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Hybrid Integrator Gain System as a Controller for Infinite Dimensional Systems

Jens Oprel

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Committee members:
prof. dr. H.J. Zwart
dr. F.L. Schwenninger
dr. F. Califano MSc
Dr. rer. nat. J. A. Iglesias Martínez

Faculty of Electrical Engineering,
Mathematics and Computer Science
University of Twente
Enschede, The Netherlands

Summary

This master thesis is about *Hybrid Integrator Gain System (HIGS)*, a nonlinear control element. HIGS is a controller that switches between proportional (P) and integral (I) control. It can be described by:

$$\mathcal{H}\colon \begin{cases} \dot{u}(t) = \omega_h e(t) & \text{(I-mode)},\\ u(t) = k_h e(t) & \text{(P-mode)},\\ u(0) = u_0, \end{cases}$$

where e is the input of the controller and u its output. The conditions that determine when each of the modes is active are defined such that the input output pair (e, u) always stays inside a sector $\mathcal{F}$ that is sketched in Figure 1. This is in particular interesting for applications where the input and output form a power bond, where HIGS could act as an active damper. Furthermore, the nonlinearity present in HIGS makes it possible to overcome limitations of linear controllers, for example in reducing or eliminating overshoot in motion control applications.

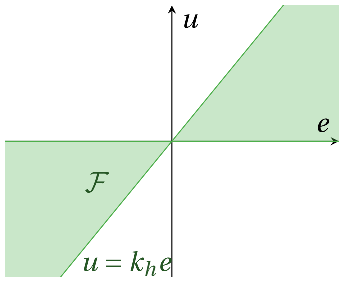


Figure 1: The region $\mathcal{F}$ in the (e, u)-plane.

The main goal of this thesis is to study the feedback connection of HIGS with a linear infinite dimensional plant $\mathcal{G}$, see Figure 2. HIGS has been studied before, but not as a controller for infinite dimensional systems. In this work, the focus is on extending results on well-posedness from the finite dimensional to the infinite dimensional case.

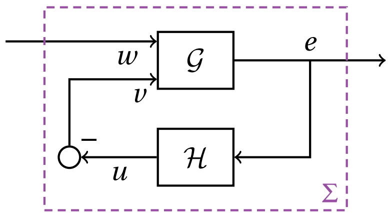


Figure 2: The closed loop system that is the main topic of study.

Different mathematical formulations of the closed loop system Σ are presented, the most useful one is to write explicitly that the system switches between two linear systems:

$$\begin{cases} \dot{x}(t) = A_1 x(t) & \text{if } \mathcal{H} \text{ in I-mode},\\ \dot{x}(t) = A_2 x(t) & \text{if } \mathcal{H} \text{ in I-mode},\\ x(0) = x_0. \end{cases} \tag{1}$$

Here $A_1 : X \to X$ and $A_2 : X \to X$ are operators on a possibly infinite dimensional Hibert space $X \ni x(t)$. In case the system would not switch and A_i can be represented by a matrix, then $\dot{x}(t) = A_i x(t);\ x(0) = x_0$ would have the solution $x(t) = \mathrm{e}^{A_i t} x_0$, where $\mathrm{e}^{A_i t}$ is the matrix exponential. For general operators on Hilbert spaces of infinite dimensions, a generalization of this matrix exponential is needed, which is the semigroup that is generated by A_i.

An earlier proof from the finite dimensional case served as an inspiration to achieve existence of solutions for the case where the operators A_1 and A_2 are bounded. This is achieved by explicitly constructing local solutions using the semigroup that is generated by A_1 or A_2. It is also shown that these local solutions can be extended to global solutions.

Many interesting applications, for example systems described by partial differential equations would result in unbounded A_1 and A_2. Some first steps are made towards achieving well-posedness also in this case, for example providing conditions under which A_1 and A_2 still generate semigroups. Establishing existence of solutions in the case where A_1 and A_2 are unbounded remains an open problem. This work provides directions that could be promising for further research to tackle this problem.

Contents

1 Introduction

This master thesis describes research on a specific nonlinear controller element: *Hybrid Integrator Gain System (HIGS)*. HIGS is an element that switches between proportional and integral control. Although these controller modes are linear, HIGS is a nonlinear system because of the switching between the modes. This nonlinear characteristic allows HIGS to overcome certain limitations of linear controllers, but also makes the analysis of mathematical properties challenging.

1.1 Hybrid Integrator Gain System

Hybrid Integrator Gain System (HIGS) is a controller element that switches between two linear controller modes: an integral (I) and a proportional gain (P) mode. Because of this switching behavior, the controller becomes a nonlinear system and can be written as a hybrid system. It was first introduced by Deenen, Heertjes, Heemels and Nijmeijer in 2017 [1]. HIGS has a single input e and a single output u (SISO), that is the input and output both take values in $\mathbb{R}$. The system is defined through:

$$\mathcal{H}\colon \begin{cases} \dot{u}(t) = \omega_h e(t) & \text{(I-mode), if } (e(t), u(t), \dot{e}(t)) \in \mathcal{F}_1, \\ u(t) = k_h e(t) & \text{(P-mode), if } (e(t), u(t), \dot{e}(t)) \in \mathcal{F}_2, \\ u(0) = u_0. \end{cases} \tag{1.1}$$

Here, k_h, ω_h are positive constants. The regions $\mathcal{F}_1$ and $\mathcal{F}_2$ where the I and P mode are active will be defined below. They are chosen such that the (e, u)-curve stays inside a set $\mathcal{F}$ which is sketched in Figure 1.1 and given by:

$$\mathcal{F} := \mathcal{F}^+ \cup -\mathcal{F}^+ \quad \text{with} \quad \mathcal{F}^+ := \left\{ (e, u) \in \mathbb{R}^2 \mid \left\langle \begin{pmatrix} e \\ u \end{pmatrix}, v_1 \right\rangle \geq 0 \wedge \left\langle \begin{pmatrix} e \\ u \end{pmatrix}, v_2 \right\rangle \geq 0 \right\}. \tag{1.2}$$

The vectors v_1 and v_2 are sketched in Figure 1.1, these are defined as:

$$v_1 := \begin{pmatrix} k_h \\ -1 \end{pmatrix}, \; v_2 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{1.3}$$

This means that the set $\mathcal{F}^+$ contains the points which lie above the horizontal e-axis and to the right of the line $u = k_h e$. Notice that v_1 and v_2 are orthogonal to these lines (see also Figure 1.1).

