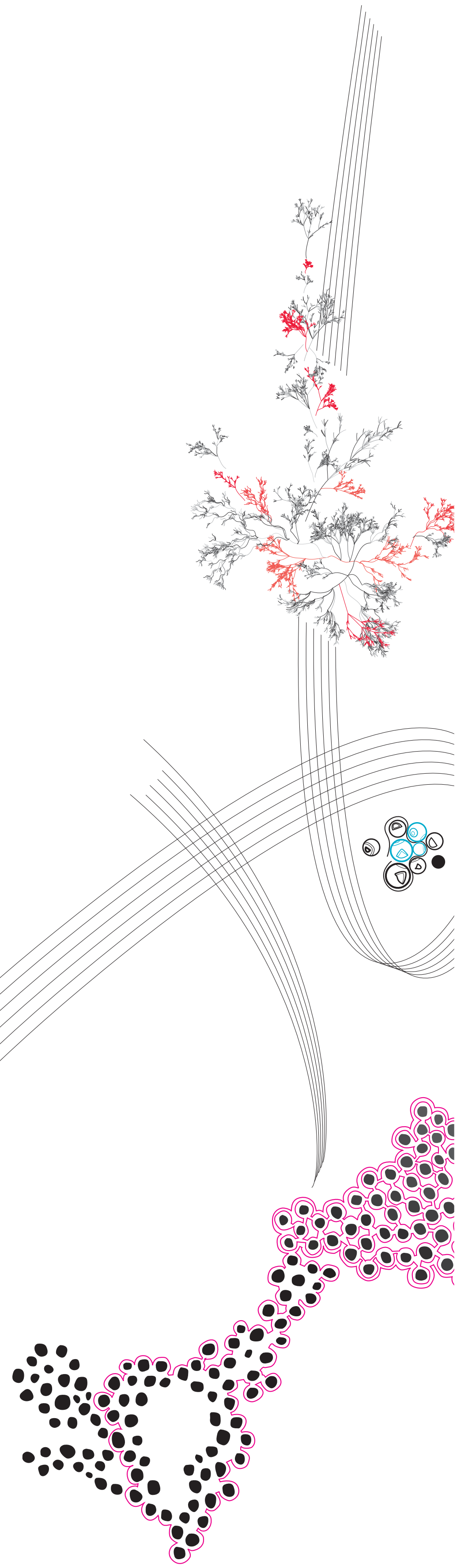

MSc Mechanical Engineering
Thesis

Robust data-driven state-feedback synthesis from data corrupted by perturbations with bounded norms and rates-of-variation

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December, 2023

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PREFACE

I am thrilled to present my Master of Science (MSc) thesis, titled *Robust data-driven state-feedback synthesis from data corrupted by perturbations with bounded norms and rates-of-variation*. This thesis represents the culmination of an academic journey that has spanned over the last 6.5 years. It reflects the commitment, hard work, and passion invested in the pursuit of knowledge within the field of mechanical engineering.

I look back at my years at the University of Twente with joy and treasure the many possibilities the university has provided to collaborate with others. In particular, I look back fondly at my year at DroneTeam Twente, which allowed me to utilize my knowledge in a practical and multidisciplinary setting. Similarly to such a project, this thesis would not have been possible without the support and guidance of many individuals.

I would like to express my profound gratitude to my supervisor, Hakan Köroğlu, whose mentorship has been invaluable throughout this research. His unwavering encouragement, patience, and expertise created an environment for this research to thrive. During our weekly meetings, time flied as we discussed the sometimes quite complex material, and I always left the meeting with a new path to investigate, keeping me motivated.

This thesis aims to synthesize $\mathcal{H}_\infty$-optimal state feedback controllers from data directly. This is achieved within a linear matrix inequality (LMI) framework. A major challenge in data-driven controller design is dealing with perturbations in the data, such as those coming from external disturbances and measurement noise. This thesis presents a framework, in which characterizations of the perturbation in terms of bounded norm and rate-of-variation can be included in the controller synthesis.

The thesis is structured as follows. The work is introduced in Chapter 1. Chapter 2 provides an introduction to LMI-based controller synthesis and provides some tools relevant to this thesis. In Chapter 3, some key concepts in LMI-based data-driven control are presented, and together with Chapter 2 forms the theoretical background for Chapter 4. This chapter is a stand-alone paper, which has been submitted to the 22nd European Control Conference (ECC) in Stockholm, Sweden. In Chapter 5, supplementary results which could not be added to the paper are presented. Here, the focus is on considerations when using the results of Chapter 4 for controller synthesis of a mechanical system. These are illustrated by simulations on a double pendulum system. Chapter 6 presents concluding remarks on the thesis.

As a final remark, I extend my heartfelt thanks to my family and friends, whose support sustained me during this academic journey. In particular I want to thank Jorn for the many hours we have spent together working on our theses, and Sanne for her constant encouragement.

Rens de Boer
December, 2023
Enschede

ABSTRACT

Data-driven $\mathcal{H}_\infty$-optimal controller synthesis is considered for unknown discrete-time linear time-invariant systems. Perturbations in the data, such as those coming from external disturbances and measurement noise, are assumed to be bounded in norm and rate-of-variation. A linear matrix inequality (LMI) based framework is presented in which these realistic bounds can be included in the synthesis. This reduces the set of systems consistent with the data and thus offers a reduction in conservatism, at the cost of increased computational complexity. The method is evaluated through simulations on a double pendulum system.

CONTENTS

1 INTRODUCTION

The world of engineering is swiftly adopting data-driven methodologies [1]. Through the integration of sensors and advanced monitoring systems, optimization of manufacturing processes and efficiency refinement are underway. Breakthroughs in predictive maintenance allow us to foresee and address equipment failures before occurrence. Engineers utilize data analytics and machine learning for tasks like quality control in production lines and structural health monitoring in infrastructure projects. This shift signifies a paradigm change in engineering practices, where the strategic use of data not only enhances efficiency but also revolutionizes approaches to precision and reliability across various applications.

Terms like “data-based” and “data-driven” have permeated the language of the control community as well. Direct data-driven control is a notable example of this, which refers to the process of designing controllers for an unknown system starting from only data collected from this system. In contrast to the classical approach of refined system identification [2] followed by controller design, which uses data indirectly, direct data-driven control is a procedure in which no intermediate model identification step is required.

Forgoing the need for system identification could position data-driven methods as a compelling alternative. The process of refined system identification is frequently time-consuming, requiring significant human intervention to ensure model quality. If the demand for such thorough identification is deemed too cumbersome, practitioners may find data-driven methods to be a viable alternative, provided that these methods are simple and quicker to execute. A recent line of work on direct data-driven control might fulfill these requirements, as the solutions of these works are formulated in the form of linear matrix inequality (LMI) optimization. This is beneficial, as LMIs have been shown to be computationally efficient and effective in a variety of analysis and synthesis problems [3].

One of the critical issues in data-driven control is how to deal with perturbations affecting the data. Not all input signals are measurable (e.g. external disturbances) and measurements are always corrupted by some noise. A recent line of work resolves this problem for linear time-invariant (LTI) systems, building on the assumption that the matrix containing the perturbation samples can be described by a known *perturbation model*, which is a characterization of disturbances and noise (through convenient set descriptions). Through this perturbation model, a set of LTI systems *consistent* with the data emerges, in that each system in this set could have generated the experimental data for some admissible perturbation sequence. In this context, direct data-driven controller synthesis amounts to finding a single controller that satisfies the performance objective when interconnected with any system consistent with the data.

In recent works, including but not limited to [4, 5, 6, 7, 8, 9, 10], the perturbation model is defined *explicitly* as the solution of a specific quadratic matrix inequality (QMI), and can capture energy bounds on the perturbation sequence and/or bounds on the noise sample covariance matrix. Considering the QMI’s less-than-ideal structure, a perturbation model may not be easily retrieved based on some characterization of disturbances and noise, yielding potentially conservative results.

In this thesis, it is shown that these perturbation models can be constructed via LMIs using an *implicit* characterization of the perturbation signal. This characterization can describe bounds in the form of quadratic inequalities in any of the elements of the perturbation samples, including energy bounds as in previous works. In addition, this framework allows for bounds on the rate-of-variation between samples. This is highly relevant for controlling mechanical systems, due to the possible and very probable correlation between perturbation samples, such as those coming from external disturbances as well as from the nonlinear system dynamics.

2 BASIC LMI THEORY FOR CONTROLLER SYNTHESIS

The aim of this chapter is to give an overview of the tools required for the controller design of subsequent chapters. The main purpose of this chapter is to introduce the Linear Matrix Inequality (LMI) framework and its application in the analysis and synthesis of control systems. Here, it will be shown that controller design problems relevant to this thesis can be reduced to convex optimization problems involving LMIs. This is beneficial, since these kinds of optimization problems can be solved numerically using very efficient algorithms. The development of these algorithms can be traced back to the late 1980's and increased the popularity of LMIs for controller design. As such, there exist a large collection of work explaining the properties and application of LMIs in analysis and synthesis of control systems, with different terminology and notation. This work mainly follows the lines of [11], as it presents LMIs in a convenient way for implementation (e.g. in MATLAB) and has clear sections on discrete-time systems, which are the interest of this work. In this chapter, LMIs for stability and $\mathcal{H}_\infty$-performance are presented, and subsequently used for synthesis of full state feedback controllers. The reader is referred to [11, 12, 13] for LMIs addressing various other control system analysis and synthesis problems, including optimal state estimation and dynamic output feedback control.

2.1 Definitions and fundamental LMI properties

In order to define LMIs, first it needs to be established what it means for a matrix to be positive or negative definite.

2.1.1 Definiteness of a matrix

Definition 1 (e.g. [11], page 8)**.** *Consider a symmetric*[1] *matrix $P \in \mathbb{S}^{\eta_x}$. P is called*

1. *positive semidefinite or non-negative definite, denoted by $P \succeq 0$, if*

$$x^\top P x \geq 0,\ \forall x \in \mathbb{R}^{\eta_x}, \tag{2.1}$$

2. *positive definite, denoted by $P \succ 0$, if*

$$x^\top P x > 0,\ \forall x \in \mathbb{R}^{\eta_x} \setminus \{0\}, \tag{2.2}$$

3. *negative semidefinite, denoted by $P \preceq 0$, if*

$$x^\top P x \leq 0,\ \forall x \in \mathbb{R}^{\eta_x}, \tag{2.3}$$

4. *negative definite, denoted by $P \prec 0$, if*

$$x^\top P x < 0,\ \forall x \neq 0 \in \mathbb{R}^{\eta_x} \setminus \{0\}. \tag{2.4}$$

[1] The same definitions hold for self-adjoint matrices in the complex space, which are not considered in this thesis.

5. *indefinite if $x^\top Px$ is neither positive nor negative definite.*

Since this specific form will be used throughout the thesis, it is defined explicitly.

Definition 2. *A function $f : \mathbb{R}^{\eta_x} \to \mathbb{R}$ is called a quadratic form if it can be written in the form*

$$f(x) = x^\top Px, \tag{2.5}$$

for some $P \in \mathbb{S}^{\eta_x}$.

An important property of quadratic forms is that the sign of $x^\top Px$ is dictated by the eigenvalues of P.

Theorem 1 (e.g. [11], page 9)**.** *Consider a symmetric matrix $P \in \mathbb{S}^{\eta_x}$. The following are equivalent*

1. *P is positive semidefinite,*

2. *$\underline{\lambda}(P) \geq 0$,*

where $\underline{\lambda}(P)$ denotes the smallest eigenvalue of P. The same can be shown for the other cases of Definition 1, so the matrix $P \in \mathbb{S}^{\eta_x}$ is

1. *positive definite if and only if $\underline{\lambda}(P) > 0$,*

2. *negative definite if and only if $\overline{\lambda}(P) < 0$,*

3. *negative semidefinite if and only if $\overline{\lambda}(P) \leq 0$,*

4. *and indefinite if and only if $\underline{\lambda}(P) < 0$ and $\overline{\lambda}(P) > 0$,*

where $\overline{\lambda}(P)$ denotes the largest eigenvalue of P.

Clearly, the definiteness of matrices is an interesting property to study for the analysis and design of control systems, as eigenvalues play a major role in the stability and dynamic behaviour of such systems.

2.1.2 Matrix inequalities and LMIs

Any form of mathematical optimization involves finding decision variables that satisfy specific constraints, aiming to determine the most favorable solution for some objective function. As will be shown in later sections, definiteness of a matrix is a relevant constraint for the optimization problems considered in this thesis. Based on the notions of definite matrices in Section 2.1.1, definiteness of a matrix can be enforced via a matrix inequality as follows:

Definition 3 (e.g. [11], page 10)**.** *A matrix inequality in the decision variable $x = \begin{bmatrix} x_1 & \dots & x_m \end{bmatrix}^\top \in \mathbb{R}^m$ is an expression of the form*

$$G(x) = G_0 + \sum_{i=1}^{p} f_i(x) G_i \preceq 0, \tag{2.6}$$

and defines the mapping $G : \mathbb{R}^m \to \mathbb{S}^n$ with $G_i \in \mathbb{S}^n, i = 0, \dots, p$.

The inequality means that $G(x)$ is a negative semidefinite matrix as by Definition 1.
The most relevant class of matrix inequalities in this thesis are *linear* matrix inequalities, which have an affine dependence on the decision variable.

Definition 4 (e.g. [11], page 10)**.** *An LMI in the decision variable* $x = \begin{bmatrix} x_1 & \dots & \dots & x_m \end{bmatrix}^\top \in \mathbb{R}^m$ *is an expression of the form*

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i \preceq 0, \tag{2.7}$$

and defines the mapping $F : \mathbb{R}^m \to \mathbb{S}^n$ *with* $F_i \in \mathbb{S}^n, i = 0, \dots, m.$

LMIs can also be defined in terms of matrix variables as follows:

Definition 5 (e.g. [11], page 11)**.** *An LMI in the matrix variables* $X_i \in \mathbb{R}^{p_i \times q_i}, i = 1, \dots, r$*, is an expression of the form*

$$F(X_1, \dots, X_r) = F_0 + \sum_{i=1}^{r} (G_i X_i H_i + H_i^\top X_i^\top G_i^\top) \preceq 0, \tag{2.8}$$

defined by the mapping $F : \mathbb{R}^{p_1 \times q_1} \times \dots \times \mathbb{R}^{p_r \times q_r}$ *with* $F_0 \in \mathbb{S}^n$*,* $G_i \in \mathbb{R}^{n \times p_i}$*, and* $H_i \in \mathbb{R}^{q_i \times n}, i = 1, \dots, r.$

Besides LMIs, relevant to this thesis are bilinear matrix inequalities, which are matrix inequalities that contain products of decision variables

Definition 6 (e.g. [11], page 10)**.** *A bilinear matrix inequality (BMI) in the variable* $x = \begin{bmatrix} x_1 & \dots & x_m \end{bmatrix}^\top$ *is an expression of the form*

$$H(x) = H_0 + \sum_{i=1}^{m} x_i H_i + \sum_{i=1}^{m} \sum_{j=1}^{m} x_i x_j H_{i,j} \preceq 0, \tag{2.9}$$

defined by the mapping $H : \mathbb{R}^m \to \mathbb{S}^n$ *with* $H_0 \in \mathbb{S}^n$ *and* $H_i, H_{i,j} \in \mathbb{S}^n, i = 1, \dots, m,\ j = 1, \dots, m.$

Next to these, relevant to this thesis is the less common quadratic matrix inequality.

Definition 7 (e.g. [14])**.** *A quadratic matrix inequality (QMI) in the matrix variable* $X \in \mathbb{R}^{p \times q}$ *is an expression of the form*

$$J(X) = \begin{bmatrix} X \\ I \end{bmatrix}^\top \underbrace{\begin{bmatrix} P_{11} & P_{12} \\ P_{12}^\top & P_{22} \end{bmatrix}}_{P} \begin{bmatrix} X \\ I \end{bmatrix} \preceq 0, \tag{2.10}$$

defined by the mapping $J : \mathbb{R}^{p \times q} \to \mathbb{S}^q$ *with some partitioned matrix* $P \in \mathbb{S}^{p+q}$*.*

2.1.3 Convex sets and functions

To demonstrate that LMIs are convex constraints, the notions of convex sets and convex functions are introduced.

Definition 8 (Convex sets)**.** *A set* **F** *in a linear vector space is convex if*

$$\{x_1, x_2 \in \mathbf{F}\} \Rightarrow \{x := \alpha x_1 + (1 - \alpha) x_2 \in \mathbf{F}, \forall \alpha \in (0, 1)\}. \tag{2.11}$$

Definition 9 (Convex function)**.** *A function* $F : \mathbf{F} \to \mathbb{R}$ *is convex if* **F** *is a non-empty convex set and*

$$F(\alpha x_1 + (1 - \alpha) x_2) \leq \alpha F(x_1) + (1 - \alpha) F(x_2). \tag{2.12}$$

2.1.4 Convexity of LMIs

From Definitions 4 and 9, it can be shown that LMIs are convex constraints. Consider $x, y \in \mathbb{R}^m$ and $\alpha \in (0, 1)$ and an LMI as in 4. Recall by Definition 8 that $\mathbb{R}^m$ is a non-empty convex set. Using the definition of an LMI as in (2.8) shows that the expression

$$\begin{aligned} F(\alpha x + (1-\alpha)y) = F_0 + \sum_{i=1}^{m}(\alpha x_i + (1-\alpha)y_i)F_i &= F_0 + \alpha \sum_{i=1}^{m} x_i F_i + (1-\alpha)\sum_{i=1}^{m} y_i F_i \\ &= F_0 + \alpha \sum_{i=1}^{m} x_i F_i + \underbrace{-\alpha F_0 + \alpha F_0}_{=0} + (1-\alpha)\sum_{i=1}^{m} y_i F_i \\ &= \alpha(F_0 + \sum_{i=1}^{m} x_i F_i) + (1-\alpha)(F_0 + \sum_{i=1}^{m} y_i F_i) = \alpha F(x) + (1-\alpha)F(y). \end{aligned} \quad (2.13)$$

Hence if $F(x) \preceq 0$ and $F(y) \preceq 0$, then also $F(\alpha x + (1-\alpha)y) \preceq 0$. According to Definition 9, this implies that an LMI $F(x) \preceq 0$ defines a convex constraint on x and the set of solutions to this LMI is convex.

2.1.5 Semidefinite Programs (SDPs)

A semidefinite program (SDP) is a convex optimization problem of the form

$$\min_{x \in \mathbb{R}^m} \; c^\top x \quad (2.14a)$$

$$\text{subject to } \; F_0 + \sum_{i=1}^{m} x_i F_i \preceq 0, \quad (2.14b)$$

where $x^\top = \begin{bmatrix} x_1 & \dots & x_m \end{bmatrix}$, $c \in \mathbb{R}^m$, $F_i \in \mathbb{S}^n, i = 0, \dots, m$, and (2.14b) is an LMI in the decision variable x. The problems in which there is no objective to minimize are referred to as feasibility problems. In such cases, the SDP answers the question if there exists an $x \in \mathbb{R}^m$ such that (2.14b) is satisfied.

2.2 LMI properties and common methods for LMI formulations

The following sections demonstrate some tricks using properties of matrix inequalities. The main objective of these is to reformulate matrix inequalities in a way to end up with a semidefinite program, such that it can be solved using convex optimization.

2.2.1 Change of variables

As demonstrated in Section 2.1.5, in order to solve problems with semidefinite programs, constraints on the decision variables are required to be LMIs. BMIs as in Definition 6, which are not convex, can thus not be solved via SDPs. However, in some cases, a BMI can be converted into an LMI using a change of variables. By introducing a new variable, the bilinear term can be replaced by this variable, yielding a linear matrix inequality. One needs to be sure that the change of variables is chosen to be a one-to-one mapping, such that the original variable can be reconstructed after the optimization. Similarly, a change of variables can be used to replace the inverse of a decision variable, which is not convex either. Here, one needs to make sure that the newly introduced variable is invertible in order to reconstruct to original variable after the optimization.

2.2.2 Congruence transformation

Another relevant property of definite matrices follows from Sylvester's law of inertia. In order to present this theorem, the congruence transformation is defined as follows:

Definition 10. *Consider $P \in \mathbb{S}^n$ and a non-singular matrix $T \in \mathbb{R}^{n\times n}$. The product $T^\top PT$ is called a congruence transformation of P.*

With this definition, Sylvester's law of inertia states the following:

Theorem 2. *If $P \in \mathbb{S}^n$ is a real matrix, then any congruence transformation of P as in Definition 10 will have the same number of positive, negative, and zero eigenvalues as P.*

Proof. See, e.g, page 154 in [15].

Following Theorem 1, this implies that definiteness of P is preserved under congruence transformations.

2.2.3 Schur complement

From the properties of Section 2.2.2, an important relation can be obtained, which is the Schur complement.

Theorem 3 (e.g. [11], page 18)**.** *Consider $P \in \mathbb{S}^n$, $S \in \mathbb{R}^{n\times m}$, and $Q \in \mathbb{S}^m$. The following statements are equivalent.*

1. $\begin{bmatrix} P & S \\ S^\top & Q \end{bmatrix} \prec 0,$
2. $Q \prec 0$ *and* $P - SQ^{-1}S^\top \prec 0,$
3. $P \prec 0$ *and* $Q - S^\top P^{-1}S \prec 0.$

2.2.4 Concatenation of LMIs

If there exist multiple LMI constraints on x as $F_1(x) \preceq 0, F_2(x) \preceq 0, \ldots, F_n(x) \preceq 0$, they can be concatenated together to form a single LMI. This LMI will take the form of

$$F(x) \triangleq \begin{bmatrix} F_1(x) & 0 & \ldots & 0 \\ 0 & F_2(x) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & F_n(x) \end{bmatrix} \preceq 0. \tag{2.15}$$

One may observe that this is an application of the Schur complement of Theorem 3, which emerges when $S = 0$.

2.2.5 The S-procedure for quadratic forms

The S-procedure shows that certain implications involving inequalities of quadratic forms can be reformulated as LMIs. Using this procedure, some non-LMI conditions can be represented as LMIs. One of these non-LMI conditions are definiteness constraints of quadratic forms over a subset. The S-procedure is stated as follows:

Theorem 4 (e.g. [13], page 24)**.** *Consider* $x \in \mathbb{R}^m$ *and the quadratic forms* $x^\top F x$ *and* $x^\top G x$ *with* $F \in \mathbb{S}^m, G \in \mathbb{S}^m$*. Then*

$$z^\top F z \geq 0,\ \forall z \in \left\{x : x^\top G x \geq 0\right\}, \tag{2.16}$$

if and only if[2] *there exists a scalar* $\tau \geq 0$ *such that*

$$F - \tau G \succeq 0. \tag{2.17}$$

Similarly, positivity of a quadratic form can be ensured over a region defined by multiple quadratic forms as

$$z^\top F z \geq 0,\ \forall z \in \left\{x : x^\top G_i x \geq 0, i = 1, \dots, p\right\}, \tag{2.18}$$

if

$$F - \sum_{i=1}^{p} \tau_i G_i \succeq 0, \tag{2.19}$$

with $\tau_i \geq 0$.

2.3 Analysis of LTI systems through LMIs

In this section, it is shown how properties of an LTI system can be analysed using LMIs. For conciseness, only discrete-time LTI systems are considered. Here, the considered formulation is

$$x_{k+1} = Ax_k + Bu_k + Hw_k, \tag{2.20a}$$
$$z_k \quad = Cx_k + Du_k + Ew_k, \tag{2.20b}$$

where $x_k \in \mathbb{R}^{\eta_x}$ denotes the state, while $w_k \in \mathbb{R}^{\eta_w}$ is the exogenous input, $u_k \in \mathbb{R}^{\eta_u}$ is the control input, and $z_k \in \mathbb{R}^{\eta_z}$ is the performance output. The exogenous inputs are those which are not measurable and/or cannot be altered by control. This includes references to be tracked, external disturbances, and measurement noise. The performance output z_k is a signal which consists of terms which ought to be minimized, such as a tracking error or the response to the exogenous inputs on certain states.

2.3.1 LMI characterization for Lyapunov stability

Consider the autonomous system obtained from (2.20) with $u_k = 0$ and $w_k = 0$ as

$$x_{k+1} = Ax_k, \tag{2.21}$$

with the equilibrium point at $x = 0$. It is well known that this system is stable if $A \in \mathbb{R}^{\eta_x \times \eta_x}$ is Schur (i.e. has all its eigenvalues inside the unit disk in the complex plane). Alternatively, one can search for a function $V(x)$ which satisfies

- $V(x) > 0$ for $x \neq 0$,
- $V(x) = 0$ for $x = 0$,
- $V(x_{k+1}) - V(x_k) < 0$ for $x_k \neq 0$.

[2] For the only if direction, it is necessary that $\left\{x : x^\top G x > 0\right\}$ has an interior point, e.g. G has at least one positive eigenvalue. This is known as the generalized Slater condition.

If such a function can be found, then the system is asymptotically stable. Using a Lyapunov function which has a quadratic form $V(x) = x^\top Px$ where $P \in \mathbb{S}^{\eta_x}$ and $P \succ 0$, the asymptotic stability can be rewritten as an LMI constraint. The first two conditions of $V(x)$ are satisfied by the quadratic form and the constraint that $P \succ 0$, while the third condition reads as

$$x_{k+1}^\top Px_{k+1} - x_k^T Px_k = (Ax_k)^\top P(Ax_k) - x_k^\top Px_k = x_k^\top(A^\top PA - P)x_k < 0, \tag{2.22}$$

which itself is a quadratic form. As such, an equivalent LMI condition to (2.22) is found as

$$A^\top PA - P \prec 0. \tag{2.23}$$

Thus, the Lyapunov stability of a discrete-time LTI can be verified by an LMI feasibility problem.

2.3.2 LMI characterization of $\mathcal{H}_\infty$ performance

System norms are of interest, as they can clearly define a performance criteria for control systems. Consequently, they can be used to find controllers which optimize these performance criteria. One of such system norms is the $\mathcal{H}_\infty$ norm, which is relevant for reference tracking and disturbance attenuation. In order to define the $\mathcal{H}_\infty$ norm, consider the discrete-time LTI system of (2.20). With $u_k = 0$, the influence of w_k on z_k is captured by the state-space realization (A, H, C, E) with the corresponding transfer function

$$\mathcal{P}(z) = C(zI - A)^{-1}H + E. \tag{2.24}$$

The $\mathcal{H}_\infty$ norm of $\mathcal{P}$ is then defined as follows:

Definition 11. *The $\mathcal{H}_\infty$ norm of $\mathcal{P}$ as in* (2.24)*, denoted by $\|\mathcal{P}\|_\infty$, is defined as*

$$\|\mathcal{P}\|_\infty = \sup_{z\in\mathbb{C}_+} \sigma_{\text{max}}(\mathcal{P}(z)), \tag{2.25}$$

where $\sigma_{\text{max}}(\mathcal{P}(z)$ represents the maximum singular value of $\mathcal{P}(z)$ evaluated at the point z in the complex z-plane.

From this definition, it can be observed that $\mathcal{P}$ needs to be analytic in the right-half plane in order for its $\mathcal{H}_\infty$ norm to be finite. Furthermore, it can be shown that

$$\|\mathcal{P}(z)\|_\infty = \sup_{\omega\in\mathbb{R}} \sigma_{\text{max}}(\mathcal{P}(e^{j\omega})), \tag{2.26}$$

i.e, the $\mathcal{H}_\infty$ norm is equal to the maximum singular value of the transfer function evaluated at points on the unit circle in the z-plane. A signal-based interpretation of the $\mathcal{H}_\infty$ can be stated as follows:

Lemma 5 (e.g. [11], page 52)**.** *The $\mathcal{H}_\infty$ norm of $\mathcal{P}$ as in* (2.24)*, denoted by $\|\mathcal{P}\|_\infty$, is equal to*

$$\|\mathcal{P}\|_\infty = \sup_{w\in\mathcal{L}_2, w\neq 0} \frac{\|z\|_2}{\|w\|_2}, \tag{2.27}$$

where $\mathcal{L}_2$ denotes the space of square-summable sequences, i.e. $w \in \mathcal{L}_2 \Rightarrow \sum_{k=0}^{\infty} w_k^\top w_k < \infty$.

Lemma 5 shows a clear motivation for for utilizing the $\mathcal{H}\infty$ norm in controller synthesis, as the squared $\mathcal{H}_\infty$ norm corresponds to the maximum (worst-case) energy amplification from the exogenous input to the performance output. Therefore, one may want to minimize this norm or ensure it is below a certain fixed value.

