BSc Thesis Applied Mathematics

Sojourn time in bulk terminals assuming deterministic, exponential and phase type distributed vessel contents

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Abstract

In this thesis, a system is analyzed that models a bulk terminal. The main focus is to obtain performance measures such as the expectation and the variance of the sojourn time of docked vessels. This thesis is divided into three main topics. The first section assumes exponentially distributed vessel contents. The second section assumes deterministically distributed vessel contents. And the last section assumes a phase type distribution for the vessel content. In the first section, an expression for both the expected value and variance of the sojourn time is given. The second section gives the expected value of the sojourn time. And the last section proposes a distribution for the sojourn time.

Keywords: Sojourn time, Fluid queue, Queuing theory, Markov Chains, Variance, Expectation, Vessel, Bulk terminal

Introduction

Energy plays a crucial role in our modern society, and according to BP statistical data [6], oil accounted for about 31% of global energy consumption in 2021. This highlights the continued reliance on oil as a major energy source worldwide. Analyzing the waiting time for vessels transporting oil globally is of great importance. The system we will be analyzing consists of a harbor. At this harbor there are unloading ports present. Here a vessel can dock and unload its cargo. In this system, all the cargo will be fluid, either oil or something else. When a vessel arrives at a port, the captain may see both another vessel already at the port and/or other vessels waiting near the port. The captain then has to wait until it can be attached to the port. When the vessel can be attached to the port, it does so, and the vessel starts unloading its cargo. The cargo runs from the vessel to the tank through an unloading arm. This unloading arm has a maximum rate at which the fluid can flow through. Let us say that there is a strong pump attached to this unloading arm, such that the unloading arm can always unload at that current rate. Then, this cargo flows into a fluid tank ashore. This tank will fill up, possibly completely, and drain to get the fluid distributed elsewhere. When the tank is completely filled up and there is still a vessel present, it takes longer for the vessel to unload all of its cargo. A simple diagram of this model can be seen in Figure 1

For simplicity, we somewhat change grammar and meaning to better align with previous research. For example, we say that a vessel *sees* a system. Of course, a vessel itself cannot see, but we will use this sentence to mean that the captain of a vessel *sees* a sytem or another vessel. In general, a vessel will be able to see, sail, wait, dock, and unload from this point on. When we say that a system is *busy*, that means that there is already a vessel present and is being unloaded. A vessel being unloaded also can be said as a vessel being *serviced*, and therefore the vessel is *in service*.

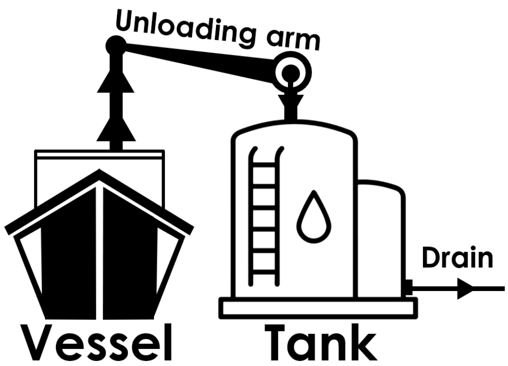


FIGURE 1: Diagram of the model

We call the entire time a vessel spends in the system the sojourn time. This includes the waiting time until the vessel can be serviced. The time it takes for a vessel to attach to the port, the unloading time, and the detaching from the port.

One of the goals could be to determine the tank's capacity to minimize the sojourn time. Also, the optimal tank's capacity could be determined that minimizes the variability in the sojourn time. Another goal could be to determine some optimal maximum rate of the unloading arm. Although this thesis does not focus on *finding* these optimal values, it does focus on the analysis of the system for later research to find these optimal values.

In the performance analysis of such systems, a fluid queue model is frequently used. This thesis focuses on an analysis of customer sojourn times in a fluid queue model. Sojourn time is defined as the waiting and service time that a customer spends in the system.

The 'classical' fluid queue model, described by Anick et al. [2], uses a Markov chain as the 'background process' that drives the content change of a buffer. Then the steady state behaviour of the buffer was analyzed. This has been done over the years by several authors [15], [20], [3], [8]. After the definition of the 'classic' fluid queue model, the *feedback* fluid queue model was proposed. This meant that the buffer content had influence on the background process [1], [18]. However, these papers primarily focus on the buffer content. In a paper by Horváth [11], the waiting time is determined for a priority queue with different types of fluids.

We assume that the proposed model uses a Poisson process for arrivals of vessels and therefore the inter-arrival times are taken from an exponential distribution. We assume that the harbor has only one place for vessels to dock. Next to that, there is no place for vessels to wait if they arrive and see the system busy. The vessel will sail away and not come back. More details of the model used can be found in Section 1.

The content of any vessel can be modelled in several ways. In this thesis we model the vessel's content in three different ways. First, in Section 3 we take an exponential distribution to model the vessel's content. In this section the expected sojourn time has already been found [17]. Thus, this section shows the derivation of the variance of the sojourn time. Secondly, we assume that the content of the vessel is the same for every vessel. This means that the corresponding distribution is deterministic. The analysis with this assumption can be found in Section 4, in which we find the distribution and expected value for the sojourn time. And finally, in Section 5, we take a general phase type distribution to model the content of any vessel. In this section an idea is given on how to find the distribution of the sojourn time.

This thesis also includes a computer simulation that uses a discrete event simulation. This approach and the results can be found in Section 6. And we end this thesis with a discussion and conclusion, which can be found in Sections 7 and 8 respectively.

1 Model Description

As introduced above, in this paper we will analyse a system that describes the unloading process of a (oil) vessel. To analyse the system, we need a model. In this section we will describe the model used. Note that this model is based on a model by Roy, Scheinhardt and Van Ommeren [17].
In this model we look at one dock with one unloading arm. We assume that the arrivals of vessels follow a Poisson process with rate λ. When a vessel arrives, and sees no one currently at the dock, it will attach to the dock. When a vessel arrives and sees already another vessel at the dock, it will sail away and not come back. We assume that the time it takes for a vessel to attach to the dock is constant for every vessel, and is zero. The same holds for the time it takes for a vessel to detach and sail away. This is because for the analysis of the sojourn time, this constant time for the attachment and detachment of the vessel does not add any complexity to the model.

When a vessel has been attached, it will immediately start unloading. The contents of the vessel are (initially) stored in a tank on the shore. This tank has a certain size in liters, let us call this $\hat{y}$. In addition, we assume that the tank drains at a constant rate independent of the presence of a vessel, let us call this d. However, the tank only unloads when there is content in the tank.

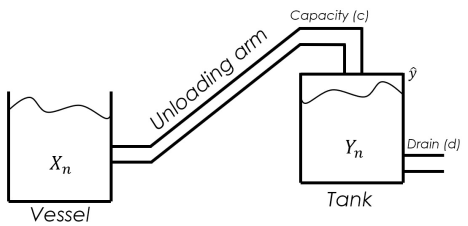


FIGURE 2: Diagram of the model [17]

As long as the tank is not completely full, the unloading of the vessel is at a certain constant rate c. When the tank is full, the unloading rate of the vessel drops to the rate of the draining rate d of the tank. We assume that the rate of unloading a vessel is much larger than the unloading rate of the tank ($c \gg d$).

In Figure 1, a diagram of the model is shown. Note that the contents of the vessel and the tank are continuous random variables. But from this point foreward we only look at these contents at the moment a vessel docks. The content of the n'th vessel that docks is called X_n and the content of the tank at that moment is called Y_n.

2 Variables, constants and general formulas

In this section, some general formulas are presented that work with all distributions for the vessel's content. These expressions have all been made in the draft of Roy, Scheinhardt and Van Ommeren [17]. Note that the aforementioned assumptions all hold.

When $n-1$ vessels have been unloaded and another vessel docks, it can be called the n'th vessel (for $n \in \mathbb{N}$). Let the content of this vessel be called X_n and the content of the tank at the moment the n'th vessel docks, Y_n. Let A_n be the residual interarrival time before the n'th arrival.

Since the draining rate is positive, when there is no vessel present and the tank is non-empty, the tank storage decreases with rate $d > 0$. And since $d \ll c$, when a vessel is present and the tank is not full, the tank storage increases with rate $c - d > 0$. When the tank is empty the increase rate is 0.

When a vessel docks, the tank can have enough room left over for the vessel's content or not. When there is enough room, the storage of the tank at the end of the unloading phase is $Y_n + \frac{c-d}{c}X_n$. When the tank is filled, the storage of the tank at the end of the unloading phase is $\hat{y}$. In the time between departure of the vessel and arrival of a new vessel (A_{n+1}) the storage drains dA_{n+1} liters. Since the tank's storage is always non-negative the recursion formula for the tank's capacity is:

$$Y_{n+1} = \left[\min(Y_n + \frac{c-d}{c}X_n, \hat{y}) - dA_{n+1}\right]^+ \tag{1}$$

With $[x]^+ = \max(0, x)$. Note that the process Y_n is a Discrete Time Markov Chain (DTMC) on $[0, \hat{y})$.

Let us define $\bar{W}_n = Y_n/d$, then:

$$\bar{W}_{n+1} = \left[\min(\bar{W}_n + \frac{c-d}{cd}X_n, \frac{\hat{y}}{d}) - A_{n+1}\right]^+ \tag{2}$$

Which is the recursive formula for the waiting time in a M/G/1 queue with bounded workload in which a customer who has a workload higher than some capacity is allowed in after removing its 'excess' workload [14].