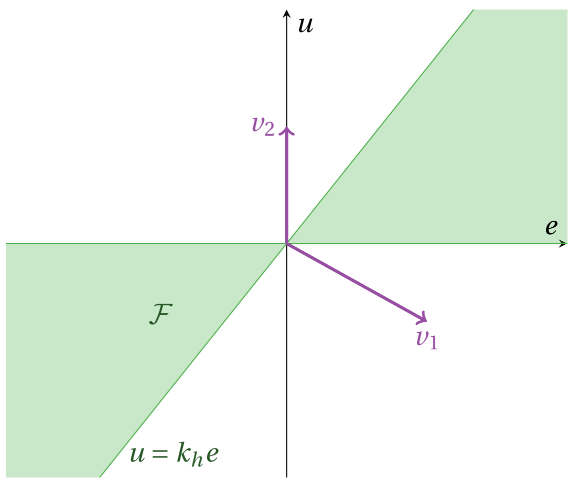


Figure 1.1: The region $\mathcal{F}$ in the $(e,\ u)$-plane.

Because equality is also allowed, the two borders are also included in the set $\mathcal{F}^+$, which is thus closed. By (1.2), $\mathcal{F}$ contains points that lie in $\mathcal{F}^+$ and also points that lie below the e-axis and to the left of the $u = k_h e$ line (thus in $-\mathcal{F}^+$).

The sets $\mathcal{F}_1$ and $\mathcal{F}_2$ that determine where the I and P mode are active can also be defined using v_1 and v_2:

$$\mathcal{F}_2 := \mathcal{F}_2^+ \cup -\mathcal{F}_2^+ \text{ with } \mathcal{F}_2^+ := \left\{(e,u,d) \in \mathbb{R}^3 \mid \left\langle \begin{pmatrix} e \\ u \end{pmatrix}, v_1 \right\rangle = 0 \wedge \left\langle \begin{pmatrix} e \\ u \end{pmatrix}, v_2 \right\rangle > 0 \wedge \left\langle \begin{pmatrix} d \\ \omega_h e \end{pmatrix}, v_1 \right\rangle < 0 \right\}. \tag{1.4}$$

Notice that this set is in $\mathbb{R}^3$ and the third coordinate is denoted by d. In the use of these sets, this third coordinate will often be equal to $\dot{e}(t)$.The $\mathcal{F}_1$ set is defined as:

$$\mathcal{F}_1 := \left\{(e,u,d) \in \mathbb{R}^3 \mid (e,u) \in \mathcal{F}\right\} \setminus \mathcal{F}_2. \tag{1.5}$$

As an illustration of the switching behavior that is achieved by having the I and P mode active in the $\mathcal{F}_1$ and $\mathcal{F}_2$ set respectively, an example is considered (inspired by [1]).

Example 1.1 (Open loop response)**.** Consider the HIGS element as defined in (1.1) with $k_h = \omega_h = 1$ and $\mathcal{F}_1$ and $\mathcal{F}_2$ as in (1.5) and (1.4). Consider the situation in which $u_0 = 0$ and the input to the HIGS element is given by

$$e(t) = \sin(t) + 0.7\sin(3t). \tag{1.6}$$

In Figure 1.2 the first part of the response is plotted, with two switching points. The starting point is $(e,u) = (0,0)$. Due to the strict inequality in the definition of $\mathcal{F}_2$ in (1.4) $(0,0) \notin \mathcal{F}_2$. As $(e,u) \in \mathcal{F}$ this means (by (1.5)) that $(e,u) \in \mathcal{F}_1$, thus the start is in I-mode. As long as $u \neq k_h e$ the integration can continue.

After some time t_1, $k_h e(t_1) = u(t_1)$. At this time, it therefore holds that $\left\langle \begin{pmatrix} e(t_1) & u(t_1) \end{pmatrix}^T, v_1 \right\rangle = 0$. As $u(t_1) > 0$ also $\left\langle \begin{pmatrix} e(t_1) & u(t_1) \end{pmatrix}^T, v_2 \right\rangle > 0$. The last condition also holds: $\left\langle \begin{pmatrix} \dot{e}(t_1) & \omega_h e(t_1) \end{pmatrix}^T, v_1 \right\rangle < 0$. This can be seen from the purple arrow in Figure 1.2b. It points in the direction that the I-mode would go, as in this mode it would hold that $\dot{u}(t_1) = \omega_h e(t_1)$. The arrow points outside the sector, to the left of the $u = k_h e$ line, meaning that also the last condition of (1.4) holds and thus $(e(t_1), u(t_1), \dot{e}(t_1) \in \mathcal{F}_2$. Therefore, a switch to P-mode is made. This switch to P-mode ensures that the (e,u)-curve stays inside $\mathcal{F}$. As indicated by the last condition that was checked, the integrator mode would have led to the trajectory leaving $\mathcal{F}$.