The bounded real lemma is an LMI characterization of the $\mathcal{H}_\infty$ performance used to ensure that $\|\mathcal{P}\|_\infty < \gamma$ for a fixed or to-be-optimized γ. For discrete-time systems as in (2.24) it reads as follows:

Lemma 6 (Discrete-time bounded real lemma)**.** *The following statements are equivalent:*

1. *A is Schur-stable and $\|\mathcal{P}\|_\infty < \gamma$.*
2. $\begin{bmatrix} P & 0 & * & * \\ 0 & \gamma I & * & * \\ PA & PH & P & 0 \\ C & E & 0 & \gamma I \end{bmatrix} \succ 0.$

Here $*$ denotes blocks which can be deduced from symmetry.

This is a well known result, and multiple equivalent forms exists (see e.g. Section 3.2.2 of [11]). Since showing the equivalence of these forms requires cumbersome algebraic manipulations, a proof for this specific form is presented in A.1, which follows the technical notes of [16] very closely.

2.4 Controller synthesis for LTI systems through LMIs

In this section it is shown how controllers for LTI systems can be designed using LMIs. Here the analytical results of Section 2.3 are extended to synthesis problems by introducing decision variables that represent the controller parameters. The standard control problem, as presented in Figure 2.1, is considered.

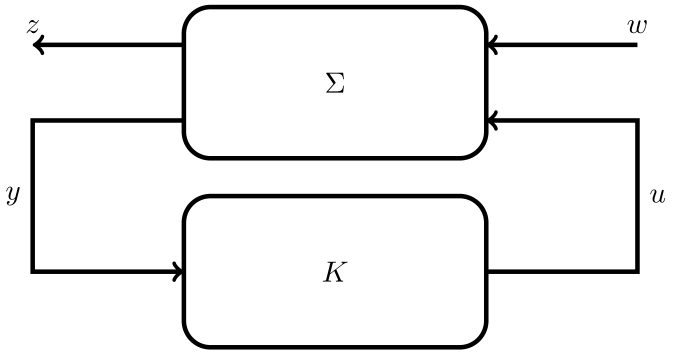

Figure 2.1: Standard control problem for a plant Σ and a static gain matrix K.

Here, the plant Σ is considered to be of the form (2.20) and full state measurements are available, so

$$x_{k+1} = Ax_k + Bu_k + Hw_k, \tag{2.28a}$$
$$z_k = Cx_k + Du_k + Ew_k, \tag{2.28b}$$
$$y_k = x_k. \tag{2.28c}$$

With the control law, $u_k = Kx_k$, the closed-loop dynamics from w_k to z_k of (2.28) is described by the state-space model

$$\left[\begin{array}{c} x_{k+1} \\ \hline z_k \end{array}\right] = \left[\begin{array}{c|c} A_K & H_K \\ \hline C_K & E_K \end{array}\right] \left[\begin{array}{c} x_k \\ \hline w_k \end{array}\right], \tag{2.29}$$

where

$$\left[\begin{array}{c|c} A_K & H_K \\ \hline C_K & E_K \end{array}\right] = \left[\begin{array}{c|c} A + BK & H \\ \hline C + DK & E \end{array}\right]. \tag{2.30}$$

It should be observed that (2.29) is a standard state-space formulation. Consequently, LMIs such as those presented in Section 2.3 can be applied to the closed-loop system through the substitution $(A, B, C, D) \rightarrow (A_K, H_K, C_K, E_K)$. To illustrate, the stability of the closed-loop is guaranteed if there exists $P \succ 0$ and $K \in \mathbb{R}^{\eta_u \times \eta_x}$ such that

$$(A + BK)^\top P(A + BK) - P \prec 0. \tag{2.31}$$

This is a direct application of the LMI used for stability analysis in (2.23), translated into a controller synthesis problem through the decision variable K. Because (2.31) contains products of the decision variables P and K it is not an LMI, hence K can not be determined via an SDP. Luckily though, (2.31) can be turned into an LMI via some of the tricks in Section 2.2. In preparation, rewrite (2.31) into

$$P - (A + BK)^\top P \underbrace{P^{-1} P}_{I}(A + BK) \tag{2.32}$$

$$= P - (PA + PBK)^\top P^{-1}(PA + PBK) \succ 0. \tag{2.33}$$

This form yields itself for application of the Schur complement of Theorem 3 into

$$\begin{bmatrix} P & PA + PBK \\ * & P \end{bmatrix} \succ 0, \tag{2.34}$$

which contains a bilinear term in the decision variables P and K. This form does not yield itself for a change of variables. Luckily, via a congruence transform with blkdiag(P^{-1}, P^{-1}), which is invertible since $P \succ 0$ implies it has no eigenvalues that are equal to zero, the constraint in (2.34) can be equivalently formulated as

$$\begin{bmatrix} P^{-1} & 0 \\ 0 & P^{-1} \end{bmatrix}^\top \begin{bmatrix} P & PA + PBK \\ * & P \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ 0 & P^{-1} \end{bmatrix} = \begin{bmatrix} P^{-1} & AP^{-1} + BKP^{-1} \\ * & P^{-1} \end{bmatrix} \succ 0, \tag{2.35}$$

which can be turned into an LMI via the change of variables $Q = P^{-1}$ and $F = KP^{-1}$ into

$$\begin{bmatrix} Q & AQ + BF \\ * & Q \end{bmatrix} \succ 0. \tag{2.36}$$

The controller gain matrix can be reconstructed from the solution of the SDP as $K = FP^{-1}$. The design of state feedback controllers for optimal $\mathcal{H}_\infty$-performance is presented in Chapter 4 and as such will not be discussed here.

2.5 Controller synthesis for LTI systems with uncertainty

In view of robustness, one must consider that the system might not be exactly known. In Section 2.4, the controller synthesis does not consider these uncertainties, potentially impacting stability and performance when uncertainties are present. Fortunately, through a characterization of the uncertainty, controllers can be designed that guarantee stability and/or performance criteria in the presence of such uncertainties.

In this thesis, only systems with an affine dependency on time-invariant parametric uncertainties are considered. To illustrate, an uncertain autonomous system could be defined as

$$x_{k+1} = (A + \Delta)x_k, \tag{2.37}$$

where $\Delta \in \mathbb{R}^{\eta_x \times \eta_x}$ denotes the uncertainty on some nominal system matrix A. Although the exact value of Δ is unknown, it is assumed that $\Delta \in \mathbf{\Delta}$ for some known set of uncertainties $\mathbf{\Delta}$. With this description, it will become possible to define LMIs which can guarantee stability and/or performance criteria for any $\Delta \in \mathbf{\Delta}$.

2.5.1 Introducing uncertainty in LMI problems

For control systems with uncertainty, any subsequent LMI-based analysis or controller synthesis should be carried out with parameter dependent LMIs. A generic description of such a parameter dependent LMI is defined as follows:

Definition 12. *A parameter dependent LMI is an expression of the form*

$$F(\Delta) \preceq 0, \ \forall \Delta \in \boldsymbol{\Delta}, \tag{2.38}$$

with uncertain parameter Δ and a known set $\boldsymbol{\Delta}$.

To demonstrate how uncertainty might be included in LMI problems[3], the notion of quadratic stability is introduced as follows:

Theorem 7. *The uncertain system in* (2.37) *is quadratically stable over $\boldsymbol{\Delta}$ if there exists a $P \succ 0$, such that*

$$(A+\Delta)^\top P(A+\Delta) - P \prec 0, \ \forall \Delta \in \boldsymbol{\Delta}. \tag{2.39}$$

One should observe that (2.39) reduces to (2.23) when $\boldsymbol{\Delta} = \{0\}$, which is used to determine the stability of systems without uncertainty.
The affine dependence on Δ in (2.37) causes the Lyapunov condition in (2.39) to have a quadratic dependency on Δ. This dependency can be rendered affine via the Schur complement, since rewriting (2.39) in the form

$$P - (A+\Delta)^\top P P^{-1} P(A+\Delta) \tag{2.40}$$

$$= P - (PA + P\Delta)^\top P^{-1}(PA+P\Delta) \succ 0, \ \forall \Delta \in \boldsymbol{\Delta} \tag{2.41}$$

yields an equivalent LMI as

$$\begin{bmatrix} P & PA + P\Delta \\ * & P \end{bmatrix} \succ 0, \ \forall \Delta \in \boldsymbol{\Delta}. \tag{2.42}$$

Thus, just as the introduction of a parameter K in LMI analysis problems, which represents the controller, transforms analysis problems into controller synthesis problems, these LMIs can be employed for uncertain LTI systems by introducing a parameter Δ to represent the uncertainty. A critical differentiation to address is that the controller parameter K is introduced as a decision variable to be searched for in an LMI problem, whereas the uncertainty Δ imposes additional constraints on an LMI problem.
In realistic scenarios, the uncertainty Δ represents some unknown physical parameters, which can take any real value and as such, $\boldsymbol{\Delta}$ will have an infinite number of elements. Thus, solving LMI problems involving this uncertainty will require satisfying an infinite number of constraints. This is known as a semi-infinite program with a finite number of variables and an infinite number of constraints, and cannot be solved directly by convex optimization. Fortunately, depending on the characterization of $\boldsymbol{\Delta}$, different strategies exist to solve such a problem, and are presented in Sections 2.5.2-2.5.4.

2.5.2 Convex hull relaxation

Structured real uncertainty blocks can often be described by the convex hull of finitely many matrices as

$$\boldsymbol{\Delta} = \text{conv}(\Delta_1, \ldots, \Delta_p). \tag{2.43}$$

[3]Although presented here for stability, a similar approach should be taken for other LMI problems involving parameter dependent matrices which can be characterized by some uncertainty set.

This description can capture polytopic or interval uncertainty regions, in which the matrices $\Delta_1, \ldots, \Delta_p$ represent the vertices of the region.
Only in the special case that (3.42) depends affinely on the uncertainty Δ, it defines a convex constraint on $\Delta \in \boldsymbol{\Delta}$. This is because the convex hull is a non-empty convex set, and as such the results from Section 2.1.4 are applicable. With this, the convex hull relaxation can be stated as follows:

Theorem 8. *A parameter dependent constraint as* (2.38) *with affine dependency on Δ is satisfied for all $\Delta \in \boldsymbol{\Delta}$, where $\boldsymbol{\Delta}$ is as in* (2.43)*, if and only if*

$$F(\Delta_i) \preceq 0, \forall i = 1, \ldots, p. \tag{2.44}$$

Here, p represents the number of vertices of the region.

2.5.3 Full-block S-procedure

In order to apply Theorem 8, it is required that the parameter dependent constraint has an affine dependency on the uncertainty. If the dependency is not affine, (2.38) does not have a convexity property (as defined in Section 2.1.4), even when the uncertainty set is defined by a convex hull. Fortunately, the full-block S-procedure of [17] can resolve this issue for parameter dependent constraints that have a rational dependency on the uncertainty.
Due to its clear implementation in the context of this thesis, a corollary of the full-block S-procedure ([18], Lemma 7) is used. For uncertainties which enter the matrix inequality rationally, the interconnection can be expressed by a Linear Fractional Transformation (LFT) as

$$\Delta \star \underbrace{\left[\begin{array}{c|c} Y_{11} & Y_{12} \\ \hline Y_{21} & Y_{22} \end{array}\right]}_{Y} \triangleq Y_{22} + Y_{21}\Delta(I - Y_{11}\Delta)^{-1}Y_{12}, \tag{2.45}$$

for a known matrix Y and the uncertainty Δ. The lemma states the following:

Lemma 9 (Full-block S-procedure)**.** *The LFT $\Delta \star Y$ is well-posed and*

$$\text{He}(\Delta \star Y) \triangleq \Delta \star Y + (\Delta \star Y)^\top \succ 0, \ \forall \Delta \in \boldsymbol{\Delta}, \tag{2.46}$$

holds if there exists a multiplier matrix Φ which satisfies

$$\begin{bmatrix} \Delta^\top \\ I \end{bmatrix}^\top \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^\top & \Phi_{22} \end{bmatrix} \begin{bmatrix} \Delta^\top \\ I \end{bmatrix} \preceq 0, \ \forall \Delta \in \boldsymbol{\Delta}, \tag{2.47}$$

and

$$\begin{bmatrix} Y_{21}\Phi_{22}Y_{21}^\top + \text{He}(Y_{22}) & Y_{21}\Phi_{22}Y_{11}^\top + Y_{21}\Phi_{12}^\top + Y_{12}^\top \\ * & \Phi_{11} + Y_{11}\Phi_{22}Y_{11}^\top + \text{He}(Y_{11}\Phi_{12}^\top) \end{bmatrix} \succ 0. \tag{2.48}$$

What should be observed is that the full-block S-procedure transforms the LMI constraint with rational dependency on the uncertainty (2.46) into a single constraint (2.48), which is independent of Δ, and a semi-infinite QMI (2.47), which has a quadratic dependency on Δ. This form is beneficial, as one only needs to find a single multiplier which satisfies (2.47) and (2.48) to solve the original problem. Furthermore, using Lemma 10, a convexity property of the mapping in (2.47) can be enforced, recovering the convex hull relaxation.

Lemma 10 ([12])**.** *Consider structured real uncertainty blocks as in* (2.43) *and the constraint in* (2.47)*. If the conditions*

$$\begin{bmatrix} I \\ 0 \end{bmatrix}^\top \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^\top & \Phi_{22} \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} \succeq 0, \tag{2.49}$$

and

$$\begin{bmatrix}\Delta_i^\top \\ I\end{bmatrix}^\top \begin{bmatrix}\Phi_{11} & \Phi_{12} \\ \Phi_{12}^\top & \Phi_{22}\end{bmatrix} \begin{bmatrix}\Delta_i^\top \\ I\end{bmatrix} \preceq 0, \forall i = 1, \ldots, p, \tag{2.50}$$

hold, then (2.47) *is satisfied.*

Proof. Consider $\Delta_1, \Delta_2 \in \boldsymbol{\Delta}$ and $\alpha \in (0,1)$. Definition 9 implies that the mapping

$$\Delta \to \begin{bmatrix}\Delta^\top \\ I\end{bmatrix}^\top \begin{bmatrix}\Phi_{11} & \Phi_{12} \\ \Phi_{12}^\top & \Phi_{22}\end{bmatrix} \begin{bmatrix}\Delta^\top \\ I\end{bmatrix}, \tag{2.51}$$

is convex if and only if

$$\begin{bmatrix}(\alpha\Delta_1 + (1-\alpha)\Delta_2)^\top \\ I\end{bmatrix}^\top \begin{bmatrix}\Phi_{11} & \Phi_{12} \\ \Phi_{12}^\top & \Phi_{22}\end{bmatrix} \begin{bmatrix}(\alpha\Delta_1 + (1-\alpha)\Delta_2)\top \\ I\end{bmatrix} \leq \\ \alpha \begin{bmatrix}\Delta_1^\top \\ I\end{bmatrix}^\top \begin{bmatrix}\Phi_{11} & \Phi_{12} \\ \Phi_{12}^\top & \Phi_{22}\end{bmatrix} \begin{bmatrix}\Delta_1^\top \\ I\end{bmatrix} + (1-\alpha) \begin{bmatrix}\Delta_2^\top \\ I\end{bmatrix}^\top \begin{bmatrix}\Phi_{11} & \Phi_{12} \\ \Phi_{12}^\top & \Phi_{22}\end{bmatrix} \begin{bmatrix}\Delta_2^\top \\ I\end{bmatrix}. \tag{2.52}$$

Well

$$\begin{bmatrix}(\alpha\Delta_1 + (1-\alpha)\Delta_2)^\top \\ I\end{bmatrix}^\top \begin{bmatrix}\Phi_{11} & \Phi_{12} \\ \Phi_{12}^\top & \Phi_{22}\end{bmatrix} \begin{bmatrix}(\alpha\Delta_1 + (1-\alpha)\Delta_2)\top \\ I\end{bmatrix} = \\ \alpha \begin{bmatrix}\Delta_1^\top \\ I\end{bmatrix}^\top \begin{bmatrix}\Phi_{11} & \Phi_{12} \\ \Phi_{12}^\top & \Phi_{22}\end{bmatrix} \begin{bmatrix}\Delta_1^\top \\ I\end{bmatrix} + (1-\alpha) \begin{bmatrix}\Delta_2^\top \\ I\end{bmatrix}^\top \begin{bmatrix}\Phi_{11} & \Phi_{12} \\ \Phi_{12}^\top & \Phi_{22}\end{bmatrix} \begin{bmatrix}\Delta_2^\top \\ I\end{bmatrix} - \\ \alpha(1-\alpha) \begin{bmatrix}(\Delta_1 - \Delta_2)^\top \\ 0\end{bmatrix}^\top \begin{bmatrix}\Phi_{11} & 0 \\ 0 & 0\end{bmatrix} \begin{bmatrix}(\Delta_1 - \Delta_2)^\top \\ 0\end{bmatrix}, \tag{2.53}$$

contains the right-hand side of (2.52). The convexity condition then reads as

$$-\alpha(1-\alpha) \begin{bmatrix}(\Delta_1 - \Delta_2)^\top \\ 0\end{bmatrix}^\top \begin{bmatrix}\Phi_{11} & 0 \\ 0 & 0\end{bmatrix} \begin{bmatrix}(\Delta_1 - \Delta_2)^\top \\ 0\end{bmatrix} \leq 0, \tag{2.54}$$

which implies that $\Phi_{11} \succeq 0$, since $-\alpha(1-\alpha) < 0$ for $\alpha \in (0,1)$. This condition on Φ_{11} is enforced by the inequality in (2.49). Since the mapping is convex and $\boldsymbol{\Delta}$ defines a convex set, (2.47) is satisfied if and only if it is satisfied at the vertices of $\boldsymbol{\Delta}$. This is enforced by (2.50). ☐

In this way, the set of admissible multipliers is described by finitely many LMI constraints.
In case $\boldsymbol{\Delta}$ describes a convex region with a smooth boundary, Lemma 10 does not yield tractable conditions. Luckily, the full-block S-procedure can still be used to turn this problem into a finite number of LMIs with guaranteed validity over the whole parameter space by constructing valid multipliers from inner approximations. To this end, introduce define the set of all admissible multipliers as

$$\boldsymbol{\Phi} \triangleq \left\{\Phi \mid \text{(2.47) is satisfied}\right\}. \tag{2.55}$$

It is possible to obtain inner approximations of this set based on finitely many LMI conditions. To illustrate, if it is known that the uncertainty satisfies a known bound on the spectral norm as

$$\|\Delta\|_2 \leq \bar{\delta}, \ \forall \Delta \in \boldsymbol{\Delta}, \tag{2.56}$$

for some $\bar{\delta} > 0$, then it can be deduced that

$$\|\Delta\|_2 \leq \bar{\delta} \Leftrightarrow \Delta\Delta^\top \preceq \bar{\delta}^2 I \Leftrightarrow \begin{bmatrix}\Delta^\top \\ I\end{bmatrix}^\top \begin{bmatrix}I & 0 \\ 0 & -\bar{\delta}^2 I\end{bmatrix} \begin{bmatrix}\Delta^\top \\ I\end{bmatrix} \preceq 0. \tag{2.57}$$

Hence a valid inner approximation of $\mathbf{\Phi}$ can be constructed as

$$\mathbf{\Phi}_{\mathsf{n}} = \left\{ \begin{pmatrix} \tau I & 0 \\ 0 & -\tau\bar{\delta}^2 I \end{pmatrix} \middle| \tau \geq 0 \right\}, \tag{2.58}$$

so checking (2.48) amounts to solving a problem with one additional decision variable τ. In order for (2.48) to be an LMI, one needs to make sure that Y_{11} and Y_{21} do not depend on any decision variables (since this would lead to a BMI problem). For other, more refined inner approximations of $\mathbf{\Phi}$ the reader is referred to Section 6.3 in [12].

2.5.4 Sum-of-squares relaxation

In Section 2.5.3, the condition on the multiplier is satisfied by constructing an inner approximation of the set of admissible multipliers. The construction relies on the deduction from an *explicit* condition (such as a bound on the spectral norm), which holds for all uncertainties Δ. In cases where the uncertainty set can only be described *implicitly*, such as through scalar-valued inequalities as

$$\mathbf{\Delta} = \left\{ \Delta : g_1(\Delta) \leq 0, \ldots, g_p(\Delta) \leq 0 \right\}, \tag{2.59}$$

or through an LMI description as

$$\mathbf{\Delta} = \left\{ \Delta : G_1(\Delta) \preceq 0, \ldots, G_p(\Delta) \preceq 0 \right\}, \tag{2.60}$$

such a deduction cannot be made. For these kinds of problems, the sum-of-squares relaxation can be used. As a preparation for this, the following notions for global positive definiteness are required.

Definition 13. *A function $f : \mathbb{R}^m \to \mathbb{R}$ is globally positive semidefinite if and only if*

$$f(x) \geq 0, \; \forall x \in \mathbb{R}^m. \tag{2.61}$$

Definition 14. *A matrix-valued function $F : \mathbb{R}^m \to \mathbb{S}^n$ is globally positive semidefinite if and only if*

$$F(x) \succeq 0 \; \forall x \in \mathbb{R}^m. \tag{2.62}$$

With these, the sum-of-squares property is defined as follows:

Definition 15. *A matrix-valued function $F : \mathbb{R}^m \to \mathbb{S}^n$ is a sum-of-squares (SOS) if it can be represented as*

$$F(x) = T(x)^\top T(x), \tag{2.63}$$

for some matrix-valued function $T : \mathbb{R}^m \to \mathbb{R}^{q \times n}$.

The significance of sum-of-squares lies in the fact it is a sufficient condition for global positive semidefiniteness. This can clearly be seen in the scalar case ($n = 1$), since $T(x)$ will be a column vector and $F(x)$ is equal to the sum of the squared elements of $T(x)$, which are all positive. As such, a parameter dependent constraint like the multiplier condition of (2.47) is clearly satisfied by finding a multiplier matrix Φ[4] such that

$$-F(\Phi, \Delta) \text{ is SOS.} \tag{2.64}$$

A sum-of-squares relaxation for (2.64), with knowledge on Δ as (2.59), can be formulated as follows. If there exists a multiplier Φ and positive semidefinite matrices $S_1, \ldots, S_p$ such that

$$-F(\Phi, \Delta) + \sum_{i=1}^{p} S_i g_i(\Delta) \text{ is SOS,} \tag{2.65}$$

[4] This holds in general for any choice of decision variables; the multiplier matrix is solely used as a practical example.

one can infer that (2.64) is satisfied. Indeed, (2.65) implies

$$F(\Phi, \Delta) \preceq \sum_{i=1}^{p} S_i g_i(\Delta), \ \forall \Delta \in \mathbf{\Delta}, \tag{2.66}$$

exploitation of the negative semidefiniteness of $g_i(\Delta), i = 1, \ldots, p$ shows the right-hand side of (2.66) to be negative semidefinite and hence $F(\Phi, \Delta) \preceq 0$.
For uncertainty regions described by (2.60), a similar relaxed condition as (2.65) can be constructed as

$$-F(\Phi, \Delta) + \sum_{i=1}^{p} (S_i, G_i(\Delta))_\nu \text{ is SOS}, \tag{2.67}$$

where $(.,.)_\nu : \mathbb{R}^{\nu q \times \nu q} \times \mathbb{R}^{\nu q \times \nu q} \rightarrow \mathrm{R}^{\nu \times \nu}$ denotes the bilinear mapping from [19], defined as

$$(S, G)_\nu = \mathrm{tr}_\nu(S^\top (I_\nu \otimes G)), \tag{2.68}$$

with

$$\mathrm{tr}_p(C) = \begin{bmatrix} \mathrm{tr}(C_{11}) & \ldots & \mathrm{tr}(C_{1\nu}) \\ \vdots & \ddots & \vdots \\ \mathrm{tr}(C_{\nu 1}) & \ldots & \mathrm{tr}(C_{\nu\nu}) \end{bmatrix}, \text{ for } C \in \mathbb{R}^{\nu q \times \nu q}, \ C_{jk} \in \mathrm{R}^{q \times q}. \tag{2.69}$$

This mapping ensures dimensional consistency without loss of definiteness. For a more thorough explanation of this, the the reader is referred to [19].
To verify the SOS property, one can resort to numerical solvers such as the SOS module in Yalmip [20]. Although very powerful, SOS solvers are computationally expensive. As will be shown in Chapter 4, for some specific forms of $F(\Phi, \Delta)$, including the multiplier condition of (2.47), and some specific matrix-valued functions $G_i(\Delta)$ in (2.60), the SOS relaxation can be transformed into a computationally efficient LMI problem.

3 INTRODUCTION TO LMI-BASED DATA-DRIVEN CONTROL

This chapter aims to highlight some key concepts in LMI-based data-driven control. Here, the primary interest lies in more recent literature on data-driven control, used to synthesize controllers directly without any intermediate system identification. For clarity, this type of approach is sometimes called *direct* data-driven controller synthesis, in contrast to *indirect* approaches using intermediate system identification (which could be seen as a data-driven alternative to first principle modelling).
Due to the lack of system identification, the designer will not have access to an explicit system representation, such as Bode plots or state space models. The recent literature on LMI-based direct data-driven control revolves around the idea that data obtained from a single open-loop experiment can be a system representation itself. This line of work is heavily inspired by research led by Willems [21], discussed in more detail in Section 3.2. This work shows how the input signal and time horizon can be chosen such that data from a single open-loop experiment captures the whole *behaviour* of the system. This entails that all possible trajectories of the system can be constructed with data from this single experiment. In consequent works, it is shown that this constitutes an informative representation of the system and can be used to design controllers.
The rest of this chapter shows how the results from [21], later coined the fundamental lemma, can be used for controller synthesis in the LMI framework. Here, the works are limited to approaches which consider discrete-time LTI systems. In correspondence with the novel implementation of the thesis, cases of special interest are those in which the data is perturbed. The word *perturbation* is used as an umbrella term for all effects which corrupt the data, such as measurement noise, external disturbances, and effects caused by slight nonlinearities (e.g. geometric terms).

Preliminaries

Similarly to Section 2.4, the exclusive focus is on the state evolution of discrete-time LTI systems which can be described by

$$x_{k+1} = A_{\text{tr}} x_k + B_{\text{tr}} u_k + H w_k, \tag{3.1}$$

where the true system matrices A_{tr} and B_{tr} are unknown. Here $x_k \in \mathbb{R}^{\eta_x}$ denotes the state, $u_k \in \mathbb{R}^{\eta_u}$ the control input, and $w_k \in \mathbb{R}^{\eta_w}$ the perturbation. The matrix H is assumed to be known and can model a priori that the perturbation w is contained in a subspace rather than affecting the entire state space. If such knowledge is not available, consider $H = I$.
From a length-T open-loop experiment of (3.1), input-state measurements are obtained. By collecting the samples of the experiment, the following matrices are defined:

$$X \;= \begin{bmatrix} x_0 & x_1 & \dots & x_{T-1} \end{bmatrix} \in \mathbb{R}^{\eta_x \times T}, \tag{3.2a}$$

$$X_+ = \begin{bmatrix} x_1 & x_2 & \dots & x_T \end{bmatrix} \in \mathbb{R}^{\eta_x \times T}, \tag{3.2b}$$

$$U \;= \begin{bmatrix} u_0 & u_1 & \dots & u_{T-1} \end{bmatrix} \in \mathbb{R}^{\eta_u \times T}. \tag{3.2c}$$

Although the perturbation signal cannot be measured, and hence its samples unknown, they can be collected in a similar way to form the matrix

$$W = \begin{bmatrix} w_0 & w_1 & \dots & w_{T-1} \end{bmatrix} \in \mathbb{R}^{\eta_w \times T}. \tag{3.3}$$

3.1 The informativity framework

In order to get a better understanding of the data-driven controller synthesis approaches in later sections, some underlying concepts and definitions are presented, mainly following the data informativity framework from [5].
Although in data-driven control, no explicit system representation is available, it is assumed that the true system can be described as in (3.1). As a general description of all such state-space models, the model class is defined as follows:

Definition 16. *A model class $\boldsymbol{M}$ is a set of systems (A, B) which satisfy the dimensional constraints*

$$\boldsymbol{M} = \left\{ (A, B) \middle| A \in \mathbb{R}^{\eta_x \times \eta_x}, B \in \mathbb{R}^{\eta_x \times \eta_u} \right\}. \tag{3.4}$$

Consider the true system as in (3.1), for which $(A_{\text{tr}}, B_{\text{tr}}) \in \boldsymbol{M}$. This system, denoted by Σ_{tr} is thus an element of the model class $\boldsymbol{M}$, as shown in Figure 3.1.

