some $n = k$ equation (13) holds true. Then for $n = k + 1$ it must also hold true. Let us call $P_{ind} = (1 - p_1)P\Big(Q_{3(i)}(t + k + 2) = i, (33\overset{\text{k+1}}{\ldots}3)\,|A(t) = 1\Big)$. Then we find that

$$\begin{aligned}
P_{ind} &= (1 - p_1)P\Big(Q_{3(i)}(t + k + 2) = i, (33\overset{\text{k+1}}{\ldots}3)\,|A(t) = 1\Big) \\
&= (1 - p_1)^2 p_3 P\Big(Q_{3(i)}(t + k + 1) = i, (33\overset{\text{k}}{\ldots}3)\,|A(t) = 1\Big) \\
&\quad + (1 - p_1)^2(1 - p_2 - p_3)P\Big(Q_{3(i+1)}(t + k + 1) = i + 1, (33\overset{\text{k}}{\ldots}3)\,|A(t) = 1\Big) \\
&= (1 - p_1)p_3\Big[p_3^{k+i}p_1^{i-1}\frac{(1 - p_1 - p_2 - p_3)}{(1 - p_1)^i(1 - p_2 - p_3)^i}\Big] \\
&\quad + (1 - p_1)(1 - p_2 - p_3)\Big[p_3^{k+i+1}p_1^{i}\frac{(1 - p_1 - p_2 - p_3)}{(1 - p_1)^{i+1}(1 - p_2 - p_3)^{i+1}}\Big] \\
&= \Big[(1 - p_1) + p_1\Big]p_3^{k+i+1}p_1^{i-1}\frac{(1 - p_1 - p_2 - p_3)}{(1 - p_1)^i(1 - p_2 - p_3)^i} \\
&= p_3^{k+i+1}p_1^{i-1}\frac{(1 - p_1 - p_2 - p_3)}{(1 - p_1)^i(1 - p_2 - p_3)^i},
\end{aligned}$$

which concludes the induction step, as the last equation is equal to the right hand side of equation (13) at $k + 1$. □

Now that we have found this probability, it is necessary to find the same probability as described in equation (13), only for the exception that $i = 0$. This leads to the following lemma:

Lemma 4.5 *Consider the probability* $P(Q(t + k + 1) = 0, (33\overset{k}{\ldots}3)\,|A(t) = 1)$*, which indicates the probability that given an occupation of type 1 at time t, another k consecutive occupations occur on time* $t + 1, t + 2, ..., t + k$*, and the queue at time* $t + k + 1$ *is empty. Then for* $k \geq 1$*:*

$$(1 - p_1)P\Big(Q(t + k + 1) = 0, (33\overset{k}{\ldots}3)\,|A(t) = 1\Big) = p_3^k(1 - p_1 - p_2 - p_3)\sum_{n=0}^{k}(1 - p_1)^n \tag{14}$$

This lemma will again be proven with mathematical induction. For the basis step we must prove that $P_{BS} = (1-p_1)P\Big(Q(t+2) = 0, A(t+1) = 3|A(t) = 1\Big) = p_3(1-p_1-p_2-p_3)\sum_{n=0}^{1}(1-p_1)^n$. Using lemma 4.2 again we find that:

$$\begin{aligned}
P_{BS} &= (1 - p_1)P\Big(Q(t + 2) = 0, A(t + 1) = 3|A(t) = 1\Big) \\
&= (1 - p_1)^2\Big[p_3P\big(Q(t) = 0|A(t) = 1\big) + (1 - p_2 - p_3)P\big(Q_{3(1)}(t + 1) = 1|A(t) = 1\big)\Big] \\
&= (1 - p_1)^2\Big[p_3\frac{1 - p_2 - p_3}{p_1}P(Q(t) = 0) + \frac{(1 - p_2 - p_3)^2}{p_1}P(Q_{3(1)}(t) = 1)\Big] \\
&= (1 - p_1)^2p_3(1 - p_2 - p_3)\frac{1 - p_1 - p_2 - p_3}{(1 - p_1)(1 - p_2 - p_3)} \\
&\quad + (1 - p_1)^2\frac{(1 - p_2 - p_3)^2}{p_1}\frac{p_1p_3(1 - p_1 - p_2 - p_3)}{(1 - p_1)^2(1 - p_2 - p_3)^2} \\
&= p_3(1 - p_1 - p_2 - p_3)(1 - p_1) + p_3(1 - p_1 - p_2 - p_3)
\end{aligned}$$

$$= p_3(1-p_1-p_2-p_3)\sum_{n=0}^{1}(1-p_1)^n,$$

which concludes the basis step. For the induction step, it is needed to prove that $P_{ind} = (1-p_1)P\Big(Q(t+k+2)=0, (33\overset{\text{k+1}}{\dots}3)\,|A(t)=1\Big)$ is equal to $p_3^{k+1}(1-p_1-p_2-p_3)\sum_{n=0}^{k+1}(1-p_1)^n$, given that the induction hypothesis, equation (14) holds true. We will use the result from Lemma 4.4 on the part with an $\oplus$ in the following calculations:

$$\begin{aligned}
P_{ind} &= (1-p_1)P\Big(Q(t+k+2)=0, (33\overset{\text{k+1}}{\dots}3)\,|A(t)=1\Big)\\
&= (1-p_1)^2p_3P\Big(Q(t+k+1)=0, (33\overset{\text{k}}{\dots}3)\,|A(t)=1\Big)\\
&\quad + (1-p_1)^2(1-p_2-p_3)P\Big(Q_{3(1)}(t+k+1)=1, (33\overset{\text{k}}{\dots}3)\,|A(t)=1\Big) \oplus\\
&= p_3^{k+1}(1-p_1)(1-p_1-p_2-p_3)\sum_{n=0}^{k}(1-p_1)^n\\
&\quad + (1-p_1)(1-p_2-p_3)\Big[p_3^{k+1}\frac{1-p_1-p_2-p_3}{(1-p_1)(1-p_2-p_3)}\Big]\\
&= p_3^{k+1}(1-p_1-p_2-p_3)\Big[1+(1-p_1)\sum_{n=0}^{k}(1-p_1)^n\Big]\\
&= p_3^{k+1}(1-p_1-p_2-p_3)\sum_{n=0}^{k+1}(1-p_1)^n.
\end{aligned}$$

This confirms the induction hypothesis and concludes the proof of Lemma 4.5. □

The next Lemma gives an understanding in the probability that given an occupation of a type 1 vehicle at time t, that there are k occupations of type 3 vehicles, and the queue at time $t+k+1$ is non-empty and the first vehicle in the queue at time $t+k+1$ is of type 3.

Lemma 4.6

$$(1-p_1)P\Big(Q_{3(1)}(t+k+1)>0, (33\overset{k}{\dots}3)\,|A(t)=1\Big) = p_3^{k+1} \tag{15}$$

The proof consists of finitely splitting the probability described in Lemma 4.6 k times, and using the result of Lemma 4.4.

Let $P(\text{begin} = (1-p_1)P\Big(Q_{3(1)}(t+k+1)>0, (33\dots3)^{\text{k}}\,|A(t)=1\Big)$. It must be noted that the term $P\Big(Q_{3(i+1)}(t+k+1)>i, (33\dots3)^{\text{k}}\,|A(t)=1\Big)$ is always equal to the sum of $(1-p_1)^2p_3P\Big(Q_{3(i+1)}(t+k)=i+1, (33\dots3)^{\text{k-1}}\,|A(t)=1\Big)$ and $(1-p_1)^2P\Big(Q_{3(i+2)}(t+k)>i+1, (33\dots3)^{\text{k-1}}\,|A(t)=1\Big)$ as long as $k\geq 1$.

$$\begin{aligned}
P_B &= (1-p_1)P\Big(Q_{3(1)}(t+k+1)>0, (33\overset{\text{k}}{\dots}3)\,|A(t)=1\Big)\\
&= (1-p_1)^2p_3P\Big(Q_{3(1)}(t+k)=1, (33\overset{\text{k-1}}{\dots}3)\,|A(t)=1\Big)
\end{aligned}$$

$$+ (1-p_1)^2 P\Big(Q_{3(2)}(t+k) > 1, (33\overset{\text{k-1}}{\ldots}3)\,|A(t)=1\Big)$$

$$= (1-p_1)^2 p_3 P\Big(Q_{3(1)}(t+k) = 1, (33\overset{\text{k-1}}{\ldots}3)\,|A(t)=1\Big)$$

$$+ (1-p_1)^3 p_3 P\Big(Q_{3(2)}(t+k-1) = 2, (33\overset{\text{k-2}}{\ldots}3)\,|A(t)=1\Big) + ...$$

$$+ (1-p_1)^3 P\Big(Q_{3(3)}(t+k-1) > 2, (33\overset{\text{k-2}}{\ldots}3)\,|A(t)=1\Big)^{\oplus}.$$

As explained, the part with the $\oplus$ can be finitely expanded upon. Doing this $k-2$ more times, and using Lemma 4.4, it is found that

$$\begin{aligned}
P_B &= \sum_{n=1}^{k}(1-p_1)^n p_3(1-p_1)P\Big(Q_{3(n)}(t+1+k-n) = n, (33\overset{\text{k-n}}{\ldots}3)\,|A(t)=1\Big)\\
&\quad + (1-p_1)^{k+1}\Big[p_3 P(Q_{3(k)}(t)=k) + P(Q_{3(k+1)}(t)>k)\Big]\\
&= p_3\sum_{n=1}^{k}(1-p_1)^n p_3^{k-n+n}p_1^{n-1}\frac{1-p_1-p_2-p_3}{(1-p_1)^n(1-p_2-p_3)^n}\\
&\quad + (1-p_1)^{k+1}\Big[p_3\frac{p_1^k p_3^k(1-p_1-p_2-p_3)}{(1-p_1)^{k+1}(1-p_2-p_3)^{k+1}} + \frac{p_1^{k+1}p_3^{k+1}}{(1-p_1)^{k+1}(1-p_2-p_3)^{k+1}}\Big]\\
&= p_3^{k+1}\frac{(1-p_1-p_2-p_3)}{p_1}\sum_{n=1}^{k}\frac{p_1^n}{(1-p_2-p_3)^n} + p_3^{k+1}p_1^k\frac{1-p_1-p_2-p_3+p_1}{(1-p_2-p_3)^{k+1}}\\
&= p_3^{k+1}\frac{(1-p_1-p_2-p_3)}{p_1}\Big[\sum_{n=0}^{k}\big(\frac{p_1}{1-p_2-p_3}\big)^n - 1\Big] + p_3^{k+1}\frac{p_1^k}{(1-p_2-p_3)^k}\\
&= p_3^{k+1}\frac{(1-p_1-p_2-p_3)}{p_1}\Big[\frac{1-(\frac{p_1}{1-p_2-p_3})^{k+1}}{1-\frac{p_1}{1-p_2-p_3}} - 1\Big] + p_3^{k+1}\frac{p_1^k}{(1-p_2-p_3)^k}\\
&= p_3^{k+1}\frac{(1-p_1-p_2-p_3)}{p_1}\Big[\frac{1-p_2-p_3-p_1(\frac{p_1}{1-p_2-p_3})^k}{1-p_1-p_2-p_3} - 1\Big] + p_3^{k+1}\frac{p_1^k}{(1-p_2-p_3)^k}\\
&= p_3^{k+1}\Big[\frac{1-p_2-p_3}{p_1} - \frac{p_1^k}{(1-p_2-p_3)^k} - \frac{1-p_1-p_2-p_3}{p_1} + \frac{p_1^k}{(1-p_2-p_3)^k}\Big]\\
&= p_3^{k+1}\frac{1-p_2-p_3-(1-p_1-p_2-p_3)}{p_1}\\
&= p_3^{k+1}.
\end{aligned}$$

This concludes Lemma 4.6. □

With Lemma's 4.5 and 4.6 it is possible to find the probability that, given an occupation of type 1 at time t, exactly k occupations of type 3 follow:

Lemma 4.7

$$P\Big((33\overset{k}{\ldots}3)\,|A(t)=1\Big) = p_3^k\Big[1 + (1-p_1-p_2-p_3)\sum_{n=0}^{k-1}(1-p_1)^n\Big] \tag{16}$$

It is straigthforward to see that $P\Big((33\ldots3)^{\text{k}}\,|A(t)=1\Big)$ is equal to $1-p_1$ times the sum of the probability that given a type 1 occurance, $k-1$ type 3 vehicles consecutively occupy the road segment, and the queue is nonempty with a type 3 vehicle in the front of the queue, and the probability that given a type 1 occurance, $k-1$ type 3 vehicles consecutively occupy the road segment, and the queue is empty at time $t+k$, times the probability that a type 3 vehicle arrives at the queue at time $t+k$. Then using Lemma 4.5 and Lemma 4.6, this yields

$$\begin{aligned} P\Big(\overset{\text{k}}{(33\ldots 3)}\,|A(t)=1\Big) &= (1-p_1)p_3 P\Big(Q(t+k+1)=0, \overset{\text{k-1}}{(33\ldots 3)}\,|A(t)=1\Big) \\ &\quad + (1-p_1)P\Big(Q_{3(1)}(t+k)>0, \overset{\text{k-1}}{(33\ldots 3)}\,|A(t)=1\Big) \\ &= p_3^k(1-p_1-p_2-p_3)\sum_{n=0}^{k-1}(1-p_1)^n + p_3^k \\ &= p_3^k\Big[1+(1-p_1-p_2-p_3)\sum_{n=0}^{k-1}(1-p_1)^n\Big], \end{aligned}$$

which concludes the proof of Lemma 4.7. □

Now that the probability is found that k type 3 vehicles consecutively occupy the road segment after a type 1 occupation, the same probability will be inspected for when a type 2 occupation occurs at time t. The following Lemma's have the same structure as Lemma's 4.4, 4.5 and 4.6.

Lemma 4.8 *Consider the probability $P(Q_{3(i)}(t+k+1)=i, \overset{k}{(33\ldots 3)}\,|A(t)=2)$, which indicates the probability that given an occupation of type 2 at time t, another k consecutive occupations occur on time $t+1, t+2, ..., t+k$, and the queue at time $t+k+1$ is equal to i with all the vehicles in the queue being of type 3. Then for $i \geq 1, k \geq 1$:*

$$P\Big(Q_{3(i)}(t+k+1)=i, \overset{k}{(33\ldots 3)}\,|A(t)=2\Big) = p_3^{k+i}p_1^i\frac{(1-p_1-p_2-p_3)}{(1-p_1)^{i+1}(1-p_2-p_3)^{i+1}} \tag{17}$$

We prove Lemma 4.8 with mathematical induction. The basis step is to prove that equation (17) holds for $k=1$. Let us call $P_{BS} = (1-p_1)P\Big(Q_{3(i)}(t+2)=i, A(t+1)=3|A(t)=2\Big)$. Using the concept that the probability $P(Q_{3(i)}(t+1)=i, A(t)=3)$ can be splitted and expressed as $(1-p_1)p_3P(Q_{3(i)}(t)=i) + (1-p_1)(1-p_2-p_3)P(Q_{3(i+1)}(t)=i+1)$, it becomes clear that

$$\begin{aligned} P_{BS} &= P\Big(Q_{3(i)}(t+2)=i, A(t+1)=3|A(t)=2\Big) \\ &= (1-p_1)p_3P\Big(Q_{3(i)}(t+1)=i|A(t)=2\Big) \\ &\quad + (1-p_1)(1-p_2-p_3)P\Big(Q_{3(i+1)}(t+1)=i+1|A(t)=2\Big) \end{aligned}$$

.