For a bounded M/G/1 queue, the stationary waiting time distribution is the same as the conditional stationary distribution in the unbounded model, as long as it is below the bound. Let W be distributed as the stationary waiting time in a M/G/1 queue with arrival rate λ and the service time distributed as $\frac{c-d}{cd}X$. Then the stationary distribution of Y_n can be given as [17]:

$$P(Y \le y) = \frac{P(W \le y/d)}{P(W \le \hat{y}/d)} \quad \text{for } 0 \le y \le \hat{y} \tag{3}$$

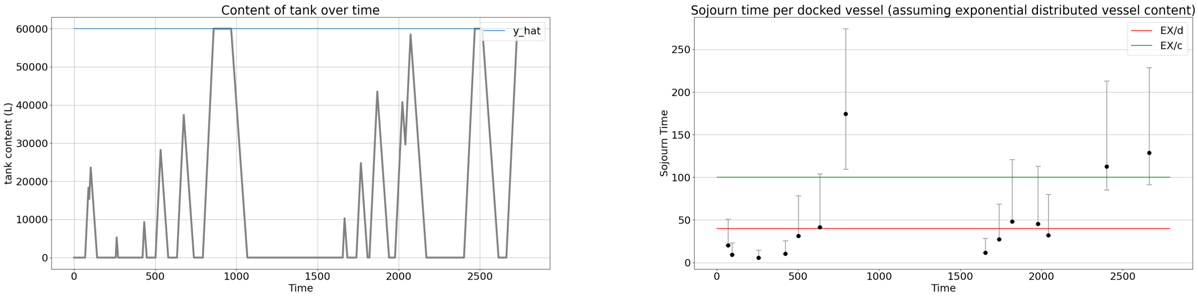
When a vessel arrives, and it's capacity (X_n) will not completely fill the tank during the unloading, then the amount in the tank will increase over X_n/c time. The net amount the tank storage would increase with during the unloading is $\frac{c-d}{c}X_n$. This means that when $Y_n + \frac{c-d}{c}X_n \le \hat{y}$ then the vessel can unload at full capacity during the whole unloading process, and the sojourn time is $S_n = X_n/c$.

However, when the vessel arrives and fills the whole tank during the unloading process, it cannot unload at full speed during the whole unloading process. Therefore, we split up the unloading time of the vessel. One part, where the vessel *can* unload at full speed. And another part in which the tank is completely full, thus the vessel can only unload at a rate of d.

In the first part the tank gets filled up with a rate of $c - d$, which takes $\frac{\hat{y}-Y_n}{c-d}$ time units. That means that the amount that the vessel unloads is $c \cdot \frac{\hat{y}-Y_n}{c-d}$. After that, the unloading of the vessel is decreased to a rate of d. Therefore, the time it takes is the residual content divided by the rate d. Adding these two together, we get: $S_n = \frac{\hat{y}-Y_n}{c-d} + \frac{X_n - c\cdot\frac{\hat{y}-Y_n}{c-d}}{d} = \frac{X_n+Y_n-\hat{y}}{d}$.

Therefore the sojourn time of a random vessel with content amount X_n is:

$$S_n = \begin{cases} \frac{X_n}{c}, & \text{for } X_n < \frac{c}{c-d}\cdot(\hat{y} - Y_n) \\ \frac{X_n+Y_n-\hat{y}}{d}, & \text{else} \end{cases} \tag{4}$$



(A) Content of the tank vs time

(B) Sojourn time per vessel vs time

Figure 3: Tank content and vessel sojourn time for an exponentially distributed vessel content model

The content of the tank over time is shown in Figure (3a). For that representation, the tank's capacity ($\hat{y}$) is chosen to be $120.000L$. In Figure 3b the sojourn time per vessel is shown. In this Figure the error bars represent the minimum and maximum sojourn time based on that vessel's content. So the upper error bar represents the sojourn time X_n/d and the lower represents X_n/c. When the dot is on the lowest end of the range, the vessel was able to completely unload at rate c. When the dot is above the lowest range, it means that for some part of the cargo it had to unload at rate d. The red line represents the average lowest sojourn time and the green line represents the average largest sojourn time. An overview of all the variables and constants used in this thesis can be found in Table 1 Lastly, note that the vessels have no waiting time. This is due to the fact that we consider no queue in this model. Therefore, the sojourn time completely equals the service time.

Variable/constant	**Description**	**Unit**
X_n	Content of the n'th docked vessel	L
Y_n	Content of the tank at the arrival of the n'th vessel	L
c	Maximum rate of unloading a vessel	L/time unit
d	Constant draining rate from tank (assuming non-empty tank)	L/time unit
$\hat{y}$	Capacity of the tank	L
λ	Arrival rate of Poisson process	1/time unit

Table 1: Variables and constant used throughout this thesis

3 Assume Exponential Distribution for vessel content

In this section we assume that the vessel's content is exponentially distributed with expected value $1/\mu$, i.e. $X \sim \text{Exp}(\mu)$. Note that the results up to and including the expected value for the sojourn time comes from [17]. The corresponding M/G/1 queue then becomes an M/M/1 queue with arrival rate λ and service times $\frac{c-d}{cd}X \sim \text{Exp}(\frac{cd}{c-d}\mu)$. Therefore, the waiting time distribution function is as follows [5]:

$$P(W \leq t) = 1 - \rho e^{\frac{-cd}{c-d}\mu(1-\rho)t} \tag{5}$$

And, using equation (3), the distribution function of the content of the tank is:

$$F_Y(y) := P(Y \leq y) = \frac{1 - \rho e^{-\gamma y}}{1 - \rho e^{-\gamma \hat{y}}} \quad \text{for } 0 \leq y \leq \hat{y} \tag{6}$$

With the two following constants:

$$\rho = \frac{\alpha}{\mu}\frac{c-d}{cd} = \alpha\left(\frac{1}{\mu d} - \frac{1}{\mu c}\right) \qquad \text{and} \qquad \gamma = \frac{cd}{c-d}\mu(1-\rho)\frac{1}{d} = \frac{c\mu}{c-d} - \frac{\alpha}{d}$$

The probability density function then is the derivative of the cumulative distribution function with respect to y:

$$f_Y(y) = \frac{d}{dy}P(Y \leq y) = \frac{\rho\gamma e^{-\gamma y}}{1 - \rho e^{-\gamma \hat{y}}}, \qquad 0 \leq y \leq \hat{y}$$

Since we want to find the expectation of the sojourn time, the Laplace Stieltjes Transform can help us. First start of with the LST of Y [17]:

$$\begin{aligned}\tilde{Y}(s) = E[e^{-sY}] &= e^{-s\cdot 0}P(Y=0) + e^{-s\cdot\hat{y}}P(Y=\hat{y}) + \int_0^{\hat{y}} \frac{\rho\gamma e^{-\gamma y}}{1-\rho e^{-\gamma\hat{y}}} e^{-sy}dy \\ &= \frac{1-\rho}{1-\rho e^{-\gamma\hat{y}}} + \frac{\rho\gamma}{\gamma+s}\frac{1-e^{-(\gamma+s)\hat{y}}}{1-\rho e^{-\gamma\hat{y}}}\end{aligned}$$

Let $B := \frac{\hat{y}-Y}{c-d}$, then the LST of B will be:

$$\tilde{B}(s) = E[e^{-sB}] = E[e^{-s\frac{\hat{y}-Y}{c-d}}] = E[e^{-\frac{s}{c-d}\hat{y}} \cdot e^{\frac{sY}{c-d}}] = e^{-\frac{s}{c-d}\hat{y}} \cdot \tilde{Y}\left(\frac{-s}{c-d}\right)$$

And by conditioning on $\frac{X}{c}$, the Laplace Stieltjes Transform of the sojourn time becomes:

$$\tilde{S}(s) = \frac{\mu c}{s+\mu c}(1 - \tilde{B}(s+\mu c)) + \frac{\mu d}{s+\mu d}\tilde{B}(s+\mu c) \tag{7}$$

Theorem 3.1. *[10] Let X be a non-negative random variable with a probability density function, that is Laplace Stieltjes transformable. Then the n'th moment of X is $E[X^n] = (-1)^n\tilde{X}^{(n)}(0)$*

The expectation of the sojourn time is equal to the first moment. Using Theorem 3.1 the expectation of the sojourn time becomes:

Lemma 3.1. *Assuming exponential distribution for the vessel size, the expected sojourn time is*

$$ES = -\tilde{S}'(0) = \frac{1}{\mu c} + \frac{(1-\rho)e^{-\gamma\hat{y}}}{1-\rho e^{-\gamma\hat{y}}}\left(\frac{1}{\mu d} - \frac{1}{\mu c}\right) = \frac{1}{\mu c} + \tilde{B}(\mu c)\left(\frac{1}{\mu d} - \frac{1}{\mu c}\right) \tag{8}$$

where $\gamma = \frac{c\mu}{c-d} - \frac{\alpha}{d}$ *and* $\rho = \alpha\left(\frac{1}{\mu d} - \frac{1}{\mu c}\right)$

From this point on in this section, we want to find the variance of the sojourn time. Note that the variance of S can be found using:

$$\text{var}(S) = E[S^2] - (E[S])^2 \tag{9}$$

In this equation the second moment is needed. This equals the second derivative of $\tilde{S}(s)$ evaluated at $s = 0$. The first derivative w.r.t. s of $\tilde{S}$:

$$\begin{aligned}
\tilde{S}'(s) &= \frac{-\mu c}{(s+\mu c)^2} + \frac{\mu c}{(s+\mu c)^2}\tilde{B}(s+\mu c) - \frac{\mu c}{s+\mu c}\tilde{B}'(s+\mu c) - \frac{\mu d}{(s+\mu d)^2}\tilde{B}(s+\mu c) + \frac{\mu d}{s+\mu d}\tilde{B}'(s+\mu c)\\
&= \frac{-\mu c}{(s+\mu c)^2} + \left(\frac{\mu c}{(s+\mu c)^2} - \frac{\mu d}{(s+\mu d)^2}\right)\tilde{B}(s+\mu c) + \left(\frac{\mu d}{s+\mu d} - \frac{\mu c}{s+\mu c}\right)\tilde{B}'(s+\mu c)
\end{aligned}$$