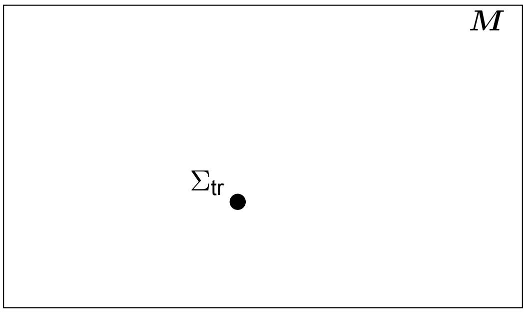

Figure 3.1: The true system can be described as an element of a model class.

Data from this system is obtained via an open-loop experiment, yielding the matrices in (3.2). With this data, it becomes possible to define a set of systems *consistent* with the data

Definition 17. *The set of consistent systems, denoted by $\boldsymbol{\Sigma}_{\text{D}} \subseteq \boldsymbol{M}$, contains all systems which could have generated the same data as the true system.*

The size of this set heavily depends on the length of the experiment and the input u, as this determines how much of the system can be learned through the experiment.
It follows naturally that the true system explains the data, e.g. $\Sigma_{\text{tr}} \in \boldsymbol{\Sigma}_{\text{D}}$ and that other systems explaining the data are (in some way) similar to the true system, as shown in Figure 3.2.

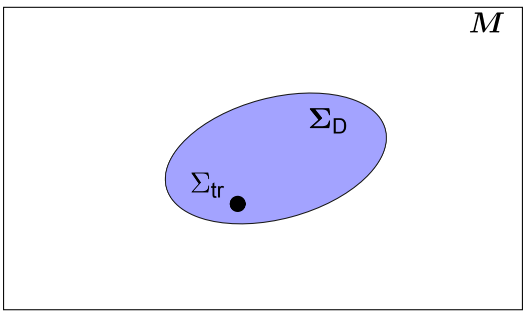

Figure 3.2: The set Σ_{D} describes all systems consistent with the data.

Suppose the data will be used to determine if the true system has a certain property (such as stability), and denote the set of all systems in $\boldsymbol{M}$ which have this property by $\boldsymbol{\Sigma}_{\mathsf{P}}$. The system Σ_{tr} can not be distinguished from any other system in $\boldsymbol{\Sigma}_{\mathsf{D}}$ solely from the data. Therefore, the only way to conclude that $\Sigma_{\mathsf{tr}} \in \boldsymbol{\Sigma}_{\mathsf{P}}$ is when *all* system consistent with the data satisfy this property, i.e. $\boldsymbol{\Sigma}_{\mathsf{D}} \subseteq \boldsymbol{\Sigma}_{\mathsf{P}}$. Hence, these methods are potentially conservative, as is shown in Figure 3.3.

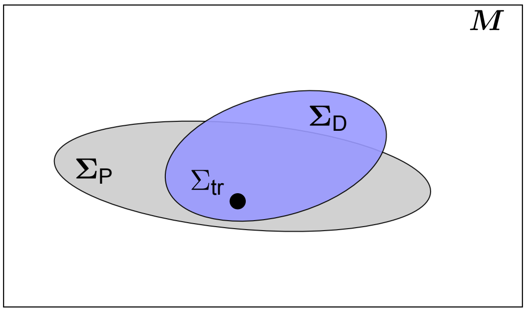

Figure 3.3: From the data, it cannot be guaranteed that the true system Σ_{tr} satisfies the desired property as $\boldsymbol{\Sigma}_{\mathsf{D}} \subsetneq \boldsymbol{\Sigma}_{\mathsf{P}}$.

By conducting proper experiments, which are sufficiently long and have sufficiently exciting inputs, it may be possible to reduce the size of the set $\boldsymbol{\Sigma}_{\mathsf{D}}$, potentially yielding less conservative results as shown in Fig. 3.4.

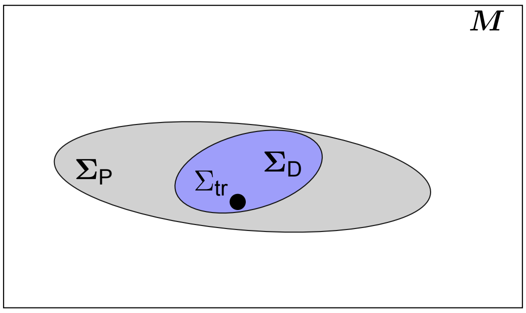

Figure 3.4: For more informative experiments, the set of consistent systems may be reduced. As a result, $\Sigma_{\mathsf{D}} \subset \Sigma_{\mathsf{P}}$ and hence the objective can be guaranteed to be satisfied for the true system Σ_{tr}.

In order to use this framework for controller synthesis, consider that the controller can be parameterized by a parameter K. With this, denote $\boldsymbol{\Sigma}_{\mathsf{D}}(K)$ as the set of all systems obtained as the interconnection of a system in $\boldsymbol{\Sigma}_{\mathsf{D}}$ with the controller K. The controller synthesis problem then reads as finding a *single* controller K that ensures $\boldsymbol{\Sigma}_{\mathsf{D}}(K) \subseteq \boldsymbol{\Sigma}_{\mathsf{P}}$.

3.2 Willems' fundamental lemma

A key consideration of data-driven controller synthesis is the choice of the time horizon T and the input sequence U of the open-loop experiment. This is because good choices can potentially reduce conservatism in the controller synthesis, as shown in Section 3.1.
Although not developed particularly for data-driven control, the main concern of [21] is whether there exist conditions on T and U such that all trajectories of (3.1) can be parameterized by the matrices in (3.2). In [21], it is shown that this can be done under some requirements. Only perturbation-free systems are considered in [21], which are captured by (3.1) with $w = 0$. In later sections, it will be shown how the results of [21] are still useful for controller synthesis

with perturbed data. In order to state Willems' lemma, introduce persistently exciting signals as follows:

Definition 18. *A finite sequence* $U = \begin{bmatrix} u_0 & u_1 & \dots & u_{T-1} \end{bmatrix} \in \mathbb{R}^{\eta_u \times T}$ *is persistently exciting of order* $L \geq 1$ *if and only if the Hankel matrix of depth* L*, defined as*

$$H_L(U) \triangleq \begin{bmatrix} u_0 & u_1 & \dots & u_{T-L} \\ u_1 & u_2 & \dots & u_{T-L+1} \\ \vdots & \vdots & \ddots & \vdots \\ u_{L-1} & u_L & \dots & u_{T-1} \end{bmatrix}, \tag{3.5}$$

has full row rank of $\eta_u \cdot L$*. This rank requirement enforces a lower bound on length of the sequence as* $T \geq (\eta_u + 1)L - 1$*.*

With this, Willems' lemma can be stated as follows:

Lemma 11. *Consider a system as in* (3.1) *with* $w = 0$ *and suppose the system is controllable and observable. Consider data as in* (3.2) *obtained by a persistently exciting input* u *of order* $\eta_x + d$ *for some* $d \geq 1$*. Then, any* d*-long input/state*[1] *trajectory of* (3.1)*, denoted by* $\begin{bmatrix} \tilde{U}_d^\top & \tilde{X}_d^\top \end{bmatrix}^\top$ *can be constructed from*

$$\begin{bmatrix} \text{vec}(\tilde{U}_d) \\ \text{vec}(\tilde{X}_d) \end{bmatrix} = \begin{bmatrix} H_d(U) \\ H_d(X) \end{bmatrix} g, \tag{3.6}$$

for some vector $g \in \mathbb{R}^{T-d+1}$*. Here,* vec$(\cdot)$ *denotes an operator which stacks the columns of its input on top of each other.*

Proof: This result is proven in [21] in the behavioural framework. For a proof in the standard state space setting, the reader is referred to [22].
Willems' fundamental lemma shows that it is possible to represent any trajectory of (3.1) as a linear combination of the collected input/state data, with an input sequence that is persistently exciting.
This result forms the basis for a specific working direction in data-driven controller synthesis. In the following sections, a few different applications are presented.

3.3 Controller synthesis from data-dependent representation

In [23], Willems' fundamental lemma is used to provide solutions to offline controller synthesis problems for state/output feedback stabilization and linear-quadratic regulation. The key point in this work is to make use of data-dependent representations of the open-loop and closed-loop state-space matrices and substitute them in LMI-based controller synthesis problems.
Integral to this approach is a special case of Lemma 11, which arises for $d = 1$. With this, (3.6) implies any input-state pair $\begin{bmatrix} \tilde{u}^\top & \tilde{x}^\top \end{bmatrix}^\top$ can be expressed as

$$\begin{bmatrix} \tilde{u} \\ \tilde{x} \end{bmatrix} = \begin{bmatrix} U \\ X \end{bmatrix} g, \tag{3.7}$$

for some vector g. By the Rouché–Capelli theorem (e.g. page 202 in [24]) this implies that $\begin{bmatrix} X^\top & U^\top \end{bmatrix}^\top$ has full row rank.
The set of systems consistent with the perturbation-free data in (3.2) is captured by

$$\mathbf{\Sigma}_\mathsf{D} = \left\{ (A, B) \in \boldsymbol{M} \middle| X_+ = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} X \\ U \end{bmatrix} \right\}, \tag{3.8}$$

[1] This result is usually presented for input/output trajectories with a state evolution as (3.1) and measured output as $y_k = Cx_k + Du_k$. However, in the context of this thesis, full state measurements are a standing assumption (e.g. $y_k = x_k$), and hence this is applied directly for clarity.

and has only one element when $\begin{bmatrix} X^\top & U^\top \end{bmatrix}^\top$ has full row rank, namely $(A_{\text{tr}}, B_{\text{tr}})$. Therefore, from an open-loop experiment with persistently exciting inputs of order $\eta_x + 1$, which requires an experiment of length $T \geq (\eta_u + 1)\eta_x + \eta_u$, the underlying model can be uniquely identified from the data as

$$\begin{bmatrix} A_{\text{tr}} & B_{\text{tr}} \end{bmatrix} = X_+ \begin{bmatrix} X \\ U \end{bmatrix}^\dagger, \tag{3.9}$$

where † denotes the Moore–Penrose inverse. Hence, the unknown system matrices $(A_{\text{tr}}, B_{\text{tr}})$ in LMI problems can be replaced by a data-dependent open-loop representation as (3.9).
For the interconnection of (3.1) with a state feedback controller $u = Kx$, a convenient data-dependent closed-loop representation can be constructed. By introducing a decision variable $G \in \mathbb{R}^{T \times \eta_u}$, which is required to satisfy

$$\begin{bmatrix} K \\ I \end{bmatrix} = \begin{bmatrix} U \\ X \end{bmatrix} G, \tag{3.10}$$

the closed-loop system matrix can be parameterized as

$$A_{\text{tr}} + B_{\text{tr}}K = \begin{bmatrix} B_{\text{tr}} & A_{\text{tr}} \end{bmatrix} \begin{bmatrix} K \\ I \end{bmatrix} = \tag{3.11a}$$

$$\begin{bmatrix} B_{\text{tr}} & A_{\text{tr}} \end{bmatrix} \begin{bmatrix} U \\ X \end{bmatrix} G = X_+ G, \tag{3.11b}$$

which can be substituted in LMI-based controller synthesis problems. When the data is obtained from a persistently exciting input signal, $\begin{bmatrix} U^\top & X^\top \end{bmatrix}^\top$ has full row rank, and as such (by the Rouché-Capelli theorem) G in (3.10) exists. In order to satisfy (3.10), any occurrence of K in the LMI needs to be substituted by

$$K = UG, \tag{3.12}$$

and the constraint

$$I = XG, \tag{3.13}$$

needs to be added. Then, after the optimization, the feedback matrix can be reconstructed as $K = UG$.
The main downside of this approach is that the direct application of Willems' lemma requires perturbation-free data. In practice this will never be the case, as data is always corrupted by some measurement noise. In [23], it is presented that the approach still works with data perturbed by a small state measurement noise, via a condition on the signal-to-noise ratio. However, this condition cannot be verified using the measured data, as direct access to the perturbation signal is required.

3.4 Robust synthesis methods for perturbed data

In [4], the work of [23] is extended to robust synthesis. In this context, robust has two meanings. On one hand, the paper presents data-dependent LMIs to achieve *robust control objectives*, such as $\mathcal{H}_\infty$-optimal performance, while on the other hand, it shows the approach is *robust to perturbations* in the data in that the performance can be guaranteed for all perturbations satisfying some known bound.
In a length-T open-loop experiment, perturbed data is captured from (3.1) as (3.2) and satisfies

$$X_+ = A_{\text{tr}}X + B_{\text{tr}}U + HW, \tag{3.14}$$

for the true perturbation W as in (3.3). Since the perturbation is not measured, W is not part of the data, and hence cannot be used to uniquely determine $(A_{\text{tr}}, B_{\text{tr}})$ from (3.14). Despite that, by constructing a set of legitimate perturbations which, e.g., are bounded in norm, a set description of systems (A, B) consistent with the data can be constructed. In [4], the set of legitimate perturbation is defined by means of a quadratic matrix inequality.

Assumption 1 (as made in [4]). *The matrix W is an element of*

$$\mathbf{W} \triangleq \left\{ W \in \mathbb{R}^{\eta_u \times T} \middle| \begin{bmatrix} W \\ I \end{bmatrix}^\top \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^\top & \Pi_{22} \end{bmatrix} \begin{bmatrix} W \\ I \end{bmatrix} \preceq 0 \right\}, \tag{3.15}$$

for some known matrices $\Pi_{11} \in \mathbb{R}^{\eta_u \times \eta_u}$, $\Pi_{12} \in \mathbb{R}^{\eta_u \times T}$, and a negative definite matrix $\Pi_{22} \in \mathbb{S}_{-}^{T}$.

This specific description is chosen for two reasons. Firstly, this form can be used to describe realistic perturbation bounds. Indeed, with $\Pi_{11} = I, \Pi_{12} = 0$, (3.15) reduces to

$$W^\top W = \sum_{k=0}^{T-1} w_k w_k^\top \preceq -\Pi_{22}, \tag{3.16}$$

which can be used to express an upper bound on the energy of w over the time interval of the experiment.

- If W has a bounded maximum singular value $\sigma_{max}(W) \leq \bar{\sigma}$, the form (3.16) can be used with $\Pi_{22} = -\bar{\sigma}^2 I$.
- Norm bounds on the samples of W such as $\|w_k\|_2 \leq \bar{w}, k = 0, \ldots, T-1$ can be captured in the form (3.16) with $\Pi_{22} = -\bar{w}^2 T I$.
- Magnitude bounds on each component of W such as $|w_{k,j}| \leq \bar{w}_j, k = 0, \ldots, T-1, j = 1, \ldots, \eta_w$ can be captured in the form (3.16) with $\Pi_{22} = -\sum_{j=0}^{\eta_w} \bar{w}_j^2 T I$.

Secondly, this form yields itself for application of the full-block S-procedure of Section 2.5.3, due to the similar structure as the multiplier coming from the procedure, and will be shown later in this section.
Using the set of legitimate perturbations as in Assumption 1, the set of systems which are consistent with the data is defined as

$$\mathbf{\Sigma}_{\mathsf{D}} = \left\{ (A, B) \in \boldsymbol{M} \mid X_+ = AX + BU + HW, W \in \mathbf{W} \right\}. \tag{3.17}$$

So $\mathbf{\Sigma}_{\mathsf{D}}$ contains the systems which could have generated the data as in (3.2) for some legitimate perturbation as in Assumption 1. Under a state feedback law $u_k = Kx_k$, the set of closed-loop matrices which are consistent with the data is defined as

$$\mathbf{\Sigma}_{\mathsf{D}}(K) = \left\{ A_K \mid A_K = A + BK, (A, B) \in \mathbf{\Sigma}_{\mathsf{D}} \right\}. \tag{3.18}$$

The goal of [4] is to find a data-dependent representation of the set $\mathbf{\Sigma}_{\mathsf{D}}(K)$ and then ensure that all systems in this set satisfy the specified performance objective. In this way, it is guaranteed that the performance objective is satisfied for the true system. To this end, they apply a similar approach as in Section 3.3 by introducing a decision variable G that is required to satisfy (3.10). In contrast to [23], the closed-loop system matrix cannot be uniquely parameterized as in (3.11), but will be dependent on the unknown perturbation matrix W as

$$A_K = A + BK = \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} K \\ I \end{bmatrix} = \tag{3.19a}$$

$$\begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} U \\ X \end{bmatrix} G = (X_+ - HW)G. \tag{3.19b}$$

With this, the closed-loop dynamics from w_k to z_k as in (2.30) (repeated in (3.20)) can be expressed by data as

$$\left[\begin{array}{c|c} A_K & H_K \\ \hline C_K & E_K \end{array}\right] = \left[\begin{array}{c|c} A+BK & H \\ \hline C+DK & E \end{array}\right] = \left[\begin{array}{c|c} (X_+ - HW)G & H \\ \hline C+DUG & E \end{array}\right]. \tag{3.20}$$

Direct substitution of this parametrization into an LMI for controller synthesis yields a parameter dependent LMI with unknown parameter W. To illustrate, the $\mathcal{H}_\infty$-optimal controller synthesis is considered, using Lemma 6.

Example 1 (LMI for $\mathcal{H}_\infty$-optimal state feedback synthesis using methods from [4])**.** *The $\mathcal{H}_\infty$-optimal state feedback controller can be designed by substitution of the data-dependent closed-loop dynamics of* (3.18) *into the LMI presented in the discrete-time bounded real lemma (Lemma 6). Finding the $\mathcal{H}_\infty$-optimal controller amounts to finding a $P \succ 0$, $G \in \mathbb{R}^{T\times\eta_u}$ and minimizing γ such that*

$$\begin{bmatrix} K \\ I \end{bmatrix} = \begin{bmatrix} U \\ X \end{bmatrix} G \tag{3.21}$$

and

$$\begin{bmatrix} P & 0 & * & * \\ 0 & \gamma I & * & * \\ P(X_+ - HW)G & PH & P & 0 \\ C+DUG & E & 0 & \gamma I \end{bmatrix} \succ 0,\ \forall W \in \boldsymbol{W}. \tag{3.22}$$

Note that (3.22) *is not linear in the decision variables P and G. Fortunately,* (3.22) *can be turned into an LMI by a congruence transform with* blkdiag(P^{-1}, I, P^{-1}, I) *and applying the change of variables $Q = P^{-1}, F = GP^{-1}$ to obtain*

$$\begin{bmatrix} Q & 0 & * & * \\ 0 & \gamma I & * & * \\ (X_+ - HW)F & H & Q & 0 \\ CQ+DUF & E & 0 & \gamma I \end{bmatrix} \succ 0,\ \forall W \in \boldsymbol{W}. \tag{3.23}$$

As such, the problem at hand is a parameter dependent LMI with parameter W.

To solve the parameter dependent LMI problem, relaxation methods as described in Section 2.5.1 are required. Since (3.23) is affine in W, the full-block S-procedure from Section 2.5.3 can be used. To reproduce the results of [4], Lemma 9 needs to be applied with $\Delta = W^\top$ and with $Y_{11} = 0$. With this, (3.23) can be expressed by a Linear Fractional Transformation as (2.45) with

$$Y = \left[\begin{array}{c|cccc} 0 & 0 & 0 & -H^\top & 0 \\ \hline F^\top & \frac{1}{2}Q & 0 & 0 & 0 \\ 0 & 0 & \frac{\gamma}{2}I & 0 & 0 \\ 0 & X_+F & H & \frac{1}{2}Q & 0 \\ 0 & CQ+DUF & E & 0 & \frac{\gamma}{2}I \end{array}\right]. \tag{3.24}$$

Application of the full-block S-procedure in this context states that (3.23) holds if there exists a multiplier matrix Φ which satisfies

$$\begin{bmatrix} W \\ I \end{bmatrix}^\top \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^\top & \Phi_{22} \end{bmatrix} \begin{bmatrix} W \\ I \end{bmatrix} \preceq 0,\ \forall W \in \mathbf{W}, \tag{3.25}$$

such that

$$\begin{bmatrix} F^\top\Phi_{22}F + Q & 0 & * & * & * \\ 0 & \gamma I & * & * & 0 \\ X_+F & H & Q & 0 & * \\ CQ+DUF & E & 0 & \gamma I & 0 \\ F\Phi_{12} & 0 & -H^\top & 0 & \Phi_{11} \end{bmatrix} \succ 0, \tag{3.26}$$

holds. The nonlinear term in the $(1,1)$ block of (3.26) can be resolved via a Schur complement. By rewriting (3.26) into

$$\begin{bmatrix} Q & 0 & * & * & * \\ 0 & \gamma I & * & * & 0 \\ X_+F & H & Q & 0 & * \\ CQ+DUF & E & 0 & \gamma I & 0 \\ F\Phi_{12} & 0 & -H^\top & 0 & \Phi_{11} \end{bmatrix} - \begin{bmatrix} F^\top \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} (-\Phi_{22})^{-1} \begin{bmatrix} F & 0 & 0 & 0 & 0 \end{bmatrix} \succ 0, \tag{3.27}$$

application of the Schur complement yields the constraint

$$\begin{bmatrix} Q & 0 & * & * & * & * \\ 0 & \gamma I & * & * & 0 & 0 \\ X_+F & H & Q & 0 & * & 0 \\ CQ+DUF & E & 0 & \gamma I & 0 & 0 \\ F\Phi_{12} & 0 & -H^\top & 0 & \Phi_{11} & 0 \\ F & 0 & 0 & 0 & 0 & -\Phi_{22}^{-1} \end{bmatrix} \succ 0. \tag{3.28}$$

After the optimization, the controller gain matrix is retrieved as

$$K = UG = UFQ^{-1}. \tag{3.29}$$

Application of the full-block S-procedure shows the clear motivation for Assumption 1. The condition on the multiplier (3.25) is satisfied by choosing

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^\top & \Phi_{22} \end{bmatrix} = \tau \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^\top & \Pi_{22} \end{bmatrix}, \tag{3.30}$$

for some decision variable $\tau \geq 0$ and $\Pi_{11}, \Pi_{12}, \Pi_{22}$ as in Assumption 1.

Remark. *In order to apply the full-block S-procedure to* (3.23) *with $\Delta = W^\top$, the decision variable F will show up in Y_{21} when constructing the LFT. As a result, the resulting constraint from the full-block S-procedure will not be an LMI (but contains the inverse term $-\Phi_{22}^{-1}$ and the bilinear terms $F\Phi_{12}$ and $\Phi_{12}^\top F^\top$). In [4] it is shown that with the choice of multiplier as* (3.30) *with $\Pi_{12} = 0$, the nonlinear problem can be solved via a line search over τ.*
Alternatively, the full-block S-procedure can be applied to (3.23) *with $\Delta = W$. In this "dual" form, decision variables will only show up in Y_{12} and hence the resulting constraint will be an LMI. As such, no line search is required, which decreases computational complexity. In turn, this requires one to formulate Assumption 1 on $W^\top$ instead of W to infer suitable multipliers as* (3.30)*. Fortunately, this form can be used to describe the same energy bounds on w as presented in this section.*

A limiting factor of this approach is that the parametrization of the closed-loop system matrices is actually not exact. Using the parametrization as in (3.19), any performance objective in an LMI controller synthesis is guaranteed for all closed-loop systems in the superset

$$\boldsymbol{\Sigma}_{\mathsf{D}}^s(K) = \left\{ A_G \middle| A_G = (X_+ - HW)G, W \in \boldsymbol{W} \right\}, \tag{3.31}$$

which contains all closed loop systems defined by a perturbation as in Assumption 1. This also contains systems that are not consistent with the data, and hence $\boldsymbol{\Sigma}_{\mathsf{D}}^s(K)$ is not necessarily equal to $\boldsymbol{\Sigma}_D(K)$ in (3.18). Therefore, designing controllers using this parametrization is potentially conservative.

3.5 Improved parametrization of the set of consistent systems

In [5], an approach is presented with which controllers can be designed to guarantee a performance objective for an exact parametrization of the closed-loop system matrices consistent with the data. In this way, it is less conservative than the methods of [4]. The central concept of this work is that QMIs in the unknown system matrices can describe both the set of consistent systems and relevant performance objectives.
To this end, consider data collected from an open-loop system such that (3.14) holds and assume $H = I$.[2] Consider a perturbation model as

$$\begin{bmatrix} W^\top \\ I \end{bmatrix}^\top \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^\top & \Pi_{22} \end{bmatrix} \begin{bmatrix} W^\top \\ I \end{bmatrix} \preceq 0, \ \forall W \in \mathbf{W}. \tag{3.32}$$

With this, the matrix containing the perturbation samples can be rewritten as

$$W = X_+ - AX - BU, \tag{3.33}$$

and substitution in (3.32) yields a QMI in A and B as

$$\begin{bmatrix} (X_+ - AX - BU)^\top \\ I \end{bmatrix}^\top \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^\top & \Pi_{22} \end{bmatrix} \begin{bmatrix} (X_+ - AX - BU)^\top \\ I \end{bmatrix} \tag{3.34}$$

$$= \begin{bmatrix} A^\top \\ B^\top \\ I \end{bmatrix}^\top \begin{bmatrix} 0 & -X \\ 0 & -U \\ I & X_+ \end{bmatrix} \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^\top & \Pi_{22} \end{bmatrix} \begin{bmatrix} 0 & -X \\ 0 & -U \\ I & X_+ \end{bmatrix}^\top \begin{bmatrix} A^\top \\ B^\top \\ I \end{bmatrix} \preceq 0. \tag{3.35}$$

With this, the set of all systems consistent with the data is clearly defined as

$$\mathbf{\Sigma}_{\mathsf{D}} = \left\{ (A, B) \in \boldsymbol{M} \middle| \text{ (3.35) is satisfied} \right\}. \tag{3.36}$$

In [5], it is shown that LMIs for the synthesis of stabilizing controllers and robust performance can be rewritten in a similar form as (3.35). Consider the following example for synthesizing stabilizing state feedback controllers.

Example 2. *As presented in Section 2.4, the stability of the closed-loop is guaranteed if there exists $P \succ 0$ and $K \in \mathbb{R}^{n_u \times n_x}$ such that*

$$P - (A + BK)^\top P (A + BK) \succ 0. \tag{3.37}$$

As shown in Section 2.4, via a Schur complement and a congruence transform with blkdiag(P^{-1}, P^{-1}), *an equivalent LMI can be formulated as*

$$\begin{bmatrix} P^{-1} & (A + BK)P^{-1} \\ * & P^{-1} \end{bmatrix} \succ 0. \tag{3.38}$$

With a change of variables $Q = P^{-1}$ and another Schur complement, the problem can be reformulated as

$$Q - (A + BK)Q\underbrace{Q^{-1}Q}_{I}(A + BK)^\top = Q - (A + BK)Q(A + BK)^\top \succ 0, \tag{3.39}$$

which can be rewritten in a similar form as (3.35) *as*

$$\begin{bmatrix} A^\top \\ B^\top \\ I \end{bmatrix}^\top \begin{bmatrix} Q & QK^\top & 0 \\ KQ & KQK^\top & 0 \\ 0 & 0 & -Q \end{bmatrix} \begin{bmatrix} A^\top \\ B^\top \\ I \end{bmatrix} \prec 0. \tag{3.40}$$

[2] In [5] it is shown how to deal with the case if $H \neq I$, by a slight alteration of the perturbation model.

In this setting, the problem of quadratic stabilization amounts to finding $Q \succ 0$ and $K \in \mathbb{R}^{\eta_u \times \eta_x}$ such that the QMI (3.40) is satisfied for all (A, B) satisfying the QMI (3.35).
The main question which arises, is under which conditions satisfying one QMI implies satisfying another QMI. To find these, the authors in [5] were heavily inspired by the standard S-procedure, such as presented in Section 2.2.5, which provides these conditions for quadratic forms. In [14] it is shown that the results of the S-procedure for quadratic forms can be extended to both strict and non-strict QMIs.