Using Lemma 4.1, we find that

$$\begin{aligned} P_{BS} &= (1-p_1)p_3P\Big(Q_{3(i)}(t)=i\Big) + (1-p_1)(1-p_2-p_3)P\Big(Q_{3(i+1)}(t)=i+1\Big) \\ &= (1-p_1)p_3\frac{p_1^ip_3^i(1-p_1-p_2-p_3)}{(1-p_1)^{i+1}(1-p_2-p_3)^{i+1}} + \frac{p_1^{i+1}p_3^{i+1}(1-p_1-p_2-p_3)}{(1-p_1)^{i+1}(1-p_2-p_3)^{i+1}} \\ &= p_3^{i+1}p_1^i\frac{1-p_1-p_2-p_3}{(1-p_1)^{i+1}(1-p_2-p_3)^{i+1}}\Big((1-p_1)+p_1\Big) \end{aligned}$$

$$= p_3^{i+1} p_1^i \frac{1-p_1-p_2-p_3}{(1-p_1)^{i+1}(1-p_2-p_3)^{i+1}},$$

concluding the basis step, as for $k = 1$ this last expression is equal to equation (17). Now suppose for some $k = n$ equation (13) holds true. Then for $k = n+1$ it must also hold true. Let us call $P_{ind} = P\Big(Q_{3(i)}(t+n+2) = i, (33\overset{\text{n+1}}{\ldots}3)\,|A(t) = 2\Big)$. Then we find that

$$\begin{aligned}
P_{ind} &= P\Big(Q_{3(i)}(t+n+2) = i, (33\overset{\text{n+1}}{\ldots}3)\,|A(t) = 2\Big) \\
&= (1-p_1)p_3 P\Big(Q_{3(i)}(t+n+1) = i, (33\overset{\text{n}}{\ldots}3)\,|A(t) = 2\Big) \\
&\quad + (1-p_1)(1-p_2-p_3)P\Big(Q_{3(i+1)}(t+n+1) = i+1, (33\overset{\text{n}}{\ldots}3)\,|A(t) = 2\Big) \\
&= (1-p_1)p_3\Big[p_3^{n+i}p_1^i \frac{(1-p_1-p_2-p_3)}{(1-p_1)^{i+1}(1-p_2-p_3)^{i+1}}\Big] \\
&\quad + (1-p_1)(1-p_2-p_3)\Big[p_3^{n+i+1}p_1^{i+1}\frac{(1-p_1-p_2-p_3)}{(1-p_1)^{i+2}(1-p_2-p_3)^{i+2}}\Big] \\
&= \Big[(1-p_1)+p_1\Big]p_3^{n+i+1}p_1^i \frac{(1-p_1-p_2-p_3)}{(1-p_1)^{i+1}(1-p_2-p_3)^{i+1}} \\
&= p_3^{n+i+1}p_1^i \frac{(1-p_1-p_2-p_3)}{(1-p_1)^{i+1}(1-p_2-p_3)^{i+1}},
\end{aligned}$$

which concludes the induction step, as the last equation is equal to the right hand side of equation (17) at $k = n+1$. □

Lemma 4.9 *Consider the probability* $P(Q(t+k+1) = 0, (33\overset{k}{\ldots}3)\,|A(t) = 2)$*, which indicates the probability that given an occupation of type 2 at time* t*, another* k *consecutive occupations of type 3 occur on time* $t+1, t+2, ..., t+k$*, and the queue at time* $t+k+1$ *is empty. Then for* $k \geq 1$*:*

$$(1-p_1)P\Big(Q(t+k+1) = 0, (33\overset{k}{\ldots}3)\,|A(t) = 2\Big) = p_3^k \frac{1-p_1-p_2-p_3}{1-p_2-p_3} \tag{18}$$

Similarly to Lemma 4.5, the method of mathematical induction will be used to prove that equation (18) holds true. For the basis step $k = 1$, we must prove that $P_{BS} = (1-p_1)P\Big(Q(t+2) = 0, A(t+1) = 3|A(t) = 2\Big) = p_3\frac{(1-p_1-p_2-p_3)}{1-p_2-p_3}$. Using lemma 4.1 again we find that:

$$\begin{aligned}
P_{BS} &= (1-p_1)P\Big(Q(t+2) = 0, A(t+1) = 3|A(t) = 2\Big) \\
&= (1-p_1)^2\Big[p_3P\big(Q(t) = 0|A(t) = 2\big) + (1-p_2-p_3)P\big(Q_{3(1)}(t+1) = 1|A(t) = 2\big)\Big] \\
&= (1-p_1)^2\Big[p_3P(Q(t) = 0) + (1-p_2-p_3)P(Q_{3(1)}(t) = 1)\Big] \\
&= (1-p_1)^2 p_3 \frac{1-p_1-p_2-p_3}{(1-p_1)(1-p_2-p_3)} + (1-p_1)^2(1-p_2-p_3)\frac{p_1p_3(1-p_1-p_2-p_3)}{(1-p_1)^2(1-p_2-p_3)^2} \\
&= p_3\frac{(1-p_1-p_2-p_3)}{1-p_2-p_3}(1-p_1) + p_3\frac{(1-p_1-p_2-p_3)}{1-p_2-p_3}p_1
\end{aligned}$$

$$= p_3 \frac{(1-p_1-p_2-p_3)}{1-p_2-p_3},$$

which concludes the basis step. For the induction step, it is needed to prove that $P_{ind} = (1-p_1)P\Big(Q(t+m+2)=0, (33\overset{\text{n+1}}{\dots}3)\,|A(t)=2\Big)$ is equal to $p_3^{n+1}\frac{(1-p_1-p_2-p_3)}{1-p_2-p_3}$, given that the induction hypothesis, equation (14) holds true for $k=n$. We will use the result of Lemma 4.8 on the part with the $\oplus$ in the following calculations:

$$\begin{aligned}
P_{ind} &= (1-p_1)P\Big(Q(t+n+2)=0, (33\overset{\text{n+1}}{\dots}3)\,|A(t)=2\Big)\\
&= (1-p_1)^2 p_3 P\Big(Q(t+n+1)=0, (33\overset{\text{n}}{\dots}3)\,|A(t)=2\Big)\\
&\quad + (1-p_1)^2(1-p_2-p_3)P\Big(Q_{3(1)}(t+n+1)=1, (33\overset{\text{n}}{\dots}3)\,|A(t)=2\Big)^{\oplus}\\
&= (1-p_1)p_3^{n+1}\frac{1-p_1-p_2-p_3}{1-p_2-p_3}\\
&\quad + (1-p_1)^2(1-p_2-p_3)\Big[p_3^{n+1}\frac{1-p_1-p_2-p_3}{(1-p_1)^2(1-p_2-p_3)^2}\Big]\\
&= p_3^{n+1}\frac{1-p_1-p_2-p_3}{1-p_2-p_3}\Big[(1-p_1)+p_1\Big]\\
&= p_3^{n+1}\frac{1-p_1-p_2-p_3}{1-p_2-p_3}
\end{aligned}$$

This confirms the induction hypothesis and concludes the proof of Lemma 4.9. □

Lemma 4.10 *Consider the probability* $P(Q_{3(1)}(t+k+1)>0, (33\overset{k}{\dots}3)\,|A(t)=2)$*, which indicates the probability that given an occupation of type 2 at time* t*, another* k *consecutive occupations of type 3 occur on time* $t+1, t+2, ..., t+k$*, and the queue at time* $t+k+1$ *is non-empty and the first vehicle in the queue is of type 3. Then for* $k \geq 1$*:*

$$(1-p_1)P\Big(Q_{3(1)}(t+k+1)>0, (33\overset{\text{k}}{\dots}3)\,|A(t)=2\Big) = p_3^{k+1}\frac{p_1}{1-p_2-p_3} \tag{19}$$

The proof of Lemma 4.10 is similar to that of the proof of Lemma 4.6. Let $P_B = (1-p_1)P\Big(Q_{3(1)}(t+k+1)>0, (33\overset{\text{k}}{\dots}3)\,|A(t)=2\Big)$. It must be noted that the term $P\Big(Q_{3(i+1)}(t+k+1)>i\,(33\overset{\text{k}}{\dots}3)\,|A(t)=2\Big)$ is always equal to $(1-p_1)^2p_3P\Big(Q_{3(i+1)}(t+k)=i+1, (33\overset{\text{k-1}}{\dots}3)\,|A(t)=2\Big) + (1-p_1)^2P\Big(Q_{3(i+2)}(t+k)>i+1, (33\overset{\text{k-1}}{\dots}3)\,|A(t)=2\Big)$ as long as $k \geq 1$.

$$\begin{aligned}
P_B &= (1-p_1)P\Big(Q_{3(1)}(t+k+1)>0, (33\overset{\text{k}}{\dots}3)\,|A(t)=2\Big)\\
&= (1-p_1)^2p_3P\Big(Q_{3(1)}(t+k)=1, (33\overset{\text{k-1}}{\dots}3)\,|A(t)=2\Big)\\
&\quad + (1-p_1)^2P\Big(Q_{3(2)}(t+k)>1, (33\overset{\text{k-1}}{\dots}3)\,|A(t)=2\Big)^{\oplus}.
\end{aligned}$$

The part with an $\oplus$ can be expanded upon $k-1$ more times. Using Lemma 4.4, it is found that

$$\begin{aligned}P_B &= (1-p_1)^2 p_3 P\Big(Q_{3(1)}(t+k)=1, (33\overset{\text{k-1}}{\ldots}3)\,|A(t)=2\Big)\\ &\quad + (1-p_1)^3 p_3 P\Big(Q_{3(2)}(t+k-1)=2, (33\overset{\text{k-2}}{\ldots}3)\,|A(t)=2\Big) + \ldots\\ &\quad + (1-p_1)^{k+1} p_3 P(Q_{3(k)}(t+1)=k|A(t)=2)\\ &\quad + (1-p_1)^{k+1} P(Q_{3(k+1)}(t+1)>k|A(t)=2)\end{aligned}$$

.

Summing over the first k elements and using Lemma 4.1 to expand on the last element yields:

$$\begin{aligned}P_B &= \sum_{n=1}^{k}(1-p_1)^n p_3(1-p_1)P\Big(Q_{3(n)}(t+1+k-n)=n, (33\overset{\text{k-n}}{\ldots}3)\,|A(t)=2\Big)\\ &\quad + (1-p_1)^{k+1}P(Q_{3(k+1)}(t)>k)\\ &= p_3\sum_{n=1}^{k}(1-p_1)^{n+1}p_3^{k-n+n}p_1^n\frac{1-p_1-p_2-p_3}{(1-p_1)^{n+1}(1-p_2-p_3)^{n+1}}\\ &\quad + (1-p_1)^{k+1}\frac{p_1^{k+1}p_3^{k+1}}{(1-p_1)^{k+1}(1-p_2-p_3)^{k+1}}\\ &= p_3^{k+1}\frac{(1-p_1-p_2-p_3)}{1-p_2-p_3}\sum_{n=1}^{k}\frac{p_1^n}{(1-p_2-p_3)^n} + p_3^{k+1}\frac{p_1^{k+1}}{(1-p_2-p_3)^{k+1}}\\ &= p_3^{k+1}\frac{(1-p_1-p_2-p_3)}{1-p_2-p_3}\Big[\sum_{n=0}^{k}\big(\frac{p_1}{1-p_2-p_3}\big)^n - 1\Big] + p_3^{k+1}\frac{p_1^{k+1}}{(1-p_2-p_3)^{k+1}}\\ &= p_3^{k+1}\frac{(1-p_1-p_2-p_3)}{1-p_2-p_3}\Big[\frac{1-(\frac{p_1}{1-p_2-p_3})^{k+1}}{1-\frac{p_1}{1-p_2-p_3}} - 1\Big] + p_3^{k+1}\frac{p_1^{k+1}}{(1-p_2-p_3)^{k+1}}\\ &= p_3^{k+1}\frac{(1-p_1-p_2-p_3)}{1-p_2-p_3}\Big[\frac{1-p_2-p_3-p_1(\frac{p_1}{1-p_2-p_3})^k}{1-p_1-p_2-p_3} - 1\Big] + p_3^{k+1}\frac{p_1^{k+1}}{(1-p_2-p_3)^{k+1}}\\ &= p_3^{k+1}\Big[1-\frac{1-p_1-p_2-p_3}{1-p_2-p_3} - \frac{p_1^{k+1}}{(1-p_2-p_3)^{k+1}} + \frac{p_1^{k+1}}{(1-p_2-p_3)^{k+1}}\Big]\\ &= p_3^{k+1}\Big[\frac{1-p_2-p_3-(1-p_1-p_2-p_3)}{1-p_2-p_3}\\ &= p_3^{k+1}\frac{p_1}{1-p_2-p_3}.\end{aligned}$$

This concludes Lemma 4.10. □

Lemma's 4.9 and 4.10 can be utilized to find the probability that, given a type 2 vehicle at time t, there will be following k type 3 vehicles:

Lemma 4.11

$$P\Big((33\overset{k}{\ldots}3)\,|A(t)=2\Big) = p_3^k \tag{20}$$

Just like in Lemma 4.7, equation (20) can be expressed as follows:

$$
\begin{aligned}
P\Big((33\overset{\text{k}}{\ldots}3)\,|A(t)=2\Big) &= p_3(1-p_1)P\Big(Q(t+k+1)=0,(33\overset{\text{k-1}}{\ldots}3)\,|A(t)=2\Big) \\
&\quad + (1-p_1)P\Big(Q_{3(1)}(t+k+1)>0,(33\overset{\text{k-1}}{\ldots}3)\,|A(t)=2\Big) \\
&= p_3\Big[p_3^{k-1}\frac{1-p_1-p_2-p_3}{1-p_2-p_3}\Big] + \Big[p_3^k\frac{p_1}{1-p_2-p_3}\Big] \\
&= p_3^k\frac{1-p_1-p_2-p_3+p_1}{1-p_2-p_3} \\
&= p_3^k
\end{aligned}
$$

This concludes Lemma 4.11. $\square$

4.2.3 Concept of proof: part 1 and 2

Theorem 4.12 *For $k \geq 1$, the following two equations hold true:*

$$P\Big((13\overset{k+1}{3\ldots}3) \,|A(t) = 1\Big) = p_1 p_3^k (1 + (1 - p_1 - p_2 - p_3) \sum_{n=0}^{k-1} (1 - p_1)^n \qquad (21)$$

$$P\Big((23\overset{k+1}{3\ldots}3) \,|A(t) = 1\Big) = p_2 p_3^k (1 + (1 - p_1 - p_2 - p_3) \sum_{n=0}^{k} (1 - p_1)^n \qquad (22)$$

For equation (21), it should be noted that

$$P\Big((13\overset{\text{k+1}}{3\ldots}3) \,|A(t) = 1\Big) = p_1 \cdot P\Big((3\overset{\text{k}}{3\ldots}3) \,|A(t) = 1\Big),$$

and by using Lemma 4.7 it is quickly found that equation (21) holds true. Equation (22) can be found by using Lemma 4.5 and and a modified version of Lemma 4.6. As it is proven that $P\Big(Q_{3(1)}(t + k + 1) > 0, (3\overset{\text{k}}{3\ldots}3) \,|A(t) = 1\Big) = p_3 p_3^k$, the argument that the first factor p_3 comes from the first customer in the queue at time $t + k + 1$ makes it feasible to argue that $P\Big(Q_{2(1)}(t + k + 1) > 0, (3\overset{\text{k}}{3\ldots}3) \,|A(t) = 1\Big) = p_2 p_3^k$. Therefore

$$\begin{aligned}
P\Big((23\overset{\text{k+1}}{3\ldots}3) \,|A(t) = 1\Big) &= p_2(1 - p_1) P\Big(Q(t + k + 1) = 0, (3\overset{\text{k}}{3\ldots}3) \,|A(t) = 1\Big) \\
&\quad + (1 - p_1) P\Big(Q_{2(1)}(t + k + 1) > 0, (3\overset{\text{k}}{3\ldots}3) \,|A(t) = 1\Big) \\
&= p_2 \Big[p_3^k (1 - p_1 - p_2 - p_3) \sum_{n=0}^{k} (1 - p_1)^n\Big] + p_2 p_3^k \\
&= p_2 p_3^k \Big[1 + (1 - p_1 - p_2 - p_3) \sum_{n=0}^{k} (1 - p_1)^n\Big],
\end{aligned}$$

Concluding theorem 4.12. □.