Continuing on this, the second derivative is found:

$$\begin{aligned}
\tilde{S}''(s) &= \frac{2\mu c}{(s+\mu c)^3} + \left(\frac{2\mu d}{(s+\mu d)^3} - \frac{2\mu c}{(s+\mu c)^3}\right)\tilde{B}(s+\mu c) + \left(\frac{\mu c}{(s+\mu c)^2} - \frac{\mu d}{(s+\mu d)^2}\right)\tilde{B}'(s+\mu c)\\
&+ \left(\frac{\mu c}{(s+\mu c)^2} - \frac{\mu d}{(s+\mu d)^2}\right)\tilde{B}'(s+\mu c) + \left(\frac{\mu d}{s+\mu d} - \frac{\mu c}{s+\mu c}\right)\tilde{B}''(s+\mu c)\\
&= \frac{2\mu c}{(s+\mu c)^3} + \left(\frac{2\mu d}{(s+\mu d)^3} - \frac{2\mu c}{(s+\mu c)^3}\right)\tilde{B}(s+\mu c) + \left(\frac{2\mu c}{(s+\mu c)^2} - \frac{2\mu d}{(s+\mu d)^2}\right)\tilde{B}'(s+\mu c)\\
&+ \left(\frac{\mu d}{s+\mu d} - \frac{\mu c}{s+\mu c}\right)\tilde{B}''(s+\mu c)
\end{aligned}$$

Evaluating $\tilde{S}''(s)$ at $s = 0$ results in the second moment, by Theorem 3.1.

$$E\left[S^2\right] = \tilde{S}''(0) = \frac{2}{(\mu c)^2} + \left(\frac{2}{(\mu d)^2} - \frac{2}{(\mu c)^2}\right)\tilde{B}(\mu c) + \left(\frac{2}{\mu c} - \frac{2}{\mu d}\right)\tilde{B}'(\mu c) \tag{10}$$

And

$$(ES)^2 = \frac{1}{(\mu c)^2} + \frac{2}{\mu c}\left(\frac{1}{\mu d} - \frac{1}{\mu c}\right)\tilde{B}(\mu c) + \left(\frac{1}{\mu d} - \frac{1}{\mu c}\right)^2\tilde{B}(\mu c)^2 \tag{11}$$

Combining both equations 10 and 11 we get the variance.

Lemma 3.2. *Assuming exponentially distributed vessel content,*

$$var(S) = \frac{1}{(\mu c)^2} + \frac{2}{\mu d}\left(\frac{1}{\mu d} - \frac{1}{\mu c}\right)\tilde{B}(\mu c) - 2\left(\frac{1}{\mu d} - \frac{1}{\mu c}\right)\tilde{B}'(\mu c) - \left(\frac{1}{\mu d} - \frac{1}{\mu c}\right)^2\tilde{B}(\mu c)^2 \tag{12}$$

with $\tilde{B}(s) = e^{-\frac{s}{c-d}\hat{y}} \cdot \tilde{Y}\left(\frac{-s}{c-d}\right)$

Note that we have the term $\tilde{B}'(\mu c)$, which is $\tilde{B}(s)$ differentiated with respect to s and then evaluated at μc. From this point on we will explicitly show what $\tilde{B}'(\mu c)$ and give an final expression for the variance of the sojourn time. Let us refresh our memories with the expressions for $\tilde{B}(s)$ and $\tilde{Y}(s)$:

$$\tilde{B}(s) = e^{-\frac{s}{c-d}\hat{y}} \cdot \tilde{Y}\left(\frac{-s}{c-d}\right) \qquad \tilde{Y}(s) = \frac{1-\rho}{1-\rho e^{-\gamma\hat{y}}} + \frac{\rho\gamma}{\gamma+s}\frac{1-e^{-(\gamma+s)\hat{y}}}{1-\rho e^{-\gamma\hat{y}}}$$

We start with the derivative of $\tilde{B}(s)$ with respect to s:

$$\begin{aligned}\frac{d}{ds}\tilde{B}(s) = \tilde{B}'(s) &= \frac{-\hat{y}}{c-d} \cdot e^{\frac{-s}{c-d}\hat{y}} \cdot \tilde{Y}\left(\frac{-s}{c-d}\right) - \frac{1}{c-d} \cdot e^{\frac{-s}{c-d}\hat{y}} \cdot \tilde{Y}'\left(\frac{-s}{c-d}\right) \\ &= \frac{-1}{c-d}\left(\hat{y} \cdot \tilde{Y}\left(\frac{-s}{c-d}\right) + \tilde{Y}'\left(\frac{-s}{c-d}\right)\right) e^{\frac{-s}{c-d}\hat{y}}\end{aligned}$$

Note that in this expression the derivative of $\tilde{Y}(s)$ plays a role. This derivative is shown below:

$$\begin{aligned}\frac{d}{ds}\tilde{Y}(s) = \tilde{Y}'(s) &= \frac{-\rho\gamma}{(\gamma+s)^2} \cdot \frac{1-e^{-(\gamma+s)\hat{y}}}{1-\rho e^{-\gamma\hat{y}}} + \rho\frac{\gamma}{\gamma+s} \cdot \frac{\hat{y}e^{-(\gamma+s)\hat{y}}}{1-\rho e^{-\gamma\hat{y}}} \\ &= \frac{-\rho\gamma}{\gamma+s}\left(\frac{1}{\gamma+s} \cdot \frac{1-e^{-(\gamma+s)\hat{y}}}{1-\rho e^{-\gamma\hat{y}}} - \frac{\hat{y}e^{-(\gamma+s)\hat{y}}}{1-\rho e^{-\gamma\hat{y}}}\right) \\ &= \frac{-\rho\gamma}{\gamma+s}\frac{1}{1-\rho e^{-\gamma\hat{y}}}\left(\frac{1}{\gamma+s}(1-e^{-(\gamma+s)\hat{y}}) - \hat{y}e^{-(\gamma+s)\hat{y}}\right) \\ &= \frac{-\rho\gamma}{\gamma+s}\frac{e^{-(\gamma+s)\hat{y}}}{1-\rho e^{-\gamma\hat{y}}}\left(\frac{e^{(\gamma+s)\hat{y}}}{\gamma+s} - \frac{1}{\gamma+s} - \hat{y}\right) \\ &= \frac{\rho\gamma}{\gamma+s} \cdot \frac{e^{-(\gamma+s)\hat{y}}}{1-\rho e^{-\gamma\hat{y}}} \cdot \frac{1+\hat{y}(\gamma+s)-e^{(\gamma+s)\hat{y}}}{\gamma+s}\end{aligned}$$

We need to evaluate these formulas at $s = \mu c$. And given that $\frac{\rho\gamma d}{\lambda} = 1-\rho$ and $\gamma - \frac{\mu c}{c-d} = -\frac{\lambda}{d}$:

$$\begin{aligned}\tilde{Y}'\left(\frac{-\mu c}{c-d}\right) &= -(1-\rho) \cdot \frac{e^{\frac{\lambda}{d}\hat{y}}}{1-\rho e^{-\gamma\hat{y}}} \cdot \frac{1-\frac{\lambda}{d}\hat{y}-e^{-\frac{\lambda}{d}\hat{y}}}{-\lambda/d} \\ &= \frac{(1-\rho)e^{\frac{\lambda}{d}\hat{y}}}{1-\rho e^{-\gamma\hat{y}}} \cdot \left(\frac{d}{\lambda} - \hat{y} - \frac{d}{\lambda}e^{-\frac{\lambda}{d}\hat{y}}\right)\end{aligned}$$

Given that

$$\hat{y} \cdot \tilde{Y}\left(\frac{-\mu c}{c-d}\right) = \hat{y}\left(\frac{1-\rho}{1-\rho e^{-\gamma\hat{y}}} - \frac{\rho\gamma d}{\lambda} \cdot \frac{1-e^{\frac{\lambda}{d}\hat{y}}}{1-\rho e^{-\gamma\hat{y}}}\right) = \frac{(1-\rho)e^{\frac{\lambda}{d}\hat{y}}}{1-\rho e^{-\gamma\hat{y}}} \cdot \hat{y}$$

That means that:

$$\hat{y} \cdot \tilde{Y}\left(\frac{-\mu c}{c-d}\right) + \tilde{Y}'\left(\frac{-\mu c}{c-d}\right) = \frac{(1-\rho)e^{\frac{\lambda}{d}\hat{y}}}{1-\rho e^{-\gamma\hat{y}}}\left(\frac{d}{\lambda} - \frac{d}{\lambda}e^{-\frac{\lambda}{d}\hat{y}}\right)$$

$$\begin{aligned}\tilde{B}'(\mu c) &= \frac{-1}{c-d}\frac{(1-\rho)e^{\frac{\lambda}{d}\hat{y}} \cdot e^{-\frac{\mu c}{c-d}\hat{y}}}{1-\rho e^{-\gamma\hat{y}}}\left(\frac{d}{\lambda} - \frac{d}{\lambda}e^{-\frac{\lambda}{d}\hat{y}}\right) \\ &= \frac{-1}{c-d}\tilde{B}(\mu c)\left(\frac{d}{\lambda} - \frac{d}{\lambda}e^{-\frac{\lambda}{d}\hat{y}}\right)\end{aligned}$$

And finally we arrive at the variance of the sojourn time.

Lemma 3.3. *Assuming exponentially distributed vessel content, the variance of the sojourn time of a docked vessel is:*

$$var(S) = \frac{1}{(\mu c)^2} + \frac{2}{c-d}\left(\frac{1}{\mu d} - \frac{1}{\mu c}\right)\left(\frac{c-d}{d}\frac{1}{\mu} + \frac{d}{\lambda} - \frac{d}{\lambda}e^{-\frac{\lambda}{d}\hat{y}}\right)\tilde{B}(\mu c) - \left(\frac{1}{\mu d} - \frac{1}{\mu c}\right)^2 \tilde{B}(\mu c)^2 \quad (13)$$

With $\tilde{B}(\mu c) = \frac{(1-\rho)e^{-\gamma\hat{y}}}{1-\rho e^{-\gamma\hat{y}}}$

And this gives shows us the final result of this section. Note that for $\hat{y} \to 0$ we get $\text{var}(S) \to \frac{1}{(\mu d)^2}$ and for $\hat{y} \to \infty$ then $\text{var}(S) \to \frac{1}{(\mu c)^2}$, which is to be expected.