Theorem 12 (Strict matrix S-lemma, [14])**.** *Suppose the QMI*

$$\begin{bmatrix} Z \\ I \end{bmatrix}^\top G \begin{bmatrix} Z \\ I \end{bmatrix} \preceq 0, \tag{3.41}$$

is satisfied for all $Z \in \boldsymbol{Z}$ *for some bounded set* $\boldsymbol{Z}$*, and that there exists some* $Z \in \boldsymbol{Z}$ *for which*

$$\begin{bmatrix} Z \\ I \end{bmatrix}^\top G \begin{bmatrix} Z \\ I \end{bmatrix} \prec 0. \tag{3.42}$$

Then another QMI

$$\begin{bmatrix} Z \\ I \end{bmatrix}^\top F \begin{bmatrix} Z \\ I \end{bmatrix} \prec 0, \tag{3.43}$$

is satisfied for all $Z \in \boldsymbol{Z}$ *if and only if*[3] *there exists* $\alpha \geq 0$ *such that*

$$F - \alpha G \prec 0. \tag{3.44}$$

From this result, it is clear that the problem of quadratic stabilization amounts to finding $Q \succ 0$ and $K \in \mathbb{R}^{\eta_u \times \eta_x}$ such that

$$\begin{bmatrix} Q & QK^\top & 0 \\ KQ & KQK^\top & 0 \\ 0 & 0 & -Q \end{bmatrix} - \alpha \begin{bmatrix} 0 & -X \\ 0 & -U \\ I & X_+ \end{bmatrix} \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^\top & \Pi_{22} \end{bmatrix} \begin{bmatrix} 0 & -X \\ 0 & -U \\ I & X_+ \end{bmatrix}^\top \prec 0. \tag{3.45}$$

Note that (3.45) is not linear in Q and K. Luckily, (3.45) can be transformed into an LMI by a rather standard change of variables and Schur complement. By rewriting (3.45) as

$$\begin{bmatrix} Q & QK^\top & 0 \\ KQ & 0 & 0 \\ 0 & 0 & -Q \end{bmatrix} - \alpha \begin{bmatrix} 0 & -X \\ 0 & -U \\ I & X_+ \end{bmatrix} \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^\top & \Pi_{22} \end{bmatrix} \begin{bmatrix} 0 & -X \\ 0 & -U \\ I & X_+ \end{bmatrix}^\top - \begin{bmatrix} 0 \\ KQ \\ 0 \end{bmatrix} Q^{-1} \begin{bmatrix} 0 & QK^\top & 0 \end{bmatrix} \succ 0, \tag{3.46}$$

the Schur complement can be applied to obtain the BMI

$$\begin{bmatrix} Q & QK^\top & 0 & 0 \\ KQ & 0 & 0 & QK^\top \\ 0 & 0 & -Q & 0 \\ 0 & KQ & 0 & Q \end{bmatrix} - \alpha \begin{bmatrix} 0 & -X \\ 0 & -U \\ I & X_+ \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^\top & \Pi_{22} \end{bmatrix} \begin{bmatrix} 0 & -X \\ 0 & -U \\ I & X_+ \\ 0 & 0 \end{bmatrix}^\top \succ 0, \tag{3.47}$$

which can be turned into the LMI

$$\begin{bmatrix} Q & F^\top & 0 & 0 \\ F & 0 & 0 & F^\top \\ 0 & 0 & -Q & 0 \\ 0 & F & 0 & Q \end{bmatrix} - \alpha \begin{bmatrix} 0 & -X \\ 0 & -U \\ I & X_+ \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^\top & \Pi_{22} \end{bmatrix} \begin{bmatrix} 0 & -X \\ 0 & -U \\ I & X_+ \\ 0 & 0 \end{bmatrix}^\top \succ 0, \tag{3.48}$$

[3]the ”if” part of the statement remains true if (3.42) is not satisfied.

with the change of variables $F = KQ$.
As this framework is not used in Chapter 4, the reader is referred to [5] for LMIs for controller synthesis problems with a performance objective, such as $\mathcal{H}_\infty$-optimal performance. Note however, that the results of Chapter 4 are applicable to this line of work by observing that the term

$$\alpha \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^\top & \Pi_{22} \end{bmatrix} \tag{3.49}$$

has the same function as a multiplier, such as the one constructed in (3.30).

3.6 Robust synthesis methods with prior knowledge

In [6], a framework is presented which allows for combining data with prior knowledge. This prior knowledge takes the form of information on the structure of the system matrices, or as bounds on the uncertain parameters. This is especially relevant for mechanical systems, whose state space system matrices often include entries that are not affected by uncertain parameters. As an example, the discrete-time state evolution of a mass-spring-damper system can be modeled by a difference equation as

$$x_{k+1} = \begin{bmatrix} 1 & \tau_s \\ \delta_1 & \delta_2 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ \delta_3 \end{bmatrix} u_k, \tag{3.50}$$

for some unknown parameters $\delta_1, \delta_2, \delta_3$ and a (known) sampling time τ_s. In such a system, all entries of the top row of (3.50) are available. Moreover, (conservative) bounds on the unknown parameters might be available based on estimates of their values. Using the results of [6], this knowledge can be used together with data collected from an open-loop system to design controllers that satisfy performance criteria for all systems consistent with the data and the prior knowledge. With this, conservatism is potentially reduced compared to purely data-driven methods.
In the case that no prior knowledge is available, the method of [6] is equivalent to [5]. The flexibility of this method (combined with the option of including prior knowledge) makes it the most favorable framework to be used. Therefore, [6] is closely followed to formulate the new contributions of this thesis in Chapter 4. This chapter will also provide a sufficiently in-depth explanation of the methods using in [6].

3.7 Other notable lines of research

This section highlights significant research directions in data-driven control that were not explicitly explored in this thesis. By briefly addressing these research directions, their significance and potential contributions to the broader field of data-driven control methodologies are emphasized. These aspects, although not the primary focus here, serve as notable directions for future exploration and advancement in the field.

3.7.1 Data-driven model predictive control

As an advanced control method, model predictive control (MPC) has a wide range of technological applications [25] and is expected to have growing industrial impact in the future [26]. The performance of the MPC is critically dependent of the quality of the model. Developing such a refined model is very time-consuming, and hence research in data-driven alternatives began to emerge. A seminal work is [27], which shows that Hankel matrices can be used to describe the controlled processes instead of a model, and hence be used for MPC. In turn, [28] popularized this working direction by providing an algorithm that is able to satisfy system constraints, yielding more practical applications. Data-driven MPC for perturbed data is considered in [29] for guaranteed open-loop properties. In [30], this work is extended to guarantee closed-loop properties, such as exponential stability.

3.7.2 Data-driven control for nonlinear systems

Since all works discussed up to this point consider LTI systems, one might wonder if Willems' fundamental lemma, and controller methods based upon it, can be extended to nonlinear systems. In [31], the authors consider an approach of linearization around an operating point, where the nonlinear remainder is treated as a perturbation. With this, robust data-driven controller synthesis is applied to obtain local stabilization. Designing stabilizing controllers for polynomial systems is considered in [32], and extended to design with perturbed data in [33]. Another notable work is the data-driven MPC for nonlinear systems in [34], which can be applied for reference tracking problems with guaranteed closed-loop stability.

3.7.3 Data-driven control in the frequency domain

In this thesis, the focus is on LMI-based controller design from time-domain data. As an alternative to this working direction, frequency-domain data can also be used for data-driven design. In these approaches, controllers are directly synthesized from data, without the need to identify a parametric model. A notable work is [35], in which automatic loop-shaping of PID controllers from frequency-domain data is considered. Here, (frequency-dependent) bounds on the gain of the four sensitivity functions can be imposed as constraints in the optimization. A fifth constraint is used to impose a requirement on set-point tracking. In [36], frequency-domain data is used to synthesize the frequency response of a controller that achieves desired closed-loop pole placements. Data-driven synthesis for fixed structure controllers for guaranteed $\mathcal{H}_\infty$ performance is considered in [37].
Similarly to this thesis, convex optimization methods are used to design robust controllers based on frequency-domain data. In [38], convex optimization is used to synthesize linearly parameterized controllers (which includes PID controllers) for SISO systems that satisfy specifications on the gain margin, phase margin, and the desired closed-loop bandwidth. Controller synthesis for loop shaping and $\mathcal{H}_\infty$ performance for SISO systems is considered in [39]. This work is extended to MIMO systems in [40].In [41], standard performance specifications such as $\mathcal{H}_2$, $\mathcal{H}_\infty$ and loop shaping are considered for robust controller synthesis of multivariable systems with respect to a multimodel uncertainty. In this work, a unified framework is presented in which these specifications can be guaranteed starting only from frequency-domain data that is collected at different operating points.

3.8 Novel working direction of this thesis

With the works of [5] and [6], a strong framework is established for data-driven control of discrete-time LTI systems. Both make use of an exact parametrization of the set of systems consistent with the data under some model of the perturbation (in the form of a QMI). For clarity, the perturbation model is repeated here as

$$\begin{bmatrix} W^\top \\ I \end{bmatrix}^\top \underbrace{\begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^\top & \Pi_{22} \end{bmatrix}}_{\Pi} \begin{bmatrix} W^\top \\ I \end{bmatrix} \preceq 0, \ \forall W \in \mathbf{W}, \tag{3.51}$$

for some known Π_{11}, Π_{12} and $\Pi_{22} \prec 0$. This model might be potentially conservative, as it relies on the designer to choose these matrices based on the knowledge about W. Considering the QMI's less-than-ideal structure, it is not obvious to see what Π should be to satisfy (3.51) based on some characterization of $\boldsymbol{W}$. To illustrate, the choices for Π for norm bounds on the samples or elements of W, as presented in Section 3.4, are based on (potentially) conservative over-approximations of the energy bound over the whole experiment. In turn, this might increase conservatism in the controller synthesis. As such, Chapter 4 aims to show that these multipliers do not need to be constructed directly by the designer, but can be used as decision variables in a separate LMI constructed from the implicit characterization of the perturbation.

4 PAPER: ROBUST DATA-DRIVEN STATE-FEEDBACK SYNTHESIS FROM DATA CORRUPTED BY PERTURBATIONS WITH BOUNDED NORMS AND RATES-OF-VARIATION

The remainder of this chapter is an unedited version of the paper which has been submitted to the 22nd European Control Conference (ECC) in Stockholm, Sweden. As such, the paper is credited to two authors. Their contribution can be distributed as follows:

1. Rens de Boer: Formal Analysis, Investigation, Methodology, Visualization, Writing - original draft, Writing - review & editing
2. Hakan Köroğlu: Conceptualization, Methodology, Writing - review, supervision

Robust data-driven state-feedback synthesis from data corrupted by perturbations with bounded norms and rates-of-variation

Rens de Boer[1] and Hakan Köroğlu[2]

***Abstract*— Data-driven $\mathcal{H}_\infty$-optimal controller synthesis is considered for unknown discrete-time linear time-invariant systems. Perturbations in the data, such as those coming from external disturbances and measurement noise, are assumed to be bounded in norm and rate-of-variation. A linear matrix inequality (LMI) based framework is introduced in which these realistic bounds can be included in the synthesis. This is done via perturbation regions, which can describe quadratic inequalities on (the elements of) individual samples and the variation between samples. Combining multiple regions allows to infer multipliers from learnt data that better describe the properties of the perturbation. This reduces the set of systems consistent with the data and thus offers a reduction in conservatism, at the cost of increased computational complexity. The proposed framework is compatible with the most recent literature, which includes prior knowledge on the system matrices or the uncertainty in the controller synthesis.**

I. INTRODUCTION

Synthesizing controllers directly from measured data is gaining more prominence in the field of systems and control [1], [2]. Compared to the sequential process of system identification and model-based control, data-driven methods could offer a promising alternative. Refined system identification [3] can be a time-consuming process, demanding significant human intervention to ensure model quality. If this is deemed too cumbersome, data-driven methods might be a viable alternative if they are simple and faster to execute.

A particular class of data-driven control, built on pioneering research led by Willems [4], might fulfill these requirements. In this research, it is demonstrated that a single open-loop experiment can parameterize all trajectories of a linear time-invariant (LTI) system, provided that the input signal is sufficiently exciting. As a direct consequence, a data-dependent representation of the underlying LTI system can be obtained. Through the use of this representation, the seminal work [5] presented solutions for offline controller synthesis in both state/output feedback stabilization and the linear quadratic regulation problem. These solutions are formulated in the form of linear matrix inequality (LMI) optimization, which is shown to be computationally efficient and effective in a variety of analysis and synthesis problems [6].

An obstacle of data-driven methods is that not all input signals are measurable (e.g. external disturbances) and that the measurements are always corrupted by some noise. Consequently, it is not possible to derive a single exact representation of the underlying system from the data. Luckily though, using upper bound estimates of of these factors, a set of LTI systems can be constructed that would produce the same data, assuming accurate estimates. This set can then be utilized for controller synthesis. This method is potentially conservative, as a single controller needs to fulfill the design requirements for all systems in this set. As such, it is beneficial to have a good *perturbation model*, which is a characterization of disturbances and noise (through convenient set descriptions).

Prior work has considered the challenge of managing perturbations in the data for LMI based controller synthesis. In [5], sufficient conditions for closed-loop stability under additive state measurement noise are provided. However, it is worth noting that these conditions rely on assumptions that cannot be verified using measured data. Several other works assume that perturbations satisfy a quadratic matrix inequality. This specific characterization facilitates dealing with perturbations using variants of the S-procedure [7], such as the full-block S-procedure in [8], the classical S-lemma in [9], the lossless matrix S-procedure of [10] in [11], and Petersen's lemma in [12]. In [8], [11], [12] this perturbation model is used to describe an energy bound of the perturbation over the whole experiment. This is potentially conservative if bounds on (the elements of) individual samples are known. As such, [9], [13] provide approaches to describe Euclidean norm bounds on the individual perturbation samples, but this cannot be generalized to quadratic inequalities on the elements of a sample. Furthermore, none of the aforementioned methods can handle bounds on the maximum rate-of-variation, although there is very probable correlation between perturbation samples, such as those coming from external disturbances. As mentioned in [13], sum-of-squares relaxation approaches from [14] can be used to describe these more general bounds, but this approach relies on computationally expensive tools.

In this paper, it is shown that multiple realistic perturbation bounds can be implemented in the controller synthesis problem via LMIs. We present a framework that allows for more flexible bounds on perturbation signals by constructing perturbation regions. These regions can describe bounds in the form of quadratic inequalities in any of the elements of the perturbation samples. As such, bounds on the rate-of-variation between samples can be included.

The paper is structured as follows. The problem is formulated in Section II. In Section III we provide LMIs

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for designing full state-feedback controllers with guaranteed $\mathcal{H}_\infty$-performance, relying on multipliers from the full-block S-procedure. In Section IV it is shown how these multipliers can be constructed in compliance with the knowledge on the perturbation bounds. Section V demonstrates the flexibility of the proposed framework by providing expressions for some relevant perturbation regions. We show the effectiveness of this framework via an illustrative example in Section VI before the concluding remarks.

II. PROBLEM FORMULATION

Consider uncertain linear time-invariant systems of the form

$$\left[\begin{array}{c} x_{k+1} \\ \hline z_k \\ p_k \end{array}\right] = \left[\begin{array}{c|ccc} A & B & H & L \\ \hline C & D & E & 0 \\ C_\mathrm{u} & D_\mathrm{u} & 0 & 0 \end{array}\right] \left[\begin{array}{c} x_k \\ \hline u_k \\ w_k \\ q_k \end{array}\right] \tag{1a}$$

$$q_k = \Delta_\mathrm{tr} p_k, \tag{1b}$$

where $x_k \in \mathbb{R}^{\eta_x}$ denotes the state, $u_k \in \mathbb{R}^{\eta_u}$ the control input, $w_k \in \mathbb{R}^{\eta_w}$ the perturbation, and $z_k \in \mathbb{R}^{\eta_z}$ the performance output. The signals $q_k \in \mathbb{R}^{\eta_q}$ and $p_k \in \mathbb{R}^{\eta_p}$ are introduced to be able to pull out the unknown parameters and thereby form an uncertainty channel. All matrices in (1) are assumed to be known, except the true uncertainty $\Delta_\mathrm{tr} \in \mathbb{R}^{\eta_q \times \eta_p}$. It is assumed that Δ_tr has a block-diagonal structure

$$\Delta_\mathrm{tr} = \mathrm{blkdiag}_{j=1}^{\ell}(\Delta_j), \tag{2}$$

where $\Delta_j \in \mathbb{R}^{\eta_{q,j} \times \eta_{p,j}}$ is a full block or a repeated scalar block $\Delta_j = \delta_j I_{\eta_{w,j}}, \delta_j \in \mathbb{R}$. Because of this, the description

$$\Delta_\mathrm{tr} = \mathrm{blkdiag}_{j=1}^{\ell}(\Delta_j) = \sum_{j=1}^{\ell} N_j \Delta_j Y_j^\top \tag{3}$$

can be used, where N_j and Y_j are the corresponding block-columns of the identity matrix.

It is assumed that the signals w_k and q_k cannot be measured. However, we assume to have prior knowledge on the perturbation signal w_k and its rate-of-variation $v_k \triangleq w_{k+1} - w_k$ i.e., $w_k \in \mathbf{W}$ and $v_k \in \mathbf{V}$ for some known sets $\mathbf{W} \subset \mathbb{R}^{\eta_w}$, $\mathbf{V} \subset \mathbb{R}^{\eta_w}$. These sets can express norm bounds on (the elements of) w_k and v_k. For the uncertainty, we assume to have prior knowledge in the form $\Delta_\mathrm{tr} \in \mathbf{\Delta}_\mathrm{pr}$ for some known compact set $\mathbf{\Delta}_\mathrm{pr}$.

From a length-T open-loop experiment of (1), perturbed state-input measurements are obtained. By collecting the samples of the experiment, we have access to the matrices

$$X = \begin{bmatrix} x_0 & x_1 & \dots & x_{T-1} \end{bmatrix} \in \mathbb{R}^{\eta_x \times T}, \tag{4a}$$

$$X_+ = \begin{bmatrix} x_1 & x_1 & \dots & x_T \end{bmatrix} \in \mathbb{R}^{\eta_x \times T}, \tag{4b}$$

$$U = \begin{bmatrix} u_0 & u_1 & \dots & u_{T-1} \end{bmatrix} \in \mathbb{R}^{\eta_u \times T}. \tag{4c}$$

The samples of the perturbation, although unknown, are collected in the matrix

$$W = \begin{bmatrix} w_0 & w_1 & \dots & w_{T-1} \end{bmatrix} \in \mathbb{R}^{\eta_w \times T}. \tag{5}$$

From the open-loop dynamics of (1), it is clear that the data needs to satisfy

$$X_+ = AX + BU + HW + L\Delta_\mathrm{tr}(C_\mathrm{u} X + D_\mathrm{u} U). \tag{6}$$

In contrast to related works such as [5], [8], [15], measurements are said to be *perturbed* rather than *noisy*. This is done to stress that W consists of *all* corrupting factors such that a linear model as

$$x_{k+1} = (A + L\Delta C_\mathrm{u})x_k + (B + L\Delta D_\mathrm{u})u_k \tag{7}$$

cannot explain the data in (4). Next to measurement- and process noise, this would also include external disturbances and disruptions caused by slight nonlinearities. From the knowledge on the perturbation signal, we define the set of admissible perturbation matrices as

$$\mathbf{W}_T = \left\{ W \,\middle|\, \begin{array}{l} w_k \in \mathbf{W}, k = 0, \dots, T-1 \\ v_k \in \mathbf{V}, k = 0, \dots, T-2 \end{array} \right\}. \tag{8}$$

For notational purposes, similarly to [13], we introduce the matrices

$$M = X_+ - AX - BU, \tag{9a}$$

$$Z = C_\mathrm{u} X + D_\mathrm{u} U, \tag{9b}$$

such that any $\Delta \in \mathbf{\Delta}_\mathrm{ln}$ is consistent with the data, where

$$\mathbf{\Delta}_\mathrm{ln} = \{\Delta \colon \exists W \in \mathbf{W} \mid M = HW + L\Delta Z\}. \tag{10}$$

Naturally, $\Delta_\mathrm{tr} \in \mathbf{\Delta}_\mathrm{ln}$ and as such it is an element of the combined uncertainty set

$$\mathbf{\Delta}_\mathrm{com} = \{\Delta \mid \Delta \in (\mathbf{\Delta}_\mathrm{pr} \cap \mathbf{\Delta}_\mathrm{ln})\}. \tag{11}$$

This paper considers the synthesis of a state feedback controller as

$$u_k = Kx_k \tag{12}$$

for (1). The closed loop transfer matrix from w_k to z_k under this controller has an LTI representation as

$$\mathcal{G}(z) = C_K(zI - A_K)^{-1}H_K + E_K = \left[\begin{array}{c|c} A_K & H_K \\ \hline C_K & E_K \end{array}\right], \tag{13}$$

where

$$\left[\begin{array}{c|c} A_K & H_K \\ \hline C_K & E_K \end{array}\right] = \left[\begin{array}{c|c} A + BK + L\Delta_\mathrm{tr}(C_\mathrm{u} + D_\mathrm{u} K) & H \\ \hline C + DK & E \end{array}\right]. \tag{14}$$

We denote the $\mathcal{H}_\infty$ norm of $\mathcal{G}(z)$ by $\|\mathcal{G}\|_\infty$.

Based on these basic notions, the formulation of the problem reads as follows:

Problem 1. *Given a system as in* (1) *and data as in* (4) *where the perturbation matrix of* (5) *satisfies* $W \in \mathbf{W}_T$, *design a state-feedback controller as in* (12) *which renders the closed loop in* (14) *stable and ensures that* $\|\mathcal{G}\|_\infty < \gamma$ *for all* $\Delta \in \mathbf{\Delta}_\mathrm{com}$.

III. $\mathcal{H}_\infty$-OPTIMAL FULL STATE-FEEDBACK SYNTHESIS FROM PERTURBED DATA

In this section, it is shown how stability of the closed loop and $\mathcal{H}_\infty$-performance of level $\gamma > 0$ can be guaranteed via LMI conditions. Furthermore, it is shown how to deal with the unknown term $\Delta_{\rm tr}$ in (14).

The bounded real lemma gives a necessary and sufficient conditions for $\mathcal{H}_\infty$-performance for LTI systems.

Lemma 1 (Discrete-time bounded real lemma)**.** *Consider a system* (13) *and a scalar* $\gamma > 0$*. Then* A_K *is Schur-stable and* $\|\mathcal{G}\|_\infty < \gamma$ *if and only if there exists* $P \succ 0$ *such that*

$$\begin{bmatrix} P & 0 & * & * \\ 0 & \gamma I & * & * \\ PA_K & PH_K & P & 0 \\ C_K & E_K & 0 & \gamma I \end{bmatrix} \succ 0. \tag{15}$$

Proof: see e.g. (3.14) in [16], Section 3.2.2.

By substituting the matrices from (14) into (15), the controller synthesis problem (for a fixed or to-be-optimized γ) reads as

find $P \succ 0$ *and* $K \in \mathbb{R}^{\eta_u \times \eta_x}$ *such that*

$$\begin{bmatrix} P & 0 & * & * \\ 0 & \gamma I & * & * \\ P(\tilde{A}(\Delta_{\rm tr}) + \tilde{B}(\Delta_{\rm tr})K) & PH & P & 0 \\ C + DK & E & 0 & \gamma I \end{bmatrix} \succ 0, \tag{16}$$

where $\tilde{A}(\Delta) = A + L\Delta C_{\rm u}$ and $\tilde{B}(\Delta) = B + L\Delta D_{\rm u}$. This problem is bilinear in the decision variables P and K, but can be turned into an LMI by performing a congruence transform on (16) with $\text{blkdiag}(P^{-1}, I, P^{-1}, I)$ and applying the change of variables $Q = P^{-1}, F = KP^{-1}$. Since $\Delta_{\rm tr}$ is unknown, we are forced to consider a parameter dependent formulation of (16) by replacing $\Delta_{\rm tr}$ with $\Delta \in \boldsymbol{\Delta}_{\rm com}$. We now introduce the LFT

$$\tilde{\Delta} \star \underbrace{\left[\begin{array}{c|c} \mathcal{Y}_{11} & \mathcal{Y}_{12} \\ \hline \mathcal{Y}_{21} & \mathcal{Y}_{22} \end{array}\right]}_{\mathcal{Y}} \triangleq \mathcal{Y}_{22} + \mathcal{Y}_{21}\tilde{\Delta}(I - \mathcal{Y}_{11}\tilde{\Delta})^{-1}\mathcal{Y}_{12}, \tag{17}$$

such that with the choices $\tilde{\Delta} = L\Delta$ and

$$\mathcal{Y} = \left[\begin{array}{c|cccc} 0 & C_{\rm u}Q + D_{\rm u}F & 0 & 0 & 0 \\ \hline 0 & \frac{1}{2}Q & 0 & 0 & 0 \\ 0 & 0 & \frac{\gamma}{2}I & 0 & 0 \\ I & AQ + BF & H & \frac{1}{2}Q & 0 \\ 0 & CQ + DF & E & 0 & \frac{\gamma}{2}I \end{array}\right], \tag{18}$$

the parameter dependent LMI of (16) is retrieved as

$$\text{He}(\tilde{\Delta} \star \mathcal{Y}) \triangleq \tilde{\Delta} \star \mathcal{Y} + (\tilde{\Delta} \star \mathcal{Y})^\top \succ 0, \forall \tilde{\Delta} \in \tilde{\boldsymbol{\Delta}}_{\rm com}. \tag{19}$$

We want to stress that the parameter set $\tilde{\boldsymbol{\Delta}}_{\rm com}$ does not have an explicit description due to the definition of $\boldsymbol{\Delta}_{\rm ln}$. It hence becomes convenient to use the full-block S-procedure of [17] to be able to develop a general and computationally efficient framework for data-driven controller synthesis. We employ a corollary of the full-block S-procedure ([18], Lemma 7) due its clear implementation in our context. This allows us to arrive at a sufficient condition for (19) as (21) by introducing a *multiplier* matrix, denoted as Φ, which is an unstructured variable that is required to satisfy

$$\begin{bmatrix} \tilde{\Delta}^\top \\ I \end{bmatrix}^\top \underbrace{\begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^\top & \Phi_{22} \end{bmatrix}}_{\Phi} \begin{bmatrix} \tilde{\Delta}^\top \\ I \end{bmatrix} \preceq 0, \forall \tilde{\Delta} \in \tilde{\boldsymbol{\Delta}}_{\rm com}, \tag{20}$$

The controller synthesis can then be formulated as follows:

Lemma 2. *If there exist* $Q \succ 0$*,* $F \in \mathbb{R}^{\eta_u \times \eta_x}$ *and a multiplier* Φ *such that* (20) *and* (21) *hold, then for any* $\Delta \in \boldsymbol{\Delta}_{\rm com}$

1) A_K *of* (14) *with* $K = FQ^{-1}$ *is Schur-stable; and*
2) $\|G(z)\|_\infty < \gamma$.

$$\begin{bmatrix} Q & 0 & * & * & * \\ 0 & \gamma I & * & * & 0 \\ AQ + BF & H & Q + \Phi_{22} & 0 & * \\ CQ + DF & E & 0 & \gamma I & 0 \\ C_{\rm u}Q + D_{\rm u}F & 0 & \Phi_{12} & 0 & \Phi_{11} \end{bmatrix} \succ 0. \tag{21}$$

Similarly to [13], the full block S-procedure is applied on $L\Delta$ instead of Δ as it allows us to characterize valid multipliers for the corresponding condition (20) from bounds on the uncertainty and perturbation.

Lemma 3. *[13] Valid multipliers for* (20) *can be constructed as*

$$\Phi = \sum_{j=1}^{l} \begin{bmatrix} N_j^\top & 0 \\ 0 & L_j^\top \end{bmatrix}^\top \Psi_j \begin{bmatrix} N_j^\top & 0 \\ 0 & L_j^\top \end{bmatrix} + \begin{bmatrix} -Z^\top & M^\top \\ 0 & H^\top \end{bmatrix}^\top \Pi \begin{bmatrix} -Z^\top & M^\top \\ 0 & H^\top \end{bmatrix}, \tag{22}$$

where Ψ_j *and* Π *are multipliers with* $\Psi_{11} \succ 0$ *and* $\Pi_{11} \succ 0$ *which need to satisfy*

$$\begin{bmatrix} \Delta_j^\top \\ I \end{bmatrix}^\top \Psi_j \begin{bmatrix} \Delta_j^\top \\ I \end{bmatrix} \preceq 0, \forall \Delta_j \in \boldsymbol{\Delta}_j, \tag{23}$$

and

$$F_p(W) = \begin{bmatrix} W^\top \\ I \end{bmatrix}^\top \Pi \begin{bmatrix} W^\top \\ I \end{bmatrix} \preceq 0, \forall W \in \mathbf{W}_T, \tag{24}$$

respectively.

Proof: See [13], Lemma $1 - 3$.

Here L_j are partitions of the matrix L according to the structure of Δ, i.e. $L = \begin{bmatrix} L_1 & L_2 & \ldots L_\ell \end{bmatrix}$ with $L_j = LY_j$. We want to stress that multipliers derived from bounds on the uncertain parameters directly, as in (23), are required to retain the structural information on $\Delta_{\rm tr}$ (in [13], Section V.A). This includes the fact that $\tilde{\Delta}$ is of the form $L\Delta$, so no knowledge is lost by performing the full-block S-procedure on $\tilde{\Delta}$ instead of Δ if these multipliers are used.