Theorem 4.13 *For $k \geq 1$, the following two equations hold true:*

$$P\Big((13\overset{k+1}{3\ldots}3) \,|A(t) = 1\Big) = p_1 p_3^k \qquad (23)$$

$$P\Big((23\overset{k+1}{3\ldots}3) \,|A(t) = 1\Big) = p_2 p_3^k \qquad (24)$$

Using Lemma 4.11, it is again straightforward to argue that

$$\begin{aligned}
P\Big((13\overset{\text{k+1}}{3\ldots}3) \,|A(t) = 1\Big) &= p_1 P\Big((3\overset{\text{k}}{3\ldots}3) \,|A(t) = 1\Big) \\
&= p_1 p_3^k,
\end{aligned}$$

concluding the first part of the proof. Lemma 4.10 can be exploited by arguing once more that the first type 3 vehicle in the queue at time $t+k+1$ in the left hand side of $(1-p_1)P\Big(Q_{3(1)}(t+k+1) >$

$0,(\overset{\text{k}}{33\ldots3})\,|A(t)=2\Big) = p_3\cdot p_3^k\frac{p_1}{1-p_2-p_3}$ accounts for a factor p_3 in the right hand side, whereas if the first vehicle in this queue would be of type 2, this factor would be equal to p_2. Therefore $(1-p_1)P\Big(Q_{2(1)}(t+k+1) > 0,(\overset{\text{k}}{33\ldots3})\,|A(t)=2\Big) = p_2\cdot p_3^k\frac{p_1}{1-p_2-p_3}$, and expanding on equation (24) yields

$$\begin{aligned} P\Big((\overset{\text{k+1}}{233\ldots3})\,|A(t)=1\Big) &= p_2(1-p_1)P\Big(Q(t+k+1)=0,(\overset{\text{k}}{33\ldots3})\,|A(t)=2\Big) \\ &\quad + (1-p_1)P\Big(Q_{2(1)}(t+k+1)>0,(\overset{\text{k}}{33\ldots3})\,|A(t)=2\Big) \\ &= p_2p_3^k\frac{1-p_1-p_2-p_3}{1-p_2-p_3} + p_2p_3^k\frac{p_1}{1-p_2-p_3} \\ &= p_2p_3^k, \end{aligned}$$

concluding the proof of theorem 4.13. □

The first step of the proof of concept is therefore found in theorems 4.12 and 4.13. For the second step of the proof of concept, it is necessary to find the probability that, given an occupation of type 1 or type 2 at time t, there are k consecutive occupations of type 3, followed by no occupation at time $t+k+1$. Therefore the following theorem becomes

Theorem 4.14

$$P\Big((\overset{k+1}{033\ldots3})\,|A(t)=1\Big) = p_3^k(1-p_1-p_2-p_3)(1-p_2-p_3)\sum_{n=0}^{k}(1-p_1)^n \tag{25}$$

$$P\Big((\overset{k+1}{033\ldots3})\,|A(t)=2\Big) = p_3^k(1-p_1-p_2-p_3) \tag{26}$$

We can argue that this probability is equal to the probability that no arrival of type 1, 2 or 3 will come at time $t+k+1$, times the probability that the queue was empty at time $t+k+1$. Using Lemma 4.5, it follows that

$$\begin{aligned} P\Big((\overset{\text{k+1}}{033\ldots3})\,|A(t)=1\Big) &= (1-p_1)(1-p_2-p_3)P\Big(Q(t+k+1)=0,(\overset{\text{k}}{33\ldots3})\,|A(t)=1\Big) \\ &= (1-p_2-p_3)\bigg((1-p_1)P\Big(Q(t+k+1)=0,(\overset{\text{k}}{33\ldots3})\,|A(t)=1\Big)\bigg) \\ &= (1-p_2-p_3)\Big[p_3^k(1-p_1-p_2-p_3)\sum_{n=0}^{k}(1-p_1)^n\Big] \\ &= p_3^k(1-p_1-p_2-p_3)(1-p_2-p_3)\sum_{n=0}^{k}(1-p_1)^n. \end{aligned}$$

Using Lemma 4.9, it follows that

$$\begin{aligned} P\Big((\overset{\text{k+1}}{033\ldots3})\,|A(t)=2\Big) &= (1-p_1)(1-p_2-p_3)P\Big(Q(t+k+1)=0,(\overset{\text{k}}{33\ldots3})\,|A(t)=2\Big) \\ &= (1-p_2-p_3)\bigg((1-p_1)P\Big(Q(t+k+1)=0,(\overset{\text{k}}{33\ldots3})\,|A(t)=2\Big)\bigg) \end{aligned}$$

$$= (1 - p_2 - p_3)\Big[p_3^k \frac{1 - p_1 - p_2 - p_3}{1 - p_2 - p_3}\Big]$$
$$= p_3^k(1 - p_1 - p_2 - p_3),$$

Concluding the proof of theorem 4.14. □

4.2.4 Concept of proof: part 3

Lemma 4.15 *Let us define the probability* $P(A(t + k + 1) = x_0, 3_m, 0_n|A(t) = 0)$ $= P\Big(\overbrace{(\mathrm{x_0 x_1 \dots x_k})}^{k+1} |A(t) = 0\Big)$ *where* $x_0 \in \{1, 2\}$ *and* $x_1, x_2, ..., x_k \in \{0, 3\}$ *in an undetermined order, such that there are m type 3 occupations and n type 0 occupations at time* $t+1, t+2, ..., t+k$*, so that* $m + n = k$*. Then*

$$P\Big(A(t + k + 1) = 1, 3_n, 0_m|A(t) = 0\Big) = \binom{k}{m} p_1(1 - p_1)^k p_3^m(1 - p_2 - p_3)^n \tag{27}$$
$$P\Big(A(t + k + 1) = 2, 3_n, 0_m|A(t) = 0\Big) = \binom{k}{m} p_2(1 - p_1)^{k+1} p_3^m(1 - p_2 - p_3)^n \tag{28}$$

For the proof of Lemma 4.15, a nice property can be exploited: given that at time t there is no occupation, it must be true that at time $t + 1$, the queue is empty, because if it would be non-empty it would have delivered a type 2 or type 3 occupation at time t, which is not possible. So $Q(t + 1) = 0$ is always true given $A(t) = 0$. Furthermore it should be noted that the queue will stay empty if no arrivals of type 1 occur. Therefore, if no type 1 arrivals take place for k timesteps, the queue will stay empty at time $t + 1, t + 2, ..., t + k$. As a consequence, the probability that a type 3 occupation occurs given that the queue is empty, is equal to $(1 - p_1)p_3$, and the probability that no occupation occurs given that te queue is empty, is equal to $(1-p_1)(1-p_2-p_3)$. So the probability that m type 3 occupations and n type 0 occupations (no occupations) *in a specific order* take place at time $t+1, t+2, ..., t+k$, given that $A(t) = 0$ is equal to $(1 - p_1)^m p_3^m \cdot (1 - p_1)^n(1 - p_2 - p_3)^n$. This probability can be defined as $P(3_n, 0_m|A(t) = 0)$ so that $P(3_n, 0_m|A(t) = 0) = (1 - p_1)^k p_3^m(1 - p_2 - p_3)^n$.

Still at time $t + k + 1$, the queue is empty because no type 1 vehicles have arrived between time t and $t + k$. Therefore, an occupation of type 2 at time $t + k + 1$ after m type 3 and n type 0 occupations in a specific order, given that $A(t) = 0$, would be equal to $(1-p_1)p_2 \cdot P(3_n, 0_m|A(t) = 0)$. And for a type 1 occupation at time $t + k + 1$ it will be $p_1 \cdot P(3_n, 0_m|A(t) = 0)$. For equation (27) this yields:

$$P^{\text{fixed order}}(A(t + k + 1) = 1, 3_n, 0_m|A(t) = 0) = p_1 \cdot P(3_n, 0_m|A(t) = 0)$$
$$= p_1(1 - p_1)^k p_3^m(1 - p_2 - p_3)^n.$$

And for equaion (28) this yields:

$$P^{\text{fixed order}}(A(t + k + 1) = 2, 3_n, 0_m|A(t) = 0) = p_2(1 - p_1) \cdot P(3_n, 0_m|A(t) = 0)$$
$$= p_2(1 - p_1)(1 - p_1)^k p_3^m(1 - p_2 - p_3)^n$$
$$= p_2(1 - p_1)^{k+1} p_3^m(1 - p_2 - p_3)^n.$$

It must be noticed that the type 3 and type 0 occupations at time $t + 1, t + 2, ...t + k$ are in fixed order, however when 'changing' the order of type 3 and type 0 occupations, the probability does not change. Therefore we find that for an undetermined order of m type 3 vehicles, and n type

0 vehicles involves the choose function, where there are k occupations to fill for all the type 3 and type 0 occupations, where m of them are of type 3. Therefore

$$\begin{aligned} P(A(t+k+1) = 1, 3_n, 0_m|A(t) = 0) &= \binom{k}{m} P^{\text{fixed order}}(A(t+k+1) = 1, 3_n, 0_m|A(t) = 0) \\ &= \binom{k}{m} p_1(1-p_1)^k p_3^m (1-p_2-p_3)^n, \end{aligned}$$

and

$$\begin{aligned} P(A(t+k+1) = 2, 3_n, 0_m|A(t) = 0) &= \binom{k}{m} P^{\text{fixed order}}(A(t+k+1) = 2, 3_n, 0_m|A(t) = 0) \\ &= \binom{k}{m} p_2(1-p_1)^{k+1} p_3^m (1-p_2-p_3)^n, \end{aligned}$$

This concludes the proof of Lemma 4.15. □

4.2.5 Proof interarrival times for k interarrivals

Consider the family of probabilities $P\Big(\overset{\text{k+1, 3(m)}}{(\mathrm{x_0 x_1 \ldots x_k})} \,|x_{k+1} \Big)$, which is equal to the probability $P\Big(\overset{\text{k+1}}{(\mathrm{x_0 x_1 \ldots x_k})} \,|x_{k+1} \Big)$ where $x_0, x_{k+1} \in \{1, 2\}$ and where $x_1, x_2, ..., x_k \in \{0, 3\}$, where the set $\{x_1, ..., x_k\}$ contains m of type 3 and $k-m$ of type 0, where $k - m \geq 1$ in a fixed order. Then this probability is equal to the product of the two probabilities $P\Big(\overset{\text{k-j, 3(m-j)}}{(\mathrm{x_0 x_1 \ldots x_{k\text{-}j\text{-}1}})} \,|x_{k-j} \Big)$ and $P\Big(\overset{\text{j+1, 3(j)}}{(\mathrm{x_{k\text{-}j} \ldots x_k})} \,|x_{k+1} \Big)$, where x_{k-j} will be the first type 0 vehicle after the type x_{k+1} vehicle, such that $x_{k-j+1}, x_{k-j+1}, ...x_k$ are all of type 3. This can be argued because after a type 0 occupation, the queue is empty again, and so the same argument holds as used in the proof of Lemma 4.15. The following four Lemma's (Lemma's 4.16 until 4.19) find a direct expression for the probability that, given a type 1 or type 2 occupation at time t, there are no type 1 or type 2 occupations at time $t+1, t+2, ..., t+k$, and the occupation at time $t+k+1$ is of type 1 or type 2 again.

Lemma 4.16 *Let $\{x_1, ...x_k\}$ be of type 3 and type 0, with at least one of type 0. Then*

$$P\Big(\overset{k+1}{(1\ \mathrm{x_1 \ldots x_k})} \,|A(t) = 1 \Big) = p_1(1-p_1-p_2-p_3)(1-p_2-p_3)\alpha^{k-1} \sum_{j=0}^{k-1} \Big((\frac{p_3}{\alpha})^j \frac{1-(1-p_1)^{j+1}}{p_1} \Big)$$

We start by arguing that when $\{x_1, ...x_k\}$ contains m type 3 occupations, that the probability that the first j of those m type 3 occupations occur before the first empty occupation (type 0) is equal to

$$\begin{aligned} &P\Big(\overset{\text{k+1, 3(m)}}{(1\ \mathrm{x_1 \ldots x_k})}, \text{first } j \text{ occupations after A(t) are type 3 vehicles}|A(t) = 1 \Big) \\ &= \sum_{m-j=0}^{k-m} P\Big(\overset{\text{k-j, 3(m-j)}}{(1\ \mathrm{x_1 \ldots x_{k\text{-}j\text{-}1}})} \,|A(t) = 0 \Big) \cdot P\Big(\overset{\text{j+1}}{(033 \ldots 3)} \,|A(t) = 1 \Big) \end{aligned}$$

Expanding on these equations on the right hand side, when substituting $x = m-j$ and $y = k-j-$, and using 4.15 that

$$\begin{aligned}
\sum_{m-j=0}^{k-j} P\Big((1 \overset{\text{k-j, 3(m-j)}}{\text{x}_1 \dots \text{x}_{\text{k-j-1}}}) \,|A(t)=0\Big) &= \sum_{m-j=0}^{k-j} P\Big(A(t+k-j+1)=1, 3_{m-j}, 0_{k-m}|A(t)=0\Big) \\
&= \sum_{m-j=0}^{k-j} \binom{k-j}{m-j} p_1(1-p_1)^{k-j} p_3^{m-j}(1-p_2-p_3)^{k-m} \\
&= p_1(1-p_1)^y \sum_{x=0}^{y} \binom{y}{x} p_3^x (1-p_2-p_3)^{y-x} \\
&= p_1(1-p_1)^y \cdot (1-p_2-p_3+p_3)^y \\
&= p_1(1-p_1)^y(1-p_2)^y \\
&= p_1\alpha^{k-j}.
\end{aligned}$$

Looking at the original expression again, and using Theorem 4.14, it is found that

$$\begin{aligned}
& P\Big((1 \overset{\text{k+1, 3(m)}}{\text{x}_1 \dots \text{x}_{\text{k}}}), \text{first } j \text{ occupations after A(t) are type 3 vehicles}|A(t)=1\Big) \\
= \ & p_1\alpha^{k-j} \cdot P\Big((0\overset{\text{j+1}}{33\dots}3) \,|A(t)=1\Big) \\
= \ & p_1\alpha^{k-j} p_3^j (1-p_1-p_2-p_3)(1-p_2-p_3) \sum_{n=0}^{j} (1-p_1)^n
\end{aligned}$$

Now summing $P\Big((1 \overset{\text{k+1, 3(m)}}{\text{x}_1 \dots \text{x}_{\text{k}}}), \text{first } j \text{ occupations after A(t) are type 3 vehicles}|A(t)=1\Big)$ over all possible j's from 0 to $k-1$, all possible combinations are found where there are k type 0 or type 3 occupations between $A(t)=1$ and $A(t+k+1)=1$. Therefore:

$$\begin{aligned}
P\Big((1 \overset{\text{k+1}}{\text{x}_1 \dots \text{x}_{\text{k}}}) \,|A(t)=1\Big) &= \sum_{j=0}^{k-1} P\Big((1 \overset{\text{k+1, 3(m)}}{\text{x}_1 \dots \text{x}_{\text{k}}}), \text{first } j \text{ after A(t) are type 3 vehicles}|A(t)=1\Big) \\
&= \sum_{j=0}^{k-1} p_1\alpha^{k-j} p_3^j (1-p_1-p_2-p_3)(1-p_2-p_3) \sum_{n=0}^{j} (1-p_1)^n \\
&= p_1(1-p_2-p_3)(1-p_1-p_2-p_3)\alpha^k \sum_{j=0}^{k-1} \big(\frac{p_3}{\alpha}\big)^j \frac{1-(1-p_1)^{j+1}}{p_1} \\
&= (1-p_2-p_3)(1-p_1-p_2-p_3)\alpha^k \sum_{j=0}^{k-1} \Bigg(\big(\frac{p_3}{\alpha}\big)^j \big(1-(1-p_1)^{j+1}\big)\Bigg).
\end{aligned}$$

This concludes the proof of Lemma 4.16. □

The key for the proof of Lemma 4.16, that

$$\sum_{m-j=0}^{k-j} P\Big((1 \overset{\text{k-j, 3(m-j)}}{\text{x}_1 \dots \text{x}_{\text{k-j-1}}}) \,|A(t)=0\Big) = p_1\alpha^{k-j} \tag{29}$$

can be exploited again in the following Lemma:

Lemma 4.17 *Let $\{x_1, ...x_k\}$ be of type 3 and type 0, with at least one type 0. Then*

$$P\Big((1\ \overset{k+1}{\mathrm{x_1 \ldots x_k}})\,|A(t)=2\Big) = p_1(1-p_1-p_2-p_3)\alpha^k \frac{1-\left(\frac{p_3}{\alpha}\right)^k}{\alpha - p_3}$$

Using the same technique as in Lemma 4.16, equation (29) and employing Theorem 4.14, the following calculations show that

$$\begin{aligned}
P\Big((1\ \overset{\text{k+1}}{\mathrm{x_1 \ldots x_k}})\,|A(t)=2\Big) &= \sum_{j=0}^{k-1}\sum_{m-j=0}^{k-j} P\big((1\ \overset{\text{k-j, 3(m-j)}}{\mathrm{x_1 \ldots x_{k\text{-}j\text{-}1}}})\,|A(t)=0\big)\cdot P\big((\overset{\text{j+1}}{033\ldots 3})\,|A(t)=2\big)\\
&= \sum_{j=0}^{k-1} p_1\alpha^{k-j} P\big((\overset{\text{j+1}}{033\ldots 3})\,|A(t)=2\big)\\
&= \sum_{j=0}^{k-1} p_1\alpha^{k-j} p_3^j (1-p_1-p_2-p_3)\\
&= p_1(1-p_1-p_2-p_3)\alpha^k \sum_{j=0}^{k-1}\Big(\frac{p_3}{\alpha}\Big)^j\\
&= p_1(1-p_1-p_2-p_3)\alpha^k \frac{1-(\frac{p_3}{\alpha})^k}{\alpha-p_3}.
\end{aligned}$$

This concludes the proof of Lemma 4.17. □

Lemma 4.18 *Let $\{x_1, ...x_k\}$ be of type 3 and type 0, with at least one type 0. Then*

$$\begin{aligned}
&P\Big((2\ \overset{k+1}{\mathrm{x_1 \ldots x_k}})\,|A(t)=1\Big)\\
= \quad & p_2(1-p_1)(1-p_1-p_2-p_3)(1-p_2-p_3)\alpha^k \sum_{j=0}^{k-1}\Big((\frac{p_3}{\alpha})^j \frac{1-(1-p_1)^{j+1}}{p_1}\Big)
\end{aligned} \tag{30}$$