Also note that if $\frac{1}{\mu} \to \infty$ then $\text{var}(S) \to \infty$, so the larger the vessel size the larger the variance of the sojourn time. In Figure 4 the relation between the variance of the sojourn time and the tank's capacity ($\hat{y}$) are shown. In Table 2 the constants used to make the graph. Another interesting result, is that the variance of the sojourn time does have a clear optimum when taken against the loadarm capacity. This is shown in Figure 5. The minimum is at $c = 1561$ at which the variance of the sojourn time is 3227.5.

A computer simulation program is written to show certain relation between parameters and variance of the sojourn time. More information on the simulation can be found in Section 6. In addition to showing possible interesting relations, the computer simulation is also used to check the formulas found. It turns out that the simulation gives the same values and relation between different parameters and the variance of the sojourn time.

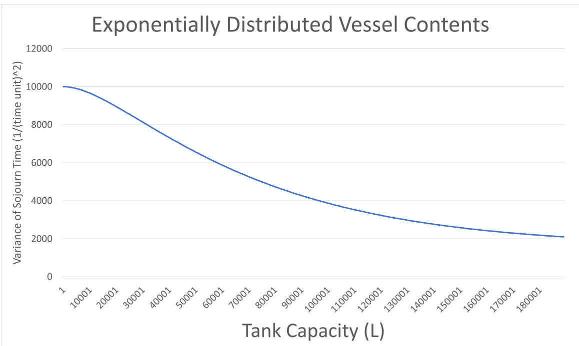


FIGURE 4: Variance of sojourn time vs tank capacity

Constant name	Value	Unit
$1/\mu$	60.000	L
c	1500	L/Time unit
d	600	L/Time unit
λ	100	Time unit
$\hat{y}$	120.000	L

TABLE 2: Table of constants

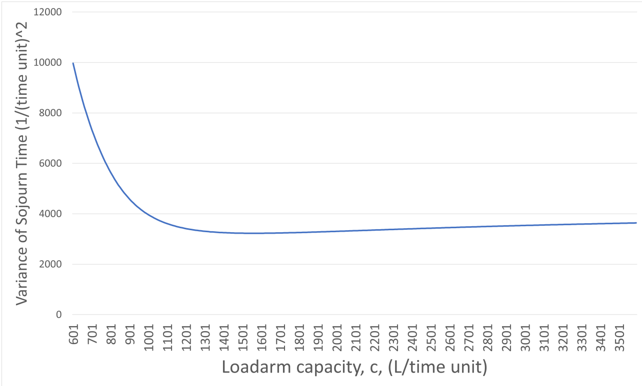


FIGURE 5: Variance of sojourn time vs loadarm capacity

4 Assume Deterministic distribution for vessel content

In this section we will assume that the content of any vessel is deterministically distributed. That means that the content of the vessel will be a value X with probability 1, and anything else with probability 0. In this case we still assume that the arrivals follow a Poisson Process with arrival rate λ. Let Y_n be the content of the tank at the arrival of the n'th docked vessel. Suppose that the tank is extremely large, then the vessel can always unload at the maximum speed. If the content of the vessel is extremely small compared to the tank size, the same holds. On the other hand, when the size of the tank is extremely small, the vessel always has to fully unload at the lower rate.

$$
\begin{aligned}
S_n &\to \frac{X}{c}, \qquad \mathrm{Var}(S_n) \to 0 \qquad \text{if } \hat{y} \to \infty \text{ or } X \ll \hat{y} \\
S_n &\to \frac{X}{d}, \qquad \mathrm{Var}(S_n) \to 0 \qquad \text{if } \hat{y} \to 0 \text{ or } X \gg \hat{y}
\end{aligned}
$$

If the tank is not extremely large, then the amount in the tank at the moment of a vessel's arrival (Y_n) determines how long it takes for the vessel to unload. We will start with the same approach to find the distribution of the tank's content as in the previous section. Let the M/G/1 queue in equation 3 be an M/D/1 queue. Meaning the arrival proces is still a Poisson process with rate λ. And suppose the service time is deterministic with value τ, with $\rho = \lambda\tau$. When we assume that the content of the vessel is always $X = 1/\mu$, the service time becomes: $\tau = \frac{c-d}{cd}\frac{1}{\mu}$. Thus, $\rho = \frac{c-d}{cd}\frac{1}{\mu} \cdot \lambda = \frac{c-d}{cd}X \cdot \lambda$. The probability of the waiting time being less than the service time is solely based on the probability of the server being busy and the probability that a new customer arrives. So for $0 < t \leq \tau$, let $f(t) = P(W < t)$:

$$f(t) = (1-\rho)e^{\lambda t} \tag{14}$$

But for any window of time between service times, as in $n\tau < t \leq (n+1)\tau$ for any non-negative integer n, $f(t)$ becomes [13]:

$$P(W < t) = e^{\lambda t}(1-\lambda\tau)\sum_{k=0}^{n(t)} e^{-\lambda k\tau}\frac{(k\tau - t)^k}{k!}\lambda^k \quad \text{for } n(t)\tau < t \leq (n(t)+1)\tau \tag{15}$$

Note that we used the term $n(t)$ which indicates that n depends on t. Let us use $\tau = \frac{c-d}{cd}X$. Then the formula becomes:

$$P(W < t) = e^{\lambda t}(1-\lambda\frac{c-d}{cd}X)\sum_{k=0}^{n(t)} e^{-\lambda k\frac{c-d}{cd}X}\frac{(k\frac{c-d}{cd}X - t)^k}{k!}\lambda^k \quad \text{for } n(t)\frac{c-d}{cd}X < t \leq (n(t)+1)\frac{c-d}{cd}X$$

$$(16)$$

We will retrieve expressions for both $P(W < y/d)$ and $P(W < \hat{y}/d)$. Substituting $t = y/d$ in equation 16 yields:

$$\begin{aligned} P(W < y/d) &= e^{\lambda y/d}(1-\rho)\sum_{k=0}^{n(y)} e^{-\lambda k \frac{c-d}{cd}X} \frac{(k\frac{c-d}{cd}X - y/d)^k}{k!}\lambda^k \quad \text{for } n(y)\frac{c-d}{cd}X < y/d \leq (n(y)+1)\frac{c-d}{cd}X \\ &= (1-\rho)\sum_{k=0}^{n(y)} e^{-\lambda(k\rho/\lambda - y/d)} \frac{(k\rho/\lambda - y/d)^k}{k!}\lambda^k \quad \text{for } n(y)\rho/\lambda < y/d \leq (n(y)+1)\rho/\lambda \end{aligned}$$

To get the expression for $P(W < \hat{y}/d)$ we define an $\hat{n}(\hat{y}) \in \mathbb{N}$ such that the following inequalities holds:

$$\hat{n}(\hat{y})\frac{c-d}{cd}X < \hat{y}/d \leq (\hat{n}(\hat{y})+1)\frac{c-d}{cd}X$$

This will result in the following expression:

$$\begin{aligned} P(W < \hat{y}/d) &= e^{\lambda \hat{y}/d}(1-\rho)\sum_{k=0}^{\hat{n}(\hat{y})} e^{-\lambda k \frac{c-d}{cd}X} \frac{(k\frac{c-d}{cd}X - \hat{y}/d)^k}{k!}\lambda^k \\ &= (1-\rho)\sum_{k=0}^{\hat{n}(\hat{y})} e^{-\lambda(k\rho/\lambda - \hat{y}/d)} \frac{(k\rho/\lambda - \hat{y}/d)^k}{k!}\lambda^k \end{aligned}$$

For numerical calculations, the expression $P(W < \hat{y}/d)$ only needs to be calculated once. We define a constant $\hat{Y}$ to be:

$$\hat{Y} := \frac{P(W < \hat{y}/d)}{1-\rho} \tag{17}$$

Where the last equation means in words, what is the probability that a ship that docks has to wait not more than the time it takes for the whole tank to unload with rate d. The constraint with it, means the following: suppose that the tank can hold between $\hat{n}$ and $\hat{n}$ amounts of vessel capacities, accounting for the constant drain of the tank. The probability of the amount in the tank then becomes:

$$\begin{aligned} P(Y \leq y) &= \frac{P(W \leq y/d)}{P(W \leq \hat{y}/d)} \\ &= \hat{Y}^{-1} \cdot e^{\lambda y/d} \cdot \sum_{k=0}^{n(y)} e^{-k\rho} \frac{(k\rho/\lambda - y/d)^k}{k!}\lambda^k \quad \text{for } n(y)\rho/\lambda < y/d \leq (n(y)+1)\rho/\lambda \\ &= \hat{Y}^{-1}\sum_{k=0}^{n(y)} e^{-\lambda(k\rho/\lambda - y/d)} \frac{(k\rho/\lambda - y/d)^k}{k!}\lambda^k \quad \text{for } n(y)\rho/\lambda < y/d \leq ((y)n+1)\rho/\lambda \\ &= \hat{Y}^{-1}\sum_{k=0}^{n(y)} e^{-\lambda(k\rho/\lambda - y/d)} \frac{(\lambda(k\rho/\lambda - y/d))^k}{k!} \quad \text{for } n(y)\rho/\lambda < y/d \leq (n(y)+1)\rho/\lambda \end{aligned}$$

When a vessel arrives (with X amount of fluid) and sees that the tank is filled with y amount, the vessel either completely fills the tank or it does not. In both cases we know for sure what the sojourn time will be. If the vessel does not fill the tank, the time it will take for the vessel to unload is exactly X/c. Then the tank would fill up with an amount $\frac{c-d}{c}X$ because of the constant unloading of the tank with rate d.