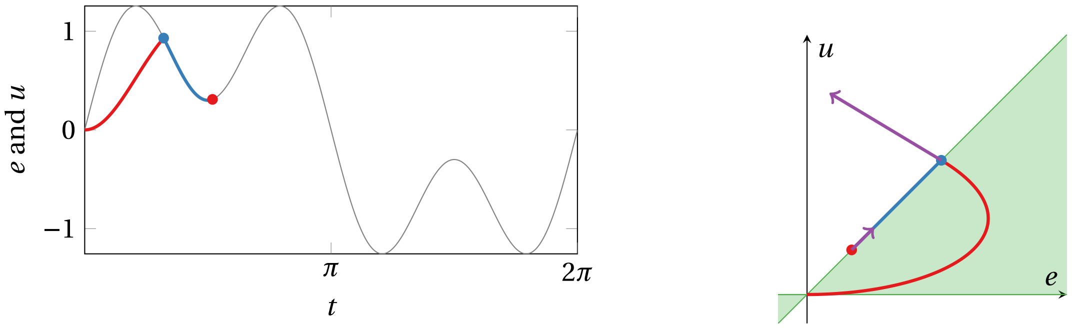
At time t_2 the second switch is made, back to I-mode, in Figure 1.2 indicated by a red dot. The first two conditions on $\mathcal{F}_2$ are still satisfied, but as can be seen from Figure 1.2b the purple arrow now does not point out of the $\mathcal{F}$ sector anymore. Hence, $(e(t_1), u(t_1), \dot{e}(t_1) \notin \mathcal{F}_2$, but $(e(t_1), u(t_1), \dot{e}(t_1) \in \mathcal{F}_1$ and thus a switch is made back to integrator mode is made.

The plot is extended in Figure 1.3. Again, the integrator mode would at some point lead to the (e,u)-curve leaving the $\mathcal{F}$ sector, thus a switch back to proportional mode is made. This time, there is no point at which a switch to I-mode can be made before the origin is reached.

Notice that by the construction of the $\mathcal{F}$ region, it is no surprise that the point $(0,0)$ is reached at $t = \pi$. This is due to the fact that $e(\pi) = 0$, and the region $\mathcal{F}$ only has a single point where $e = 0$, which is the origin.

From the origin, the trajectory is continued using the I-mode, see Figure 1.4. The switching behavior in the second half of the period is very similar as in the first half. The only difference is that $e(t)$ is negative for $t \in (\pi, 2\pi)$. Therefore, in this second half of the period $(e(t), u(t), \dot{e}(t) \notin \mathcal{F}_2^+$. The P-mode is therefore only active if $(e(t), u(t), \dot{e}(t) \in -\mathcal{F}_2^+$. By (1.4), that is when the following three conditions hold:

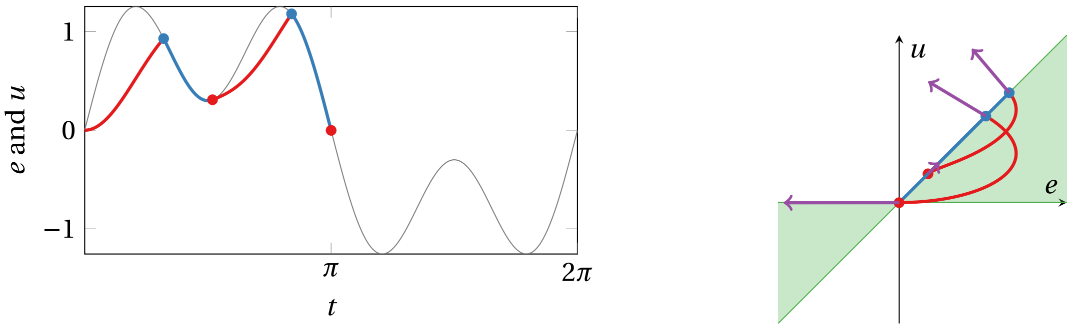
$$\left\langle \begin{pmatrix} e(t) & u(t) \end{pmatrix}^T, v_1 \right\rangle = 0 \text{ and } \left\langle \begin{pmatrix} e(t) & u(t) \end{pmatrix}^T, v_2 \right\rangle < 0 \text{ and } \left\langle \begin{pmatrix} \dot{e}(t) & \omega_h e(t) \end{pmatrix}^T, v_1 \right\rangle > 0.$$



(a) In grey the input e. In red and blue the resulting output u of HIGS. The color indicates the mode of operation: I-mode is drawn in red, P-mode in blue.

(b) (e, u)-curve in red (I-mode) and blue (P-mode). In purple the $\begin{pmatrix}\dot{e} & \omega_h e\end{pmatrix}^T$ vector at the switching points.

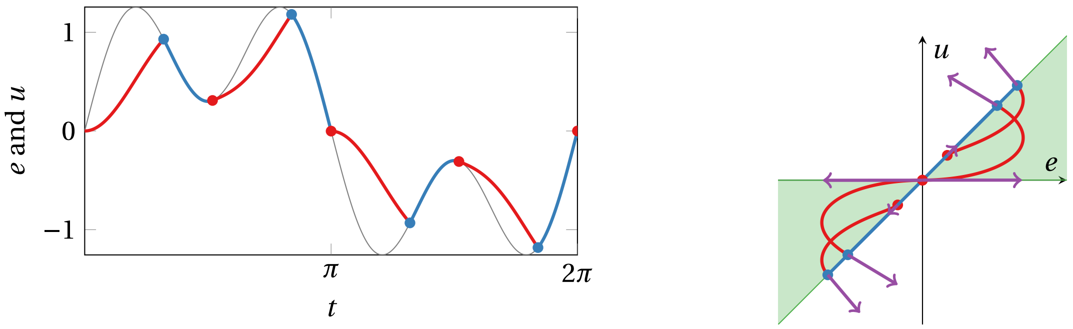
Figure 1.2: First part of the open loop response of the HIGS to the input given by (1.6).



(a) In grey the input e. In red and blue the resulting output u of HIGS. The color indicates the mode of operation: I-mode is drawn in red, P-mode in blue.

(b) (e, u)-curve in red (I-mode) and blue (P-mode). In purple the $\begin{pmatrix}\dot{e} & \omega_h e\end{pmatrix}^T$ vector at the switching points.

Figure 1.3: Extended open loop response of the HIGS to the input given by (1.6).



(a) In grey the input e. In red and blue the resulting output u of HIGS. The color indicates the mode of operation: I-mode is drawn in red, P-mode in blue.