Like in Lemma 4.17, it is argued that

$$P\Big((2\ \overset{\text{k+1}}{\mathrm{x_1 \ldots x_k}})\,|A(t)=1\Big) = \sum_{j=0}^{k-1}\sum_{m-j=0}^{k-j} P\big((2\ \overset{\text{k-j, 3(m-j)}}{\mathrm{x_1 \ldots x_{k\text{-}j\text{-}1}}})\,|A(t)=0\big)\cdot P\big((\overset{\text{j+1}}{033\ldots 3})\,|A(t)=1\big).$$

Using equation (28) of Lemma 4.15, and once more performing the substitution $x = m - j$ and $y = k - j$, it can be obtained that

$$\begin{aligned}
\sum_{m-j=0}^{k-m} P\big((2\ \overset{\text{k-j, 3(m-j)}}{\mathrm{x_1 \ldots x_{k\text{-}j\text{-}1}}})\,|A(t)=0\big) &= \sum_{m-j=0}^{k-j} P\Big(A(t+k-j+1)=1, 3_{m-j}, 0_{k-m}|A(t)=0\Big)\\
&= \sum_{m-j=0}^{k-j}\binom{k-j}{m-j} p_2(1-p_1)^{k-j} p_3^{m-j}(1-p_2-p_3)^{k-m}\\
&= p_2(1-p_1)(1-p_1)^y \sum_{x=0}^{y}\binom{y}{x} p_3^x (1-p_2-p_3)^{y-x}
\end{aligned}$$

$$
\begin{aligned}
&= p_2(1-p_1)(1-p_1)^y \cdot (1-p_2-p_3+p_3)^y \\
&= p_2(1-p_1)(1-p_1)^y(1-p_2)^y \\
&= p_2(1-p_1)\alpha^{k-j}.
\end{aligned}
$$

Therefore, using Theorem 4.14,

$$
\begin{aligned}
& P\Big(\,(2\ \mathrm{x_1} \overset{\text{k+1}}{\ldots} \mathrm{x_k})\,|A(t)=1\Big) \\
= \quad & \sum_{j=0}^{k-1} p_2(1-p_1)\alpha^{k-j} P\big(\,(03\overset{\text{j+1}}{3}\ldots 3)\,|A(t)=1\big) \\
= \quad & \sum_{j=0}^{k-1} p_2(1-p_1)\alpha^{k-j} p_3^j (1-p_1-p_2-p_3)(1-p_2-p_3)\sum_{n=0}^{j}(1-p_1)^n \\
= \quad & p_2(1-p_1)(1-p_1-p_2-p_3)(1-p_2-p_3)\alpha^k \sum_{j=0}^{k-1}\left(\left(\frac{p_3}{\alpha}\right)^j \frac{1-(1-p_1)^{j+1}}{p_1}\right),
\end{aligned}
$$

which concludes the proof of Lemma 4.18. □

Lemma 4.19 *Let* $\{x_1, ...x_k\}$ *be of type 3 and type 0, with at least one type 0. Then*

$$
P\Big(\,(2\ \mathrm{x_1} \overset{k+1}{\ldots} \mathrm{x_k})\,|A(t)=2\Big) = p_2(1-p_1)(1-p_1-p_2-p_3)\alpha^k \frac{1-\left(\frac{p_3}{\alpha}\right)^k}{\alpha - p_3}
$$

Once more the key for the proof of Lemma 4.18, that

$$
\sum_{m-j=0}^{k-j-1} P\big(\,(2\ \mathrm{x_1} \overset{\text{k-j, 3(m-j)}}{\ldots} \mathrm{x_{k\text{-}j\text{-}1}})\,|A(t)=0\big) = p_2(1-p_1)\alpha^{k-j}, \tag{31}
$$

can be used again. Using the same technique as in Lemma 4.18, equation (31) and employing Theorem 4.14, the following calculations show that

$$
\begin{aligned}
P\Big(\,(2\ \mathrm{x_1} \overset{\text{k+1}}{\ldots} \mathrm{x_k})\,|A(t)=2\Big) &= \sum_{j=0}^{k-1}\sum_{m-j=0}^{k-j} P\big(\,(2\ \mathrm{x_1} \overset{\text{k-j, 3(m-j)}}{\ldots} \mathrm{x_{k\text{-}j\text{-}1}})\,|A(t)=0\big)\cdot P\big(\,(03\overset{\text{j+1}}{3}\ldots 3)\,|A(t)=2\big) \\
&= \sum_{j=0}^{k-1} p_2(1-p_1)\alpha^{k-j} P\big(\,(03\overset{\text{j+1}}{3}\ldots 3)\,|A(t)=2\big) \\
&= \sum_{j=0}^{k-1} p_2(1-p_2)\alpha^{k-j} p_3^j (1-p_1-p_2-p_3) \\
&= p_2(1-p_1)(1-p_1-p_2-p_3)\alpha^k \sum_{j=0}^{k-1}\left(\frac{p_3}{\alpha}\right)^j \\
&= p_2(1-p_1)(1-p_1-p_2-p_3)\alpha^k \frac{1-(\frac{p_3}{\alpha})^k}{\alpha-p_3}.
\end{aligned}
$$

□

Which concludes the last Lemma 4.19. Combining Lemma's 4.16, 4.17, 4.18 and 4.19, and the earlier proven Theorems 4.12 and 4.13, the interarrival times for type 1 or type 2 vehicles can be found:

Theorem 4.20 *Consider the discrete random variables X(t), Y(t) and Z(t) to be geometrically distributed, the interarrival times of the merging process of X(t) and* $\Big[Y(t)+Z(t)\Big]$ *follow the probability mass function* $P(I_{X+Y+Z})$ *according to Lemma 3.2. Then the splitting of the process* $\Big[X(t)+Y(t)\Big]$ *and Z(t) leaves the interarrival times* I_{exit}*, where the probability mass function takes the following form:*

$$P(I_{exit}=k)=\begin{cases}1-\alpha+\phi, & k=0\\ \alpha^k\cdot(1-\alpha+\phi)-\phi(1-p_3)\frac{\alpha^k-p_3^k}{\alpha-p_3}, & k\in\mathbb{Z}_+,\end{cases} \tag{32}$$

where $\alpha=(1-p_1)(1-p_2)$ *and* $\phi=\frac{p_1p_2}{p_1+p_2}(1-p_3)$

Proof:

For $k=0$, Theorem 4.3 has proven the correctness already. For $k\in\mathbb{Z}_+$, the desired answer is found in the conditional probability

$$P(I_{exit}=k)=\frac{P\Big(A(t)\in\{1,2\},A(t+i)\in\{0,3\},A(t+k+1)\in\{1,2\},i=1,2,...,k\Big)}{P\Big(A(t)\in\{1,2\}\Big)}. \tag{33}$$

This probability is equal to the sum of all the equations of Lemma's 4.16 , 4.18, and Theorem 4.12 times the probability that a type 1 occupation occurs, added by the equations from Lemma's 4.17, 4.19 and Theorem 4.13 times the probability that a type 2 occupation occurs, divided by $P\Big(A(t)\in\{1,2\}\Big)$. This yields that k interarrivals between two occupations of type 1 or type 2, the numerator of equation (33) is equal to the sum of the following 8 components:

1. $p_1p_1(1-p_1-p_2-p_3)(1-p_2-p_3)\alpha^{k-1}\sum_{j=0}^{k-1}\Big((\frac{p_3}{\alpha})^j\frac{1-(1-p_1)^{j+1}}{p_1}\Big)$ Lemma 4.16
2. $p_1p_2(1-p_1)(1-p_1-p_2-p_3)(1-p_2-p_3)\alpha^{k-1}\sum_{j=0}^{k-1}\Big((\frac{p_3}{\alpha})^j\frac{1-(1-p_1)^{j+1}}{p_1}\Big)$ Lemma 4.18
3. $p_1p_1p_3^k\Big[1+(1-p_1-p_2-p_3)\sum_{n=0}^{k-1}(1-p_1)^n\Big]$ Theorem 4.12
4. $p_1p_2p_3^k\Big[1+(1-p_1-p_2-p_3)\sum_{n=0}^{k}(1-p_1)^n\Big]$ Theorem 4.12
5. $p_2p_1(1-p_1-p_2-p_3)\alpha^{k-1}\Big(\frac{1-\frac{p_3}{\alpha}^k}{\alpha-p_3}\Big)$ Lemma 4.17
6. $p_2p_2(1-p_1)(1-p_1-p_2-p_3)\alpha^{k-1}\Big(\frac{1-\frac{p_3}{\alpha}^k}{\alpha-p_3}\Big)$ Lemma 4.19
7. $p_2p_1p_3^k$ Theorem 4.13
8. $p_2p_2p_3^k$ Theorem 4.13

What follows is a combination of some pairs of the components. The sum of component 1 and 2 is equal to

$$\Big(1-\alpha\Big)(1-p_1-p_2-p_3)(1-p_2-p_3)\alpha^k\left(\frac{1-\big(\frac{p_3}{\alpha}\big)^k}{\alpha-p_3}-(1-p_1)\frac{1-\big(\frac{(1-p_1)p_3}{\alpha}\big)^k}{\alpha-(1-p_1)p_3}\right), \tag{34}$$

and the sum of components 5 and 6 is equal to

$$p_2\Big(1-\alpha\Big)(1-p_1-p_2-p_3)\alpha^k\Big(\frac{1-\frac{p_3}{\alpha}^k}{\alpha-p_3}\Big). \tag{35}$$

Adding components 1, 2, 5 and 6, and using that $\alpha - (1-p_1)p_3 = (1-p_1)(1-p_2-p_3)$, this equals

$$\begin{aligned}
\text{eq. (34)} + \text{eq. (35)} &= \Big(1-\alpha\Big)(1-p_1-p_2-p_3)\cdot\left[p_2\frac{\alpha^k - p_3^k}{\alpha - p_3}\right] \\
&+ \Big(1-\alpha\Big)(1-p_1-p_2-p_3)\cdot\left[(1-p_2-p_3)\Big(\frac{\alpha^k-p_3^k}{\alpha-p_3} - (1-p_1)\frac{\alpha^k-(1-p_1)^k p_3^k}{\alpha-(1-p_1)p_3}\Big)\right] \\
&= \Big(1-\alpha\Big)(1-p_1-p_2-p_3)\cdot\left[(1-p_3)\Big(\frac{\alpha^k-p_3^k}{\alpha-p_3}\Big) - \alpha^k + (1-p_1)^k p_3^k\right]
\end{aligned}$$

Furthermore components 3 and 4 must be rewritten:

$$p_1^2 p_3^k\Big[1+(1-p_1-p_2-p_3)\sum_{n=0}^{k-1}(1-p_1)^n\Big] = p_1^2 p_3^k\Big[1+(1-p_1-p_2-p_3)\frac{1-(1-p_1)^k}{p_1}\Big] \tag{36}$$

and

$$p_1p_2p_3^k\Big[1+(1-p_1-p_2-p_3)\sum_{n=0}^{k}(1-p_1)^n\Big] = p_1p_2p_3^k\Big[1+(1-p_1-p_2-p_3)\frac{1-(1-p_1)^{k+1}}{p_1}\Big]. \tag{37}$$

Summing components 3 and 4 yields

$$p_1(p_1+p_2)p_3^k + p_3^k(1-p_1-p_2-p_3)\Big[p_1\big(1-(1-p_1)^k\big) + p_2\big(1-(1-p_1)^{k+1}\big)\Big].$$

And of course summing components 7 and 8 gives $(p_1+p_2)p_2p_3^k$. Therefore summing components 3, 4, 7 and 8 gives

$$(p_1+p_2)^2p_3^k + p_3^k(1-p_1-p_2-p_3)\Big[(p_1+p_2)+(1-\alpha)(1-p_1)^k\Big]. \tag{38}$$

Multiplying both sides of equation (33) by $P\Big(A(t)\in\{1,2\}\Big) = p_1+p_2$, we find that $(p_1+p_2)P(I_{exit}=k) = Eq.(34) + Eq.(35) + Eq.(38)$ for $k \in \mathbb{Z}_+$:

$$\begin{aligned}
(p_1+p_2)P(I_{exit}=k) &= \Big(1-\alpha\Big)(1-p_1-p_2-p_3)\cdot\left[(1-p_3)\Big(\frac{\alpha^k-p_3^k}{\alpha-p_3}\Big) - \alpha^k + (1-p_1)^k p_3^k\right] \\
&\quad + (p_1+p_2)^2p_3^k + p_3^k(1-p_1-p_2-p_3)\Big[(p_1+p_2)-(1-\alpha)(1-p_1)^k\Big] \\
&= \Big(1-\alpha\Big)(1-p_1-p_2-p_3)\left[(1-p_3)\Big(\frac{\alpha^k-p_3^k}{\alpha-p_3}\Big) - \alpha^k\right] \\
&\quad + (p_1+p_2)^2p_3^k + p_3^k(1-p_1-p_2-p_3)(p_1+p_2) \\
&= \Big(1-\alpha\Big)(1-p_1-p_2-p_3)\left[(1-p_3)\Big(\frac{\alpha^k-p_3^k}{\alpha-p_3}\Big) - \alpha^k\right] \\
&\quad + (p_1+p_2)(1-p_3)p_3^k \\
&= (p_1+p_2)(1-\alpha)\left[\alpha^k - (1-p_3)\Big(\frac{\alpha^k-p_3^k}{\alpha-p_3}\Big)\right] + (p_1+p_2)(1-p_3)p_3^k
\end{aligned}$$

$$
\begin{aligned}
&+ (1-\alpha)(1-p_3)\left[(1-p_3)\Big(\frac{\alpha^k - p_3^k}{\alpha - p_3}\Big) - \alpha^k\right] \\
&= (p_1+p_2)(1-\alpha)\left[\alpha^k - (1-p_3)\Big(\frac{\alpha^k - p_3^k}{\alpha - p_3}\Big)\right] + (p_1+p_2)(1-p_3)p_3^k \\
&+ (p_1+p_2-p_1p_2)(1-p_3)\left[(1-p_3)\Big(\frac{\alpha^k - p_3^k}{\alpha - p_3}\Big) - \alpha^k\right] \\
&= (p_1+p_2)(1-\alpha)\left[\alpha^k - (1-p_3)\Big(\frac{\alpha^k - p_3^k}{\alpha - p_3}\Big)\right] + (p_1+p_2)(1-p_3)p_3^k \\
&+ (p_1+p_2)(1-p_3)\left[(1-p_3)\Big(\frac{\alpha^k - p_3^k}{\alpha - p_3}\Big) - \alpha^k\right] \\
&- p_1p_2(1-p_3)\left[(1-p_3)\Big(\frac{\alpha^k - p_3^k}{\alpha - p_3}\Big) - \alpha^k\right]
\end{aligned}
$$

.

Dividing both sides by $p_1 + p_2$ yields

$$
\begin{aligned}
P(I_{exit} = k) &= (1-\alpha)\alpha^k - (1-\alpha)(1-p_3)\Big(\frac{\alpha^k - p_3^k}{\alpha - p_3}\Big) + (1-p_3)p_3^k + (1-p_3)^2\Big(\frac{\alpha^k - p_3^k}{\alpha - p_3}\Big) \\
&- (1-p_3)\alpha^k - \frac{p_1p_2}{p_1+p_2}(1-p_3)\left[(1-p_3)\Big(\frac{\alpha^k - p_3^k}{\alpha - p_3}\Big) - \alpha^k\right] \\
&= (1-\alpha)\alpha^k + (1-p_3)\left[-(1-\alpha)\frac{\alpha^k - p_3^k}{\alpha - p_3} + p_3^k + (1-p_3)\frac{\alpha^k - p_3^k}{\alpha - p_3} - \alpha^k\right] \\
&- \phi(1-p_3)\frac{\alpha^k - p_3^k}{\alpha - p_3} + \phi\alpha^k \\
&= (1-p_3)\left[(\alpha - p_3)\frac{\alpha^k - p_3^k}{\alpha - p_3} + p_3^k - \alpha^k\right] + \alpha^k(1-\alpha+\phi) - \phi(1-p_3)\frac{\alpha^k - p_3^k}{\alpha - p_3} \\
&= (1-p_3)\left[\alpha^k - p_3^k + p_3^k - \alpha^k\right] + \alpha^k(1-\alpha+\phi) - \phi(1-p_3)\frac{\alpha^k - p_3^k}{\alpha - p_3} \\
&= \alpha^k(1-\alpha+\phi) - \phi(1-p_3)\frac{\alpha^k - p_3^k}{\alpha - p_3}.
\end{aligned}
$$

This concludes the proof of Theorem 4.20.