If it does fill the tank, we can split up the time in two parts. Until the tank is full it gets filled at rate $c-d$ in which $c\frac{\hat{y}-y}{c-d}$ amount of fluid is put in the tank. This takes $\frac{\hat{y}-y}{c-d}$ time. After that it takes $\frac{X-c\frac{\hat{y}y}{c-d}}{d}$ time units until the vessel is empty. The sojourn time in this case is $\frac{\hat{y}-y}{c-d}+\frac{X-c\frac{\hat{y}y}{c-d}}{d}=\frac{X+y-\hat{y}}{d}$. Since we will use the quantity $\hat{y}-\frac{c-d}{c}X$ a lot, we will call it $\hat{y}_{af}$ with af standing for *almost full.* In a probability function, this is:

$$P(S \leq s|Y=y)=\begin{cases}\mathbb{1}\left(s-\frac{X}{c}\right) & \text{if } y \leq \hat{y}-\frac{c-d}{c}X=\hat{y}_{af}\\ \mathbb{1}\left(s-\left(\frac{X+y-\hat{y}}{d}\right)\right) & \text{if } y > \hat{y}-\frac{c-d}{c}X=\hat{y}_{af}\end{cases}$$

Let $F_Y(y) := P(Y \leq y)$, then (for $s \geq 0$) the cumulative distribution function for the sojourn time is shown as:

$$P(S \leq s) = P(S \leq s|Y=\hat{y})P(Y=\hat{y}) + P(S \leq s|Y=0)P(Y=0) + \int_0^{\hat{y}} P(S \leq s|Y=y)\cdot f_Y(y)dy$$

The quantity $P(Y=\hat{y})$ means, what is the probability that the tank is at full capacity at the moment a vessel docks. In this model we do not have a queue, therefore the tank is always not full at the moment a vessel docks. So $P(Y=\hat{y})=0$. Let us split the integral of the previous equation into two parts. One where there is enough room in the tank for the vessel to unload completely at rate c, and one where the tank is too full.

$$\begin{aligned}
&= P(S \leq s|Y=0)\cdot F_Y(0) + \int_0^{\hat{y}_{af}} \mathbb{1}\left(s-\frac{X}{c}\right)\cdot f_Y(y)dy + \int_{\hat{y}_{af}}^{\hat{y}} \mathbb{1}\left(s-\left(\frac{X+y-\hat{y}}{d}\right)\right)\cdot f_Y(y)dy\\
&= \mathbb{1}\left(s-\frac{X}{c}\right)F_Y(0) + \mathbb{1}\left(s-\frac{X}{c}\right)\left(F_Y(\hat{y}_{af})-F_Y(0)\right) + \int_{\hat{y}_{af}}^{\hat{y}} \mathbb{1}\left(-y+(\hat{y}+ds-X)\right)f_Y(y)dy\\
&= \mathbb{1}\left(s-\frac{X}{c}\right)F_Y\left(\hat{y}_{af}\right) + \begin{cases}\int_{\hat{y}_{af}}^{\hat{y}+ds-X} f_Y(y)dy & \text{if } \frac{X}{c} < s < \frac{X}{d}\\ \int_{\hat{y}_{af}}^{\hat{y}} f_Y(y)dy & \text{if } s \geq \frac{X}{d}\end{cases}\\
&= \mathbb{1}\left(s-\frac{X}{c}\right)F_Y\left(\hat{y}_{af}\right) + \begin{cases}F_Y(\hat{y}+ds-X)-F_Y\left(\hat{y}_{af}\right) & \text{if } \frac{X}{c} < s < \frac{X}{d}\\ F_Y(\hat{y})-F_Y\left(\hat{y}_{af}\right) & \text{if } s \geq \frac{X}{d}\end{cases}\\
&= \begin{cases}0 & \text{if } 0 \leq s < \frac{X}{c}\\ F_Y(\hat{y}+ds-X) & \text{if } \frac{X}{c} \leq s < \frac{X}{d}\\ F_Y(\hat{y}) & \text{if } s \geq \frac{X}{d}\end{cases}
\end{aligned}$$

The interpretation of $F_Y(\hat{y})$ is when a vessel arrives, what is the probability that the tank is less than completely full. Since we do not have a queue, this probability is 1.

Theorem 4.1. *Assume a deterministic vessel content distribution with parameter X. Then the distribution of the sojourn time is given by:*

$$P(S \leq s) = \begin{cases} 0 & \text{if } 0 \leq s < \frac{X}{c} \\ F_Y(\hat{y} + ds - X) & \text{if } \frac{X}{c} \leq s < \frac{X}{d} \\ 1 & \text{if } s \geq \frac{X}{d} \end{cases} \tag{18}$$

Where

$$F_Y(y) = P(Y \leq y) = \hat{Y}^{-1} \sum_{k=0}^{n(y)} e^{-\lambda(k\rho/\lambda - y/d)} \frac{(\lambda(k\rho/\lambda - y/d))^k}{k!} \quad \text{for } n(y)\rho/\lambda < y/d \leq (n(y)+1)\rho/\lambda$$

The first approach of calculating the expected value of the sojourn time is solely using the cumulative distribution function. For any non-negative random variable X, the expectation can be calculated using [12]:

$$EX = \int_0^\infty (1 - F_X(x))dx \tag{19}$$

Let us define $F_S(s) := P(S \leq s)$. Then $ES = \int_0^\infty (1 - F_S(s))ds$:

$$\begin{aligned} ES &= \int_0^{\frac{X}{c}} (1-0)ds + \int_{\frac{X}{c}}^{\frac{X}{d}} (1 - F_Y(\hat{y} + ds - X))ds + \int_{\frac{X}{d}}^\infty (1-1)ds \\ &= \frac{X}{c} + \frac{1}{d} \int_{\hat{y}_{af}}^{\hat{y}} (1 - F_Y(\tilde{y}))d\tilde{y} \end{aligned}$$

Note that $\hat{y}_{af}$ could be less than $\hat{n}\frac{c-d}{c}X$. This is shown in Figure 6. Therefore we need to split up the integral into two parts.

$$ES = \frac{X}{c} + \frac{1}{d} \int_{\hat{y}_{af}}^{\hat{n}\frac{c-d}{c}X} (1 - F_Y(\tilde{y}))\, d\tilde{y} + \frac{1}{d} \int_{\hat{n}\frac{c-d}{c}X}^{\hat{y}} (1 - F_Y(\tilde{y}))\, d\tilde{y}$$

In the continuation of this section, we need the anti-derivative of the expression for $P(Y \leq y)$. We start of with the definition of the Gamma function [7], the Lower Incomplete Gamma function, [19], and the Lower Incomplete Gamma function for negative second variable [9].

Definition 4.1. *For any $s \in \mathbb{C}$ such that $Re(s) > 0$ the* ***Gamma function*** *is*

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt \tag{20}$$

Definition 4.2. *For any $z, \alpha \in \mathbb{C}$, such that $|arg(\alpha)| < \pi$ and $Re(z) > 0$ the* ***Lower incomplete gamma function*** *is*

$$\gamma(\alpha, z) = \int_0^z t^{\alpha-1} e^{-t} dt \tag{21}$$

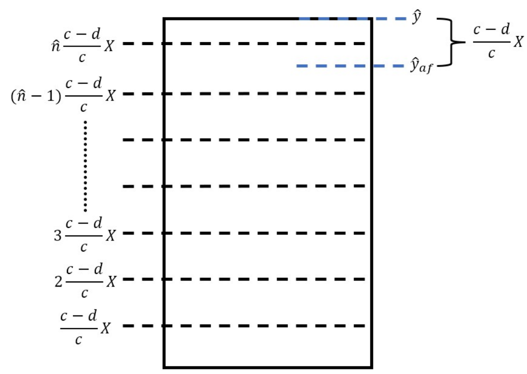


FIGURE 6: Diagram of the tank subdivided into sections of length $\frac{c-d}{c}X$ from the bottom up

Definition 4.3. *Let $a, z \in \mathbb{R}$ and such that $a > 0$ and is bounded, $z > 0$. Then the* ***Lower Incomplete Gamma function for negative second variable*** *is*

$$\gamma^*(a,-z) \sim \frac{e^z}{z\Gamma(a)} \sum_{n=0}^{\infty} \frac{(1-a)_n}{z^n} \tag{22}$$

Where $(a)_n$ is the rising factorial:

$$(a)_n = a(a+1)(a+2)\dots(a+n-1) = \prod_{k=1}^{n-1}(a+k)$$

Let us look at the first integral of the previous expression:

$$\frac{1}{d}\int_{\hat{y}_{af}}^{\hat{n}\frac{c-d}{c}X} (1-F_Y(\tilde{y}))\,d\tilde{y} = \frac{1}{d}\int_{\hat{y}_{af}}^{\hat{n}\frac{c-d}{c}X} \left(1 - \hat{Y}^{-1}\sum_{k=0}^{\hat{n}-1} e^{-\lambda(k\rho/\lambda - \tilde{y}/d)} \frac{(\lambda(k\rho/\lambda-\tilde{y}/d))^k}{k!}\right) d\tilde{y}$$

Since we let $\tilde{y}$ run from $\hat{y}_{af}$ until $\hat{n}\frac{c-d}{c}X$ this is less than one length of the service time. Since we used $\hat{n}$ for the definition of $\hat{Y}$, we will use it again here since we run over the same block of time. Note that for all values $k = 0, 1, \dots, \hat{n}-1$, we have $\lambda(k\rho/\lambda - \tilde{y}/d) < 0$. Firstly we split the integrand into two seperate integrals. Then we use the fact that if $\tilde{y}$ runs between $\hat{y}_{af}$ and $\hat{n}\frac{c-d}{cd}X$, then the n for the summation does not change. Therefore we can take the summation out of the integral, and perform a change of variables.