(b) (e, u)-curve in red (I-mode) and blue (P-mode). In purple the $\begin{pmatrix}\dot{e} & \omega_h e\end{pmatrix}^T$ vector at the switching points.

Figure 1.4: Full period of the open loop response of the HIGS to the input given by (1.6).

1.2 Advantages of HIGS

The sector that HIGS is constrained to has the idea to keep the sign of the input and the output of HIGS the same. This is powerful, for example because this sign equivalence can ensure that HIGS only acts as a damping element. If the input and output form a power bond, then HIGS only extracts energy from the plant. This makes HIGS interesting for active damping, as applied for example by Heertjes et al. [2]. Another example where this sign equivalence is useful, is in motion applications where the input to HIGS is a position and the output a force. The sector boundedness in this case ensures that HIGS does not contribute to overshoot of the system. This is useful for example in positioning systems such as wafer scanners [3–6].

The idea of constraining the input and output of a controller is not new. The idea stems from reset control. Reset control was first introduced by Clegg in 1958 [7] and is described by:

$$\mathcal{C}: \begin{cases} \dot{u}(t) = \omega e(t) & \text{if } eu \geq 0, \\ u(t) = 0 & \text{otherwise.} \end{cases}$$

The Clegg integrator thus performs hard resets to 0 to ensure sign equivalence. Reset control was an inspiration for the developers of HIGS. Whilst also guaranteeing the sign equivalence, HIGS has a continuous output signal by using the proportional mode to stay away from the u axis of the (e,u)-plane. This is in contrast to reset control, where the hard reset causes a discontinuity in the output signal. The continuity of the output signal is important in certain applications where discontinuities in the controller signal can lead to excitation of higher harmonics [1].

By switching between the two linear controllers, HIGS becomes a nonlinear controller. Likewise, the Clegg integrator is also nonlinear because of the resets of the state. It is this nonlinearity that makes it possible to overcome limitations of linear controllers and thus makes it interesting for control applications where linear controllers do not provide the desired performance. However, the nonlinearity also makes it harder to understand and design the controller, as tools that are used in linear time-invariant (LTI) controllers are not available. For example, the frequency domain tools cannot be used as for LTI control. Furthermore, it is not trivial which mathematical properties HIGS has. Analysis for HIGS specifically is needed to show for example well-posedness and stability.

1.3 Earlier research on HIGS

The research into HIGS has only started recently, with the first publications dating from 2017 [1]. Most of the related publications are from the Control System Technology group at the Eindhoven University of Technology, but also authors affiliated with industry (e.g. ASML [1–3, 5, 6, 8–19], ASM Pacific Technology [20]) and other academia (e.g. from Australian National University [4, 21, 22]) have published research and applications of HIGS.

Research results for HIGS as a controller for finite dimensional systems are available in the published literature. Two approaches have been studied to establish well-posedness of the HIGS as a controller [9, 23]. One approach is to explicitly construct solutions [1], another approach is to extend existing theory on projection based systems [23]. These proofs provide inspiration for the well-posedness proofs in this thesis where an infinite dimensional setting is considered (Chapters 4 and 5).

For stability, more published literature is available, with multiple approaches being used to establish rigorous mathematical proofs, see for example [2, 9, 14, 16, 18]. These results are all assuming that the system to be controlled is finite dimensional, but it is likely that HIGS can be used for infinite systems as well. For example, for a system that is passive and has a power bond between input and output (e.g. a port-Hamiltonian system [24]) the fact that the HIGS ensures that the input and output have equal sign already ensures that the energy in the system can only decrease, which ensures stability. A similar argument is used in the finite dimensional case for example by Heertjes et al. [2].

1.4 Goals

The main goal of this thesis is to establish conditions under which a closed loop system of HIGS with an infinite dimensional plant is well-posed. This is relevant for example if HIGS is used as a controller for a system described by a partial differential equation.

The focus is on the following sub-goals:

1. Analyze existing proofs for the finite dimensional case and provide conditions how these can be generalized to require less restrictive assumptions.
2. Investigate under which conditions the well-posedness results achieved for the finite dimensional case can be extended to infinite dimensional systems in feedback connection with HIGS as a controller.
3. Investigate to what extent HIGS outperforms linear controllers for infinite dimensional plants.

1.5 Outline of this Thesis

The rest of this thesis is structured as follows: In Chapter 2, different mathematical formulations of HIGS are introduced and shown to be equivalent. The most important are the discontinuous representation and the representation as an extended Projected Dynamical System, as these play a key role in two existing well-posedness result. It is also shown that HIGS fits into the more general framework of Hybrid Systems. Chapter 3 introduces theory of (operator) semigroups. Semigroups can be seen as a generalization of (matrix) exponentials, denoted as e^{At}. Semigroups provide solutions to differential equations of the form $\dot{x}(t) = Ax(t)$. The theory introduced in this chapter is used in Chapters 4 and 5. In Chapter 4, local and global existence of solutions is shown for a feedback connection of HIGS with an infinite dimensional linear plant which has bounded operators. In Chapter 5, an attempt is made to further generalize this result to cases where these operators are not bounded. In Chapter 6, simulations of a HIGS controlled closed loop system are shown. The example that is considered in this chapter shows that the HIGS can overcome a limitation of LTI control.

2 Representations of HIGS in open and closed loop

In this chapter, different representations of HIGS are presented. First, the HIGS in open loop as introduced in the previous chapter will be revisited in Section 2.1. Thereafter, a feedback connection (Σ) of HIGS ($\mathcal{H}$) with a plant ($\mathcal{G}$) as in Figure 2.1 is studied.