4.3 Differential equation for interarrival times

Further expanding on the interarrival times, it can be proven that the interarrival follow a distribution containing a differential equation:

Theorem 4.21 *Consider the process described in Theorem 4.20. Then the interarrival times can be expressed as*

$$P(I_{exit} = k) = \begin{cases} 1 - \alpha + \phi, & k = 0 \\ \alpha \cdot P(I_{exit} = k-1) - \phi(1-p_3) \cdot p_3^{k-1}, & k \in \mathbb{Z}_+, \end{cases} \tag{39}$$

where $\alpha = (1-p_1)(1-p_2)$*, and* $\phi = \frac{p_1 p_2}{p_1+p_2}(1-p_3)$*.*

The method of mathematical induction will be used to prove the correctness of equation (39). The induction hypothesis is that

$$\alpha^n(1-\alpha+\phi) - \phi(1-p_3)\frac{\alpha^n - p_3^n}{\alpha - p_3} = \alpha \cdot P(I_{exit} = n-1) - \phi(1-p_3) \cdot p_3^{n-1}, \tag{40}$$

given that the entities $P(I_{exit} = 0) = 1-\alpha+\phi$ and $P(I_{exit} = k) = \alpha^k(1-\alpha+\phi) - \phi(1-p_3)\frac{\alpha^k - p_3^k}{\alpha - p_3}$ for $k \in \mathbb{Z}_+$ hold true. The basis step requires that that equation (40) holds for $n = 1$:

$$\alpha(1-\alpha+\phi) - \phi(1-p_3)\frac{\alpha - p_3}{\alpha - p_3} = \alpha P(I_{exit} = 0) - \phi(1-p_3),$$

which proves to be true using the entity $P(I_{exit} = 0) = 1-\alpha+\phi$. Then follows the induction step. Assume that equation (40) is true for $n = k$. Then it must be shown that it is also true for $n = k+1$. Let us call $P_{ind} = \alpha^{k+1}(1-\alpha+\phi) - \phi(1-p_3)\frac{\alpha^{k+1} - p_3^{k+1}}{\alpha - p_3}$. Then

$$\begin{aligned} P_{ind} &= \alpha^{k+1}(1-\alpha+\phi) - \phi(1-p_3)\frac{\alpha^{k+1} - p_3^{k+1}}{\alpha - p_3} \\ &= \alpha^{k+1}(1-\alpha+\phi) - \phi(1-p_3)\frac{\alpha(\alpha^k - p_3^k) + p_3^k(\alpha - p_3)}{\alpha - p_3} \\ &- \alpha\alpha^k(1-\alpha+\phi) - \alpha\Big(\phi(1-p_3)\frac{\alpha^k - p_3^k}{\alpha - p_3}\Big) - \phi(1-p_3)\frac{p_3^k(\alpha - p_3)}{\alpha - p_3} \\ &= \alpha\Big(\alpha^k(1-\alpha+\phi) - \phi(1-p_3)\frac{\alpha^k - p_3^k}{\alpha - p_3}\Big) - \phi(1-p_3) \cdot p_3^k \\ &= \alpha P(I_{exit} = k) - \phi(1-p_3) \cdot p_3^k \end{aligned}$$

This last identity, $\alpha P(I_{exit} = k) - \phi(1-p_3) \cdot p_3^k$, is equal to the right hand side of equation (40) at $n = k+1$. Therefore, the proof of theorem 4.21 is completed as it is shown that induction hypothesis holds. □

The interpretation of equation (39) for $k > 0$ is the following: with probability α, the interarrival time $I_{exit} = k$ is delayed by one timestep when there is no queue of type 2 and type 3 vehicles, as α represents the probability that no type 1 or type 2 vehicle arrives. This is mainly equivalent to the interarrival times in equation (6) for $k > 0$. Like with $k = 0$, ϕ stands for the probability that if there is a queue of type 2 and type 3 vehicles, that the first vehicle in the queue is of type 2. However, there is a probability that the first k vehicles of Q_2 are of type 3 after the last arrival, and this probability is equal to $(1-p_3)p_3^{k-1}$, which can be interpreted as a geometric distribution with parameter $(1 - p_3)$. This is necessary, as the queue hits a 'success' when the arrival of a type 3 vehicle is not happening. So the factor $\phi(1-p_3) \cdot p_3^{k-1}$ signifies the probability that if there is a queue of k vehicles, the first $k-1$ are of type 3 and the last vehicle is of type 2.

5 Simulations and results

In this section the theoretical model from section 4 will be analyzed by the means of a simulation study. The method of sample batching will be used and subsequently confidence intervals can be determined to compare the theoretical model with. Firstly, a cross-simulation study will be executed, where the difference between interarrivals for independent flows on the roundabout will be compared with dependent flows on the roundabout. Secondly, a heuristic will be used to more accurately compare the theoretical framework with simulations, and the accuracy of this heuristic will be challenged. Furthermore, as an extension of the three-way roundabout, a simulation of a four-way roundabout will be carried out and will be compared to a modified version of the analytical model of a three-way roundabout. Lastly, the final research question will be treated regarding the influence of Bernoulli arrivals on a chain of roundabouts with two short scenario's in SUMO.

For the simulation study the program SUMO (Simulation of Urban MObility) is used. SUMO is an open source microscopic traffic simulator that can handle large traffic networks. The main interest in SUMO though is the realistic behavior of drivers in the simulations. Furthermore SUMO is able to generate Bernoulli flows in discrete time. Hence for tiny networks, like the simulation of a three-way or a four-way roundabout, or a chain of roundabouts, SUMO is an appropriate tool to conduct simulations to verify the findings of section 4. Incidentally it is possible to adjust the step-length of the discrete time in SUMO. As mentioned in the introduction of section 2, an average vehicle will need 2 seconds to move from its cell to the next; therefore the step-length of the simulations will be set to 2 seconds in all simulations. Additionally the maximum speed on all road segments, entrances and exits is 20 km/h, as the average speed on a roundabout is measured to be around this speed. The radius of a roundabout in all simulations is 13 meters. In Figure 5, a representation of the SUMO user interface is showed and connected to this visualisation the terminology of specific denominations of the three-way roundabout is chosen.

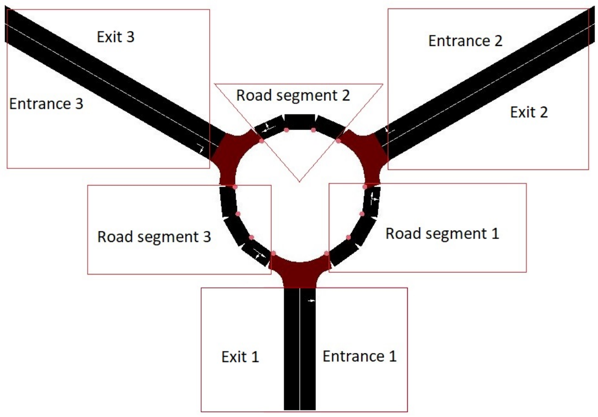

Figure 5: Visualisation of a three-way roundabout

Let the data size be the amount of interarrival times measured in a simulation. For the confidence intervals calculated for the simulations, the method of batched sampling is used. This means: the amount of interarrival times is evenly sampled in n batches, where each batch contains m interarrival times, such that the data size is equal to $n \cdot m$. In this study, the amount of batches will always be equal to $n = 30$. The data size will vary between the different simulations. Recall that if an interarrival time is equal to 1, it means that a consecutive occupation has occured, meaning that at time t and at time $t + 1$ a vehicle has occupied a road slot. If an interarrival is equal to k, it means that if at time t and at time $t + k$ an occupation has occured, but at all times between t and $t + k$, no occupation has occured.

An annotation should be given to Figure 5 with reference to Figure 1, as a first measured occupation on road segment 1 in SUMO corresponds to an occupation of S_1 in the MCA model, a first measured occupation on road segment 2 in SUMO corresponds to an occupation of S_3 in the MCA Model, and a first measured occupation on road segment 3 in SUMO corresponds to an occupation of S_5 in the MCA Model. Road segments S_2, S_4 and S_6 do not have a place in SUMO, but since no merging or splitting happens on these road segments, they are not really interesting to analyse anyway.

5.1 Simulation of Bernoulli streams in SUMO

Consider Figure 5. To test the accuracy of SUMO's Bernoulli trials, the input is given that only at entrance 1 vehicles can generate according to Bernoulli trials. Then they are all routed to exit 2, such that they leave the system there. The simulation has a data size of $\approx 1.25 \cdot 10^5$ interarrival times. Therefore the size of a batch will be equal to $\frac{1.25 \cdot 10^5}{30} \approx 4.2 \cdot 10^3$.

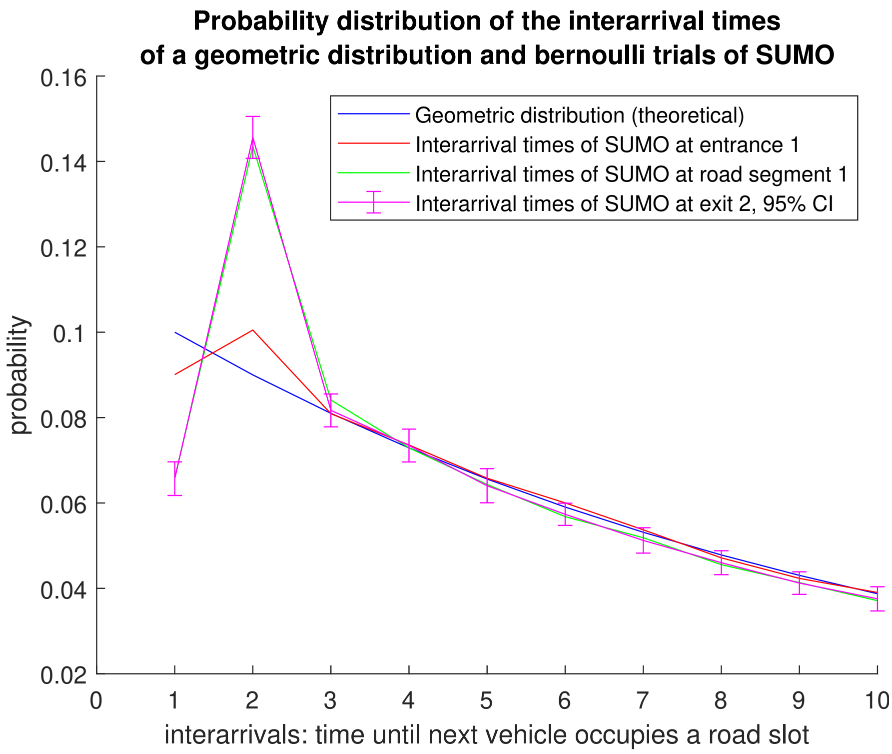

Figure 6: Bernoulli trial for $\rho = \frac{1}{10}$ in SUMO at different locations on a roundabout as depicted in Figure 5.

At a couple of positions in Figure 5, the interarrival times of the Bernoulli stream is measured. All measurements for the three positions stem from the same simulation. The three positions are the beginning of entrance 1, the beginning of road segment 1, and the beginning of exit 2. In Figure 6, it is visible that for a low traffic load of 10%, the geometric distribution follows the discrete function $\rho^{k-1}(1-\rho)$ for $k \in \mathbb{Z}_+$ with $\rho = \frac{1}{10}$, and at entrance 1 the interarrival times in SUMO approximate this geometric distribution really well, except for the first and second interarrival time. However, when vehicles move from entrance 1 to road segment 1, the first two interarrival times on road segment 1 are very excessive compared to the interarrival times of entrance 1. The behavior of the interarrival times that are measured between road segment 1 and exit 2 follow a distribution that is closely similar to the interarrival times at road segment 1, with a similar confidence interval as well. Furthermore the interarrivals for $k > 5$ at road segment 1 turn out slightly lower than that of the geometric distribution; however the 95% CI of the interarrivals at exit 2 still allow for the simulation to fall within the bounds of the geometric distribution.

Now let us consider Bernoulli arrivals at entrance 1 and entrance 2, where all arrivals exit the roundabout at exit 3. The interarrival times in SUMO are recorded at road segment 2 and exit 3. The probability distribution of the interarrival times should then follow the probability distribution according to equation (6), the theoretical framework for the probability distribution of the interarrival times of two merging Bernoulli streams where one has priority over the other. The simulation has a size of $1.5 \cdot 10^5$ interarrival times.

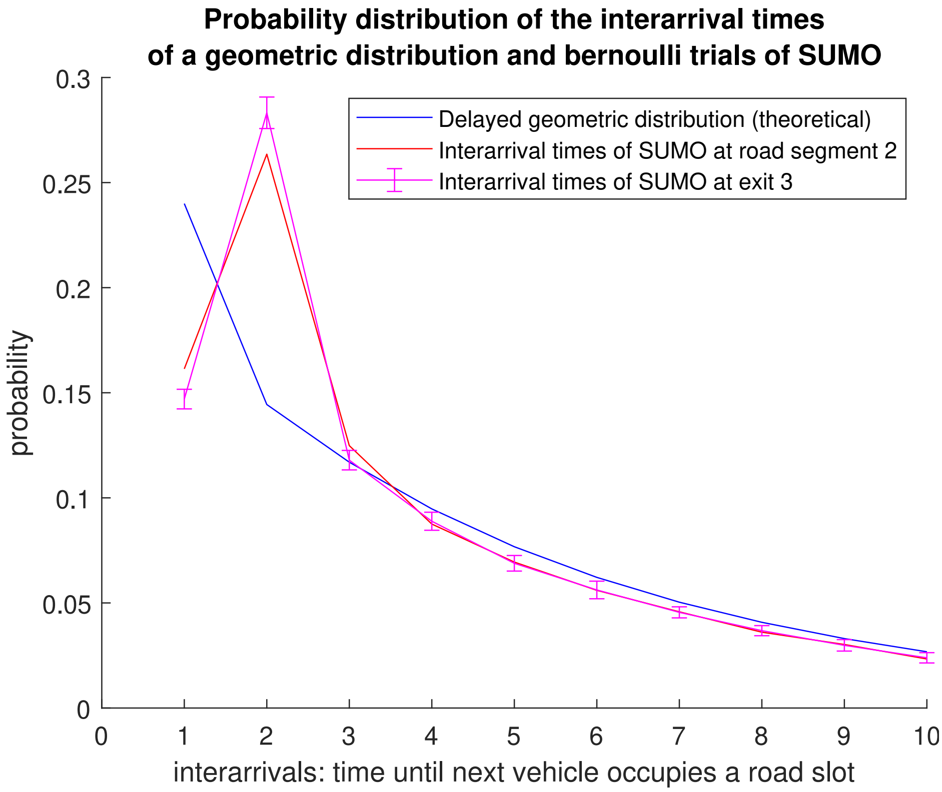

Figure 7: Bernoulli trial for arrival rate $\lambda_1 = \frac{1}{10}$ at entrance 1 and $\lambda_2 = \frac{1}{10}$ at entrance 2 in SUMO at road segment 2 and exit 3 of a roundabout as depicted in Figure 5.

Keeping in mind the discrepencies in the early interarrival times in Figure 6, in Figure 7 it can be seen that the interarrival times of the simulation at road segment 2 and exit 3 demonstrate similar behavior in comparison to the delayed geometric distribution of equation (6): the early interarrival times differ a lot, and the higher interarrivals for $k > 3$ are slightly lower again. The interarrival times at road segment 2 and exit 3 behave similarly again. However, in comparison with a single Bernoulli stream at entrance 1, in this study the theoretical framework does not fall within the 95% confidence interval of the interarrival times at exit 3 for higher interarrivals.

To conclude, from the single Bernoulli stream at entrance 1 and the double Bernoulli streams at entrance 1 and 2 both leaving at exit 3, it is observed that the behavior of drivers does not exactly fall in line with that of the theoretical frameworks. Especially for short interarrival times, the interaction between drivers seems more complex than suggested in the Markov chain in section 3. This makes an analysis to compare the theoretical framework with simulations a direct challenge.

5.2 Self-dependent and self-independent traffic flows

Consider Figure 5 again. In the formalization of the analytical model, the assumption is made that traffic flows only come from entrance 1 and entrance 2, where both flows have Bernoulli arrivals, but not from entrance 3. Let us define the arrivals at entrance 1 to be type a vehicles, arrivals at entrance 2 to be type b vehicles and arrivals at entrance 3 to be type c vehicles. Furthermore the assumption is made that on a three-way roundabout, vehicles exit the roundabout at the first or second turn. Because of the Bernoulli arrivals, the interarrival times on both entrances are geometrically distributed. As a result, the flow on road segment 1 also have Bernoulli arrivals, because no vehicle can possibly 'block' entrance 1 if no vehicles are arriving at entrance 3. Hence, the interarrival times on road segment 1 are geometrically distributed as well. On road segment 2, the interarrival times for the analytical model follow equation (6) as found in section 3. The queue that forms at entrance 2 is a $Geo/Geo/1$-queue, where the arrival times are geometrically distributed with parameter p_2, which is the probability that a type b vehicle arrives at entrance 2, and the service times are geometrically distributed with parameter $1 - p_1$, which is the probability that a type a vehicle does not block a vehicle from entrance 2. This leads to the following definition:

Definition 5.1 (queue-independency) *A queue at the entrance of a roundabout is said to be independent if the service times and the arrival times are independent from each other.*

This definition leads to the result in the specific case of the analytical model: at entrance 2 the queue follows a $Geo(1-p_1)/Geo(p_2)/1$-queue, and since parameters p_1 and p_2 do not depend on each other, therefore the queue is independent.