In [13] it is shown that norm bounds on the uncertainty as $\Delta_j\Delta_j^\top \preceq \overline{\delta}_j^2 I$ or $\delta_j^2 \leq \overline{\delta}_j^2$ with $\delta_j > 0$ can be transformed into valid classes of multipliers for (23) as

$$\mathbf{\Psi_j} = \left\{ \lambda \begin{bmatrix} I & 0 \\ 0 & -\bar{\delta}_j I \end{bmatrix} \middle| \lambda \geq 0 \right\}, \tag{25a}$$

$$\mathbf{\Psi_j} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -\bar{\delta}_j \end{bmatrix} \otimes \Lambda \middle| \Lambda \succeq 0 \right\}, \tag{25b}$$

respectively. For perturbations which satisfy a pointwise-in-time Euclidean norm bound $\|w_k\|_2 \leq \bar{w}\ \forall k = 0, \ldots, T-1$ for $\bar{w} > 0$, a valid class of multipliers for (24) is

$$\mathbf{\Pi}_\mathrm{n} = \left\{ \begin{bmatrix} \mathrm{diag}_{i=1}^{T}(\lambda_i) & 0 \\ 0 & -\sum_{i=1}^{T} \lambda_i \bar{w}^2 I_{\eta_w} \end{bmatrix} \middle| \lambda_i \geq 0 \right\} \tag{26}$$

Though valid and easy to construct, these multipliers are not very descriptive for more elaborate perturbation characterizations, such as bounds on the rate-of-variation or bounds on the individual components of a sample. This is especially relevant for bounds on the perturbation as some states might be affected by larger external disturbances, whereas others are only influenced by small measurement noise.

IV. CONSTRUCTING MULTIPLIERS FROM KNOWLEDGE ON THE PERTURBATION

In this section, we show how valid multipliers for (24) can be obtained in which full knowledge of $\mathbf{W}_T$ is taken into account. We take inspiration from sum-of-squares (SOS) programming techniques [19]. These techniques allow for solving general polynomial optimization problems via semidefinite programming by rewriting positive definiteness constraints as SOS constraints [20]. Being SOS is a sufficient condition for global positive semi definiteness and can be verified using available software tools such as the SOS module in Yalmip [21].

A. Sum-of-squares relaxation

By using a sum-of-squares based relaxation approach, (24) can be guaranteed by ensuring $F_p(W) \preceq T_p(W)$ where $T_p(W)$ is a polynomial matrix which is known to be negative semidefinite for $W \in \mathbf{W}_T$. For an effective relaxation, $T_p(W)$ ought to be constructed from the knowledge on the perturbation characterization (see e.g. [14],[19]). With this, (24) can be solved by ensuring that

$$-F_p(W) + T_p(W) \text{ is SOS.} \tag{27}$$

B. LMI-based SOS relaxation for specific perturbation bounds

Although SOS programming is a powerful tool, it can be computationally expensive. For specific, yet highly relevant, characterizations of $\mathbf{W}_T$, the SOS condition (27) can be turned into an LMI constraint with much less computational complexity. The quadratic dependence of (24) on W is particularly convenient in this sense since common uncertainty descriptions also read as quadratic inequalities. Based on this, a simple SOS relaxation was introduced in [22] within the context of robust stability analysis, particularly for rectangular and ellipsoidal uncertainty regions. Aiming for a more generic framework, we now introduce *perturbation regions* as a set of polynomial functions which characterize $\mathbf{W}_T$ as

$$\mathbf{W}_T \subseteq \{W : \beta(W)^\top R_i \beta(W) \preceq 0, i = 1, \ldots, n\}, \tag{28}$$

where $R_i \in \mathbb{S}^{T\cdot\eta_w+1}$. The function $\beta(W)$ acts as a basis function and is defined as[1]

$$\beta(W) \triangleq \begin{bmatrix} \mathrm{vec}(W) \\ 1 \end{bmatrix} \tag{29}$$

With this, any quadratic polynomial function in the elements of W can be described by some choice of R_i. In Section V, it is shown that this includes norm bounds on (the elements of) w_k as well as v_k and thus $\mathbf{W}_T$ can be described by an intersection of these regions. As preparation for the derivation, we introduce the matrix

$$\Lambda = \text{blkdiag}\left(I_T \otimes \Gamma, I_{\eta_w}\right) \tag{30}$$

where

$$\Gamma = \begin{bmatrix} 1_{1\times(\eta_w-1)} \otimes \begin{bmatrix} 1 & 0_{1\times\eta_w} \end{bmatrix} & 1 \end{bmatrix} \tag{31}$$

such that

$$\Lambda \cdot (\beta(W) \otimes I_{\eta_w}) = \begin{bmatrix} W^\top \\ I \end{bmatrix}. \tag{32}$$

With this, (24) can be rewritten to the condition that

$$-(\beta(W) \otimes I_{\eta_w})^\top \Lambda^\top \Pi \Lambda (\beta(W) \otimes I_{\eta_w}) \text{ is SOS.} \tag{33}$$

Remark. *In the case of scalar perturbations (*$\eta_w = 1$*),* $\begin{bmatrix} W & I \end{bmatrix}^\top = \beta(W)$ *and thus is already in the preferred form of* (29)*. As a result,* $\Lambda = I$ *in this case.*

In accordance with the SOS relaxation approach of Section IV-A, we will construct $T_p(W)$ to reflect the knowledge on $\mathbf{W}_T$, using perturbation regions as (28). To this end, we introduce the following result:

Lemma 4. *Let* $S \in \mathbb{S}_+^{T\cdot\eta_w+1}$ *be a positive semidefinite matrix and* $\beta^\top(W) R \beta(W) <= 0$*. Then*

$$(\beta(W) \otimes I)^\top (R \otimes S)(\beta(W) \otimes I) \preceq 0. \tag{34}$$

Proof: Using the Kronecker product rules (e.g. [23], page 118) for transposition and applying the mixed-product property twice yields

$$\begin{aligned} &(\beta(W) \otimes I)^\top (R \otimes S)(\beta(W) \otimes I) \\ &\quad = (\beta(W)^\top \otimes I^\top)(R \otimes S)(\beta(W) \otimes I) \\ &\quad = (\beta(W)^\top \otimes I^\top)(R\beta(W)) \otimes S \\ &\quad = (\beta(W)^\top R \beta(W)) \otimes S. \end{aligned} \tag{35}$$

Denote the eigenvalues of $\beta(W)^\top R \beta(W)$ by μ_i and the eigenvalues of S by ζ_j. The eigenvalues of $\beta(W)^\top R \beta(W) \otimes S$ are then all the products $\mu_i \zeta_k$ (e.g. [23], page 234). Since S is positive semidefinite and $\beta(W)^\top R \beta(W) \preceq 0$, this implies $(\beta(W)^\top R \beta(W)) \otimes S \preceq 0$. □

[1] Given an $\eta_w \times T$ matrix W, $\mathrm{vec}(W)$ produces a vector of length $\eta_w \cdot T$ that contains the columns of W, stacked below each other.

Based on this result, we can construct multiple perturbation regions and combine them to relax (33) into

$$
\begin{aligned}
&-(\beta(W)\otimes I_{\eta_w})^\top \Lambda^\top \Pi \Lambda (\beta(W)\otimes I_{\eta_w}) \\
&\quad + \sum_{i=1}^{n} (\beta(W)\otimes I_{\eta_w})^\top (R_i \otimes S_i)(\beta(W)\otimes I_{\eta_w}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{is SOS.} \quad (36)
\end{aligned}
$$

In (36), we find a sufficient LMI condition as

$$\Lambda^\top \Pi \Lambda - \sum_{i=1}^{n} (R_i \otimes S_i) \preceq 0. \tag{37}$$

This brings us to the main result of the paper stated as follows:

Theorem 5. *Consider Problem 1 with prior information captured by multiplier sets $\mathbf{\Psi}_j, j = 1,..,\ell$; and with $\mathbf{W}_T$ characterized by n polynomial inequalities of the form in* (28)*. A state-feedback controller* (12) *which satisfies the stability and performance requirements is retrieved as $K = FQ^{-1}$ by solving* (21)*, where Φ needs to be obtained as in Lemma 3 with $\Psi_j \in \mathbf{\Psi}_j, 1, \ldots, \ell$ and Π required to satisfy* (37)*.*

In Section V, some relevant polynomial inequalities of the form in (28) are presented.

Remark. *Computational complexity can be reduced by choosing $S_i \in \mathbb{S}_+^{\eta_w}$ as a diagonal matrix or even as $s_i I_{\eta_w}$ with $s_i \geq 0$ at the cost of potential conservatism.*

V. DESCRIPTIONS FOR RELEVANT PERTURBATION REGIONS

In this section, some relevant polynomial inequalities of the form in (28) are presented as convenient descriptions of the perturbation signals. By combining these, a more accurate perturbation characterization may be obtained, yielding potentially less conservative results in synthesis.

A. Regions for norm-bounded perturbations

Consider that W is bounded pointwise-in-time as $\|w_k\|_2 \leq \bar{w}, k = 0, \ldots, T-1$. This is equivalent to

$$w_{k,1}^2 + w_{k,2}^2 + \cdots + w_{k,\eta_w}^2 - \bar{w}^2 \leq 0, \tag{38}$$

which can be expressed as the intersection of T perturbation regions of the form (28) with

$$R_{norm,k} = \text{blkdiag}(0_{k\cdot\eta_w}, I_{\eta_w}, 0_{(T-k-1)\cdot\eta_w}, -\bar{w}^2). \tag{39}$$

B. Regions for element-wise bounded perturbations

Consider that each element of W is bounded as $|w_{k,i}| \leq \overline{w_i}, k = 0, \ldots, T-1, i = 1, \ldots, \eta_w$. This is equivalent to

$$w_{k,i}^2 - \overline{w_i}^2 \leq 0, \tag{40}$$

which can be expressed as the intersection of $T \cdot \eta_w$ perturbation regions of the form (28) with

$$R_{mag,k,i} = \text{blkdiag}(0_{k\cdot\eta_w}, J_i, 0_{(T-k-1)\cdot\eta_w}, -\overline{w_i}^2), \tag{41}$$

where $J_i \in \mathbb{S}^{\eta_w}$ are diagonal matrices with a single 1 at position (i,i) and zeros elsewhere.

C. Regions for perturbations with norm-bounded rates-of-variation

Consider that the variation between samples is bounded as $\|v_k\|_2 = \|w_{k+1} - w_k\|_2 \leq \bar{v}, k = 0, \ldots, T-2$. This is equivalent to

$$\begin{bmatrix} w_k \\ w_{k+1} \end{bmatrix}^\top \underbrace{\begin{bmatrix} I & -I \\ -I & I \end{bmatrix}}_{R_v} \begin{bmatrix} w_k \\ w_{k+1} \end{bmatrix} - \bar{v}^2 \leq 0, \tag{42}$$

which can be expressed as the intersection of $(T-1)$ perturbation regions of the form (28) with

$$R_{var,k} = \text{blkdiag}(0_{k\cdot\eta_w}, R_v, 0_{(T-k-2)\cdot\eta_w}, -\bar{v}^2). \tag{43}$$

D. Regions for perturbations with element-wise bounded rates-of-variation

Consider that each element of v_k is bounded as $|v_{k,i}| \leq \overline{v_i}, k = 0, \ldots, T-2, i = 1, \ldots, \eta_w$. This is equivalent to

$$\begin{bmatrix} w_{k,i} \\ w_{k+1,i} \end{bmatrix}^\top \underbrace{\begin{bmatrix} J_i & -J_i \\ -J_i & J_i \end{bmatrix}}_{R_{v,i}} \begin{bmatrix} w_{k,i} \\ w_{k+1,i} \end{bmatrix} - \bar{v}_i^2 \leq 0, \tag{44}$$

and can be expressed as the intersection of $(T-1)\cdot\eta_w$ perturbation regions of the form (28) with

$$R_{var,k,i} = \text{blkdiag}(0_{k\cdot\eta_w}, R_{v,i}, 0_{(T-k-2)\cdot\eta_w}, -\bar{v}^2). \tag{45}$$

E. Ellipsoidal regions for perturbations with magnitude and rate bounded elements

Using both regions of Section V-B and V-D requires $(2T-1)\eta_w$ perturbation regions, and for each region at least a scalar decision variable s_i. Note that multipliers which satisfy a pointwise-in-time Euclidean norm bound such as (26) require only T additional decision variables. To reduce computational complexity, the combined region as shown in Fig. 1 can be used. This region can be described by a rotated ellipse and as such, only $(T-1)\eta_w$ perturbation regions are required. For this, we make use of the following lemma.

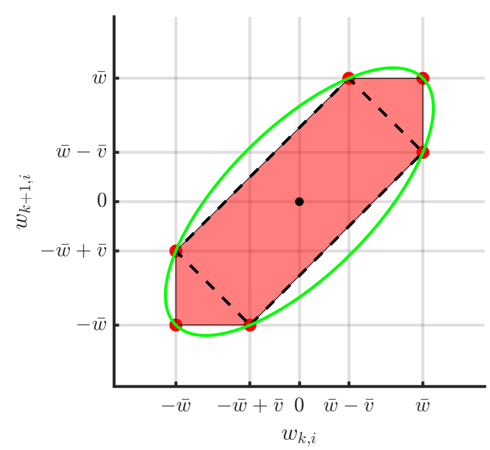

Fig. 1. Perturbation region for bounded element norm and rate-of-variation. This region can be described by a single ellipse.

Lemma 6. *Assume that $|w_{k,i}| \leq \bar{w}$ and $|w_{k+1,i} - w_{k,i}| \leq \bar{v}$ with $0 \leq \bar{v} \leq 2\bar{w}$ being constant for sequential samples. The rotated ellipse is parameterized in the (x, y) plane as*

$$\frac{(x\mathrm{c}(\theta) + y\mathrm{s}(\theta))^2}{r_x^2} + \frac{(x\mathrm{s}(\theta) - y\mathrm{c}(\theta))^2}{r_y^2} \leq 1, \qquad (46)$$

where $\mathrm{s}(\theta) \triangleq \sin\theta, \mathrm{c}(\theta) \triangleq \cos\theta$ and $(x, y) = (w_{k,i}, w_{k+1,i})$. A region for bounded element norm and rate-of-variation can be described by a minimally bounding ellipse of the form (46) *with*

$$\begin{aligned} r_x^2 &= 2\bar{w}^2, \\ r_y^2 &= \frac{2\bar{v}\bar{w}^2}{4\bar{w} - \bar{v}}, \\ \theta &= \frac{\pi}{4}. \end{aligned} \qquad (47)$$

Proof: Since $\bar{w}$ is constant over samples, it is clear that the rotation angle is $\theta = \arctan(\bar{w}/\bar{w}) = \pi/4$ and the semi-major radius is $r_x = \sqrt{\bar{w}^2 + \bar{w}^2} = \bar{w}\sqrt{2}$. The semi-minor radius r_y is obtained by enforcing that the ellipse goes through all corners of the inner rectangle (shown with dotted lines in Fig. 1). As a result of symmetry, this is the case when it goes through one of the corners. The upper-right corner is located at $(x, y) = (\bar{w} - \bar{v}, \bar{w})$ at the boundary of the ellipse. Hence for (46) with equality and substituting $\theta = \pi/4$ and $r_x = \bar{w}\sqrt{2}$, we can isolate r_y^2 to find its expression in (47). $\square$.

Based on Lemma 6, combined magnitude-bounded elements and rate-of-variation can be captured by the intersection of $(T-1)\eta_w$ perturbation regions of the form (28) with

$$R_{ell,k,i} = \mathrm{blkdiag}(0_{k\cdot\eta_w}, R_{ell,i}, 0_{(T-k-2)\cdot\eta_w}, -1), \qquad (48)$$

for $k = 0, \ldots, T-1, i = 1, \ldots, \eta_w$ and where

$$R_{ell,i} = \begin{bmatrix} \frac{1}{\bar{w}\cdot\bar{v}} & \frac{1}{2\bar{w}^2} - \frac{1}{\bar{w}\cdot\bar{v}} \\ \frac{1}{2\bar{w}^2} - \frac{1}{\bar{w}\cdot\bar{v}} & \frac{1}{\bar{w}\cdot\bar{v}} \end{bmatrix} \otimes J_i. \qquad (49)$$

Remark. *To maintain the form of* (46)*, the lower right block of* (48) *is normalised. This representation cannot be used when $\bar{v}_i = 0$.*

Remark. *Regions for norm-bounded perturbations and rates-of-variation can be captured by $T-1$ perturbation regions of the form* (28) *by replacing J_i by I in* (49) *(and replacing magnitude bounds by respective norm bounds). The resulting expression can be substituted for R_v in* (43) *to ensure correct dimensions. It is yet unclear if this extension yields the minimally bounding ellipsoid.*

VI. NUMERICAL EXAMPLE

In this section, we showcase the potential of our framework with a numerical example. To this end, we explore the influence of different perturbation multipliers, constructed from the regions of Section V, on the closed-loop performance. Here we will examine the influence of including bounds on the rate-of-variation more closely.

Let us consider the academic example from [13]:

$$x_{k+1} = \begin{bmatrix} 0 & 0.5 & -0.3 \\ \delta_1 & \Delta_{11} & \Delta_{12} \\ 0.1 & \Delta_{21} & \Delta_{22} \end{bmatrix} x_k + \begin{bmatrix} \delta_1 \\ 1 \\ 0.5 \end{bmatrix} u_k + w_k, \quad (50a)$$

$$z_k = \begin{bmatrix} I_3 \\ 0_{1\times3} \end{bmatrix} x_k + \begin{bmatrix} 0_{3\times1} \\ 0.2 \end{bmatrix} u_k + 0_{4\times3} w_k, \qquad (50b)$$

where $\Delta = \mathrm{blkdiag}_{j=1}^2(\Delta_j)$ with a repeated scalar block $\Delta_1 = \delta_1 I_2$ with true value $\delta_{1,\mathrm{tr}} = 0.2$ and a full scalar block $\Delta_2 = \begin{bmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{bmatrix}$ with $\Delta_{2,\mathrm{tr}} = \begin{bmatrix} 0.5 & -0.2 \\ -0.1 & 0.3 \end{bmatrix}$. With the choice of matrices

$$\begin{aligned} A &= \begin{bmatrix} 0 & 0.5 & -0.3 \\ 0 & 0 & 0 \\ 0.1 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \\ 0.5 \end{bmatrix}, H = I, \\ L &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, C_\mathrm{u} = \begin{bmatrix} 0_{1\times3} \\ I_3 \end{bmatrix}, D_\mathrm{u} = \begin{bmatrix} 1 \\ 0_{3\times1} \end{bmatrix}, \end{aligned} \qquad (51)$$

system (50) can be put in the form of (1). From an open-loop experiment with $T = 20$, we collect input-state data from (50). The data is generated by an input sampled uniformly as $u_k \in [-1, 1]$ and a perturbation satisfying $w_k \in \{w \in \mathbb{R}^3 \mid \|w\|_\infty \leq \bar{w}\}$ and $v_k \in \{v \in \mathbb{R}^3 \mid \|v\|_\infty \leq \bar{v} = \bar{w}\}$ for some $\bar{w} \geq 0$. The choice that $\bar{v} = \bar{w}$ corresponds to a rate-of-variation which is 50% of the maximum possible variation. We assume that the perturbed input-state data is available and consider the following scenarios:

1) An Euclidean bound on the perturbation is known as $\|w_k\|_2 \leq \bar{w}\sqrt{\eta_w}$.
2) The value of $\bar{w}$ is known.
3) The values of $\bar{w}$ and $\bar{v}$ are both known.
4) The true values of δ_1 and Δ_2 are known, i.e. (50) is known exactly without any uncertainty.

In scenarios $1-3$ we assume to have access to uncertainty bounds $\delta_{1,\mathrm{tr}}^2 \leq 1000$ and $\Delta_{2,\mathrm{tr}}\Delta_{2,\mathrm{tr}}^\top \preceq 1000I$. Although these are very conservative bounds, they inform us about the structure of Δ_tr and hence are beneficial. In scenario 1, we make use of the multiplier class in (26), while scenario 2 uses multipliers constructed from perturbation regions of Section V-B. Scenario 3 also adds the perturbation regions from Section V-D.

The optimal $\mathcal{H}_\infty$-performance levels that can be guaranteed for each scenario, are presented in Fig. 2 as a function of $\bar{w}$. Solutions are obtained using Yalmip [21] with Mosek as the LMI solver. Using multipliers constructed from perturbation regions (Scenarios 2 and 3) reduces conservatism if compared to existing methods with bounded Euclidean norm (Scenario 1). Values for $\gamma_{opt} \geq 100$ are deemed futile, and hence are not shown in the figure.

Since scenario 3 requires significantly more decision variables than scenario 1, the effectiveness of using ellipsoidal regions of Section V-E is investigated. The optimal $\mathcal{H}_\infty$-performance which can be guaranteed, depending on the rate-of-variation is presented in Fig. 3. Arbitrarily fast variation is captured by $\bar{v}/\bar{w} = 2$. Using ellipsoidal regions reduces

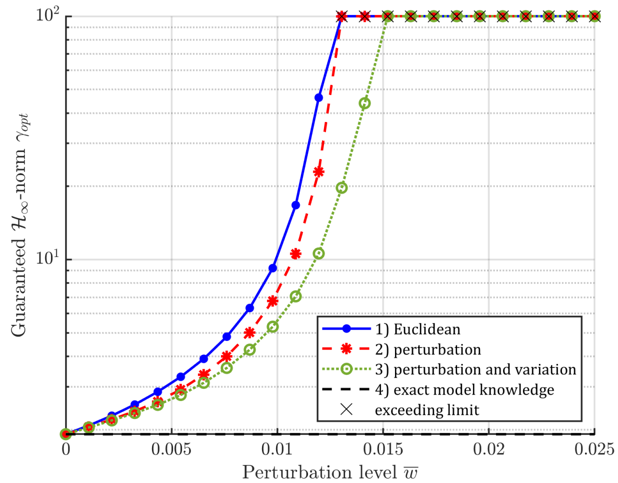

Fig. 2. Guaranteed closed-loop $\mathcal{H}_\infty$-norm for the four scenarios with different perturbation levels.

computational complexity, but this is at the cost of conservatism. This is especially prevalent for high rates of variation.

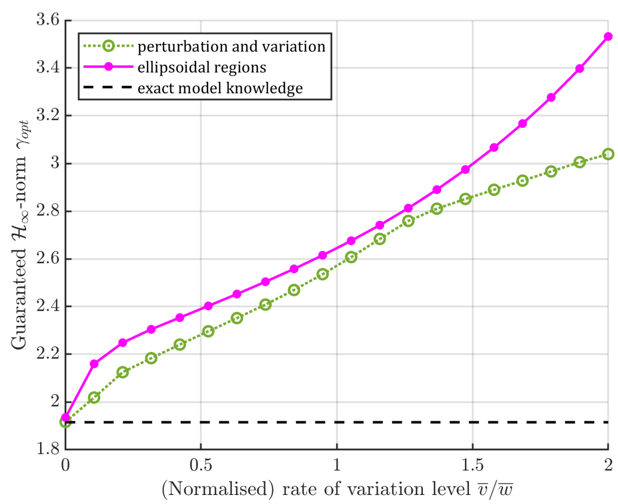

Fig. 3. Guaranteed closed-loop $\mathcal{H}_\infty$-norm for different bounds on the rate-of-variation and $\bar{w} = 0.01$.

Remark. *In order to vary $\bar{v}$ while keeping $\bar{w}$ fixed, a different data set needs be generated for each instance in Fig. 3. Results may vary due to randomness in the data generation.*

VII. Concluding Remarks

In this paper, an LMI-based framework is presented by which knowledge on perturbation samples can be utilized in data-driven robust controller synthesis. It has been shown that this framework allows for more flexible descriptions on bounds than previous research, such as magnitude bounds on individual elements of the perturbation and rate-of-variation. With this, we have shown that conservatism in the optimization procedure is reduced. A possible future working direction is extension to robust data-driven output-feedback control and developing methods to determine perturbation bounds experimentally.

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5 SUPPLEMENTARY RESULTS AND ANALYSIS

As discussed in Chapter 4, known bounds on the norm and rate-of-variation of the perturbation can be incorporated into the controller synthesis. This chapter presents supplementary results that, due to page limitations, were not included in the paper. This chapter primarily focuses on considerations when translating theoretical results from Chapter 4 into real-world control applications.
In Chapter 4, the performance output is defined as a linear combination of the physical states and the input. However, for real-world control applications like reference tracking, this is not a suitable performance output. Luckily, similarly to model-based design, state augmentation can be used to include the error dynamics, and will be presented in Section 5.1. Furthermore, it is shown how integral action and weighting filters can be included in the controller synthesis. This enhances the refinement of performance criteria.
In Section 5.2, systems with a large workspace-to-footprint ratio are considered. These systems may have dynamics that depend heavily on the position. In order to design a single controller that satisfies a performance criteria for the whole workspace, data needs to be collected from multiple open-loop experiments at different operating points. Using the synthesis methods as presented in Chapter 4, one may be forced to make the prevailing assumption that there exists a *common* Lyapunov function for all operating points in order to reconstruct the controller after optimization. In turn, this might yield conservative results. In this section, dilated LMIs are presented, which remove the necessity of this assumption.
SSection 5.3 presents LMIs for multi-objective controller synthesis, specifically considering mixed $\mathcal{H}_\infty$ and generalized $\mathcal{H}_2$ optimization. In this way, $\mathcal{H}_\infty$-optimal controllers can be designed, while respecting peak bounds on the control effort to accommodate for actuator limitations.
Section 5.4 provides considerations that need to be made to design experiments for data-driven controller synthesis. These considerations are demonstrated in Section 5.5 through a reference tracking problem involving a double pendulum.

5.1 Controller synthesis for reference tracking

In Chapter 4, uncertain linear time-invariant systems of the form

$$\begin{bmatrix} x_{k+1} \\ \hline z_k \\ p_k \end{bmatrix} = \left[\begin{array}{c|ccc} A & B & H & L \\ \hline C & D & E & 0 \\ C_{\mathsf{u}} & D_{\mathsf{u}} & 0 & 0 \end{array}\right] \begin{bmatrix} x_k \\ \hline u_k \\ w_k \\ q_k \end{bmatrix}, \tag{5.1a}$$

$$q_k = \Delta_{\mathsf{tr}} p_k, \tag{5.1b}$$

are considered. Recall all matrices except for the true uncertainty Δ_{tr} are known. From this framework, and the numerical example in Chapter 4, it should be observed that the matrices A, B represent partial model knowledge on the unknown system matrices A_{tr} and B_{tr}. However, these matrices are not limited to partial knowledge on unknown system matrices. By augmenting the state vector, prior knowledge on these additional states can be included. As a relevant example, weighting filters represent a natural form of prior knowledge, and can be added to the framework by extending the matrices in (5.1).

5.1.1 Error dynamics

Since reference signals are inputs which cannot be altered by control, they are a part of the exogenous inputs w in the framework of (5.1). Consider that a position reference r needs to be tracked. To this end, define $C_p \in \mathbb{R}^{\eta_r \times \eta_x}$ as a matrix which which picks the position entries of x, such that the positional error can be defined as

$$e_k = r_k - C_{\mathsf{p}} x_k. \tag{5.2}$$

The positional error can then be included in the framework via state augmentation as

$$\begin{bmatrix} x_{k+1} \\ e_k \\ \hline z_k \\ p_k \end{bmatrix} = \left[\begin{array}{cc|c:cc:c} A & 0 & B & H & 0 & L \\ -C_{\mathsf{p}} & 0 & 0 & 0 & I & 0 \\ \hline C & C_{\mathsf{e}} & D & E & E_{\mathsf{r}} & 0 \\ C_{\mathsf{u}} & 0 & D_{\mathsf{u}} & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} x_k \\ e_{k-1} \\ \hline u_k \\ \hdashline w_k \\ r_k \\ \hdashline q_k \end{bmatrix}, \tag{5.3a}$$

$$q_k = \Delta_{\mathsf{tr}} p_k, \tag{5.3b}$$

where C_e and E_r can be used to include the newly introduced signals e and r in the performance objective. What should be observed is that (5.3) is again of the form (5.1) where all matrices are known except the true uncertainty Δ_{tr}, hence the same tools can be used for controller synthesis.