For the structure of the plant $\mathcal{G}$, the following assumption is made throughout the rest of this thesis:

Assumption 2.1. *The plant $\mathcal{G}$ is linear and has no direct feed-trough. That is, it can be written in the form:*

$$\mathcal{G}\colon \begin{cases} \dot{x}_g(t) = A_g x_g(t) + B_{gv} v(t) + B_{gw} w(t), \\ e(t) = C_g x_g(t). \end{cases} \tag{2.1}$$

The state vector $x_g(t)$ is an element of a Hilbert space $X_{\mathcal{G}}$ (the state space) and the input $w(t)$ is in a Hilbert space X_w (the input space). The control input $v(t)$ and the output of the plant $e(t)$ are assumed to be in $\mathbb{R}$. The following are linear operators $A_g\colon X_{\mathcal{G}} \to X_{\mathcal{G}}$, $B_{gv}\colon \mathbb{R} \to X_{\mathcal{G}}$, $B_{gw}\colon X_w \to X_{\mathcal{G}}$ and $C_g\colon X_{\mathcal{G}} \to \mathbb{R}$. In this thesis, B_{gv}, B_{gw} and C_g are always assumed to be bounded.

For the closed loop system Σ as in Figure 2.1, first the representation of the system as a discontinuous system is discussed in Section 2.2. This representation is in some sense the most intuitive form from the idea of the switching between the integral and proportional control. In Section 2.3, the observation is made that such switching systems fit into the framework of Hybrid Systems. In recent literature, the HIGS is often seen as a particular case of an (extended) Projected Dynamical system. This representation is presented in Section 2.4.

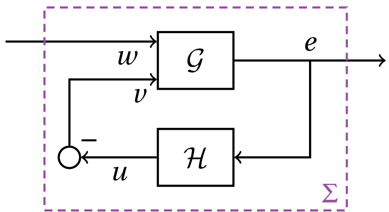


Figure 2.1: The closed-loop interconnection of the HIGS with a plant.

2.1 Open loop system and closing $\mathcal{F}_1$ and $\mathcal{F}_2$

The following definition of HIGS $\mathcal{H}$ will be used:

$$\mathcal{H}\colon \begin{cases} \dot{x}_h(t) = \omega_h e(t) & \text{(I-mode), if } (e(t), u(t), \dot{e}(t)) \in \mathcal{F}_1, \\ \dot{x}_h(t) = k_h \dot{e}(t) & \text{(P-mode), if } (e(t), u(t), \dot{e}(t)) \in \mathcal{F}_2, \\ x_h(0) = x_{h,0}, & \\ u(t) = x_h(t) & \text{and } (e(t), u(t)) \in \mathcal{F}. \end{cases} \tag{2.2}$$

The input $e(t)$, output $u(t)$ and the state $x_h(t)$ of $\mathcal{H}$ are assumed to be in $\mathbb{R}$, thus HIGS is a single input, single output (SISO) system that has a single internal state. The constant $k_h > 0$ is the gain in proportional mode and $\omega_h > 0$ is the integrator gain.

Remark 2.2. *The notation used to define HIGS in this chapter in* (2.2) *differs slightly from the notation used in* (1.1)*. By writing the proportional mode with a derivative on both sides, it becomes clearer that the output of HIGS is continuous as long as the input is continuous. The variable x_h represents the internal state of the HIGS, that is used for the integration.*

By (2.2), the input-output pair (e,u) of $\mathcal{H}$ curve stays inside the set $\mathcal{F}$ that was defined in Chapter 1 as:

$$\mathcal{F} := \mathcal{F}^+ \cup -\mathcal{F}^+ \text{ with } \mathcal{F}^+ := \left\{(e,u) \in \mathbb{R}^2 \mid \left\langle \begin{pmatrix} e \\ u \end{pmatrix}, v_1 \right\rangle \geq 0 \wedge \left\langle \begin{pmatrix} e \\ u \end{pmatrix}, v_2 \right\rangle \geq 0 \right\}, v_1 := \begin{pmatrix} k_h \\ -1 \end{pmatrix},\ v_2 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.3}$$

By substituting v_1 and v_2, the set $\mathcal{F}^+$ can also be written as:

$$\mathcal{F}^+ = \left\{(e,u) \in \mathbb{R}^2 \mid k_h e - u \geq 0 \wedge u \geq 0\right\}.$$

Because $\mathcal{F}$ is the union of $\mathcal{F}^+$ and $-\mathcal{F}^+$, the set $\mathcal{F}$ is the set of points where $k_h e - u$ and u have the same sign, or one of these terms is zero. This means that $\mathcal{F}$ can also be written as:

$$\mathcal{F} = \left\{(e,\ u) \in \mathbb{R}^2 \mid k_h e u - u^2 \geq 0\right\}. \tag{2.4}$$

The sets $\mathcal{F}_1$ and $\mathcal{F}_2$ that determine when the I and P mode are active are defined in Chapter 1 as:

$$\mathcal{F}_2 := \mathcal{F}_2^+ \cup -\mathcal{F}_2^+ \text{ with } \mathcal{F}_2^+ := \left\{(e,u,d) \in \mathbb{R}^3 \mid \left\langle \begin{pmatrix} e \\ u \end{pmatrix}, v_1 \right\rangle = 0 \wedge \left\langle \begin{pmatrix} e \\ u \end{pmatrix}, v_2 \right\rangle > 0 \wedge \left\langle \begin{pmatrix} d \\ \omega_h e \end{pmatrix}, v_1 \right\rangle < 0 \right\}, \tag{2.5}$$

$\mathcal{F}_1 := \left\{(e,u,d) \in \mathbb{R}^3 \mid (e,u) \in \mathcal{F}\right\} \setminus \mathcal{F}_2.$

Substitution of v_1 and v_2 gives:

$$\mathcal{F}_2^+ = \left\{(e,u,d) \in \mathbb{R}^3 \mid k_h e = u \wedge u > 0 \wedge k_h d - \omega_h e < 0\right\}.$$

Because $k_h > 0$, the equality in the first condition implies that e and u have the same sign. Therefore, to lie in $\mathcal{F}_2$, it must hold that $k_h d - \omega_h e$ and e have opposite sign. That is:

$$\mathcal{F}_2 = \left\{(e,u,d) \in \mathbb{R}^3 \mid k_h e = u \wedge \omega_h e^2 > k_h d e\right\}. \tag{2.6}$$

Using (2.4) and (2.6), $\mathcal{F}_1$ can be written as:

$$\mathcal{F}_1 = \left\{(e,u,d) \in \mathbb{R}^3 \mid k_h e u \geq u^2 \wedge \left(k_h e \neq u \vee \omega_h e^2 \leq k_h d e\right)\right\}. \tag{2.7}$$

At the points where a switch of modes occurs, there might be a discontinuity in the derivative as the modes prescribe different derivatives. Therefore, it is most likely not possible to find a classical solution i.e., a differentiable function that satisfies the differential equation everywhere. Therefore, the following weakened definition of a solution is used [25].