However, it would be closer to reality if there would also be a traffic flow at entrance 3. Therefore let us assume Bernoulli arrivals at entrance 3 as well. The problem that now arises when all three entrances generate Bernoulli arrivals, is that the individual type b vehicles that enter the roundabout on road segment 2 no longer arrive in an independent queue: a type b vehicle can cause waiting time for a type c vehicle at entrance 3; this same delayed type c vehicle can cause waiting time for a type a vehicle at entrance 1. Lastly this delayed type a vehicle can delay a type b vehicle again at entrance 2. Therefore any type b vehicle can receive possible delay because of an earlier arrived type b vehicle. It is complex to find the queueing discipline of the queue that is formed at entrance 2, but it is certain that it is not an independent queue, as the type b vehicles have an influence on the service times. And when the service times of the queue at entrance 2 change, the analytical model can no longer provide for the interarrival times on the roundabout and at the exits of the roundabout.

Through a simulations it is possible to detect the difference between a threeway roundabout with an independent queue and a dependent queue:

- Interarrival times at exit 3 in the case of an **independent** queue at entrance 2, where there are only Bernoulli arrivals at entrance 1 and entrance 2

- Interarrival times at exit 3 in the case of a **dependent** queue at entrance 2, where there are Bernoulli arrivals at all entrances.

A distinction is made between low traffic load, medium traffic load, and high traffic load. The definition of the traffic loads is:

- Low traffic load is an arrival rate of 10% at an entrance, where half of the traffic at an entrance takes the first turn, and half of the traffic at an entrance takes the second turn.
- Medium traffic load is an arrival rate of 20% at an entrance, where half of the traffic at an entrance takes the first turn, and half of the traffic at an entrance takes the second turn.
- High traffic load is an arrival rate of 30% at an entrance, where half of the traffic at an entrance takes the first turn, and half of the traffic at an entrance takes the second turn.

Although it is possible to vary the ratio of the traffic flow at entrances that take a first or second turn, in these simulations only the fifty-fifty ratio is used.

5.2.1 Low traffic load

Consider low traffic load of 10% at entrance 1 and entrance 2. This means: according to a Bernoulli trial with probability $\frac{1}{10}$, a vehicle joins entrance 1 or 2, where half of this 10% takes the first possible turn and half takes the second possible turn.

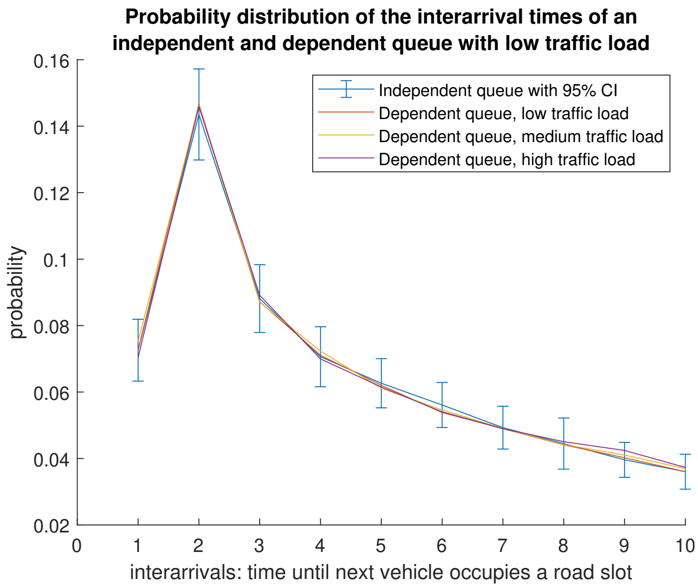

Figure 8: Probabilities of interarrival times for the four different models with a simulation size of 10^5 for the four different models.

For the independent queue at entrance 2, no vehicles arrive at entrance 3. For the dependent queue at entrance 2, three additional instances are looked at: one where the traffic load at entrance 2 is low, one where the traffic load at entrance 3 is medium, and one where the traffic load at entrance 3 is high. In Table 1, an overview of the probabilities for the different dependent queues is given.

Type queue	Traffic load entrance 1	Traffic load entrance 2	Traffic load entrance 3
Indep. queue	10%	10%	0%
Dep. queue low	10%	10%	10%
Dep. queue med	10%	10%	20%
Dep. queue high	10%	10%	30%

Table 1: Traffic load probabilities at all entrances for dependent and independent queues.

It can be seen in Figure 8 that for the first 10 interarrival times at exit 3, the difference between the self-dependent and self-independent queueing models vastly lies within the 95% confidence intervals of the independent queue. All higher interarrival times for interarrivals $k > 10$ fall within the 95% confidence interval as well.

5.2.2 Medium traffic load

Consider medium traffic load of 20% at entrance 1 and entrance 2. This means: according to a Bernoulli trial with probability $\frac{2}{10}$, a vehicle joins entrance 1 or 2. For the independent queue at entrance 2, no vehicles arrive at entrance 3. For the dependent queue at entrance 2, three additional instances are looked at: one where the traffic load at entrance 2 is low, one where the traffic load at entrance 3 is medium, and one where the traffic load at entrance 3 is high. In Table 2, an overview of the probabilities for the different dependent queues is given.

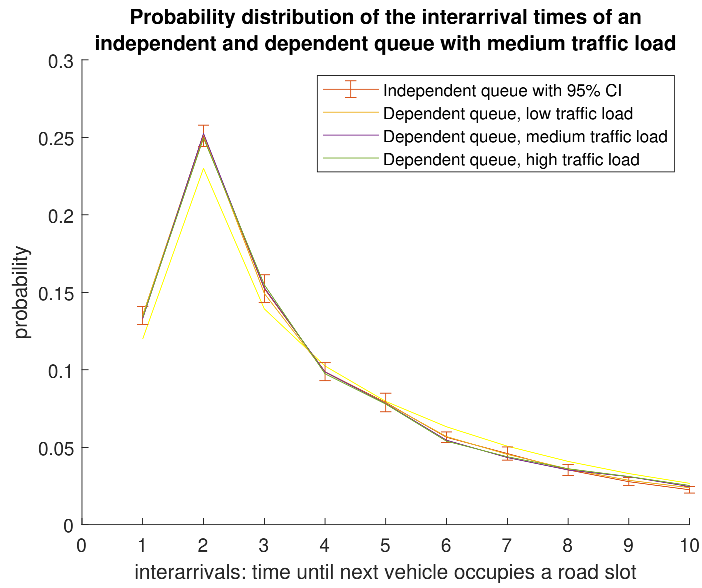

Figure 9: Probabilities of interarrival times for the four different models with a simulation size of 10^5 for the four different models.

Type queue	Traffic load entrance 1	Traffic load entrance 2	Traffic load entrance 3
Indep. queue	20%	20%	0%
Dep. queue low	20%	20%	10%
Dep. queue med	20%	20%	20%
Dep. queue high	20%	20%	30%

Table 2: Traffic load probabilities at all entrances for dependent and independent queues.

It can be seen in Figure 9 that for the interarrival times at exit 3, the difference between the self-dependent and self-independent queueing models again lies within the 95% confidence intervals of the independent queue, except for interarrival times $k = 9$ and $k = 10$. This may be due to the moderate simulation size. All higher interarrival times for $k > 10$ fall within the 95% confidence interval as well.

5.2.3 Medium traffic load

Consider high traffic load of 30% at entrance 1 and entrance 2. This means: according to a Bernoulli trial with probability $\frac{3}{10}$, a vehicle joins entrance 1 or 2. For the independent queue at entrance 2, no vehicles arrive at entrance 3. For the dependent queue at entrance 2, three additional instances are looked at: one where the traffic load at entrance 2 is low, one where the traffic load at entrance 3 is medium, and one where the traffic load at entrance 3 is high. In Table 3, an overview of the probabilities for the different dependent queues is given.

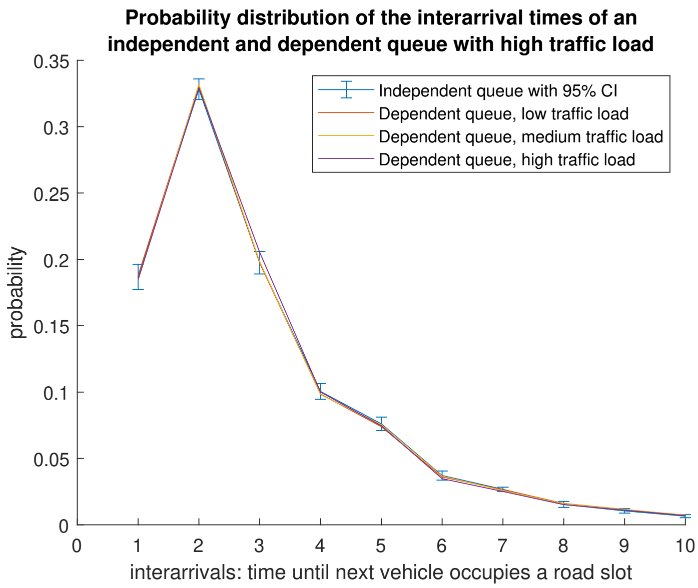

Figure 10: Probabilities of interarrival times for the four different models with a simulation size of 10^5 for the four different models.

Type queue	Traffic load entrance 1	Traffic load entrance 2	Traffic load entrance 3
Indep. queue	30%	30%	0%
Dep. queue low	30%	30%	10%
Dep. queue med	30%	30%	20%
Dep. queue high	30%	30%	30%

Table 3: Traffic load probabilities at all entrances for dependent and independent queues.

It can be seen in Figure 10 that for the first 10 interarrival times at exit 3, the difference between the self-dependent and self-independent queueing models lies within the 95% confidence intervals of the independent queue yet again, as well as all higher interarrival times for interarrivals $k > 10$ which fall within the 95% confidence interval.

Lastly, a two-sampled t-test can be carried out to determine if the independent and dependent queues have a significant difference in their probability mass function. For each interarrival time k, this test can be carried out. As the batch size $n = 30$, the result of a two-tailed t-test on low traffic load can be find in Figure 11.

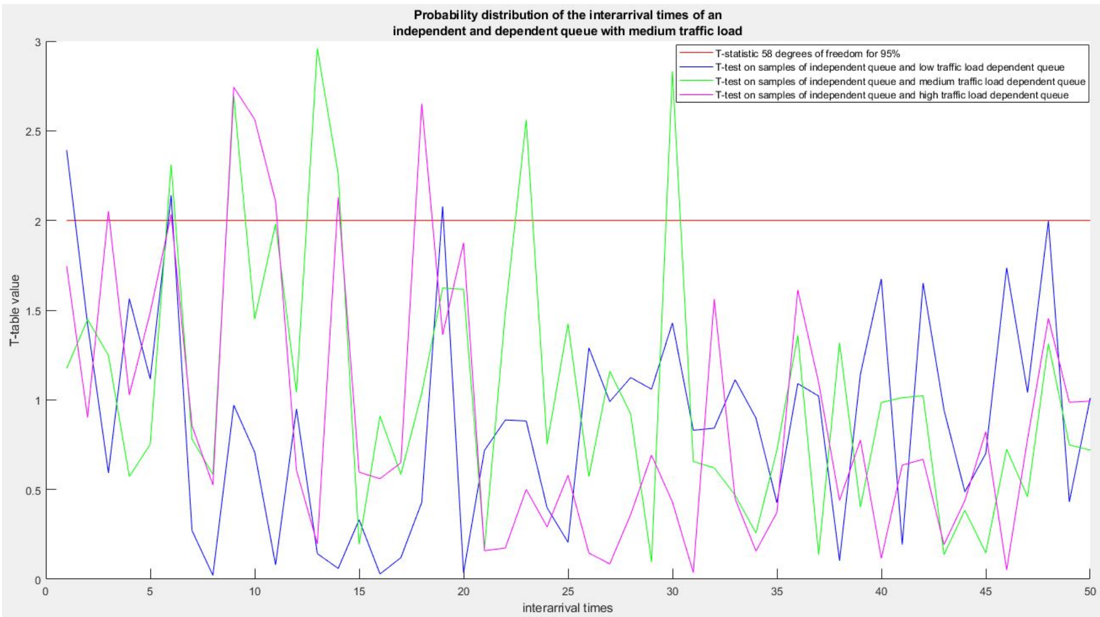

Figure 11: Two sampled t-test results for the first 50 interarrival times of an independent queue with low traffic load compared to a dependent queue with low, medium and high traffic load at entrance 3. The data size of all the samples are $4.0 \cdot 10^5$. Equivalence of the traffic loads of the samples can be found in Table 1.

Figure 11 must be interpreted as follows: if for an interarrival time the sample test is lower than the t-statistic, there is a 95% probability that the interarrival time of the independent queue and the dependent queue match with each other. In fact, the lower the outcome of the t-test, the more evidence there is to assume no significant difference between the two interarrival times of the independent and dpeendent queue. The blue line represents the t-test between the independent queue with low traffic load and the dependent queue with low traffic load at entrance 3 as well, corresponding to the traffic loads of 'indep. queue' and 'dep. queue low' in Table 1 respectively. Almost all the interarrival times of the t-test lie below the t-statistic of the test. Only for no consecutive arrivals and 5 interarrivals, the t-statistic is lower than the t-test of the interarrival times. This can be seen as a really strong correlation between the interarrival times of the low traffic load dependent queue and the independent queue. For the dependent queues with medium and high traffic load, some more interarrival times lie above the t-statistic of 95% CI, but those still give a really strong connection as well. We can therefore conclude that for independent queues and dependent queues, the departure process at the exit of the roundabout follow almost the exact same interarrival times. For the medium and high traffic load, the same t-tests can be performed on the coherence of the interarrival times for independent and dependent queues, and the results of these t-tests are similar to the one in Figure 11. This reinforces the conjecture that the interarrival times at the exit of roundabouts coincide for independent and dependent queues.

The conclusion of this specific comparison study is that systems of three-way roundabouts with self-dependent and self-independent queues have a near 95% probability that the probability distribution of the interarrival times are equal to each other. Near, because not all interarrival times of the dependent queues fall within the 95% CI, but yet approach the border of the interval. However, the t-tests let us believe that there is no distinct difference between the interarrival times at the exit of a roundabout for independent and dependent queues at the entrance of the roundabout.

5.3 Theoretical framework and SUMO comparison

5.3.1 Heuristic for theoretical framework

As Figure 7 suggests, the theoretical framework from equation (6) and the simulations from SUMO do not match. A practical explanation is that at entrance 2, whenever a vehicle is blocked, it needs to accelerate from a total or partial stop. This means it takes more time for a vehicle to reach the cell, resulting in a higher probability that not a consecutive occupation will happen ($I_{exit} = 0$), but that one empty spot will be between a blocked vehicle at entrance 2 and its predecessor from entrance 1 ($I_{exit} = 1$).

The introduction of an heuristic on the analytic model is therefore needed. Let $\pi_k = P(I_{exit} = k)$. Then the following linear transformation will be made, where $\hat{\pi}$ is the new probability mass function:

$$\pi_0 = \hat{\pi}_0 + \frac{1}{2}\hat{\pi}_1$$
$$\pi_k = (1 - \frac{1}{2^k})\hat{\pi}_k + \frac{1}{2^{k+1}}\hat{\pi}_{k+1}$$

This transformation can be solved using a matrix expression $A\hat{\pi} = \pi$, only if k is finite. Then A is the two-diagonal matrix with $A(1,1) = 1$; diagonal entries $A(i+1, i+1) = 1 - \frac{1}{2^i}$ for $i = 1, 2, ..., k$ and super-diagonal entries $A(j, j+1) = 2^{-j}$ for $j = 1, 2, ..., k-1$. Since the first interarrival times are of significantly more importance than the later interarrival times, it is sufficient to set $k = 100$, and solve the linear system of equations.

This transformation is a guess that coincidentally fits well for low, medium and high traffic load. However, a second linear transformation can be laid over the first heuristic to more accurately approach the simulation data. This second transformation will be called $\tilde{\pi}$ and the transformation looks as follows:

$$\begin{aligned} \tilde{\pi}_0 &= \hat{\pi}_0 \\ \tilde{\pi}_1 &= \hat{\pi}_1 + \frac{1}{2}\hat{\pi}_2 \\ \tilde{\pi}_k &= \frac{1}{2}\hat{\pi}_k + \frac{1}{2}\hat{\pi}_{k+1} \qquad , \text{for } k \geq 2 \end{aligned}$$

In Figure 12, the results of this heuristic can be measured. Whereas the probability of the first 4 interarrival times do improve for $\hat{\pi}$ and $\tilde{\pi}$ compared to π, but not improve so much that it lies within the 95% CI, it can be observed that for higher interarrivals, $\tilde{\pi}$ approximates the simulation data much better than $\hat{\pi}$ and π. Furthermore π underestimates the probabilities found in the simulation for interarrivals $k = 1$, $k = 2$ and $k = 3$ and for higher traffic loads also $k = 4$, but it overestimates at $k = 0$ and all higher interarrival times $k \geq 5$.