$$\begin{aligned}
&= \frac{1}{d}\int_{\hat{y}_{af}}^{\hat{n}\frac{c-d}{c}X} 1 d\tilde{y} - \frac{1}{d}\sum_{k=0}^{\hat{n}-1}\int_{\lambda((k+1)\frac{c-d}{cd}X - \hat{y}/d)}^{\lambda(\frac{c-d}{cd}X(k-\hat{n}))} e^{-x}\cdot\frac{x^k}{k!}dx \\
&= \frac{1}{d}\left(\hat{n}\frac{c-d}{c}X - \hat{y}_{af}\right) \\
&- \frac{1}{d}\sum_{k=0}^{\hat{n}-1}\frac{1}{k!}\left(\gamma^*\left(k+1, -\lambda\left((\hat{n}-k)\frac{c-d}{cd}X\right)\right) - \gamma^*\left(k+1, -\lambda\left(\hat{y}/d - (k+1)\frac{c-d}{cd}X\right)\right)\right)
\end{aligned}$$

Similarly for the other integral we get:

$$\frac{1}{d}\int_{\hat{n}\frac{c-d}{c}X}^{\hat{y}} (1 - F_Y(\tilde{y}))\,d\tilde{y} = \frac{1}{d}\left(\hat{y} - \hat{n}\frac{c-d}{c}X\right)$$
$$- \frac{1}{d}\sum_{k=0}^{\hat{n}} \frac{1}{k!}\left(\gamma^*\left(k+1, -\lambda\left(\hat{y}/d - k\frac{c-d}{cd}X\right)\right) - \gamma^*\left(k+1, -\lambda\Big((\hat{n}-k)\frac{c-d}{cd}X\Big)\right)\right)$$

And finally the expression for the expectation becomes:

$$ES = \frac{X}{c} + \frac{c-d}{cd}X + \frac{1}{d}\sum_{k=0}^{\hat{n}-1}\frac{1}{k!}\left(\gamma^*\left(k+1, -\lambda\left(\hat{y}/d - (k+1)\frac{c-d}{cd}X\right)\right) - \gamma^*\left(k+1, -\lambda\left(\hat{y}/d - k\frac{c-d}{cd}X\right)\right)\right)$$
$$+ \frac{1}{\hat{n}!}\cdot\gamma^*\left(\hat{n}+1, -\lambda\left(\hat{y}/d - \hat{n}\frac{c-d}{cd}X\right)\right)$$

Let us define $Y_k := \lambda(\hat{y}/d - k\frac{c-d}{cd}X)$ then

Lemma 4.1. *Assuming deterministic vessel contents the expectated sojourn is given by:*

$$ES = \frac{X}{d} + \frac{1}{d\cdot\hat{n}!}\gamma^*\left(\hat{n}+1, -Y_{\hat{n}}\right) + \frac{1}{d}\sum_{k=0}^{\hat{n}-1}\gamma^*(k+1, -Y_{k+1}) - \gamma^*(k+1, -Y_k) \tag{23}$$

With

$$Y_K = \lambda(\hat{y}/d - k\frac{c-d}{cd}X)$$

And $\gamma^*(k+1, -Y_K)$ as defined in Definition 4.3

Let us take a closer look at the expression $\gamma^*(k+1, -Y_{k+1}) - \gamma^*(k+1, -Y_k)$. Let us create two new variables A_k and B_k to be:

$$A_k = \frac{e^{Y_{k+1}}}{Y_{k+1}\cdot\Gamma(k+1)}$$
$$B_k = \frac{e^{Y_k}}{Y_k\cdot\Gamma(k+1)}$$

Note that for all finite values of $\hat{y}, c, d, \lambda, X$ these two expressions are finite. We now get a new expression:

$$\gamma^*(k+1, -Y_{k+1}) - \gamma^*(k+1, -Y_k) = A_k\sum_{i=0}^{\infty}\frac{(k)_i}{Y_{k+1}^i} - B_k\sum_{j=0}^{\infty}\frac{(k)_j}{Y_k^j}$$
$$= \sum_{i=0}^{\infty}\frac{(k)_i\left(A_kY_k^i - B_kY_{k+1}^i\right)}{Y_k^iY_{k+1}^i}$$

Which gets us to a new expression for the expectation:

Lemma 4.2.

$$ES = \frac{X}{d} + \frac{1}{d \cdot \hat{n}!}\gamma^*\left(\hat{n}+1, -Y_{\hat{n}}\right) + \frac{1}{d}\sum_{k=0}^{\hat{n}-1}\sum_{i=0}^{\infty}\binom{k+i-1}{i}\left(\frac{e^{Y_{k+1}}}{Y_{k+1}^{i+1}} - \frac{e^{Y_k}}{Y_k^{i+1}}\right) \quad (24)$$

The convergence, and therefore the conditions of convergence has to be determined.

Let us continue to the variance of the sojourn time. To get an expression we start with the second moment of the sojourn time:

$$\begin{aligned}
E\left[S^2\right] &= 2\int_0^{\infty} s(1-F_S(s))ds \\
&= 2\int_0^{\frac{X}{c}} s(1-0)ds + 2\int_{\frac{X}{c}}^{\frac{X}{d}} s(1-F_Y(\hat{y}+ds-X))ds + 2\int_{\frac{X}{d}}^{\infty} s(1-1)ds \\
&= \left(\frac{X}{c}\right)^2 + \frac{1}{d^2}\int_{\hat{y}_{af}}^{\hat{y}} (\tilde{y}-\hat{y}+X)(1-F_Y(\tilde{y}))d\tilde{y} \\
&= \left(\frac{X}{c}\right)^2 + \frac{1}{d^2}\int_{\hat{y}_{af}}^{\hat{y}} \Big(\tilde{y} - \tilde{y}F_Y(\tilde{y}) - \hat{y} + \hat{y}F_Y(\tilde{y}) + X - XF_Y(\tilde{y})\Big)d\tilde{y} \\
&= \left(\frac{X}{c}\right)^2 + \frac{1}{d^2}(X-\hat{y})(\hat{y}-\hat{y}_{af}) + \frac{1}{d^2}\int_{\hat{y}_{af}}^{\hat{y}} \Big(\tilde{y} - \tilde{y}F_Y(\tilde{y}) + \hat{y}F_Y(\tilde{y}) - XF_Y(\tilde{y})\Big)d\tilde{y} \\
&= \left(\frac{X}{c}\right)^2 - \frac{1}{d^2}\frac{c-d}{c}X(\hat{y}-X) + \frac{1}{d^2}\int_{\hat{y}_{af}}^{\hat{y}} \tilde{y}(1-F_Y(\tilde{y}))d\tilde{y} + \frac{1}{d^2}(\hat{y}-X)\int_{\hat{y}_{af}}^{\hat{y}} F_Y(\tilde{y})d\tilde{y}
\end{aligned}$$

This expression is too complex for the scope of this thesis. Therefore the variance of the sojourn time in this section will not be expressed.

In Secion 6, it will be shown what the expected sojourn time is in relation to several parameters given by the computer simulation. Since the computer simulation corresponded to the relations given in Section 3, the correctness of the computer simulation is granted. In this way, numerical values can be found for the expectation of the sojourn time. But due to the complexity of the analytical expression of the expectation of the sojourn time, we could not determine the correctness of the expressions numerically. Which we were able to do in the case of the assumption of exponentially distributed vessel contents. Similarly, the analytical expression for the variance of the sojourn time was too complex to get and therefore be checked numerically. The variance of the sojourn can however be looked at numerically using the computer simulation.

5 Assume Phase Type Distribution for vessel content

If we want to get more grip on the distribution of the content of the vessel, a phase type distribution is used. This section is also based on the draft [17]. This is up until Theorem 5.1.

Suppose the content of the vessel has a phase type distribution. Let X be the content of the vessel, then $X \sim \text{PH}(\boldsymbol{m}, \boldsymbol{M})$ on a phase space $\mathcal{M} = \{1, \dots, m\}$. Thus, X is distributed as the time it spends in this system until it reaches an absorbing state. This system is a Markov process which moves according to a transient generator matrix M. The probability of starting in phase i is denoted by the i'th component of $\boldsymbol{m}$.

While the tank is not full, the background process moves with a generator matrix Q:

$$Q = \begin{bmatrix} -\alpha & \alpha \boldsymbol{m} \\ -c\boldsymbol{Me} & c\boldsymbol{M} \end{bmatrix}$$

When the tank is full, the generator matrix changes. Since there does not exists a queue of vessels, when the tank is full there cannot arrive a new vessel. This is because when a previous vessel leaves, there always exists some time before the next vessel arrives. When the state is not 0 (there is a ship present) and the tank is full, the background process moves with generator matrix $\hat{Q}$:

$$\hat{Q} = \begin{bmatrix} 0 & 0 \\ -d\boldsymbol{Me} & d\boldsymbol{M} \end{bmatrix}$$

Lastly, the rate of change of the tank is given by matrix R. When the tank is not empty, and the state is 0 (there is no ship present) the rate of change is $-d$. When the tank is not empty and there is a ship present, the content of the tank increases with rate $c - d$.

$$R = \begin{bmatrix} -d & \\ & (c-d)\boldsymbol{I} \end{bmatrix}$$

Let the stationary content of the tank be denoted by Y and the state of the background process ϕ. We introduce two vectors of probabilities that denote the stationary distribution:

$$\begin{aligned} \boldsymbol{F}(y) &= (F_0(y), F_1(y), \dots, F_m(y)) \quad \text{where} \quad F_i(y) = P(\phi = i, Y \le y), \quad i = 0, 1, \dots, m, 0 \le y \le \hat{y} \\ \boldsymbol{p} &= (p_0, p_1, \dots, p_m) \quad \text{where} \quad p_i = P(\phi = i, Y = \hat{y}), \quad i = 1, 2 \dots, m \end{aligned}$$

From feedback fluid queue theory it is known that the following differential equation holds:

$$\boldsymbol{F}'(y)R = \boldsymbol{F}(y)Q$$

Solving this differential equation yields the following:

Theorem 5.1. *For model 1 with phase-type vessel size distribution, the stationary distribution of the joint background/storage process is given by the vector* $\boldsymbol{p}$ *and the vector function:*

$$\boldsymbol{F}(y) = \boldsymbol{F}(0)e^{QR^{-1}y} = \sum_{k=0}^{m} c_k \boldsymbol{v}_k e^{\lambda_k y}, \quad 0 \le y < \hat{y} \tag{25}$$

where the pairs $(\lambda_k, \boldsymbol{v}_k)$ *are the eigenvalue/eigenvector pairs satisfying* $\boldsymbol{v}_k QR^{-1} = \lambda_k \boldsymbol{v}_k$ *and* c_k *are coefficients.*

The $2m + 2$ unknowns p_i and c_k can be determined using the following boundary conditions:

$$\begin{aligned} F_i(0) &= 0, \quad i = 1, \dots, m \\ p_0 &= 0, \\ \boldsymbol{F}'(\hat{y}-)R &= -\boldsymbol{p}\hat{Q}, \\ (\boldsymbol{F}'(\hat{y}-) + \boldsymbol{p})\boldsymbol{e} &= 1 \end{aligned}$$

Distribution of the sojourn time

To calculate the variance of the sojourn time in this system, we want to determine the distribution of the sojourn time. At the moment a vessel arrives the phase type process 'starts'. Let S be the random variable for the sojourn time when a ship arrives and docks. Since we will first determine the sojourn time, conditioned on that there is a vessel present, and that the tank is not full, the generator matrix for the background process is as follows:

$$\boldsymbol{Q} = \begin{bmatrix} 0 & 0 \\ -c\boldsymbol{M}\boldsymbol{e} & c\boldsymbol{M} \end{bmatrix} \tag{26}$$

When a vessel is present, and the tank is full, the background process changes. The generator matrix for this new background process is given by:

$$\hat{\boldsymbol{Q}} = \begin{bmatrix} 0 & 0 \\ -d\boldsymbol{M}\boldsymbol{e} & d\boldsymbol{M} \end{bmatrix} \tag{27}$$

Let us condition on the content of the tank Y and the phase the phase type distribution is in when it arrives. And let $f_{Y,\phi}(y,i) = \frac{d}{dy}F_i(y)$. Then the distribution for the sojourn time is shown as:

$$P(S \leq s) = \frac{1}{1 - P(\phi = 0)} \left(\sum_{i=1}^{m} \Bigg(P(S \leq s | \phi = i, Y = 0) f_{Y,\phi}(0,i) \cdot P(\phi = i) \right.$$
$$\left. + \int_0^{\hat{y}-} P(S \leq s | \phi = i, Y = y) f_{Y,\phi}(y,i) \cdot P(\phi = i) dy \Bigg) \right)$$

Which leads us to the following expression:

$$P(S \leq s) = \frac{1}{1 - m_0} \left(\sum_{i=1}^{m} \Bigg(P(S \leq s | \phi = i, Y = 0) f_{Y,\phi}(0,i) \cdot m_i + \int_0^{\hat{y}-} P(S \leq s | \phi = i, Y = y) f_{Y,\phi}(y,i) \cdot m_i dy \Bigg) \right) \tag{28}$$

The term $1/(1 - P(\phi = 0))$ makes sure that this distribution runs from 0 to 1. When a vessel docks the moment right after we know for sure that the phase is not 0 anymore. However, the way that $F_i(y)$ is defined, the case of $\phi = 0$ is included. Therefore, we need to compensate in this formula. Note that the probability of the tank being completely full when a ship arrives is 0. This is because in this system there is no queue. That means that there always exists some time between the departure of a vessel and the arrival of a new one. During this time the tank empties continuously, and therefore the tank is not full. Thus, $P(Y = \hat{y}) = 0$.

Note that $P(\phi = i) = m_i$ where m was the probability vector that denotes the probability that the phase i will begin at the moment of arrival. An expression for $f_{Y,\phi}(y,i)$ is needed and given below:

$$\boldsymbol{F}(y)' = \frac{d}{dy}\boldsymbol{F}(y) = \frac{d}{dy}\left(\sum_{k=0}^{m} c_k \boldsymbol{v}_k e^{\lambda_k y} \right) = \sum_{k=0}^{m} c_k \boldsymbol{v}_k \lambda_k e^{\lambda_k y}$$

Thus

$$f_{Y,\phi}(y,i) = \frac{d}{dy}F_i(y) = \sum_{k=0}^{m} c_k v_{k,i} \lambda_k e^{\lambda_k y}$$

Where $v_{k,i}$ is the i'th component of the k'th eigenvector of QR^{-1}.

Suppose that some $\tau \sim PH(\boldsymbol{\pi}, \boldsymbol{T})$. Here $\boldsymbol{\tau}$ is a vector representing the probabilities of starting in a certain phase. And $\boldsymbol{T}$ is a matrix that describes the phase type process. hen the distribution function F of τ is given by: [4]:

$$F(s) = P(\tau \leq s) = 1 - \boldsymbol{\pi} \exp(\boldsymbol{T}s)\boldsymbol{e} \tag{29}$$

Suppose that the beginning state is known, i, and the content of the tank is also known, y. Then the sojourn time at the moment that a vessel arrives has a phase type distribution. Namely $S \sim \mathrm{PH}(\mathbf{1}_i, \boldsymbol{Q})$, where $\mathbf{1}_i$ is a row vector filled with zeros except on place i where it has a 1. And $\boldsymbol{Q}$ is defined as in equation 26. While the content of the vessel does not exceed the room that is left in the tank ($s < \hat{y} - y$), then we can say the following:

$$P(S \leq s | \phi = i, Y = y) = 1 - \mathbf{1}_i \exp(\boldsymbol{Q}s)\boldsymbol{e} \quad 0 \leq y < \hat{y} \quad \text{and} \quad 0 \leq s < \frac{\hat{y} - y}{c - d} \tag{30}$$

When the content of the vessel *does* exceed the room that is left in the tank ($s > (\hat{y} - y)/(c - d)$), then the generator matrix for the background process changes. From this point on we give a suggestion to how this derivation should go on. Suppose that the tank becomes full at the moment a vessel is present. Then we know for sure that the phase type process has run at least for some time. The amount of time is known, this is because we condition on the amount in the tank at the moment of arrival. Now there needs to be a factor which denotes the probability of being in phase j after t time given that the starting phase was i. Generally this probability is known as the transition probability function $P_{ij}(t)$ of a continuous time Markov Chain [16]. Finding these transition probabilities tends to be quite complex.

Suppose these $P_{ij}(t)$ have been found. Then we can more easily condition on the phase the background process is in after t time. When the tank is full the background process changes and the generator matrix becomes $\hat{\boldsymbol{Q}}$ as defined in Equation 27.

$$P(S \leq s | \phi = i, Y = y) = \frac{1}{1 - m_0} \sum_{j=1}^{m} \left(1 - \mathbf{1}_j \exp\left(\hat{\boldsymbol{Q}} \cdot \frac{s + y - \hat{y}}{c - d}\right)\boldsymbol{e}\right) P_{ij}\left(\frac{\hat{y} - y}{c}\right) \quad \text{for } s > \frac{\hat{y} - y}{c}$$

Because we condition on the phase the background process is in at the moment the tank becomes full, we can view the extra time the vessel spends after as a new and independent process.

6 Simulation

For two cases mentioned in this thesis, we have made a simulation of the sojourn time. We simulated the exponentially distributed vessel content and the deterministically distributed vessel content. We used a discrete event simulation to retrieve an unbiased estimator for the expectation and variance of the sojourn time.

The simulation was built up using a vessel class and a system class. The flow chart of a vessel arriving is shown in Figure 7. The system class keeps track of all the performance measures. These are for example the waiting times of the vessels over time, the content of the tank over time and the amount of vessels seeing an empty system.

On first look, there is a problem because of the dependence of the values. The sojourn time of a vessel is dependent on the arrival time and content of the previous vessels. However, calculating the expectation will not be a problem. This is because of the following. Let $X_1, X_2, \dots, X_n$ be dependent, identically distributed random variable. Then we have the following expression: $\bar{X} = \frac{1}{n}(X_1 + X_2 + \cdots + X_n)$ and

$$\frac{1}{n} \cdot E(X_1 + X_2 + \cdots + X_n) = \frac{1}{n} \cdot \sum_{i=1}^{n} EX_i \tag{31}$$

shows that the sum of expectations is a linear operator [16]. But the variance of these different random variables also depends on the covariance between the two. Suppose that the random variables are the data points. Then the random variables are 'in order' in terms of time stamps. It is reasonable to assume that the n'th observation has close to no dependence with the first observation. This idea of decreasing covariance of the observations based on the time is a subject of Time Series Analysis. In this thesis we will not go further in depth in this side of mathematical statistics. But we can conclude that the variance of a single random variable can be well estimated when using enough data points.

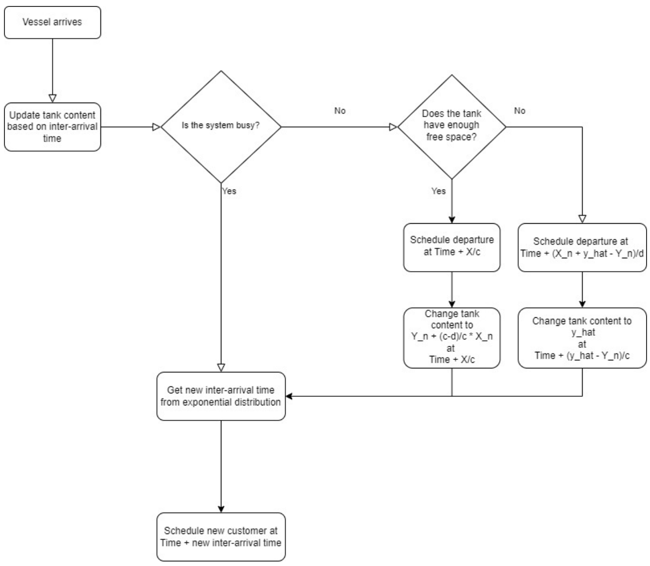


FIGURE 7: Flow chart describing the vessel class in the computer simulation

Using the constants shown in Table 3, we were able to get numerical results that line up with the expressions derived in this thesis. For the case that the vessel content is exponentially distributed, the 95% confidence interval for both the expectation as the variance for the sojourn time are given in Table 8a. For the case that the vessel contents are deterministically distributed, the values are given in Table 8b.