Definition 2.3. *Let $\mathcal{X}$ be a Hilbert space and let $f : \mathcal{X} \times \mathbb{R}_{\geq 0} \to \mathcal{X}$. Then a function $x : \mathbb{R}_{\geq 0} \to \mathcal{X}$ is called a Carathéodory solution to the equation $\dot{x}(t) = f(x(t), t)$ if x an absolutely continuous function that satisfies the integral version of the differential equation:*

$$x(t) = x(0) + \int_0^t f(x(\tau), \tau)\, \mathrm{d}\tau.$$

In other words, the solution x is continuous on $\mathbb{R}_{\geq 0}$ and differentiable almost everywhere on $\mathbb{R}_{\geq 0}$, and the differential equation is satisfied almost everywhere.

In case of the HIGS $\mathcal{H}$ as in (2.2), the differential equation can be written in the form $\dot{x}_h(t) = f(x_h(t), t)$ from Definition 2.3 by defining f as:

$$\dot{x}_h(t) = f(x_h(t), t) := \begin{cases} \omega_h e(t) & \text{if } (e(t), x_h(t), \dot{e}(t) \in \mathcal{F}_1, \\ k_h \dot{e}(t) & \text{if } (e(t), x_h(t), \dot{e}(t) \in \mathcal{F}_2, \end{cases} \tag{2.8}$$

where e is a given input function that needs to be absolutely continuous. The function e is thus incorporated in the function $f : \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ using its time dependence.

In existing frameworks, such as Hybrid Systems introduced later in this chapter in Section 2.3, it is common to assume that the sets that prescribe where equations are active are closed. However, in the definition of HIGS as given before in (2.2) the sets $\mathcal{F}_1$ and $\mathcal{F}_2$ are not closed. Therefore, it is natural to propose the following alternative definition of HIGS:

$$\overline{\mathcal{H}}\colon \begin{cases} \left(\dot{x}_h(t) = \omega_h e(t) \;\wedge\; (e(t), u(t), \dot{e}(t)) \in \overline{\mathcal{F}_1}\right) \quad \vee \quad \left(\dot{x}_h(t) = k_h \dot{e}(t) \;\wedge\; (e(t), u(t), \dot{e}(t)) \in \overline{\mathcal{F}_2}\right), \\ x_h(0) = x_{h,0}, \\ u(t) \;\; = x_h(t). \end{cases} \tag{2.9}$$

This system cannot be described by a single differential equation anymore, as was the case for $\mathcal{H}$ in (2.8). As the $\overline{\mathcal{F}_1}$ and $\overline{\mathcal{F}_2}$ overlap, the system is instead described by a differential inclusion:

$$\dot{x}_h(t) \in F(x_h(t), t) := \left\{\omega_h e(t) \mid (e(t), x_h(t), \dot{e}(t) \in \overline{\mathcal{F}_1}\right\} \cup \left\{k_h \dot{e}(t) \mid (e(t), x_h(t), \dot{e}(t) \in \overline{\mathcal{F}_2}\right\} \tag{2.10}$$

where e is again an absolutely continuous function that is incorporated into the set valued mapping $F : \mathbb{R} \times \mathbb{R}_{\geq 0} \rightrightarrows \mathbb{R}$. The symbol '$\rightrightarrows$' denotes that F maps to sets in $\mathbb{R}$. Differential inclusions generalize differential equations. For example, the differential equation in Equation (2.8) can be written as a differential inclusion of the same form as in (2.10), where the set valued mapping always maps to a set with one element. Also the definition of a solution as in Definition 2.3 can be extended to differential inclusions.

Definition 2.4. *Let $\mathcal{X}$ be a Hilbert space and let $F : \mathcal{X} \times \mathbb{R}_{\geq 0} \rightrightarrows \mathcal{X}$. Then a function $x : \mathbb{R}_{\geq 0} \to \mathcal{X}$ is called a Carathéodory solution to the differential inclusion $\dot{x}(t) \in F(x(t), t)$ if x an absolutely continuous function that satisfies the differential inclusion almost everywhere.*

The following result gives that the formulation of HIGS as in (2.9) is equivalent to (2.2) in the sense that these formulations have the same Carathéodory solutions.

Theorem 2.5. *Consider $\mathcal{H}$ and $\overline{\mathcal{H}}$ as defined in* (2.2) *and* (2.9) *respectively. Let $e(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be absolutely continuous. Then, a function $x_h(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is a Carathéodory solution to $\mathcal{H}$ if and only if $x_h(t)$ is a Carathéodory solution to $\overline{\mathcal{H}}$.*

Proof. First consider a Carathéodory solution x_h to the system $\mathcal{H}$. As $(e(t), x_h(t)) \in \mathcal{F}$ for every t, it holds that either $(e(t), x_h(t), \dot{e}(t)) \in \mathcal{F}_1$ or $(e(t), x_h(t), \dot{e}(t)) \in \mathcal{F}_2$ for times t where $\dot{e}(t)$ is defined. Therefore, $x_h(t)$ satisfies one of the two differential relations for almost every $t \in \mathbb{R}_{\geq 0}$:

$$(\dot{x}_h(t) = \omega_h e(t) \;\wedge\; (e(t), x_h(t), \dot{e}(t)) \in \mathcal{F}_1) \quad \vee \quad (\dot{x}_h(t) = k_h \dot{e}(t) \;\wedge\; (e(t), x_h(t), \dot{e}(t)) \in \mathcal{F}_2).$$

And thus, as the sets are contained in their closure, it is clear that this implies that $x_h(t)$ is a Carathéodory solution to $\overline{\mathcal{H}}$.