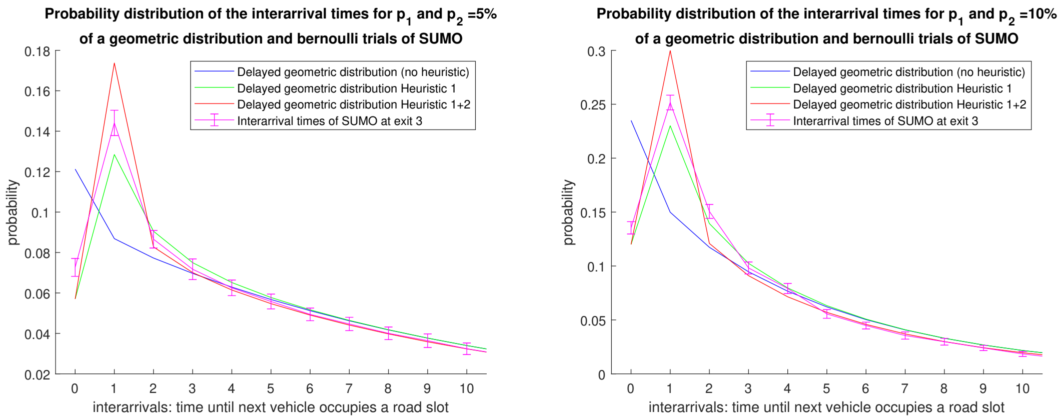

Figure 12: Probability distribution of the interarrival times for low (5%) and medium (10%) traffic load of simulations with size $4 \cdot 10^5$, the analytical model π, the analytical model with the first heuristic, $\hat{\pi}$, and the analytical model with the first and second heuristic, $\tilde{\pi}$.

A more robust heuristic to approximate the simulation can be found by looking at the behavior of the probability distribution of the interarrival times of the analytical model π compared to the simulation: let s_k be the k'th proability of an interarrival of the simulation, and then analyse the proportion between the two values. Figure 13 shows us a plot of $y = \frac{\pi_k}{s_k}$ on interarrivals k for 7 different traffic loads. Furthermore it contains a simple red line at $y = 1$. Firstly the interarrival times that are underestimated by definition of the function $y = \frac{\pi_k}{s_k}$ are located below the red line $y = 1$. We observe that for values above the red line, the interarrival times for traffic loads are arranged from highest traffic load for high values of y to lowest traffic load on low values of y. For values of y lower than 1, the interarrival times are arranged in reverse, so that the values of the low traffic loads are closest to $y = 1$ and the values of the high traffic loads are furthest away from $y = 1$. Furthermore the values of y for higher interarrivals, that is $k > 5$, seem to be equal to some constant, apart from the highest traffic load of 15% which comes close to an overloaded system. This allows us to perform some form of regression, in our case an exponential one. Let us call p_0 the average traffic load of p_1 and p_2, such that $p_0 = \frac{p_1+p_2}{2}$. An approximation of the

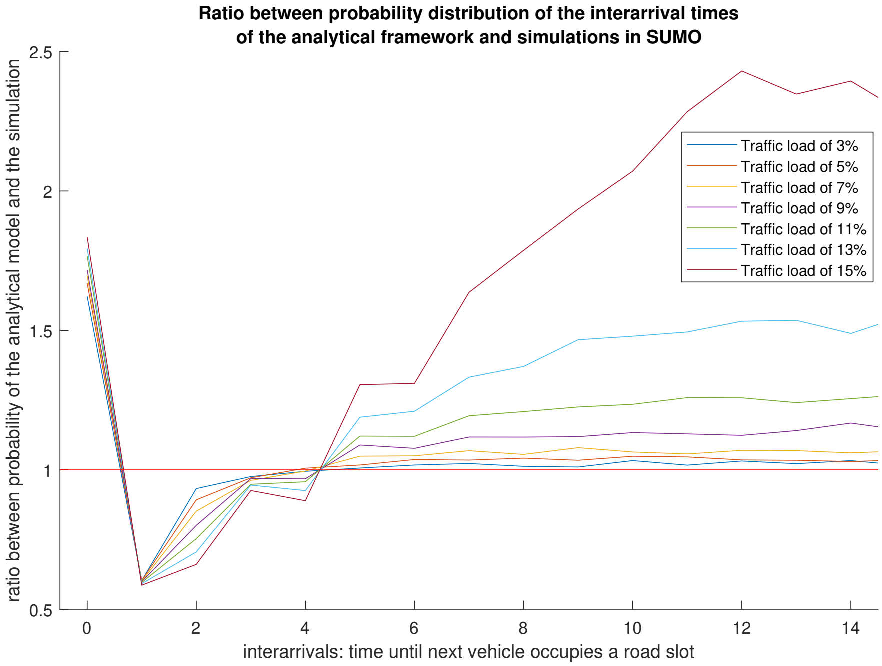

Figure 13: Ratio of the interarrival times between analytical framework and SUMO. A traffic load of $x\%$ means: $p_1 = p_2 = p_3 = x\%$

model for a non-overloaded system,

$$s_k \approx \pi_k \cdot \frac{1}{1 + e^{29p_0 - 4.605}} \qquad \text{for } k > 5$$

For the interarrivals $k \leq 3$, a linear regression seems to fit on the difference between the interarrival times for the various traffic loads. The following set of equations can therefore be found:

$$\begin{aligned} s_0 &\approx \pi_0 \cdot \frac{5}{7.75 + 9 \cdot p_0} \\ s_1 &\approx \pi_1 \cdot \frac{10}{6.07 - p_0} \\ s_2 &\approx \pi_2 \cdot \frac{9}{9 - 20 \cdot p_0} \\ s_3 &\approx \pi_3 \cdot \frac{5}{5 - 3 \cdot p_0}. \end{aligned}$$

For $k = 4$, the non-overloaded system approaches the line $y = 1$, so no change is needed on π_4. For $k = 5$, most values of their corresponding value y lie somewhere between the red line $y = 1$ and the earlier mentioned constant that is used for $k > 5$. Therefore,

$$s_5 \approx \pi_5 \cdot \frac{2}{2 + e^{29p_0 - 4.605}}.$$

And so we end up with a third heuristic for the probability distribution of the interarrival times, called ξ_k, where we use that the approximation s_k is equal to the heuristic ξ_k:

$$\begin{aligned}
\xi_0 &= \pi_0 \cdot \frac{5}{7.75 + 9 \cdot p_0} \\
\xi_1 &= \pi_1 \cdot \frac{10}{6.07 - p_0} \\
\xi_2 &= \pi_2 \cdot \frac{9}{9 - 20 \cdot p_0} \\
\xi_3 &= \pi_3 \cdot \frac{5}{5 - 3 \cdot p_0} \\
\xi_4 &= \pi_4 \\
\xi_5 &= \pi_5 \cdot \frac{2}{2 + e^{29p_0 - 4.605}} \\
\xi_k &= \pi_k \cdot \frac{1}{1 + e^{29p_0 - 4.605}} \qquad \text{for } k > 5
\end{aligned} \tag{41}$$

In Figure 14, the approximation of ξ_k in comparison with SUMO can be seen for low (5%) and medium (10%) traffic load. Both the low and medium traffic load are extremely close to the distribution of their respective simulations.

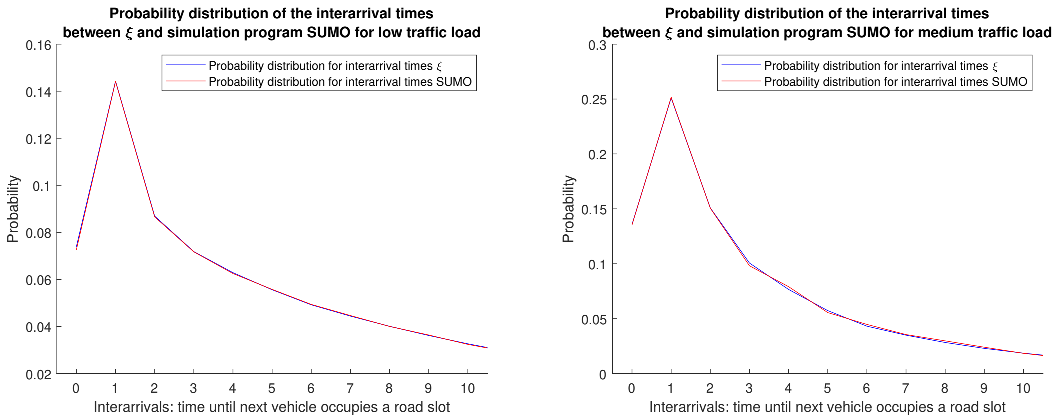

Figure 14: Probability distribution of the interarrival times for low (5%) and medium (10%) traffic load of simulations with size $4 \cdot 10^5$ and the analytical model with heuristic ξ.

To test the probability distribution ξ, we look at cases where $p_1 = p_2 = p_3$ is not necessarily true. Let us look at cases where $(p_1, p_2, p_3) = (0.10, 0.05, 0.10)$ and $(p_1, p_2, p_3) = (0.05, 0.10, 0.10)$. We find that $p_0 = \frac{p_1 + p_2}{2} = 0.075$ for both of the transformation of π_k to ξ_k. Since the probability distribution of the interarrival times of the analytical model are symmetric in p_1 and p_2, ξ_k applies to both simulations. Figure 15 shows the approximation of both interarrival time distribution compared with a simulation of size $2.0 \cdot 10^5$.

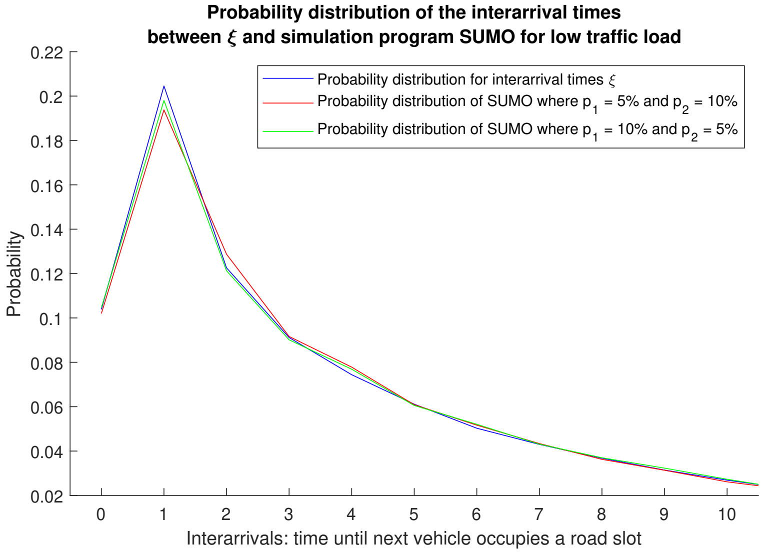

Figure 15: Probability distribution of the interarrival times of ξ_k and SUMO for $p_2 = p_3 = 10\%, p_1 = 5\%$ (red) and $p_1 = p_3 = 10\%, p_2 = 5\%$ (green)

It can be seen that the approximation ξ for a mixed traffic load still applies to a great extend when compared with the simulations for light to moderate traffic loads.

5.4 Case study 1: multiple connected roundabout

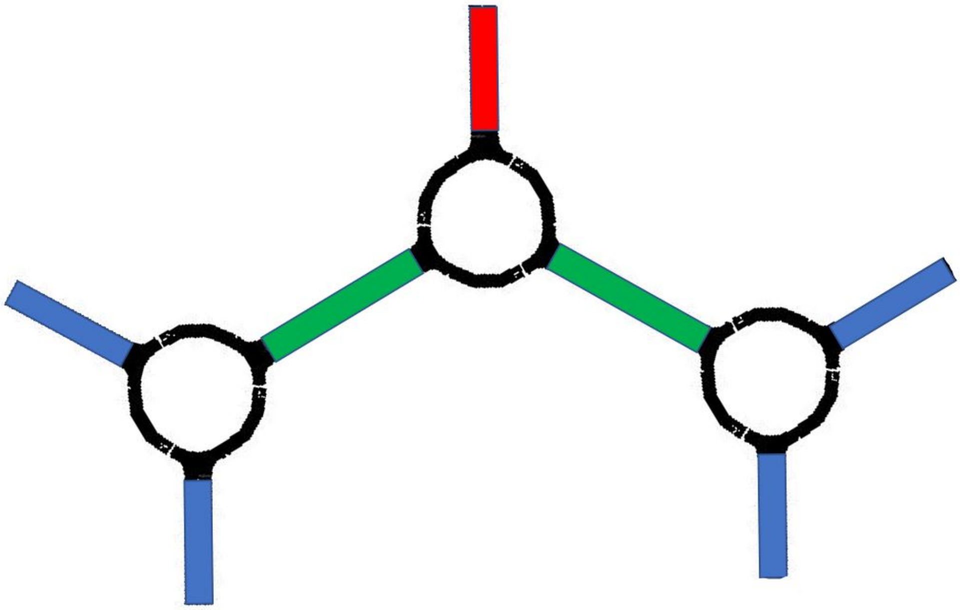

Figure 16: A chain of three-way roundabouts.

Consider Figure 16, where Bernoulli arrivals occur on the blue entrances, and the departure process of the red exit is investigated. The theoretical framework from section 4 shows that the departure processes on the green road segments, which is the same as the arrival processes of the roundabout in the middle, both follow the probability mass function for the interarrival times ξ_k of equation (41), where $\pi_k = P(I_{X,Y} = k)$. Whereas the assumption always has been that bernoulli arrivals occur on the entrance of a roundabout, now arrivals according to ξ will happen at the entrance of the middle roundabout of Figure 16. However, we try to use the same theoretical framework for the departure process on the exit of the middle roundabout, so that ξ_k will be compared to the probability distribution of the interarrival times that are found by the simulation of SUMO. Following this framework, the probabilities for interarrival times are found in Figures 17 and 18. The approximations of the heuristic ξ once again approximate the simulation well for the low and medium traffic load, with a small underestimate

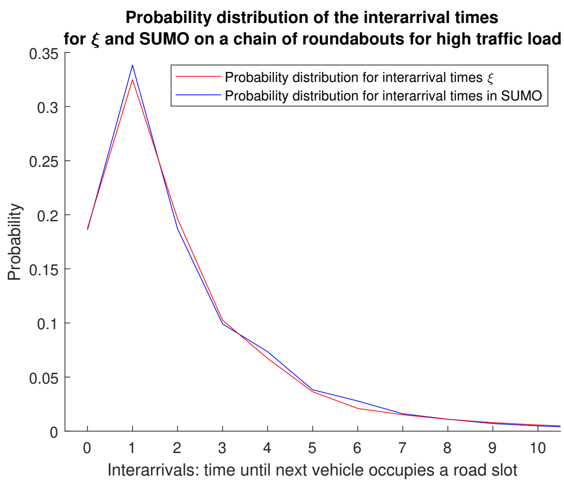

Figure 17: Probability distribution of interarrivals at the exit of a chain of roundabouts for high traffic load of ξ and SUMO with simulation size $2.0 \cdot 10^5$.

at $k = 1$. For high traffic load, Figure 17 shows a couple of clear underestimates at $k = 1$, $k = 4$ and $k = 6$. An important observation is that the shape of the probability distributions are similar in all different kinds of traffic load.

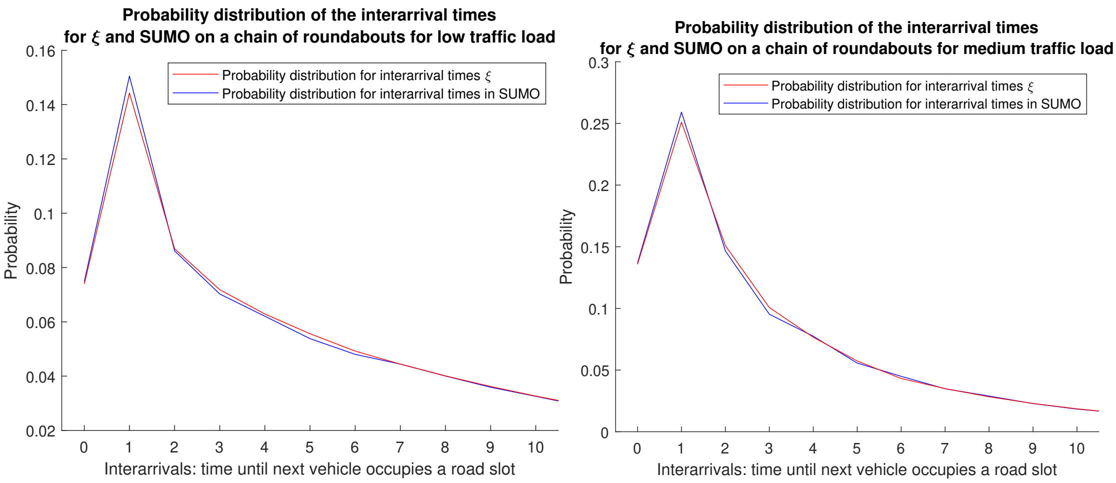

Figure 18: Probability distribution of the interarrival times for low (5%) and medium (10%) traffic load of simulations with size $2 \cdot 10^5$ and the analytical model with heuristic ξ.