5.1.2 Including integral action

In (5.3), the state vector is augmented by the error signal. In this form, the error signal can be chosen as a performance output, such that e.g. $\mathcal{H}_\infty$-optimal controllers are synthesized which aim to minimize the positional error. In certain tracking problems, it is desired to have zero steady-state error, which can be obtained by integral action. To this end, introduce the integral of the error state as ξ, whose evolution is described by

$$\xi_{k+1} = \xi_k + e_k = \xi_k + r_k - C_p x_k. \tag{5.4}$$

This state can be included in the framework via state augmentation as

$$\begin{bmatrix} x_{k+1} \\ \xi_{k+1} \\ \hline z_k \\ p_k \end{bmatrix} = \left[\begin{array}{cc|c:cc:c} A & 0 & B & H & 0 & L \\ -C_{\mathsf{p}} & I & 0 & 0 & I & 0 \\ \hline C & C_{\xi} & D & E & E_{\mathsf{r}} & 0 \\ C_{\mathsf{u}} & 0 & D_{\mathsf{u}} & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} x_k \\ \xi_k \\ \hline u_k \\ \hdashline w_k \\ r_k \\ \hdashline q_k \end{bmatrix}, \tag{5.5a}$$

$$q_k = \Delta_{\mathsf{tr}} p_k. \tag{5.5b}$$

To obtain controllers with integral action, one needs to choose the integral of the error state as one of the performance outputs by proper choice of C_ξ.

5.1.3 Including weighting filters

In real-world control applications, the direct minimization of the $\mathcal{H}_\infty$-norm is rarely practical. Instead, it is common to incorporate weighting filters to shape the frequency response of the closed loop system. As a relevant case, weighting filters on the control effort can be used to reduce high frequency control effort. This specific scenario can be realised with a high-pass filter on the input. Suppose its state evolution is described by

$$\zeta_{k+1} = A_\mathsf{f}\zeta_k + B_\mathsf{f}u_k, \tag{5.6}$$

then the filter can be included in the framework as

$$\left[\begin{array}{c} x_{k+1} \\ \zeta_{k+1} \\ \xi_{k+1} \\ \hline z_k \\ p_k \end{array}\right] = \left[\begin{array}{ccc|c:cc:c} A & 0 & 0 & B & H & 0 & L \\ 0 & A_\mathsf{f} & 0 & B_\mathsf{f} & 0 & 0 & 0 \\ -C_\mathsf{p} & 0 & I & 0 & 0 & I & 0 \\ \hline C & C_\mathsf{f} & C_\mathsf{e} & D & E & E_\mathsf{r} & 0 \\ C_\mathsf{u} & 0 & 0 & D_\mathsf{u} & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} x_k \\ \zeta_k \\ \xi_k \\ \hline u_k \\ \hdashline w_k \\ r_k \\ \hdashline q_k \end{array}\right] \tag{5.7a}$$

$$q_k = \Delta_\mathsf{tr} p_k. \tag{5.7b}$$

Here, C_f can be used to include the newly introduced state ζ_k in the performance objective. With proper selection of the performance channel, a trade-off between small tracking errors and control inputs can be realized.

5.2 Dilated LMIs for data-driven control

Using the dilation technique, as suggested by [42], a decoupling between the Lyapunov and system matrices can be realised. This decoupling is accomplished through the introduction of an auxiliary matrix variable, taking on the multiplication with the state-space matrices. The usage of dilated LMIs has been demonstrated to be beneficial for reducing conservatism in robust stability and performance analysis [43, 44], multi-objective controller synthesis [45, 46], as well as parameter-dependent controller synthesis [47].

To illustrate, consider a system with position dependent dynamics for which a single state feedback controller needs to be designed. To ensure good performance over the whole workspace, data is obtained from n open-loop experiments at different operating points. Each of these experiments will generate different data that is affected by a different perturbation realization. Furthermore, one may have prior knowledge on this perturbation and on the uncertainty, which differ from experiment to experiment. From these, valid multipliers Φ^i, $i = 1, \ldots, n$ are constructed for each experiment.

With this, Lemma 2 from Chapter 4 can be utilized to find an $\mathcal{H}_\infty$-optimal controller by finding $Q \succ 0$, $F \in \mathbb{R}^{\eta_u \times \eta_x}$ such that

$$\begin{bmatrix} Q & 0 & * & * & * \\ 0 & \gamma I & * & * & 0 \\ AQ + BF & H & Q + \Phi^i_{22} & 0 & * \\ CQ + DF & E & 0 & \gamma I & 0 \\ C_\mathsf{u}Q + D_\mathsf{u}F & 0 & \Phi^i_{12} & 0 & \Phi^i_{11} \end{bmatrix} \succ 0, \; i = 1, \ldots, n. \tag{5.8}$$

The controller is then reconstructed as

$$K = FQ^{-1}. \tag{5.9}$$

What should be observed is that a common Lyapunov matrix Q is used for each experiment, which is required to reconstruct a single controller K. Therefore, this method is potentially conservative. A dilated form of (5.8) (using the methods of A.2) can be constructed as

$$\begin{bmatrix} Q_i & 0 & * & * & * \\ 0 & \gamma I & * & * & 0 \\ AG+BN & H & G+G^\top - Q_i + \Phi_{22}^i & 0 & * \\ CG+DN & E & 0 & \gamma I & 0 \\ C_\mathrm{u}G + D_\mathrm{u}N & 0 & \Phi_{12}^i & 0 & \Phi_{11}^i \end{bmatrix} \succ 0,\ i = 1, \ldots, n, \tag{5.10}$$

where $Q_i' s$ are different Lyapunov matrices for each experiment. The decision variable $G \in \mathbb{R}^{\eta_x \times \eta_x}$ is a slack variable, which is an unstructured matrix, and $N = KG$ is a decision variable that takes the function of F in (5.8). Hence, the controller is reconstructed as

$$K = NG^{-1}. \tag{5.11}$$

It should be observed that (5.10) reduces to (5.8) when $G = Q$ and $Q_i = Q, i = 1, \ldots, n$. Since (5.10) allows for different Lyapunov matrices per experiment, this approach may be less conservative than using (5.8).

Remark. *Note that this approach facilitates the design of controllers which yield $\mathcal{H}_\infty$-optimal performance around specific operating points, and only ensures stability around others. This can be achieved by changing the LMI constraints for certain operating points to those for stability. Additionally, by making γ operating-point-dependent, the $\mathcal{H}_\infty$-performance over the workspace can be "shaped" by minimizing the objective function $\sum_{i=1}^n \alpha_i \gamma_i$, with $\sum_{i=1}^n \alpha_i = 1$, where each $\alpha_i \in \mathbb{R}$ acts as a weight for that specific operating point. Here, larger weights will be assigned to operating points in which the tracking error needs to be kept low.*

5.3 Mixed $\mathcal{H}_\infty$ and generalized $\mathcal{H}_2$ optimization

The $\mathcal{H}_\infty$-performance guarantees bounds on the worst-case $\mathcal{L}_2$-gain, and thus the worst-case energy gain from w to z. However, in certain cases it is much preferred to ensure that the peak value of z does not exceed a certain value for any finite energy signal w. A particularly relevant case is the effect from w on the control effort u, which has a peak bound due to actuator limitations. To this end, the $\mathcal{H}_2$-norm of a discrete-time system is defined as follows:

Definition 19 (e.g. [48]). *The $\mathcal{H}_2$-norm of an asymptotically stable system Σ with corresponding transfer function $\mathcal{T}$, denoted by $\|T\|_2$, is defined as*

$$\|\mathcal{T}_2\|_2 = \sqrt{\frac{1}{2\pi} \int_{-\pi}^{\pi} \text{trace}|T(e^{j\omega})^\top T(e^{j\omega})| d\omega}. \tag{5.12}$$

In the case that $\|\mathcal{T}_2\|_2 < \infty$, then for any input w with $\sum_{k=0}^\infty w_k^\top w_k < \infty$, the corresponding output v_k has a finite amplitude [12], i.e.

$$\|v_k\|_\infty = \sqrt{\sup_{k \geq 0} v_k^\top v_k} < \infty. \tag{5.13}$$

With this, the so-called generalized $\mathcal{H}_2$ norm provides a suitable characterization of the problem described at the start of this section. It offers a quantitative measure for the $\mathcal{L}_2 \to \mathcal{L}_\infty$-gain[1]. The generalized $\mathcal{H}_2$ norm is defined as follows:

[1] The generalized $\mathcal{H}_2$ norm squared is equal to the energy to squared-peak gain, which is the $\mathcal{L}_2 \to \mathcal{L}_\infty$-gain

Definition 20 (e.g. [12], page 78). *The generalized $\mathcal{H}_2$ norm of a transfer function $\mathcal{T}(z)$, denoted by $\|\mathcal{T}\|_{2\to\infty}$, is defined as*

$$\|\mathcal{T}\|_{2\to\infty} = \sup_{w\in\mathcal{L}_2, w\neq 0} \frac{\|z\|_\infty}{\|w\|_2}. \tag{5.14}$$

From this definition, it can be observed that $\mathcal{T}(z)$ needs to be Lyapunov stable in order for its generalized $\mathcal{H}_2$ norm to be finite.
For mixed $\mathcal{H}_\infty$ and generalized $\mathcal{H}_2$ optimization, dilated LMIs are presented in the technical notes of [49]. It is shown that a single LMI can be added to the $\mathcal{H}_\infty$-optimal problem (Lemma 1 in Chapter 4) to turn this into a mixed $\mathcal{H}_\infty$ and generalized $\mathcal{H}_2$ optimization problem. To this end, introduce a second performance output of signals whose peak bound should be regulated as

$$v_k = Ux_k + Vu_k. \tag{5.15}$$

The state feedback controller synthesis using dilated LMIs for mixed $\mathcal{H}_\infty$ and generalized $\mathcal{H}_2$ optimization then reads as follows:

Lemma 13. *If there exist $Q \succ 0$, $G \in \mathbb{R}^{\eta_x\times\eta_x}$, $N \in \mathbb{R}^{\eta_u\times\eta_x}$ such that*

$$\begin{bmatrix} Q & 0 & * & * \\ 0 & \gamma I & * & * \\ AG+BN & H & G+G^\top - Q & 0 \\ CG+DN & E & 0 & \gamma I \end{bmatrix} \succ 0, \tag{5.16}$$

$$\begin{bmatrix} Q & * \\ UG+VN & \sigma^2 I \end{bmatrix} \succ 0, \tag{5.17}$$

then the closed-loop system is stable, has $\mathcal{H}_\infty$ performance as

$$\|z\|_2^2 < \gamma^2\|w\|_2^2, \tag{5.18}$$

and generalized $\mathcal{H}_2$ performance as

$$\|v\|_\infty^2 < \gamma\cdot\sigma^2\|w\|_2^2, \tag{5.19}$$

via the controller $u_k = Kx_k$, with controller gain matrix computed as $K = NG^{-1}$.

A proof for this specific form is presented in A.2, which follow the the technical notes of [16] and [49] closely.

Remark. *In the context of limiting peak values of the control effort, one may set a desired value for the generalized $\mathcal{H}_2$-performance prior to the optimization. To this end, the value of σ is fixed and the remaining parameters are determined to minimize γ. Therefore, the optimum value of γ will change depending on the choice of σ. Using the LMIs as presented in Lemma 13, this will indirectly affect the guaranteed generalized $\mathcal{H}_2$-performance, which is defined as $\gamma\cdot\sigma^2$. In order to obtain the desired generalized $\mathcal{H}_2$-performance, one should be prepared to to iteratively adjust σ to regulate the product $\gamma\cdot\sigma^2$.*[2]

The data-driven version of Lemma 5.3 is stated as follows:

Lemma 14. *If there exist $Q \succ 0$, $G \in \mathbb{R}^{\eta_x\times\eta_x}$, $N \in \mathbb{R}^{\eta_u\times\eta_x}$ and a multiplier Φ satisfying*

$$\begin{bmatrix} \tilde{\Delta}^\top \\ I \end{bmatrix}^\top \underbrace{\begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^\top & \Phi_{22} \end{bmatrix}}_{\Phi} \begin{bmatrix} \tilde{\Delta}^\top \\ I \end{bmatrix} \preceq 0, \forall \tilde{\Delta} \in \tilde{\boldsymbol{\Delta}}_{\text{com}}, \tag{5.20}$$

[2]The technical notes of [49] present LMIs which decouple γ and σ, and hence they are more convenient to use if the generalized $\mathcal{H}_2$-performance needs to be regulated precisely. These LMIs are not used in this chapter to keep consistency with Chapter 4.

such that

$$\begin{bmatrix} Q & 0 & * & * & * \\ 0 & \gamma I & * & * & 0 \\ AG+BN & H & G+G^\top - Q + \Phi_{22} & 0 & * \\ CG+DN & E & 0 & \gamma I & 0 \\ C_{\mathsf{u}}G + D_{\mathsf{u}}N & 0 & \Phi_{12} & 0 & \Phi_{11} \end{bmatrix} \succ 0, \tag{5.21}$$

$$\begin{bmatrix} Q & * \\ UG+VN & \sigma^2 I \end{bmatrix} \succ 0, \tag{5.22}$$

then the closed-loop system is stable, has $\mathcal{H}_\infty$ performance as

$$\|z\|_2^2 < \gamma^2 \|w\|_2^2, \tag{5.23}$$

and generalized $\mathcal{H}_2$ performance as

$$\|v\|_\infty^2 < \gamma \cdot \sigma^2 \|w\|_2^2, \tag{5.24}$$

via the controller $u_k = Kx_k$, where the controller gain matrix computed as $K = NG^{-1}$.

The proof follows from the observation that only the matrices A and B are dependent on Δ, hence the LMI in (5.22) remains unchanged. The dilated LMI for $\mathcal{H}_\infty$-performance is equivalent to (5.10) evaluated at a single operating point. The condition on the multiplier follows from (20) in Chapter 4.

5.4 Designing experiments for data-driven control

A crucial factor for real-world application of the methods described in this thesis, is that bounds on the norm and rate-of-variation should be determined experimentally. This section aims to provide some of the choices and trade-offs that need to be made to design experiments, and how one could systematically determine perturbation bounds.

5.4.1 Constructing input signals for experiments

A handful of aspects are important when constructing the input signal. Firstly, the input signal should be persistently exciting of a sufficiently high order such that data is obtained which is informative for controller design. The theoretical minimal order of persistent excitation is $\eta_x + 1$ and hence can be determined only from a (correct) assumption on the number of states of the system.

Remark. *In simulated examples on data-driven control from the literature, the input signal is often sampled from a uniform distribution (e.g. [4, 6, 23]), since this signal is guaranteed to be persistently exciting. However, such signals do not allow one to set the order of persistent excitation. For a more systematic approach one could generate pseudorandom binary sequences (PRBS), in which this order can be set by the designer [50].*

Secondly, one should carefully establish a proper signal amplitude. While a higher amplitude will increase the signal-to-noise ratio with respect to measurement noise and external disturbances, it might also increase the effect of nonlinearities of the system in the data. Since noise and the effect of nonlinear terms are both considered as perturbations in the controller synthesis, neither of them should be considerably high.

5.4.2 Acquiring data from the system

The followup step is to capture data from the open-loop system using the constructed input signal. Since there is a certain component of randomness involved in the construction of the input signal, one should ensure that the system remains in a small region surrounding the operating point to minimize the effect of nonlinear terms. If this is not the case, the signal amplitude should be decreased or a new input signal should be generated.
The length of the experiment introduces a trade-off between conservatism reduction and computational complexity. Extending the experiment might reduce the set of systems consistent with the data and hence result in less conservative results from the controller synthesis. However, since perturbation regions are constructed per sample, this increases computational complexity.
In order to synthesize full state feedback controllers from the data using methods described in this thesis, full state measurements are required. Therefore, all states ought to be measured or need to be accurately reconstructed from the available measurements.

5.4.3 Construct prior knowledge on the system

While the goal of data-driven control is to avoid the need for refined system identification, some prior knowledge on the underlying system might be available. This could be in the form of estimated parameter values or knowledge about the plant structure. Especially plant structure might be straightforward to determine as the equations of motion of many mechanical systems can be described in the generic form

$$\tau = M(q)\ddot{q} + C(\dot{q}, q)\dot{q} + Kq. \tag{5.25}$$

Here, M, C, and K are the mass, damping, and stiffness matrices, τ is the external input and q represents the generalized coordinates. Equations of motion of this form can be converted to a state space model as

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & I \\ -M^{-1}(q)K & -M^{-1}(q)C(\dot{q}, q) \end{bmatrix}}_{A_\text{c}} x + \underbrace{\begin{bmatrix} 0 \\ M^{-1}(q) \end{bmatrix}}_{B_\text{c}} u, \tag{5.26}$$

where $x = \begin{bmatrix} q^\top & \dot{q}^\top \end{bmatrix}^\top$. The top row of (5.26) is known for any linearization of the plant, even when no prior knowledge on the uncertain parameters is available. The same holds for any Euler discretization of such as model, as the top row of the discretized state-space matrices

$$A = I + \tau_s \cdot A_\text{c}, \tag{5.27}$$

$$B = \tau_s \cdot B_\text{c}, \tag{5.28}$$

with sampling time τ_s is not affected. For other discretization methods, such as Tustin or zero-order-hold, the top row is approximately known and can be captured by (5.27) together with a small uncertainty.
Next to this, the designer should decide on the inclusion of prior knowledge through integral action and weighting filters as presented in Sections 5.1.2 and 5.1.3.

5.5 Reference tracking of a double pendulum system

In this section, the data-driven controller synthesis methods are tested by applying them to a simulated ideal double pendulum model. Here, the goal is to design a mixed $\mathcal{H}_\infty$ and generalized $\mathcal{H}_2$-optimal state feedback controller from perturbed data.

5.5.1 System model

The ideal double pendulum, shown in Figure 5.1, consists of two point masses connected by two weightless rotating rigid beams. The beams are connected to each other, and to the fixed world, by ideal springs and dampers. The double pendulum will operate in the horizontal plane. Consequently, the influence of gravity can be neglected.

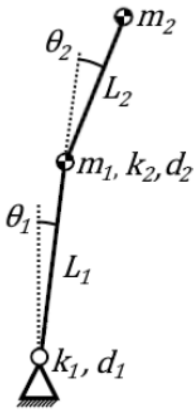


Figure 5.1: Double pendulum.

The system parameters are defined as follows: L_i are the lengths of the arms, m_i represent the weight of a point mass, θ_i is the (relative) angular displacement of an arm, and k_i and d_i represent the stiffness and damping of the joint corresponding to the degree of freedom θ_i respectively. The double pendulum is in its nominal position when it is fully vertical. Table 5.1 shows the values of the parameters used in the simulations.

Table 5.1: Values for the simulation parameters of the double pendulum

Parameter	Value
τ_s	0.005 s
L_1	0.4 m
L_2	0.3 m
m_1	0.5 kg
m_2	0.6 kg
k_1	4.0 N rad^{-1}
k_2	3.5 N rad^{-1}
d_1	0.2 N s rad^{-1}
d_2	0.1 N s rad^{-1}

The equations of motion for this system are derived in [51] in the form of (5.25) with generalized coordinates and its derivatives

$$q = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}, \; \dot{q} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}, \; \ddot{q} = \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2, \end{bmatrix} \tag{5.29}$$

and the matrices

$$M(q) \quad = \begin{bmatrix} (m_1 + m_2)L_1^2 + m_2L_2^2 + 2m_2L_1L_2 \cos(\theta_2) & m_2L_2^2 + m_2L_1L_2 \cos(\theta_2) \\ m_2L_2^2 + m_2L_1L_2 \cos(\theta_2) & m_2L_2^2 \end{bmatrix}, \tag{5.30a}$$

$$C(q, \dot{q}) = \begin{bmatrix} -2m_2L_1L_2 \sin(\theta_2)\dot{\theta}_2 + d_1 & -m_2L_1L_2 \sin(\theta_2)\dot{\theta}_2 \\ m_2L_1L_2 \sin(\theta_2)\dot{\theta}_1 & d_2 \end{bmatrix}, \tag{5.30b}$$

$$K \quad = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}. \tag{5.30c}$$

The input is defined as

$$\tau = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}, \tag{5.31}$$

and acts on the two joints of the double pendulum.
Obviously, the double pendulum is a nonlinear system and hence its trajectories may not be described by a standard LTI model. Inspired by [31], linearization around an operating point is considered, where the nonlinear remainder is treated as a perturbation in the controller synthesis. If the experiment is carried out such that the double pendulum remains sufficiently close to this operating point, sensible results may be obtained. Specifically, guarantees on the $\mathcal{H}_\infty$ and generalized $\mathcal{H}_2$ norm of the linearized system interconnected with this controller can be computed. In return, the controller might yield good tracking performance when interconnected with the nonlinear system, provided that the reference stays sufficiently close to this operating point.
To this end, the model is linearized around its nominal position using a first-order Taylor approximation, and subsequently converted to a continuous time state-space model via (5.26). This procedure is followed in [51] and yields the state-space matrices

$$A_c = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1}{m_1 L_1^2} & \frac{(L_1+L_2)k_2}{m_1 L_1^2 L_2} & -\frac{d_1}{m_1 L_1^2} & \frac{(L_1+L_2)d_2}{m_1 L_1^2 L_2} \\ \frac{(L_1+L_2)k_1}{m_1 L_1^2 L_2} & -\frac{k_2}{m_2 L_2^2} - \frac{(L_1+L_2)^2 k_2}{m_1 L_1^2 L_2^2} & \frac{(L_1+L_2)d_1}{m_1 L_1^2 L_2} & -\frac{d_2}{m_2 L_2^2} - \frac{(L_1+L_2)^2 d_2}{m_1 L_1^2 L_2^2} \end{bmatrix}, \tag{5.32a}$$

$$B_c = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{m_1 L_1^2} & -\frac{L_1+L_2}{m_1 L_1^2 L_2} \\ -\frac{L_1+L_2}{m_1 L_1^2 L_2} & \frac{1}{m_2 L_2^2} + \frac{(L_1+L_2)^2}{m_1 L_1^2 L_2^2} \end{bmatrix}. \tag{5.32b}$$

The model is discretized using a zero-order hold with sampling time τ_s and yields the true system matrices A_{tr} and B_{tr} of the linearized model.
In order to obtain sufficient tracking performance, integral action is included in the controller synthesis. Therefore, a modified framework of (5.5) will be used to describe the linearized uncertain double pendulum system as

$$\begin{bmatrix} x_{k+1} \\ \xi_{k+1} \\ \hline z_k \\ v_k \\ p_k \end{bmatrix} = \left[\begin{array}{cc|c:ccc:c} \hat{A} & 0 & \hat{B} & H & 0 & L \\ -C_{\text{p}} & I & 0 & 0 & I & 0 \\ \hline C & C_\xi & D & E & E_{\text{r}} & 0 \\ U & 0 & V & 0 & 0 & 0 \\ C_{\text{u}} & 0 & D_{\text{u}} & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} x_k \\ \xi_k \\ \hline u_k \\ \hdashline w_k \\ r_k \\ \hdashline q_k \end{bmatrix} \tag{5.33a}$$

$$q_k = \Delta_{\text{tr}} p_k, \tag{5.33b}$$

which includes the second performance channel v_k. Here, $\hat{A}$ and $\hat{B}$ represent a generic form of the system matrices, which may differ depending on the available prior knowledge.
The perturbation is assumed to affect the entire state space, which is captured by $H = I$. Since the tracking of a position reference is considered, the matrix C_{p} should pick the angular deflection states of x, which is achieved by $C_{\text{p}} = \begin{bmatrix} I & 0 \end{bmatrix}$.
The performance output z_k describes the signals that should be minimized in a $\mathcal{H}_\infty$ sense, which in this case is the integral of the error, hence $C_\xi = I$ and $D = E = E_r = 0$. The performance output v_k describes the signals that should be minimized in a generalized $\mathcal{H}_2$ sense, which in this case is the control input, hence $V = I$ and $U = 0$.
The goal of the controller synthesis is for the second point mass to track the smooth profile shown in Figure 5.2, which has a duration of 20s.

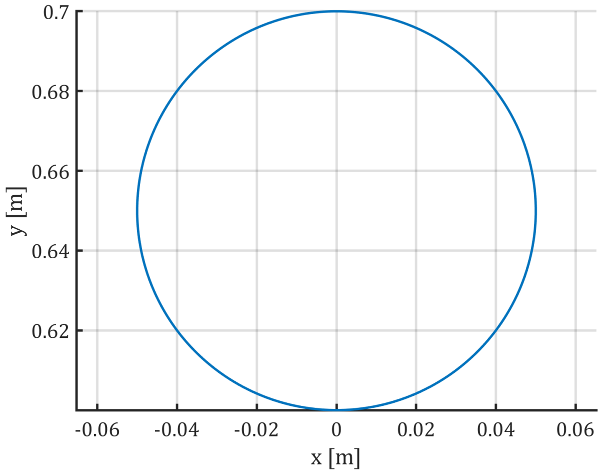


Figure 5.2: Reference profile to be tracked.

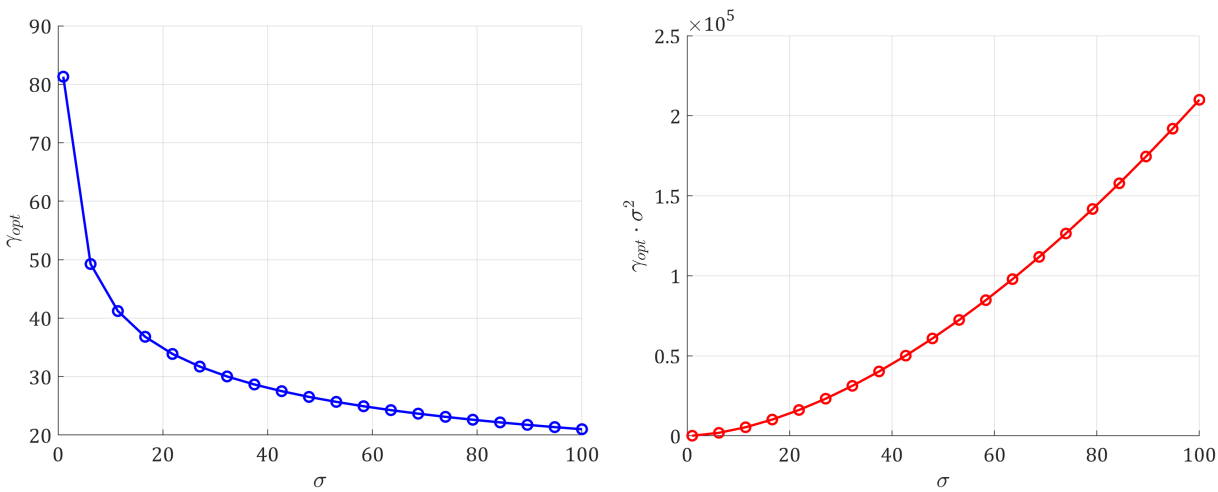
This profile is translated to references for the joint angles using the inverse kinematic relationships

$$r_1(x,y) = \tan^{-1}\left(\frac{x}{y}\right) - \frac{\sin^{-1}(L_2 \sin(r_2(x,y)))}{\sqrt{x^2+y^2}}, \tag{5.34a}$$

$$r_2(x,y) = \cos^{-1}\left(\frac{x^2+y^2-L_1^2-L_2^2}{2L_1L_2}\right). \tag{5.34b}$$

5.5.2 Model-based controller synthesis

In order to establish a limit on the achievable $\mathcal{H}_\infty$ and generalized $\mathcal{H}_2$-performance, a model-based controller synthesis is carried out. Consequently, the system matrices A_{tr} and B_{tr} are exactly known and hence $\Delta_{\text{tr}} = 0$. This case is captured by (5.33) with $\hat{A} = A_{\text{tr}}$, $\hat{B} = B_{\text{tr}}$, and with L, C_u, D_u being zero matrices of appropriate dimensions.
With this, Lemma 13 is used to generate state feedback controllers for this system. Solutions are obtained using Yalmip [52] with Mosek as the LMI solver. The guaranteed $\mathcal{H}_\infty$ and generalized $\mathcal{H}_2$-performance of the closed-loop system for different values of σ are presented in Figure 5.3.



(a) Guaranteed $\mathcal{H}_\infty$-norm.

(b) Guaranteed generalized $\mathcal{H}_2$-norm.

Figure 5.3: Guaranteed $\mathcal{H}_\infty$ (a) and generalized $\mathcal{H}_2$-performance (b) of the closed-loop as function of σ. The controller is synthesized with exact model knowledge.