To prove the other direction of the equivalence, now assume that x_h is a Carathéodory solution to $\overline{\mathcal{H}}$. This gives that $(e(t), x_h(t), \dot{e}(t)) \in \overline{\mathcal{F}_2} \cup \overline{\mathcal{F}_2} = \mathcal{F}_2 \cup \mathcal{F}_2$ for a.e. $t \in \mathbb{R}_{\geq 0}$. By continuity of e and x_h, it even holds that $(e(t), x_h(t)) \in \overline{\mathcal{F}} = \mathcal{F}$ for all $t \in \mathbb{R}_{\geq 0}$. What remains to show is that for almost every $t \in \mathbb{R}_{\geq 0}$ one of the following two statement holds:

$$\dot{x}_h(t) = \omega_h e(t) \;\text{ and }\; (e(t), x_h(t), \dot{e}(t)) \in \mathcal{F}_1, \quad \text{or} \tag{2.11a}$$

$$\dot{x}_h(t) = k_h \dot{e}(t) \;\text{ and }\; (e(t), x_h(t), \dot{e}(t)) \in \mathcal{F}_2. \tag{2.11b}$$

As x_h is a Carathéodory solution to $\overline{\mathcal{H}}$, for almost every $t \in \mathbb{R}_{\geq 0}$ at least one of the following statement holds:

$$\dot{x}_h(t) = \omega_h e(t) \;\text{ and }\; (e(t), x_h(t), \dot{e}(t)) \in \overline{\mathcal{F}_1}, \quad \text{or} \tag{2.12a}$$

$$\dot{x}_h(t) = k_h \dot{e}(t) \;\text{ and }\; (e(t), x_h(t), \dot{e}(t)) \in \overline{\mathcal{F}_2}. \tag{2.12b}$$

Fix a τ such that (2.12a) or (2.12b) holds for $t = \tau$, e and u are differentiable at τ and it does not hold that $(e(\tau) = 0 \wedge \dot{e}(\tau) \neq 0)$. As $(e(\tau), x_h(\tau), \dot{e}(\tau)) \in \overline{\mathcal{F}_1} \cup \overline{\mathcal{F}_2} = \mathcal{F}_1 \cup \mathcal{F}_2$, one of the following four cases must hold:

Case 1: $(e(\tau), x_h(\tau), \dot{e}(\tau)) \in \mathcal{F}_1 \setminus \overline{\mathcal{F}_2}$
In particular as $(e(\tau), x_h(\tau), \dot{e}(\tau)) \notin \overline{\mathcal{F}_2}$, (2.12b) cannot hold for $t = \tau$. Therefore, (2.12a) must hold. Combined with the assumption that $(e(\tau), x_h(\tau), \dot{e}(\tau)) \in \mathcal{F}_1$, it follows that also (2.11a) holds for $t = \tau$.

Case 2: $(e(\tau), x_h(\tau), \dot{e}(\tau)) \in \mathcal{F}_2 \setminus \overline{\mathcal{F}_1}$
As $(e(\tau), x_h(\tau), \dot{e}(\tau)) \notin \overline{\mathcal{F}_1}$, (2.12a) cannot hold and thus (2.12b) must hold for $t = \tau$. Hence, also (2.11b) holds for $t = \tau$.

Case 3: $(e(\tau), x_h(\tau), \dot{e}(\tau)) \in \mathcal{F}_2 \cup \overline{\mathcal{F}_1}$
As $\mathcal{F}_2 \cap \mathcal{F}_1 = \emptyset$, it is already clear that (2.11a) cannot hold. Thus the aim must be to show that (2.11b) holds for $t = \tau$.

Towards a contradiction assume that it holds that

$$\dot{x}_h(\tau) = \omega_h e(\tau). \tag{2.13}$$

By definition of $\mathcal{F}_2$,

$$k_h e(\tau) = x_h(\tau), \quad \text{and} \tag{2.14}$$

$$k_h \dot{e}(\tau) e(\tau) < \omega_h e(\tau)^2. \tag{2.15}$$

Now consider the function $\varphi(t) := k_h e(t) x_h(t) - x_h^2(t)$. By (2.14), $\varphi(\tau) = 0$. By assumption, $e(t)$ and $x_h(t)$ are differentiable at $t = \tau$, and thus φ is also differentiable at $t = \tau$ and the derivative is given by:

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t)|_{t=\tau} = k_h \dot{e}(\tau) x_h(t) + k_h e(\tau) \dot{x}_h(\tau) - 2x_h(\tau)\dot{x}_h(\tau).$$

Subsititution of (2.14) gives:

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t)|_{t=\tau} = k_h \left(k_h \dot{e}(\tau) e(\tau) - \omega_h e(\tau)^2\right) =: L.$$

By definition of the derivative and (2.15):

$$\lim_{h \to 0} \frac{\varphi(\tau + h)}{h} = L < 0.$$

And thus there exists an $\epsilon > 0$ such that for $0 < h < \epsilon$ $\varphi(\tau + h) < 0$. But this implies that $(e(\tau+h), x_h(\tau+h)) \notin \mathcal{F} = \overline{\mathcal{F}}$ and thus neither (2.12a) nor (2.12b) can hold for $t \in (\tau, \tau+h)$. As the measure of this interval is $\epsilon > 0$ this contradicts that for almost every t (2.12a) or (2.12b) holds. Hence, the assumption made in (2.13) cannot hold, meaning that for $t = \tau$ (2.12a) cannot hold, thus by (2.12b) it follows that:

$$\dot{x}_h(\tau) = k_h \dot{e}(\tau).$$

From $(e(\tau), x_h(\tau), \dot{e}(\tau)) \in \mathcal{F}_2$, it then follows that (2.11a) holds.