Constant name	**Value**	**Unit**
$\hat{y}$	120.000	L
$1/\mu$	60.000	L
c	1500	L/Time unit
d	600	L/Time unit
λ	100	Time unit
Amount of runs	300	Runs
Amount of vessels per run	400	Vessels

TABLE 3: Values used for the discrete event simulation

Measure	**CI (95%)**	**Middle**
ES	$[47.456; 48.396]$	47.926
$\text{var}(S)$	$[3935.03; 4221.178]$	4078.11

(A) Assuming exponentially distributed vessel content, 95% confidence interval for the expectation and variance

Measure	**CI (95%)**	**Middle**
ES	$[40.123; 40.145]$	40.13
$\text{var}(S)$	$[3.981; 4.869]$	4.4248

(B) Assuming deterministically distributed vessel content, 95% confidence interval for the expectation and variance

FIGURE 8: Numerical results from simulation

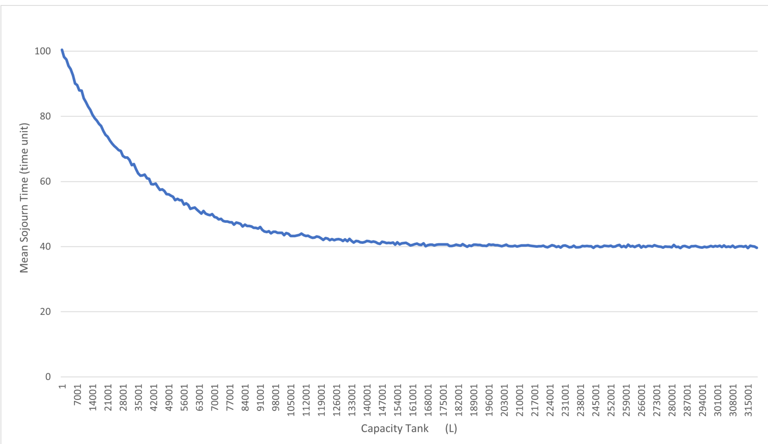


FIGURE 9: Mean waiting time over $\hat{y}$. Assuming exponentially distributed vessel content. With $1/\mu = 60.000$, $c = 1500$, $d = 500$, $1/\lambda = 100$

7 Discussion

In this thesis, we have looked at three different approaches, i.e. different assumptions. In this section we will highlight a few points which could be further looked at for this thesis or future research.

In the section assuming the exponentially distributed vessel contents, a new expression was found for the variance of the sojourn time. The quantities within the expression could be further looked into. On first hand, we could not see a clear interpretation of all the terms.

The second section assumed a deterministic distribution for the vessel content. Although the choice of using a deterministic distribution made the model less complex, the probability function did not. The biggest issue is that the quantity $k\rho/\lambda - y/d$ is negative for all terms up until $k = n - 1$. This arose a serious problem when using the lower incomplete gamma function. therefore we used a different expression for the lower incomplete gamma function. Under what conditions the sum converges is still to be determined.

Thirdly, the section assuming phase type distributed vessel content. Although we tried various methods of solving the problem, we could not give a answer to what the distribution is for the sojourn time. In the not published paper by Roy, Scheinhardt and Van Ommeren, an expression is given for the expectation. But that was using an alternative approach. This approach did not need the distribution of the sojourn time. In this section, an approach is given for future research to correctly work out the distribution.

Lastly the computer simulation. The variance of the sojourn time has to be more accurately determined. This is because the sample sojourn times that are taken near each other have dependence on each other. Using the fact that the dependence and correlation strongly tend to zero as the distance between the moments the values are taken, one could look into time series analysis. This is a facet of statistical analysis that deals with these types of situations. Loosely speaking, when the decay of the dependence is very strong, we can approximate the variance quite well. This is only if the amount of samples is sufficiently high.

We end this section with thoughts about future research. When expressions are found for each of the methods described above, some things can be done with them. The first thing that comes to mind is optimizing. In each case the optimal value(s) could be determined to have the lowest expected sojourn time while also minimizing the variance of the sojourn time. For future research, the phase type distribution could be used to model a wide range of distributions. When we take only one phase in the phase space, it models an exponential distribution. When we increase the amount of phases approaching infinity, the phase type distribution approximates a deterministic distribution. When an expression for the variance, using the phase type distribution for the vessel's content, is found it can be used to check both other cases discussed in this thesis.

8 Conclusion

In this thesis expressions for the expected sojourn time and the variance have been determined for the case we assume exponentially distributed vessel contents. Both have explicit formulas based on various parameters. Assuming a deterministic distribution for the vessel content, the expected value for the sojourn time has been shown. The expression for the variance turned out to be too complex for the time we had to work on this thesis. Lastly, the assumption was made that the vessels content was drawn from a phase type distribution. An attempt was made on showing the distribution of the sojourn time. The expectation and variance could be calculated from this in future research.

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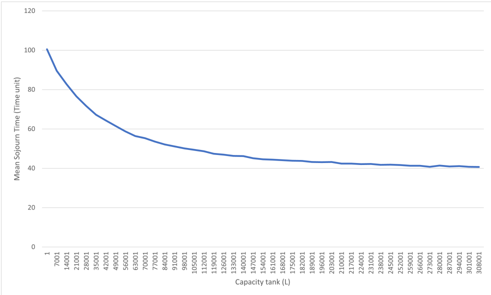
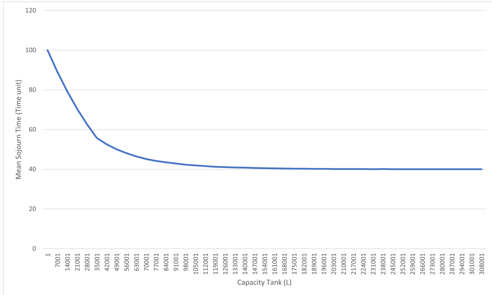
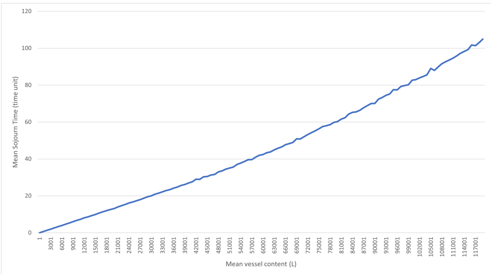
A Some more figures



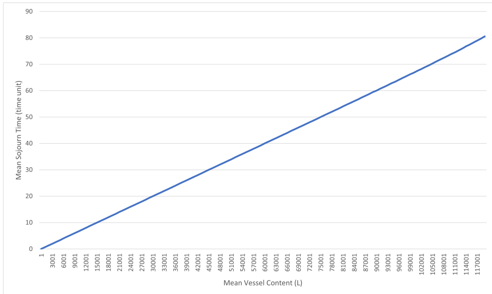
(a) Mean Sojourn Time over $\hat{y}$ (Exponential)



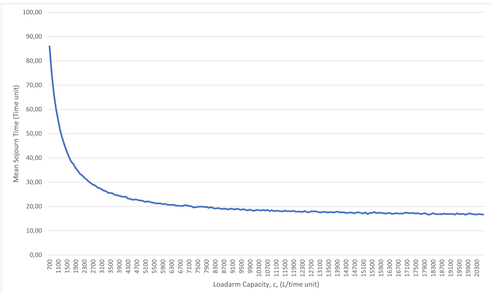
(b) Mean Sojourn Time over $\hat{y}$ (Deterministic)



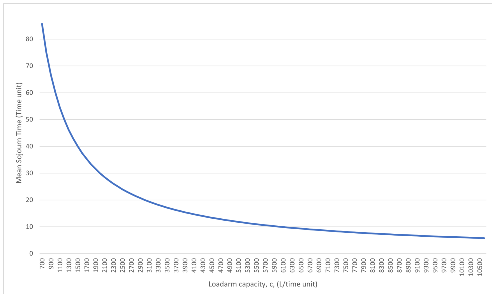
(c) Mean Sojourn time over EX (Exponential)



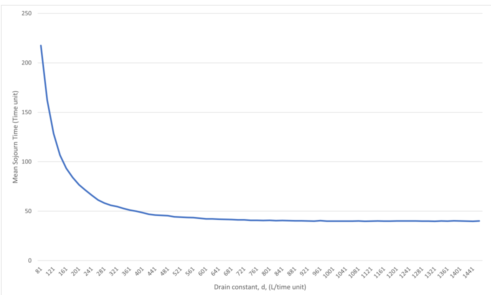
(d) Mean Sojourn Time over EX (Deterministic)



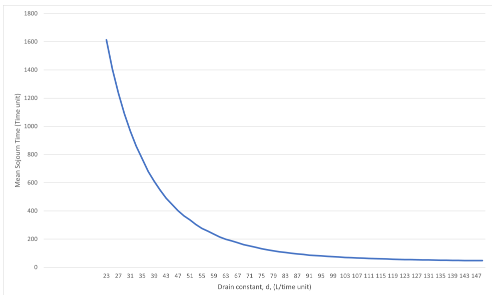
(e) Mean Sojourn Time over c (Exponential)



(f) Mean Sojourn Time over c (Deterministic)



(g) Mean Sojourn Time over d (Exponential)



(h) Mean Sojourn Time over d (Deterministic)

FIGURE 10: Relations between Mean Sojourn Time and different parameters. Assume exponentially and deterministically distributed vessel content