From Figure 5.3, two observations emerge. As expected, larger values of σ enhance the $\mathcal{H}_\infty$-performance at the expense of the generalized $\mathcal{H}_2$-performance. The more unexpected observation are the large values for the guaranteed performance indices, which are of several order of magnitude. This can be explained by the order of the sequence in which new states are introduced and the discretization of the linearized model. By introducing the integral of the error after discretization, the performance channel z_k contains the state corresponding to this error, but is not representative of the integral of the error itself. In order to get more intuitive values for the guaranteed performance indices, the integral of the error should be included as a state before discretization as

$$\dot{\xi} = r - C_{\mathsf{p}} x. \tag{5.35}$$

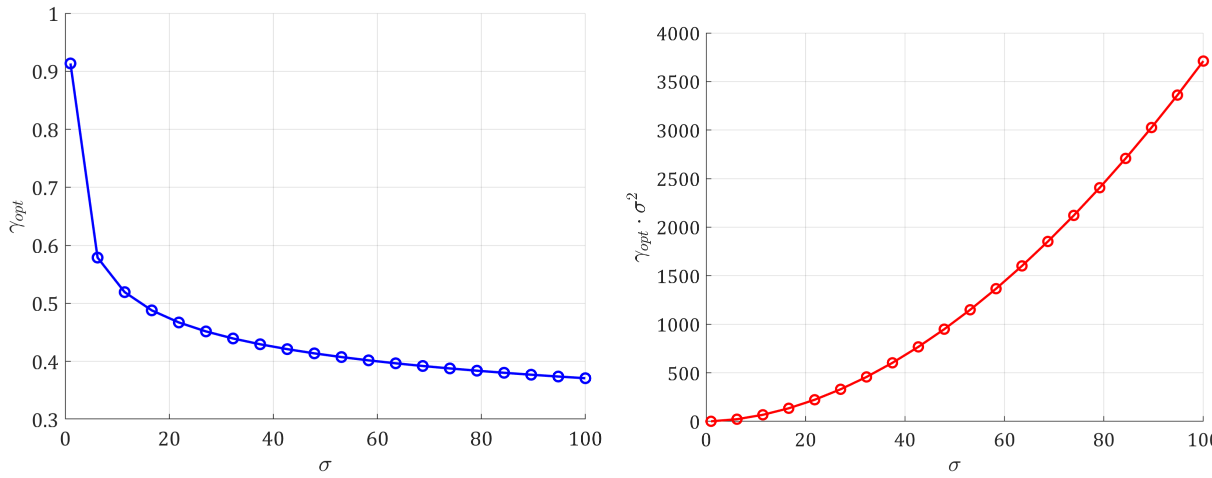
The discrete-time state evolution can then be approximated via Euler discretization as

$$\xi_{k+1} = \xi_k + (r_k - C_{\mathsf{p}} x_k)\tau_s, \tag{5.36}$$

which is a better approximation of the integral of the error. In a similar way, including the perturbation signal w in the continuous time as

$$\dot{x} = A_{\mathsf{c}} x + B_{\mathsf{c}} u + H_{\mathsf{c}} w, \tag{5.37}$$

with $H_{\mathsf{c}} = I$ before discretization shows that in discrete-time the Euler approximation of $H = \tau_s I$ fits better. With these changes, more intuitive values for the performance guarantees are obtained, as shown in Figure 5.4. It is important to emphasize that these choices do not affect the actual tracking performance, just the calculation and representation of the performance indices.



(a) Guaranteed $\mathcal{H}_\infty$-norm. (b) Guaranteed generalized $\mathcal{H}_2$-norm.

Figure 5.4: More intuitive values for the guaranteed $\mathcal{H}_\infty$ (a) and generalized $\mathcal{H}_2$-performance (b) are obtained by introducing new states before discretization.

In this framework, a proper choice of σ can be determined to balance tracking performance and guaranteed bounds on peaks in the control effort. This can be done systematically by starting with a low value for σ and iteratively increasing it for better tracking performance. This requires evaluation the tracking performance and corresponding control effort for multiple controllers.
From Figure 5.4b it can be observed that the guaranteed generalized $\mathcal{H}_2$-performance grows quickly with σ and hence the guaranteed peak bound on the control effort is not that meaningful for larger values of σ. However, one can still use these larger values for systems which need to track a finite number of known references, by assessing the required control effort for each reference signal in separate experiments.

5.5.3 Data-driven synthesis with perturbation-free data

Data-driven controller synthesis from perturbation-free data is considered as an initial step towards synthesis with perturbations. In this way, it can be shown that this problem is equivalent to the model-based synthesis when the input signal is constructed with regards to Willems' fundamental lemma. To this end, consider that the system matrices A_{tr} and B_{tr} are fully unknown, so $\Delta_{tr} = \begin{bmatrix} A_{\text{tr}} & B_{\text{tr}} \end{bmatrix}$. This case is captured by (5.33) with

$$\hat{A} = 0, \hat{B} = 0, L = I, C_u = \begin{bmatrix} I & 0 \end{bmatrix}^\top, D_u = \begin{bmatrix} 0 & I \end{bmatrix}^\top. \tag{5.38}$$

The performance- and perturbation channels are defined as in Section 5.5.2.
In accordance with Willems' fundamental lemma (Lemma 11) with $d = 1$, data from an open-loop experiment generated by an uniformly sampled input as $u_k \in [-1, 1]$ is acquired for $T = (\eta_u + 1)\eta_x + \eta_u = 14$ samples.
Lemma 14 is applied to synthesize controllers, here $\sigma = 30$ is chosen. The multiplier matrix Φ is constructed according to Lemma 3 of Chapter 4 as

$$\Phi = \begin{bmatrix} -Z^\top & M^\top \\ 0 & H^\top \end{bmatrix}^\top \Pi \begin{bmatrix} -Z^\top & M^\top \\ 0 & H^\top \end{bmatrix}, \tag{5.39}$$

with a multiplier Π from learnt data only. The multiplier Π is constructed following Theorem 5 of Chapter 4 via the LMI

$$\Lambda^\top \Pi \Lambda - \sum_{i=1}^{n} (R_i \otimes S_i) \preceq 0, \tag{5.40}$$

where R_i are regions for norm-bounded perturbations with $\bar{w} = 0$.
Under these conditions, it is expected that the data-driven and model-based yield the same performance. This is because the condition on the multiplier from learnt data

$$\begin{bmatrix} W^\top \\ I \end{bmatrix}^\top \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^\top & \Pi_{22} \end{bmatrix} \begin{bmatrix} W^\top \\ I \end{bmatrix} \preceq 0, \forall W \in \mathbf{W}_T, \tag{5.41}$$

is satisfied for $\Pi_{11} \succeq 0, \Pi_{12} = 0$ and $\Pi_{22} = 0$ when $W = 0$. Consequently, the data-dependent LMI (5.21) in Lemma 14 reduces to its model-based equivalent in Lemma 13.
With these choices, the controller synthesis verifies that the problem is feasible with $\gamma_{opt} = 0.4450$ and a corresponding controller gain matrix is found as

$$K = \begin{bmatrix} -12.2482 & -3.6930 & -2.6649 & -0.9088 & 38.4254 & 8.1850 \\ -4.7048 & -2.4512 & -1.0880 & -0.4235 & 11.2376 & 14.6357 \end{bmatrix}, \tag{5.42}$$

which yields to a closed-loop $\mathcal{H}_\infty$-norm of $\gamma_{CL} = 0.4316$. This is slightly larger than the model-based case, which yields $\gamma_{opt} = 0.4445$ and $\gamma_{CL} = 0.4311$ via the controller gain matrix

$$K = \begin{bmatrix} -12.0319 & -4.5866 & -2.6929 & -0.8869 & 38.0700 & 9.9762 \\ -5.1697 & -0.1501 & -1.0065 & -0.4755 & 11.9608 & 10.0027 \end{bmatrix}. \tag{5.43}$$

This is likely due to numerical challenges in the optimization process, which terminates before achieving an even more precise solution for γ_{opt}.
The tracking performance is evaluated using the closed-loop simulation setup presented in Figure 5.5.

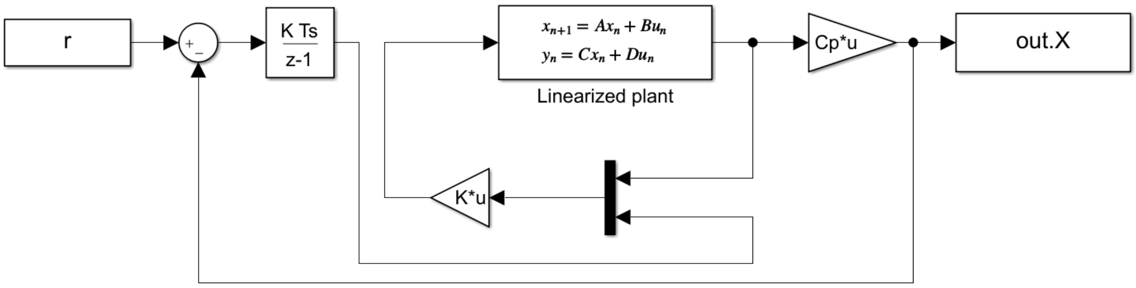
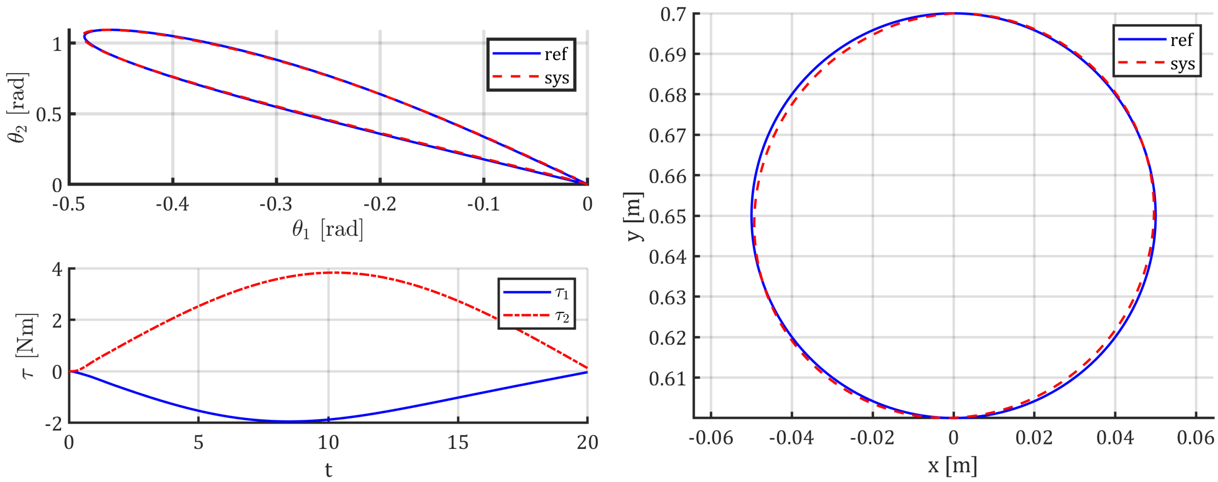
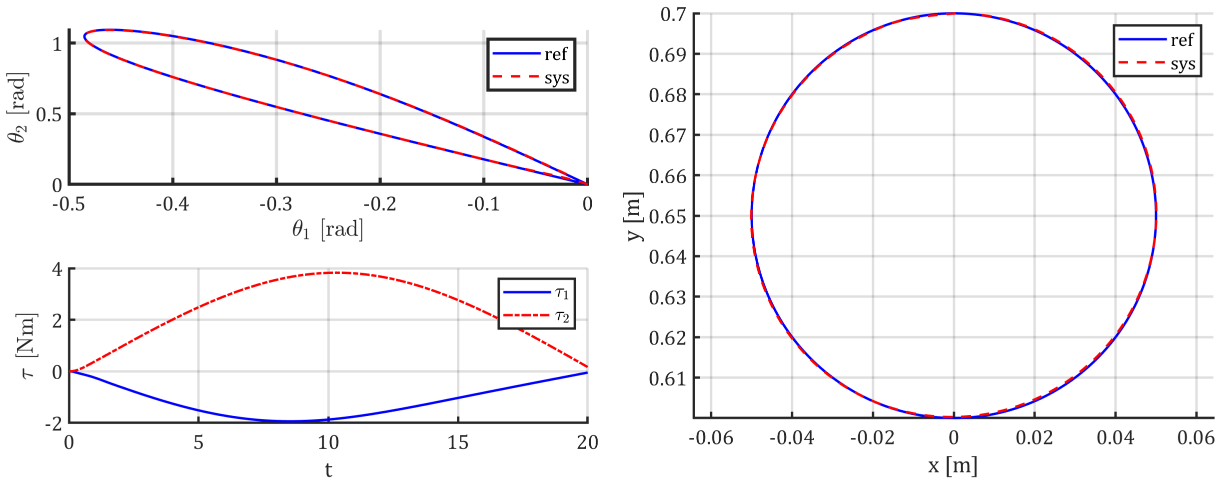


Figure 5.5: Simulink model for reference tracking.

The obtained tracking performance using the controllers as in (5.42) and (5.43) are presented in Figure 5.6. Here, it can be seen that both methods yield approximately the same tracking performance and that the control effort is smooth and does not have large peaks.



(a) Controller synthesized with exact model knowledge.



(b) Controller synthesized from perturbation-free data.

Figure 5.6: Tracking performance in the model-based case (a) and the data-driven case with perturbation-free data (b) are approximately equivalent.

5.5.4 Data-driven synthesis with perturbed data

In this section, data-driven controller synthesis from perturbed data is considered. The perturbation is considered to be a measurement noise acting on all the states. For illustrative purposes, the data is captured from the discretized linearized system. The same matrices as in Section 5.5.3 are used to describe the controller synthesis framework. This time, the data is captured following the state evolution

$$x_{k+1} = A_{\text{tr}} x_k + B_{\text{tr}} u_k + H w_k, \tag{5.44}$$

with a uniformly sampled input as $u_k \in [-1, 1]$ and a uniformly sampled perturbation as $w_k \in \left[-5 \cdot 10^{-3}, 5 \cdot 10^{-3}\right]$.

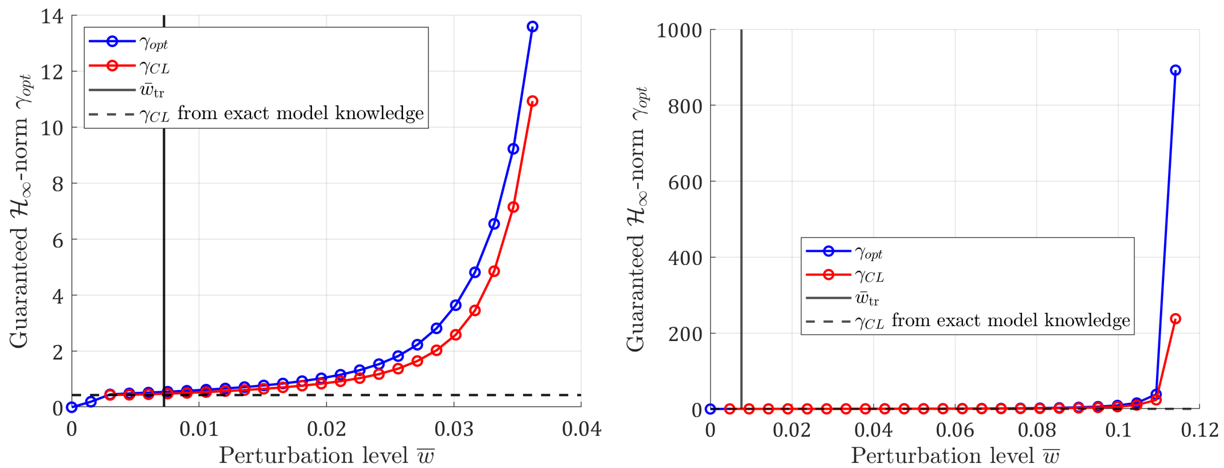
With the data captured, Lemma 14 is applied to synthesize controllers, here $\sigma = 30$ is chosen as well. The multiplier Φ is constructed following Lemma 3 of Chapter 4 with a multiplier Π from learnt data only. In contrast to the case of perturbation-free data as in Section 5.5.3, the proper perturbation regions and their parameters are not clearly defined.

In this case, regions for norm-bounded perturbations and regions for perturbations with norm-bounded rates-of-variation are considered. These regions are described by the upper bound of norm-bound as $\bar{w}$ and the upper bound on the rate-of-variation as $\bar{v}$. Since their true values cannot be determined from the experiment itself, they act as tuning parameters in the controller synthesis. A general approach is outlined as follows:

1. Start with a high value for $\bar{w}$ and let $\bar{v} = 2\bar{w}$, which corresponds to an arbitrary variation.
2. Synthesise a controller for these bounds and denote the guaranteed $\mathcal{H}_\infty$-performance from the optimization as γ_{opt}.
3. Evaluate the resulting closed-loop $\mathcal{H}_\infty$-performance for this controller as γ_{CL}.
4. If $\gamma_{CL} < \gamma_{opt}$, decrease $\bar{w}$ and repeat until this statement is false.
5. Use the lowest value of $\bar{w}$ for which $\gamma_{CL} < \gamma_{opt}$, and repeat the process by decreasing $\bar{v}$ until the statement is false.

As an alternative approach, one could swap the order in which the parameters are tuned, so by first tuning $\bar{v}$ instead of $\bar{w}$. If there is reason to think that the perturbation signal is (almost) constant, one could start with $\bar{v} = 0$ (which describes a constant perturbation) and iteratively increase its value.

Following the outlined procedure, varying $\bar{w}$ yields the results presented in Figure 5.7.



(a) $\bar{w}$ is close to its true value.

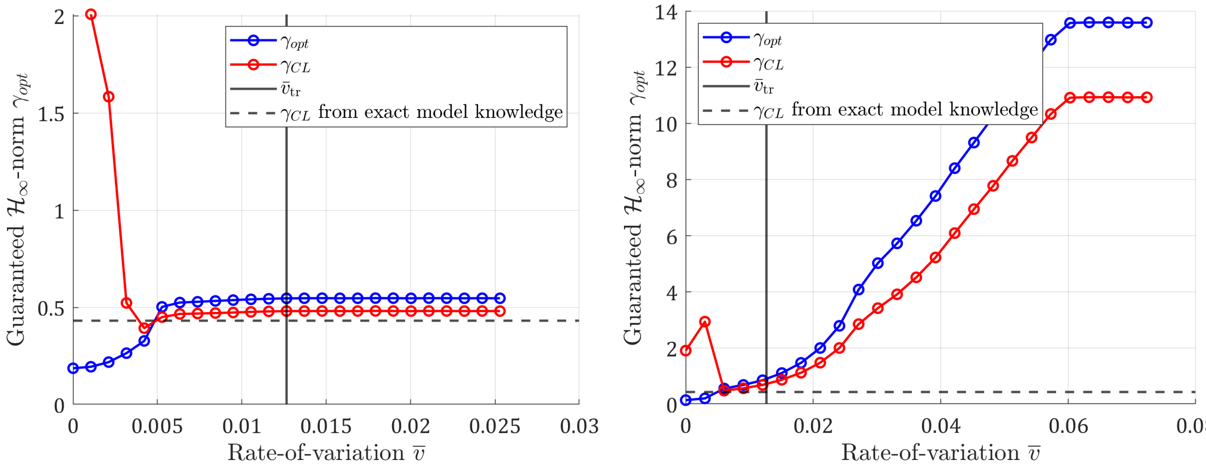
(b) With larger values for $\bar{w}$.

Figure 5.7: Guaranteed $\mathcal{H}_\infty$-performance for perturbed data with true norm bound $\bar{w}_{tr}$.

From Figure 5.7, three distinct regions can be observed

1. The assumed norm bound of the perturbation is below the true value, i.e. $\bar{w} < \bar{w}_{\text{tr}}$. In this scenario, the true system is not an element of the set of consistent systems. Consequently, it cannot be guaranteed that $\gamma_{opt} > \gamma_{CL}$, or even that γ_{CL} is finite, as seen in Figure 5.7a.
2. The assumed perturbation norm bound is larger but close to the true value. For these values, the best closed-loop performance can be guaranteed.
3. The assumed perturbation bound is much larger than the true value. The set of systems consistent with the data is too large to generate controllers that guarantee a sufficient tracking performance.

For even larger values of the assumed perturbation bound, the problem may become infeasible. From Figure 5.7a, the true perturbation level can be estimated. Subsequently, a bound on the rate-of-variation is introduced and decreased from $\bar{v} = 2\bar{w}$ to 0. The effect on the resulting performance is presented in Figure 5.8a

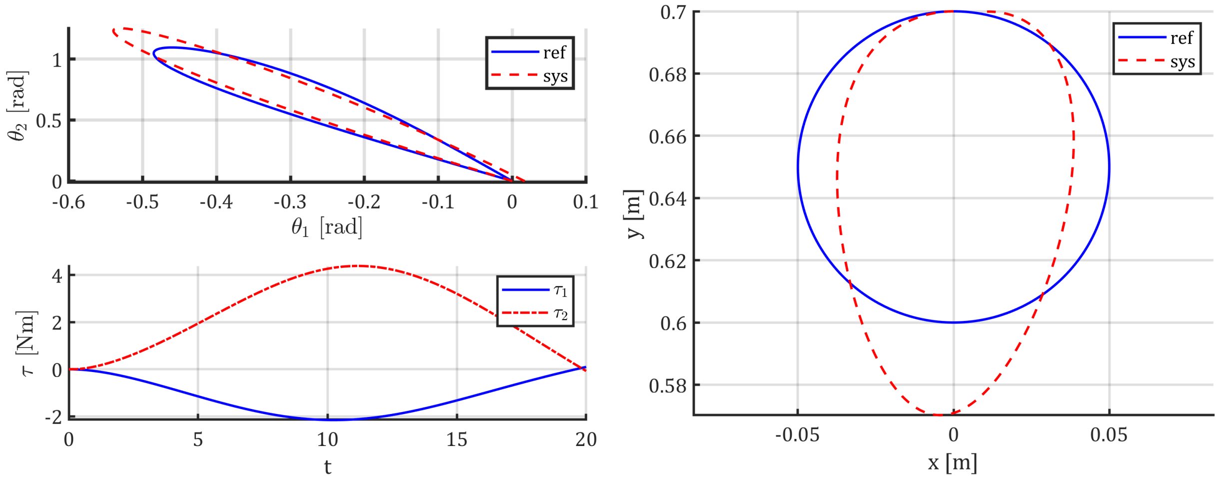


(a) $\bar{w} = \bar{w}_{\text{tr}}$.

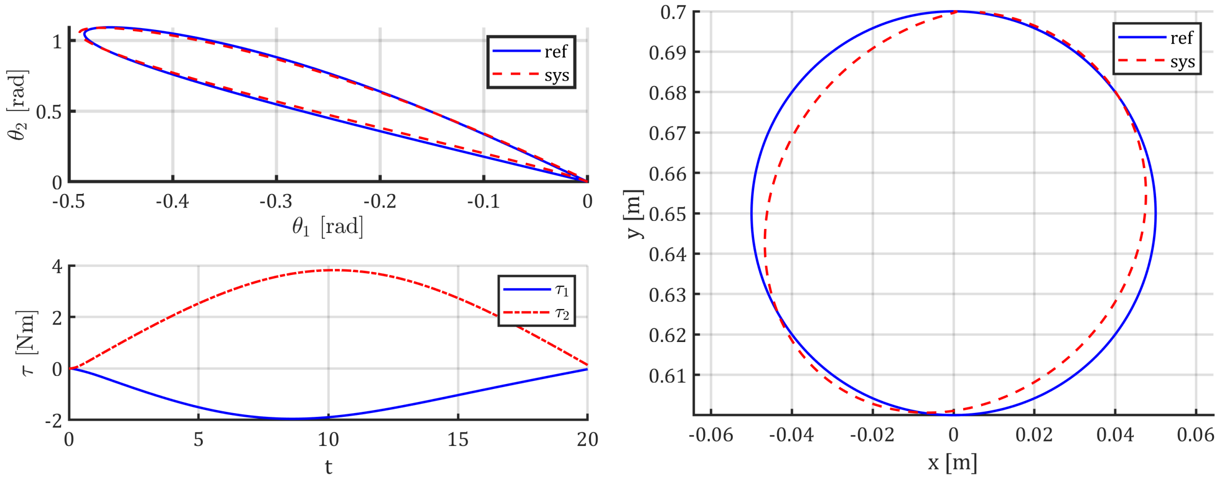
(b) $\bar{w} = 0.0361$

Figure 5.8: Including bounds on the rate-of-variation improves the guaranteed $\mathcal{H}_\infty$-performance.

Again, it can be seen that it cannot be guaranteed that $\gamma_{opt} > \gamma_{CL}$ when $\bar{v} < \bar{v}_{\text{tr}}$, as the true system is not an element of the set of systems consistent with the data. In this case, the inclusion of bounds on the rate-of-variation yield a minimal performance increase, since model-like performance could already be achieved by the inclusion of norm bounds only. Repeating the same procedure with $\bar{w} = 0.0361$ shows that the inclusion of the bound on the rate-of-variation can improve the performance significantly, as can be seen in Figure 5.8b.
Comparing Figure 5.7a to Figure 5.8b, it can be seen that good performance can be achieved by introducing a bound on the rate-of-variation, even when the bound on the norm is not fully tuned, as performance improves more rapidly with this choice.



(a) With $\bar{w} = 0.0361$ and $\bar{v} = 2\bar{w}$ (arbitrary rate-of-variation).



(b) With $\bar{w} = 0.0361$ and $\bar{v} = \bar{v}_{\text{tr}}$.

Figure 5.9: The tracking performance using a sub-optimal norm bound (a) can be improved by introducing a bound on the rate-of-variation (b).

In Figure 5.9 it can be seen that the tracking performance, although improved significantly, might still not fulfill the requirements on the tracking accuracy. It should be noted that only 14 samples are being used in the optimization. Increasing the number of samples decreases the amount of systems that are consistent with the data, and hence might decrease conservatism in the optimization. To illustrate, the controller synthesis used to generate Figure 5.7a is repeated for $T = 30$ samples. What should be observed is that good performance can be obtained without needing to tune the norm bound as precisely as for the case with 14 samples. However, this is at the expense of computational complexity.

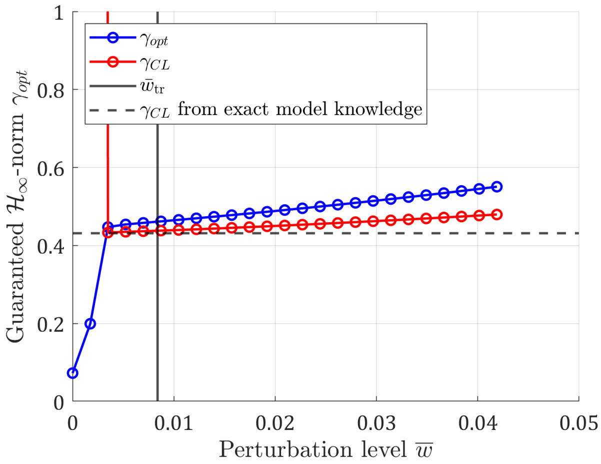


Figure 5.10: Increasing the number of samples reduces conservatism.