5.5 Case study 2: chain of roundabouts

In section 5.4, it is found that the departure process of three linked roundabouts as depicted in Figure 16 behaves strongly like the departure process of a single roundabout with bernoulli arrivals at its entrances. We will now look at a chain of five roundabouts as depicted in Figure 19.

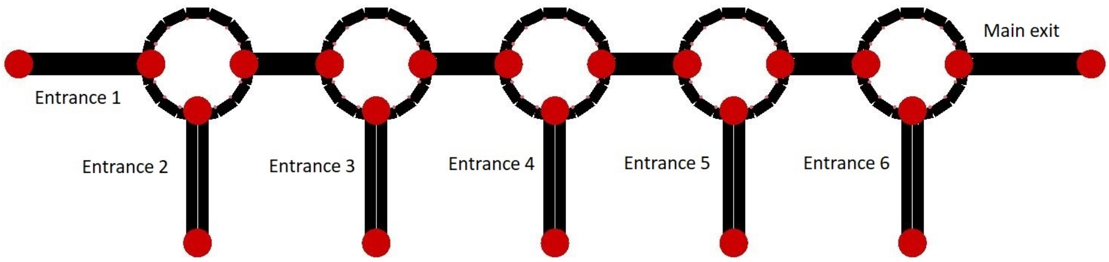

Figure 19: A chain of five roundabouts

The traffic flow on this chain of roundabouts is defined as follows: on the first roundabout, the traffic flows from entrance 1 according to a Bernoulli process with probability p_1 taking the second turn towards the second roundabout, and at entrance 2 the same process occurs such that with probability p_2 vehicles arrive that take the first turn towards the second roundabout, and with probability p_3 vehicles arrive that take the second turn towards the exit on the opposite road direction of entrance 1. This is exactly the same process as an independent roundabout with low, medium or high traffic load as discussed in section 5.2. When the traffic flow arrives at the second roundabout, half of the traffic load leaves the roundabout, and half of the traffic load moves on to the third roundabout. More specifically, half of the vehicles that come from entrance 1 leave the second roundabout, and half of the vehicles that come from entrance 2 leave the second roundabout, such that the traffic flow that can block vehicles from entrance 3 is equal to $\frac{p_1+p_2}{2}$. Meanwhile, at entrance 3 the same process occurs as at entrance 2: with probability p_2 a vechile arrives that is taking the first turn towards the third roundabout, and with probability p_3 a vehicle arrives that is taking the second turn towards the first roundabout. This same arrival process applies to all the remaining entrances 4, 5 and 6 as well. The routes of all the vehicles are defined as follows:

- Routes for vehicles that arrive at entrance 1 with probability p_1:
  - 50% Enters at entrance 1 and leaves at the opposite side road of entrance 3.
  - 50% Enters at entrance 1 and leaves at the main exit.
- Routes for vehicles that arrive at entrance 2 with probability p_2:
  - 50% Enters at entrance 2 and leaves at the opposite side road of entrance 3.
  - 50% Enters at entrance 2 and leaves at the opposite side road of entrance 4.
- Routes for vehicles that arrive at entrance 3 with probability p_2:
  - 50% Enters at entrance 3 and leaves at the opposite side road of entrance 4.
  - 50% Enters at entrance 3 and leaves at the opposite side road of entrance 5.
- Routes for vehicles that arrive at entrance 4 with probability p_2:
  - 50% Enters at entrance 4 and leaves at the opposite side road of entrance 5.
  - 50% Enters at entrance 4 and leaves at the opposite side road of entrance 6.
- Routes for vehicles that arrive at entrance 5 with probability p_2:
  - 50% Enters at entrance 5 and leaves at the opposite side road of entrance 6.
  - 50% Enters at entrance 5 and leaves at the main exit
- Vehicles that arrive at entrance 6 with probability p_2 all leave the main exit.
- Vehicles that arrive at entrance 2-6 with probability p_3 all leave at the opposite side road of entrance 1.

For the interarrival times, not only the main exit is observed, but also the interarrival times at the section between two roundabouts are recorded. The probabilities p_1, p_2 and p_3 are once again evenly distributed so that $p_1 = p_2 = p_3$. In Figure 20 and the results of the interarrival times are showed, compared with the analytical approach for the interarrival times ξ_k as formulated in section 5.3.

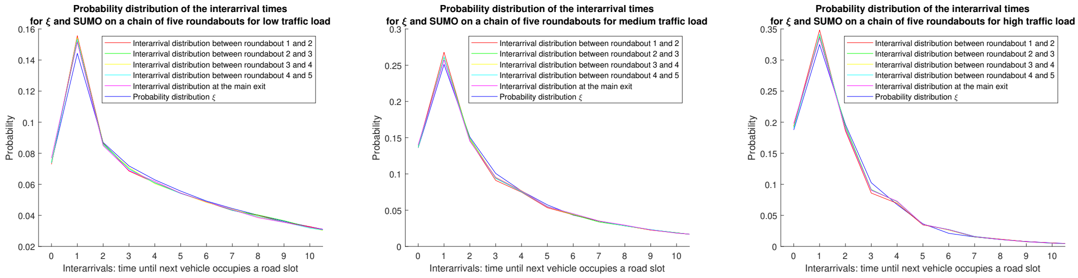

Figure 20: Probability distribution of the interarrival times for low (5%), medium (10%) and high (15%) traffic load of simulations with size $2 \cdot 10^5$ and the analytical model ξ for a chain of five roundabouts.

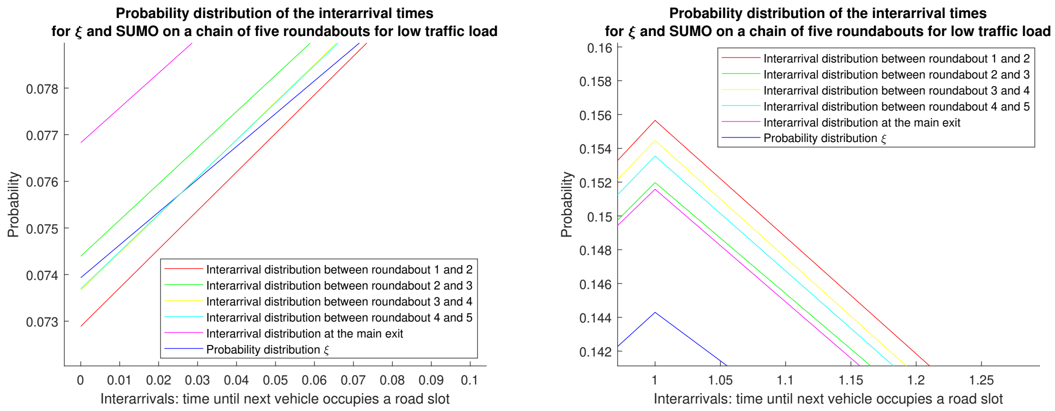

Figure 21: Figure 20, only zoomed in on the plot with the low traffic load. When zoomed in on medium and high traffic load, similar plots are obtained.

The first conclusion that can be drawn from Figure 20 is that the five interarrival distributions between the roundabouts and at the exit all take on the same probability distribution for the interarrival times of vehicles. Furthermore the analytical approximation ξ_k once again takes on the form of the distribution of the interarrival times of all SUMO-results, which can be seen as a nice result. Zooming in on the first ($k = 0$) and second ($k = 1$) interarrival times, two details stand out: firstly, as can be seen in the left side of Figure 21, the interarrival $k = 0$ at the main exit seem to have a slightly higher probability than other segments between roundabouts, and in the right side of the same figure it can be seen that the interarrival $k = 1$ between the first and second roundabout has a slightly higher probability than the other interarrival times. For the interarrival $k = 1$ the explanation can be given that the departure process from the first roundabout to the second is distinctly different as it has bernoulli arrivals on both entrances, whereas the other roundabouts have a modified arrival process at the entrance that is connected to the previous roundabout. The anomaly for $k = 0$ at the main exit of the roundabouts however can not be explained by the same logic. In fact, there should be no cause for the interarrival times at the main exit to behave different than the interarrival times on the road segments between roundabouts. It might therefore just be a coincidence in the simulations.

Another detail that stands out is that it would have made sense if the interarrival times between the first and second roundabout would more closely follow the analytical approximation ξ: after all the traffic load has not changed for the three-way roundabouts in Figures 19 and 5. The only difference that can be observed is the geometry of the roundabouts is different: in Figure 19 there is a 90° angle between the first and second entrance, and another 90° angle between the second entrance and the road segment that connects the first and second roundabout. In Figure 5 however, these angles are 120°. Therefore, even though it is a three-way roundabout, the structure of the roundabout sligthly matters as well for the interarrival times at the exit of a roundabout. This difference can also be observed in Figure 22

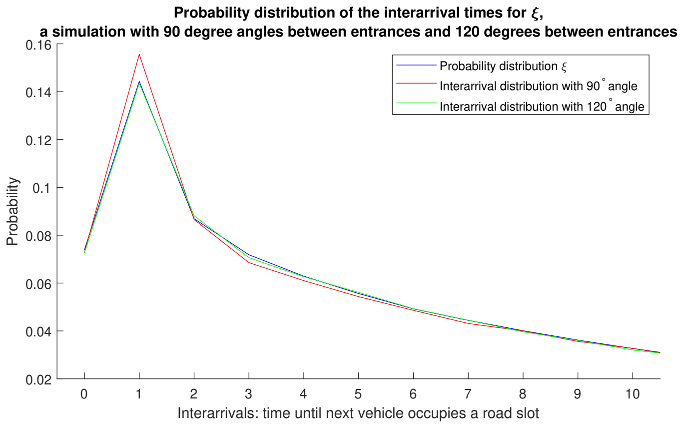

Figure 22: Difference between the simulations for three-way roundabouts with a different structure, together with distribution ξ.

Altogether, the conclusion that can be drawn from this case study is that the approximation ξ still is a solid approximation. Furthermore a small difference is observed between the interarrival times of a roundabout with only Bernoulli arrivals and a roundabout linked in a chain of roundabouts as can be seen in Figure 19. Lastly it is observed that, given Bernoulli arrivals, the structure of the roundabout matters as well for the departure process at the exit of a roundabout.

5.6 Simulation of a four-way roundabout

Let us consider a classical four-way roundabout. From all four sides, traffic flow equals each other: at entrance 1 the probability that a vehicle arrives is p_c according to a bernoulli process, and each vehicle has the same probability to leave the roundabout at exit 2, 3 or 4. For entrance 2, 3 and 4, the same principle holds. Furthermore the traffic load at entrance 1, 2, 3 and 4 is evenly divided, so that at each entrance the probability that a vehicle arrives is equal to p_c. This means that the traffic load at one of the exits of the roundabout will also be p_c. For the theoretical framework $\pi_k = P(I_{X,Y} = k)$, the probabilities p_1 and p_2 are symmetrical, and will therefore be the same in that of the approximation of a four-way roundabout. Furthermore $p_1 + p_2$ must equal the traffic load that leaves an exit of the roundabout. Therefore we find that $p_1 = p_2 = \frac{1}{2}p_c$. For convenience $p_3 = \frac{1}{2}p_c$, since it does not substantially influence the probability distribution of the interarrival times, but it is a factor that is present. We will look at different traffic loads again:

- low traffic load means $p_c = 0.06$.
- medium traffic load means $p_c = 0.12$.
- high traffic load means $p_c = 0.18$.

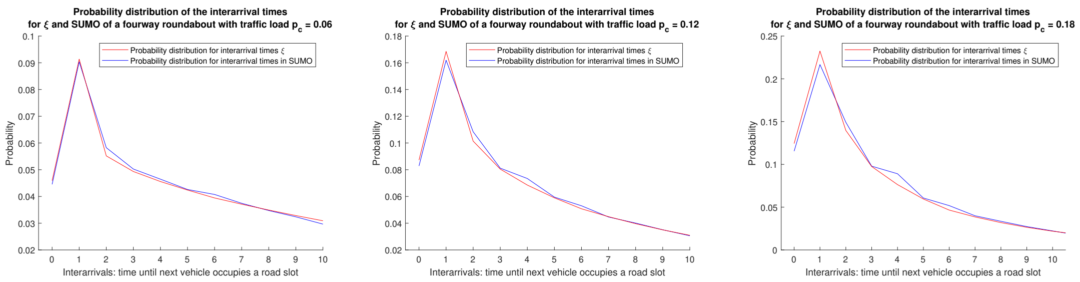

Figure 23: Probability distribution of the interarrival times of a fourway roundabout with simulation size $2.0 \cdot 10^5$ together with the analytical framework with interarrival times ξ_k.

In Figure 23, one can observe that the analytical framework with distribution ξ_k is more accurate when the traffic load decreases. The larger the traffic load, the higher the more the first two interarrival times $k = 0$ and $k = 1$ overestimate ξ_k, which will be compensated with an underestimation on later interarrival times, for example $k = 2$, $k = 4$ and $k = 6$. However the form of the probability distribution is preserved and therefore the approximation can be assessed decently.

6 Discussion

The theoretical approach of sections 3 and 4 is too simplified to accurately describe the departure process of a roundabout with Bernoulli arrivals. Therefore the heuristic ξ as described in subsection 5.3 is needed. Moreover, the results of case study 2 of section 5.6 suggest that the design of the roundabout plays a part in the interarrival times of the departure process. Another argument can be given however, that the discrete measurement of vehicle positions enables for a wider range of the probability distribution of the interarrival times. In SUMO, vehicles can only 'spawn' on a certain timestep, unlike in continuous time, and that can cause an alteration in the design of a roundabout to have a small impact on the probability distributions.

Furthermore the importance of a precise probability distribution of the interarrival times can be questioned. In the context of linking roundabouts and its departure process in the bigger picture of traffic networks, it certainly is of use. However, the arrival process towards roundabouts is not always Bernoulli, for instance during rush-hours or congestions. Likewise the connection between the arrival and departure processes of roundabouts and traffic-lights is missing, which is an important link in traffic networks.

Other topics of interest are the fourway roundabout, specifically a theoretical framework equivalent to the probability distributions (6) and (32). A direct approach is to find the queue length distribution of the queue for a Geo/DGeo/1-queue, where the distribution of DGeo refers to the interarrrivals of the delayed geometric distribution (6). Similar techniques that are used in section 4 to find the interruptive vehicles that do not exit the roundabout at the place where the departure process is measured can be performed, and as such the four-way roundabout can be analyzed. As observed in section 5.2, the interference of other Bernoulli streams do not have an effect on the departure process for three-way roundabouts, and it can be cautiously presumed that for four-way roundabouts, this same principle holds.

A point of improvement could be to look at vehicular flow that moves at a higher speed than 20 km/h. This speed was chosen conveniently as it makes a discretization of $\Delta t = 2$ seconds possible. However, for practical use this does not translate to reality. Therefore a continuous equivalent of the departure process would be a step in the right direction for future research.

7 Conclusion and Recommendations

A basic DTMC is formulated to describe the behavior of vehicular flow on a roundabout and the queues at the entrances of a roundabout for Bernoulli arrivals, which is used to find the probability mass function of the interarrival times of vehicles on the roundabout in equation (6). More importantly, the PMF of the interarrival times at the exit of a roundabout is described in equation (32), allowing for a complete description of the departure process of a roundabout given Bernoulli arrivals. Simulations show that the numerically obtained heuristic ξ_k is needed to accurately approximate the interarrival times of a three-way roundabout.

Simulations additionally demonstrate that, given Bernoulli arrivals at each entrance, the departure process at the exit of a roundabout exhibits near exact interarrival times for self-dependent and self-independent traffic flows, meaning that the departure process of a roundabout is equivalent to the departure process of a priority road with Bernoulli arrivals. Furthermore the structure of the departure process of a chain of three-way roundabouts with Bernoulli arrivals resembles the departure process of a single three-way roundabout with Bernoulli arrivals. However, the design of roundabouts have an effect on the departure times as well.

Future research should focus on the expansion of the theoretical framework on the classical four-way roundabout. Additionally it is recommended to formulate the analogous continuous time departure process at roundabouts. A necessity to fully understand traffic networks is to explore the relationship between roundabouts and (actuated) traffic lights, which can be achieved by looking at a combination of Bernoulli arrivals, batch arrivals or hyperexponential arrivals rather than just Bernoulli arrivals.

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