5.5.5 Synthesis with data from the nonlinear system with prior knowledge

In Section 5.5.4, the data is collected from a linearized system that is discretized using a zero-order hold. For this reason, the entries in A_{tr} and B_{tr} cannot be exactly known. As presented in Section 5.4.3, the Euler discretization can provide approximate values for the top two rows of A_{tr} and B_{tr}.
This case is captured using the approximate system dynamics

$$\hat{A} = \left[\begin{array}{cccc} 1 & 0 & \tau_s & 0 \\ 0 & 0 & 0 & \tau_s \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \hat{B} = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \hline 0 & 0 \\ 0 & 0 \end{array}\right], \tag{5.45}$$

and an uncertainty description as

$$\Delta_{\text{tr}} = \text{blkdiag}_{j=1}^{4}(\Delta_j). \tag{5.46}$$

Here, each Δ_j is a full block uncertainty corresponding to one of the four blocks in (5.45), i.e.

$$A = \hat{A} + \left[\begin{array}{c} \Delta_1 \\ \hline \Delta_2 \end{array}\right], B = \hat{B} + \left[\begin{array}{c} \Delta_3 \\ \hline \Delta_4 \end{array}\right]. \tag{5.47}$$

This can be put in the framework of (5.33) with

$$L = \begin{bmatrix} I_4 & I_4 \end{bmatrix}, C_{\text{u}} = \begin{bmatrix} I_4 & I_4 & 0_4 \end{bmatrix}^\top, D_{\text{u}} = \begin{bmatrix} 0_{2\times 8} & I_2 & I_2 \end{bmatrix}^\top. \tag{5.48}$$

The performance- and perturbation channels are defined as in Sections 5.5.2-5.5.4.
Again, Lemma 14 is applied to synthesize controllers, with $\sigma = 30$. In this case the multiplier matrix Φ is constructed according to Lemma 3 of Chapter 4, including the multiplier on the prior knowledge as

$$\Phi = \sum_{j=1}^{4} \begin{bmatrix} N_j^\top & 0 \\ 0 & L_j^\top \end{bmatrix}^\top \Psi_j \begin{bmatrix} N_j^\top & 0 \\ 0 & L_j^\top \end{bmatrix} + \begin{bmatrix} -Z^\top & M^\top \\ 0 & H^\top \end{bmatrix}^\top \Pi \begin{bmatrix} -Z^\top & M^\top \\ 0 & H^\top \end{bmatrix}, \tag{5.49}$$

where Ψ_j and Π are multipliers with $\Psi_{11} \succ 0$ and $\Pi_{11} \succ 0$ which need to satisfy

$$\begin{bmatrix} \Delta_j^\top \\ I \end{bmatrix}^\top \Psi_j \begin{bmatrix} \Delta_j^\top \\ I \end{bmatrix} \preceq 0, \forall \Delta_j \in \mathbf{\Delta}_j, \tag{5.50}$$

and

$$\begin{bmatrix} W^\top \\ I \end{bmatrix}^\top \Pi \begin{bmatrix} W^\top \\ I \end{bmatrix} \preceq 0, \forall W \in \boldsymbol{W}_T, \tag{5.51}$$

respectively. The multipliers Ψ_j from the uncertainty description are constructed from norm bounds on the corresponding uncertainty block. The uncertainty blocks Δ_1 and Δ_3 describe the numerical differences between discretization methods (zero-order hold and Euler), and as such can are captured by the bounds $\Delta_{1,\text{tr}}\Delta_{1,\text{tr}}^\top \preceq I$ and $\Delta_{3,\text{tr}}\Delta_{3,\text{tr}}^\top \preceq I$. The uncertainty blocks Δ_2 and Δ_4 correspond to the unknown entries of A_{tr} and B_{tr} related to the parameters of the system. These parameters are not (approximately) known, and hence are captured by the much more conservative bounds $\Delta_{2,\text{tr}}\Delta_{2,\text{tr}}^\top \preceq 1000I$ and $\Delta_{4,\text{tr}}\Delta_{4,\text{tr}}^\top \preceq 1000I$.
The data is collected from the nonlinear system using the Simulink presented in Figure 5.11. Here, the continuous-time dynamics are described by

$$\ddot{q} = M(q)^{-1}(\tau - C(\dot{q}, q)\dot{q} - Kq). \tag{5.52}$$

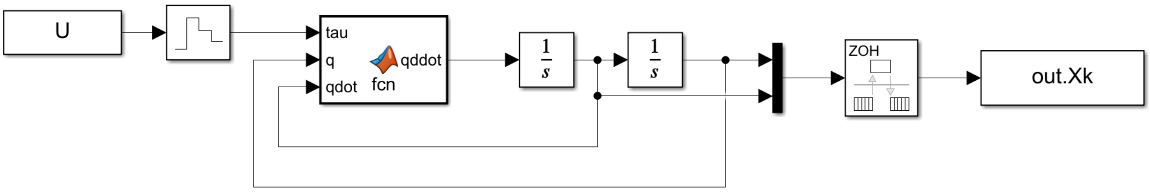


Figure 5.11: Simulink model to capture data from the nonlinear system.

In order to ensure that the system remains in a small region surrounding the operating point, the input signal is sampled uniformly as $u_k \in [-0.05, 0.05]$.
With the data captured, the synthesis procedure as described in Section 5.5.4 is followed. Its results are presented in Figure 5.12.

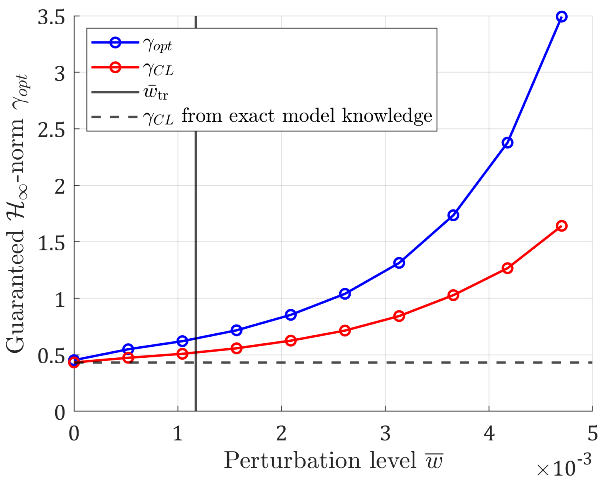


Figure 5.12: Synthesis from data of the nonlinear system.

From Figure 5.12, it can be seen that good performance can also be guaranteed in case the perturbation is coming from nonlinear terms of the system. Again, including bounds on the rate-of-variation can enhance the performance, as shown in Figure 5.13.

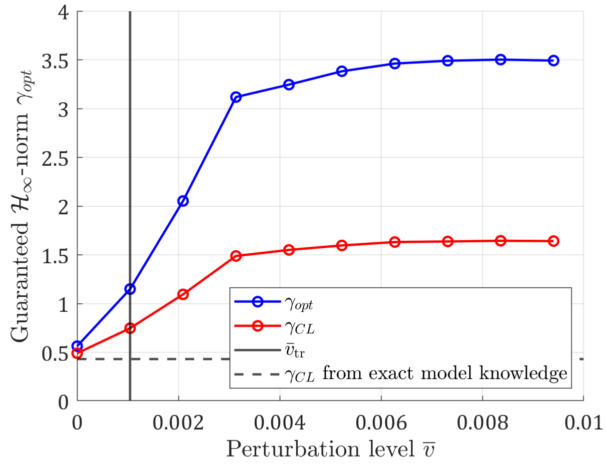


Figure 5.13: Including bounds on the rate-of-variation can improve the performance. Here, $\bar{w}$ corresponds to the largest value in Figure 5.12.

5.5.6 Proneness to numerical issues

Although Sections 5.5.2-5.4.3 show promising results, it should be noted that this is partly due to careful selection of the simulation parameters. With a different selection of parameters, the results from the optimization are rather inconsistent. To illustrate, the data-driven controller synthesis with perturbation-free data of Section 5.5.3 is repeated for different sampling times. A smaller sampling time is preferred in the optimization, as the discrepancy in the system matrices caused by different discretization methods decreases, which allows for tighter bounds on the uncertainty blocks when introducing prior knowledge. Furthermore, small sampling times for controllers are desirable to quickly and accurately respond to dynamic system changes, ensuring effective real-time control and stability. The result of this synthesis is presented in Figure 5.14.

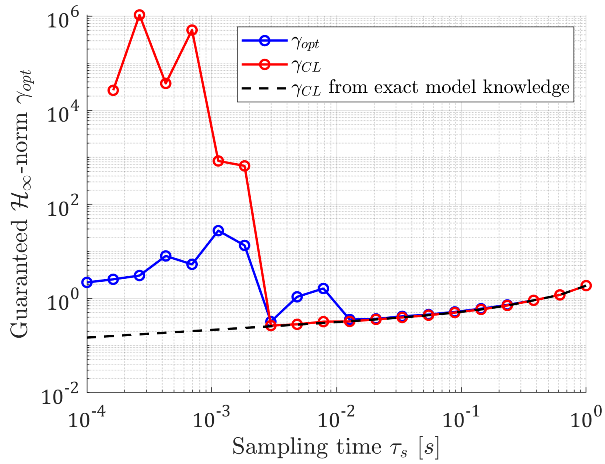


Figure 5.14: The optimization fails for smaller sampling times.

From Figure 5.14, it should be observed that the synthesis fails for lower sampling times. The value of the guaranteed $\mathcal{H}_\infty$-performance is significantly less than the corresponding closed-loop performance, i.e. $\gamma_{opt} < \gamma_{CL}$. Moreover, there is a significant discrepancy between the model-based and data-driven syntheses. This seems to be caused by numerical issues, in the sense that the variation between data samples is too little to identify the underlying system for appropriate controller synthesis. One could say that the system is not sufficiently excited for good controller synthesis even though the input signal is persistently exciting. Indeed, scaling the data matrices U, X and X_+ with a constant value before the optimization seems to alleviate some of these issues. From Figure 5.15 it can be seen that the model-based and data-driven solutions are also equivalent for smaller sampling times as compared to Figure 5.14.

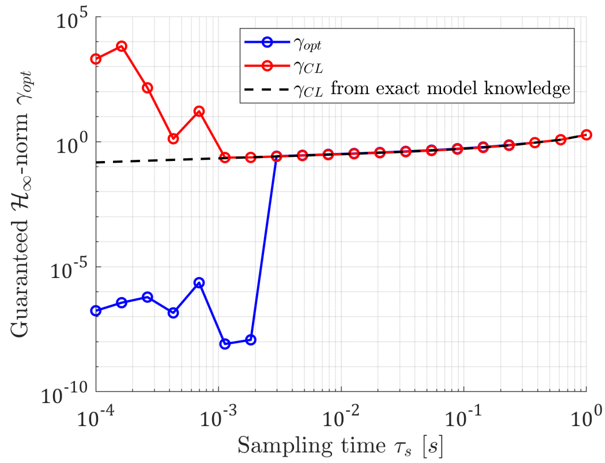


Figure 5.15: Data is scaled by a factor 100.

Unfortunately, this approach cannot be used for the controller synthesis problems in Sections 5.5.4-5.5.5 as this implies that the matrix containing the perturbation samples should be scaled as well. Accordingly, the assumed bounds on the norm and rate-of-variation should be scaled to accommodate for this scaling. Consequently, the set of systems consistent with the data grows, and hence the problem does not accurately reflect the original problem.

6 CONCLUSIONS AND RECOMMENDATIONS

In this thesis, full state feedback controllers for unknown discrete-time linear time-invariant systems were aimed to be synthesized from perturbed data directly. In Chapter 4, an LMI based framework was presented that shows how bounds on the norm and rate-of-variation of perturbation samples can be included in the synthesis. This method was subsequently applied to an academic example and to a reference tracking problem of a double pendulum system. This chapter provides concluding remarks and recommendations on future research directions.

6.1 Conclusions

The presented framework was shown to be flexible, in that multiple realistic bounds could be included in the controller synthesis. Moreover, it was shown that prior knowledge on the system matrices and its uncertain parameters could be included, making the framework compatible with the most recent literature. The method was applied to an academic example successfully in that conservatism was reduced compared to contemporary methods due to a tighter characterization of the perturbation signal. Ellipsoidal regions were introduced to reduce computational complexity compared to using separate bounds on the norm and rate-of-variation, albeit with a trade-off of increased conservatism.
Chapter 5 demonstrated the use of state augmentation to formulate reference tracking problems. It was then demonstrated how this framework could be extended to systems with mixed data-driven and model-based components, enabling $\mathcal{H}_\infty$-loop shaping through weighting filters and integral action.
A dilated LMI formulation was presented in Section 5.2, with the intent of reducing conservatism for systems with a large workspace-to-footprint and position dependent dynamics. The dilated form allows for a decoupling of the Lyapunov matrices and the system matrices between different different experiments.
LMIs for mixed $\mathcal{H}_\infty$ and generalized $\mathcal{H}_2$ optimization were presented in Section 5.3, which could be used to guarantee upper bounds on the peak of the control effort.
In Section 5.5, the methods of this thesis were applied to a reference tracking problem of a double pendulum system. Although good tracking performance was achieved for perturbed data using these methods, Section 5.5.6 revealed susceptibility to numerical issues under varying simulation parameters.

6.2 Recommendations on future research directions

In view of the findings presented in this thesis, several recommendations can be made.
It is strongly recommended to prioritize experimental validation in future studies. Rigorous real-world experiments will not only substantiate the theoretical findings, but also offer practical insights into the applicability of data-driven control methods in diverse engineering scenarios.
In light of the numerical issues identified in the simulations, a critical evaluation of available algorithms for solving semidefinite programs and their tuning parameters is advised. Increasing the robustness and reliability of these algorithms is essential for ensuring the effectiveness and

consistency of data-driven control methodologies. With this, potential challenges encountered in practical applications can be expressed.

The presented framework is used to construct multipliers for the perturbation, but could also be used to construct multipliers for the uncertainty for systems with prior knowledge. Currently, prior knowledge on uncertain parameters is characterized by norm bounds. Using the presented framework, also knowledge on the sign of an uncertain parameter could be included in the synthesis. It should be investigated to what extent this application reduces conservatism.

A possible extension of the framework involves exploring systems that have a rational dependency on the uncertain parameters, as the full-block S-procedure provides the robust framework to address uncertainties in such systems. Illustrative examples showcasing the application of this extended approach could greatly enhance the practical applicability of the proposed data-driven control methods.

In the simulations for the double pendulum systems, bounds on the norm and rate-of-variation are tuning parameters with constant values. Real mechanical systems often introduce position-dependent perturbations due to influence of nonlinear terms. The norm and rate-of-variation of such perturbations might vary significantly from sample to sample. Therefore, a possible route of exploration lies in replacing the norm and rate-of-variation tuning parameters by position-dependent functions, such as low-degree polynomials. With a good characterization of the position-dependent perturbations, conservatism in the optimization may be reduced.

In view of generalizing the results, it might be interesting to extend and investigate the effectiveness of the proposed methodology for descriptor systems. In view of this, [53] provides a variant of Willems' fundamental lemma for descriptor systems, which can be combined with the bounded real lemma for descriptor systems (e.g. [11], page 56) to design state feedback controllers for unknown descriptor systems. In [54], such an approach is used for the design of $\mathcal{H}_\infty$-optimal state feedback controllers. One can build upon this work by integrating the methods from this thesis for multiplier construction, and including prior knowledge in the synthesis.

Additionally, it should be investigated how the presented methods can be extended to linear parameter-varying systems. In [55] it is shown how Willems' fundamental lemma can be extended to capture the behaviour of an LPV system from a single experiment. This work is subsequently used in [56] to synthesize state feedback controllers that guarantee stability and performance on the generalized $\mathcal{H}_2$-norm and the $\mathcal{L}_2$-gain of the closed-loop system. However, the case of perturbed data is not considered, and hence could be a potential working direction.

Collectively, these recommendations aim to advance the robustness, applicability, and theoretical underpinnings of the proposed data-driven control framework. Such advancements might contribute to the broader adoption and effectiveness of data-driven control methodologies in diverse engineering contexts.

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A APPENDIX

A.1 Proof of the $\mathcal{H}_\infty$-norm and discrete-time bounded real lemma

Consider a discrete-time LTI system Σ with state-space representation

$$x_{k+1} = Ax_k + Hw_k, \tag{A.1a}$$

$$z_k \quad = Cx_k + Ew_k. \tag{A.1b}$$

Consider an $\epsilon > 0$ and a positive definite matrix P for which $0 \prec \underline{\lambda} I \preceq P \preceq \overline{\lambda} I$ with smallest and largest eigenvalue as $\underline{\lambda}$ and $\overline{\lambda}$ respectively. Under these conditions, it can be shown that if the Lyapunov condition

$$\frac{1}{1+\epsilon} x_k^\top P x_k - x_{k+1}^\top P x_{k+1} + \frac{\gamma}{1+\epsilon} w_k^\top w_k \geq \frac{1}{\gamma} z_k^\top z_k, \tag{A.2}$$

holds for all $k \geq 0$, then the system is stable and has $\mathcal{L}_2$-gain performance of level $\gamma > 0$. Let $\rho \triangleq \frac{1}{\sqrt{1+\epsilon}} \in (0,1)$. To prove stability, consider that $w_k = 0, \forall k \geq 0$. Then (A.2) implies

$$\rho^2 x_{k-1}^\top P x_{k-1} - x_k^\top P x_k \geq \frac{1}{\gamma} z_{k-1}^\top z_{k-1}, \forall k \geq 1. \tag{A.3}$$

Since $\gamma > 0$, this implies

$$\rho^2 x_{k-1}^\top P x_{k-1} - x_k^\top P x_k \geq 0, \forall k \geq 1. \tag{A.4}$$

Repeated exploitation of (A.4) yields

$$x_k^\top P x_k \leq \rho^2 x_{k-1}^\top P x_{k-1} \leq \rho^4 x_{k-2}^\top P x_{k-2} \leq \cdots \leq \rho^{2k} x_0^\top P x_0. \tag{A.5}$$

From the properties of P, it can be inferred that $x_k^\top P x_k \geq \underline{\lambda} x_k^\top x_k$ and $x_0^\top P x_0 \leq \overline{\lambda} x_0^\top x_0$ and as such

$$x_k^\top x_k \leq \frac{\overline{\lambda}}{\underline{\lambda}} x_0^\top x_0 \rho^k. \tag{A.6}$$

This implies the norm $x_k^\top x_k = \|x_k\|^2$ is never increasing with k and hence the system is stable.

For $\mathcal{L}_2$-gain performance, consider a perturbation signal w with $0 < \|w\|_2^2 \triangleq \sum_{k=0}^{\infty} w_k^\top w_k < \infty$. Since $\epsilon > 0$, infer from (A.2) that

$$\begin{aligned} \frac{1}{1+\epsilon} \sum_{k=0}^{\kappa} x_k^\top P x_k - \sum_{k=0}^{\kappa} x_{k+1}^\top P x_{k+1} &\leq \sum_{k=0}^{\kappa} x_k^\top P x_k - \sum_{k=0}^{\kappa} x_{k+1}^\top P x_{k+1} \\ &= x_0^\top P x_0 - x_{\kappa+1}^\top P x_{\kappa+1} \geq 0. \end{aligned} \tag{A.7}$$

Then (A.2) implies

$$x_0^\top P x_0 - x_{\kappa+1}^\top P x_{\kappa+1} + \frac{\gamma}{1+\epsilon} \sum_{k=0}^{\kappa} w_k^\top w_k \geq \frac{1}{\gamma} \sum_{k=0}^{\kappa} z_k^\top z_k. \tag{A.8}$$

Taking the limit of $\kappa \to \infty$ and setting $x_0 = 0$ implies

$$\|z_k\|_2 \leq \rho\gamma\|w_k\|_2 < \gamma\|w_k\|_2, \tag{A.9}$$

so the $\mathcal{L}_2$-gain from w_k to z_k is bounded from above by γ.

To transform (A.2) into an LMI condition, substitute (A.1) into the Lyapunov condition to obtain

$$\frac{1}{1+\epsilon}x^\top Px-(Ax+Hw)^\top P(Ax+Hw)+\frac{\gamma}{1+\epsilon}w^\top w-\frac{1}{\gamma}(Cx+Ew)^\top(Cx+Ew) \geq 0, \forall x, w, \tag{A.10}$$

where the subscripts are dropped to imply that the condition needs to be satisfied for arbitrary x and w. Rewriting (A.10) into the form

$$\begin{bmatrix} x \\ w \end{bmatrix}^\top \left(\begin{bmatrix} \frac{1}{1+\epsilon}P & 0 \\ 0 & \frac{\gamma}{1+\epsilon}I \end{bmatrix} - \begin{bmatrix} PA & PH \\ C & E \end{bmatrix}^\top \begin{bmatrix} P & 0 \\ 0 & \gamma I \end{bmatrix}^{-1} \begin{bmatrix} PA & PH \\ C & E \end{bmatrix} \right) \begin{bmatrix} x \\ w \end{bmatrix} \geq 0, \ \forall \begin{bmatrix} x \\ w \end{bmatrix}, \tag{A.11}$$

yields an equivalent LMI condition as

$$\begin{bmatrix} \frac{1}{1+\epsilon}P & 0 \\ 0 & \frac{\gamma}{1+\epsilon}I \end{bmatrix} - \begin{bmatrix} PA & PH \\ C & E \end{bmatrix}^\top \begin{bmatrix} P & 0 \\ 0 & \gamma I \end{bmatrix}^{-1} \begin{bmatrix} PA & PH \\ C & E \end{bmatrix} \succeq 0. \tag{A.12}$$

Since $\epsilon > 0$, this this reads as a strict matrix inequality expressed as

$$\begin{bmatrix} P & 0 \\ 0 & \gamma I \end{bmatrix} - \begin{bmatrix} PA & PH \\ C & E \end{bmatrix}^\top \begin{bmatrix} P & 0 \\ 0 & \gamma I \end{bmatrix}^{-1} \begin{bmatrix} PA & PH \\ C & E \end{bmatrix} \succ 0, \tag{A.13}$$

which can be turned into an LMI by applying the Schur complement from Section 2.2.3

$$\begin{bmatrix} P & 0 & * & * \\ 0 & \gamma I & * & * \\ PA & PH & P & 0 \\ C & E & 0 & \gamma I \end{bmatrix} \succ 0. \tag{A.14}$$

A.2 Proof of dilated LMIs for mixed $\mathcal{H}_\infty$ and generalized $\mathcal{H}_2$ optimization

Consider a discrete-time LTI system Σ with state-space representation

$$x_{k+1} = Ax_k + Bu_k + Hw_k, \tag{A.15a}$$
$$z_k = Cx_k + Du_k + Ew_k, \tag{A.15b}$$
$$v_k = Ux_k + Vu_k \tag{A.15c}$$

The interconnection of Σ with the state feedback controller

$$u_k = Kx_k, \tag{A.16}$$

can be described by the the closed-loop state-space system

$$x_{k+1} = (A + BK)x_k + Hw_k, \tag{A.17a}$$
$$z_k = (C + DK)x_k + Ew_k, \tag{A.17b}$$
$$v_k = (U + VK)x_k. \tag{A.17c}$$

Consider an $\epsilon > 0$ and a positive definite matrix P for which $0 \prec \underline{\lambda}I \preceq P \preceq \overline{\lambda}I$ with smallest and largest eigenvalue $\underline{\lambda}$ and $\overline{\lambda}$ respectively. Under these conditions, it can be shown that if the

Lyapunov condition in (A.2) holds, the closed-loop system is stable and has $\mathcal{H}_\infty$-performance of level γ.
To ensure generalized $\mathcal{H}_2$-performance of level σ, add a second condition to the Lyapunov condition as

$$x_k^\top P x_k \geq \frac{1}{\sigma^2} v_k^\top v_k. \tag{A.18}$$

From (A.8) it can be inherited for $\kappa \to \kappa - 1$ that

$$x_0^\top P x_0 - x_\kappa^\top P x_\kappa + \frac{\gamma}{1+\epsilon} \sum_{k=0}^{\kappa-1} w_k^\top w_k \geq \frac{1}{\gamma} \sum_{k=0}^{\kappa-1} z_k^\top z_k \geq 0,\ \kappa \geq 0. \tag{A.19}$$

With this, it is implied that

$$\frac{1}{\sigma^2} v_\kappa^\top v_\kappa \leq x_\kappa^\top P x_\kappa \leq x_0^\top P x_0 + \frac{\gamma}{1+\epsilon} \sum_{k=0}^{\kappa-1} w_k^\top w_k. \tag{A.20}$$

With zero initial conditions, $x_0 = 0$, this implies

$$\frac{1}{\sigma^2} v_\kappa^\top v_\kappa \leq \frac{\gamma}{1+\epsilon} \sum_{k=0}^{\kappa-1} w_k^\top w_k \leq \frac{\gamma}{1+\epsilon} \|w\|_2^2, \tag{A.21}$$

and hence

$$\|v\|_\infty^2 \triangleq \sup_{\kappa \geq 0} v_\kappa^\top v_\kappa \leq \frac{\gamma \cdot \sigma^2}{1+\epsilon} \|w\|_2^2 < \gamma \cdot \sigma^2 \|w\|_2^2. \tag{A.22}$$

Thus, the performance criteria will be satisfied for (A.17) if the following conditions are ensured:

$$\frac{1}{1+\epsilon} x^\top P x - x_+^\top P x_+ + \frac{\gamma}{1+\epsilon} w^\top w - \frac{1}{\gamma} z^\top z \geq 0, \text{ and } x^\top P x - \frac{1}{\sigma^2} v^\top v \geq 0,$$
$$\forall x, x_+, w, z, v : \begin{cases} (A+BK)x + Hw - x_+ & = 0 \\ (C+DK)x + Ew - z & = 0 \\ (U+VK)x \quad - v & = 0 \end{cases}, \tag{A.23}$$

where the subscripts are omitted to imply that the condition needs to be satisfied for arbitrary x,w,z and v.
In preparation for the dilated LMI formulation, introduce an invertible matrix G and $\theta = G^{-1}x$, as well as $Q = G^\top P G$ and $N = KG$. With these, (A.23) is thus re-expressed as follows:

$$\frac{1}{1+\epsilon} \theta^\top Q \theta - \theta_+^\top Q \theta_+ + \frac{\gamma}{1+\epsilon} w^\top w - \frac{1}{\gamma} z^\top z \geq 0, \text{ and } \theta^\top Q \theta - \frac{1}{\sigma^2} v^\top v \geq 0,$$
$$\forall \theta, \theta_+, w, z, v : \begin{cases} (AG+BN)\theta + Hw - G\theta_+ & = 0 \\ (CG+DN)\theta + Ew - z & = 0 \\ (UG+VN)\theta \quad - v & = 0 \end{cases}. \tag{A.24}$$

In preparation for a Schur complement, observe in reference to the system dynamics of (A.24) that

$$\text{He}((-\theta_+)^\top)((AG+BN)\theta + Hw - G\theta_+)) = 0, \tag{A.25}$$
$$z = (CG+DN)\theta + Ew, \tag{A.26}$$
$$v = (UG+VN)\theta \tag{A.27}$$

and as such the first expression in (A.24) is equivalent to

$$\frac{1}{1+\epsilon} \theta^\top Q \theta - \theta_+^\top Q \theta_+ + \text{He}((-\theta_+)^\top)((AG+BN)\theta + Hw - G\theta_+))$$
$$+ \frac{\gamma}{1+\epsilon} w^\top w - \frac{1}{\gamma}((CG+DN)\theta + Ew)^\top((CG+DN)\theta + Ew) \geq 0. \tag{A.28}$$

Rewriting (A.28) into the form

$$\begin{bmatrix}\theta\\ w\\ -\theta_+\end{bmatrix}^\top \left(\begin{bmatrix}\frac{1}{1+\epsilon}Q & 0 & *\\ 0 & \frac{\gamma}{1+\epsilon}I & *\\ AG+BN & H & G+G^\top-Q\end{bmatrix} - \begin{bmatrix}(CG+DN)^\top\\ E^T\\ 0\end{bmatrix}(\gamma I)^{-1}\begin{bmatrix}(CG+DN)^\top\\ E^T\\ 0\end{bmatrix}^\top\right)\begin{bmatrix}\theta\\ w\\ -\theta_+\end{bmatrix} \geq 0, \quad \text{(A.29)}$$

yields an equivalent LMI condition as

$$\begin{bmatrix}\frac{1}{1+\epsilon}Q & 0 & *\\ 0 & \frac{\gamma}{1+\epsilon}I & *\\ AG+BN & H & G+G^\top-Q\end{bmatrix} - \begin{bmatrix}(CG+DN)^\top\\ E^T\\ 0\end{bmatrix}(\gamma I)^{-1}\begin{bmatrix}(CG+DN)^\top\\ E^T\\ 0\end{bmatrix}^\top \succeq 0. \quad \text{(A.30)}$$

Since $\epsilon > 0$, this implies a strict inequality as

$$\begin{bmatrix}Q & 0 & *\\ 0 & \gamma I & *\\ AG+BN & H & G+G^\top-Q\end{bmatrix} - \begin{bmatrix}(CG+DN)^\top\\ E^T\\ 0\end{bmatrix}(\gamma I)^{-1}\begin{bmatrix}(CG+DN)^\top\\ E^T\\ 0\end{bmatrix}^\top \succ 0, \quad \text{(A.31)}$$

which can be rewritten as an LMI by applying the Schur complement from Section 2.2.3

$$\begin{bmatrix}Q & 0 & * & *\\ 0 & \gamma I & * & *\\ AG+BN & H & G+G^\top-Q & 0\\ CG+DN & E & 0 & \gamma I\end{bmatrix} \succ 0. \quad \text{(A.32)}$$

Substitution of (A.27) into the second condition of (A.24) yields

$$\theta^\top Q\theta - \frac{1}{\sigma^2}((UG+VN)\theta)^\top((UG+VN)\theta) = \theta^\top(Q-(UG+VN)^\top(\sigma^2 I)^{-1}(UG+VN))\theta \geq 0, \quad \text{(A.33)}$$

in which a sufficient condition is found as

$$Q-(UG+VN)^\top(\sigma^2 I)^{-1}(UG+VN) \succeq 0, \quad \text{(A.34)}$$

which can be rewritten as an LMI by applying the Schur complement

$$\begin{bmatrix}Q & *\\ UG+VN & \sigma^2 I\end{bmatrix} \succeq 0. \quad \text{(A.35)}$$