Case 4: $(e(\tau), x_h(\tau), \dot{e}(\tau)) \in \mathcal{F}_1 \cup \overline{\mathcal{F}_2}$
As the intersection of $\mathcal{F}_1$ and $\mathcal{F}_2$ is empty, $(e(\tau), x_h(\tau), \dot{e}(\tau)) \notin \mathcal{F}_2$. Combined with the assumption, it follows from the definition of $\mathcal{F}_1$ and $\mathcal{F}_2$ that:

$$k_h e(\tau) = x_h(\tau) \quad \wedge \quad k_h \dot{e}(\tau) e(\tau) = \omega_h e(\tau)^2.$$

And thus:

$$e(\tau) = 0 \quad \vee \quad k_h \dot{e}(\tau) = \omega_h e(\tau).$$

If $k_h \dot{e}(\tau) = \omega_h e(\tau)$ then the two differential equations in (2.12a) and (2.12b) prescribe the same differential equation. Hence, it is clear that $\dot{x}_h(\tau) = \omega_h e(\tau)$. Since $(e(\tau), x_h(\tau), \dot{e}(\tau)) \in \mathcal{F}_1$, (2.11a) holds.

If, on the other hand, $e(\tau) = 0$, then it follows from the assumption that $\neg(e(\tau) = 0 \wedge \dot{e}(\tau) \neq 0)$ that also $\dot{e}(\tau) = 0$. Thus again the same differential equation is prescribed, from which it can be concluded again that (2.11a) holds.

In the above, it was shown that for $\tau \in \mathcal{T}$, (2.11a) or (2.11b) holds, where $\mathcal{T}$ is the set:

$$\mathcal{T} := \mathbb{R}_{\geq 0} \setminus \left(\left\{t \in \mathbb{R}_{\geq 0} \mid \text{Both (2.12a) and (2.12b) do } \underline{\text{not}} \text{ hold}\right\} \cup \left\{t \in \mathbb{R}_{\geq 0} \mid e(t) = 0 \wedge \dot{e}(t) \neq 0\right\}\right).$$

Both sets that are subtracted are sets with measure zero, the first set because x_h is a Carathéodory solution to $\overline{\mathcal{H}}$ and the second set from the fact that e is absolutely continuous (by Theorem 1 of [26]). □

2.2 Discontinuous representation of the HIGS controlled system

In this section, the goal is to describe the closed loop system Σ as in Figure 2.1 of HIGS with a plant $\mathcal{G}$. Recall that it is assumed that the plant is of the form of (2.1). Note that the interconnection of a linear controller with the linear plant would be a linear system. As HIGS is switching between two linear controllers, it is also possible to write the combined closed-loop system as a switching linear system with states $x := (x_g, x_h) \in X_\Sigma := (X_\mathcal{G} \times \mathbb{R})$:

$$\Sigma : \begin{cases} \dot{x}(t) = \begin{pmatrix} A_g & -B_{gv} \\ \omega_h C_g & 0 \end{pmatrix} x(t) + \begin{pmatrix} B_{gw} \\ 0 \end{pmatrix} w(t) & \text{if } \mathcal{H} \text{ in I-mode,} \\ \dot{x}(t) = \begin{pmatrix} A_g & -B_{gv} \\ k_h C_g A_g & -k_h C_g B_{gv} \end{pmatrix} x(t) + \begin{pmatrix} B_{gw} \\ k_h C_g B_{gw} \end{pmatrix} w(t) & \text{if } \mathcal{H} \text{ in P-mode,} \\ e(t) = \begin{pmatrix} C_g & 0 \end{pmatrix} x(t). \end{cases}$$

The following operators are defined to be the block operators that appear in the above closed loop system:

$$A_1 : X_\Sigma \to X_\Sigma, \quad A_1 := \begin{pmatrix} A_g & -B_{gv} \\ \omega_h C_g & 0 \end{pmatrix}, \qquad B_1 : X_w \to X_\Sigma, \quad B_1 := \begin{pmatrix} B_{gw} \\ 0 \end{pmatrix}, \tag{2.16}$$

$$A_2 : X_\Sigma \to X_\Sigma, \quad A_2 := \begin{pmatrix} A_g & -B_{gv} \\ k_h C_g A_g & -k_h C_g B_{gv} \end{pmatrix}, \qquad B_2 : X_w \to X_\Sigma, \quad B_2 := \begin{pmatrix} B_{gw} \\ k_h C_g B_{gw} \end{pmatrix}, \tag{2.17}$$

$$C : X_\Sigma \to \mathbb{R}, \qquad C := \begin{pmatrix} C_g & 0 \end{pmatrix}. \tag{2.18}$$

What remains is to describe when the two modes are active, in terms of the states and inputs. As $x_h = u$, it holds that:

$$u = \left\langle \begin{pmatrix} e \\ u \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle_{\mathbb{R}^2} = \left\langle \begin{pmatrix} x_g \\ x_h \end{pmatrix}, \begin{pmatrix} 0_{X_\mathcal{G}} \\ 1 \end{pmatrix} \right\rangle_{X_\Sigma}. \tag{2.19}$$

Because C_g is a bounded linear functional, by Riesz-Fréchet [27, Sec. 8.3], there is an element $c_g \in X_\mathcal{G}$ such that $C_g(x_g) = \langle x_g, c_g \rangle$ for all $x_g \in X_\mathcal{G}$. As $e = C_g(x_g)$, this gives the following result:

$$k_h e - u = \left\langle \begin{pmatrix} e \\ u \end{pmatrix}, \begin{pmatrix} k_h \\ -1 \end{pmatrix} \right\rangle_{\mathbb{R}^2} = \left\langle \begin{pmatrix} x_g \\ x_h \end{pmatrix}, \begin{pmatrix} k_h c_g \\ -1 \end{pmatrix} \right\rangle_{X_\Sigma}. \tag{2.20}$$

Comparing the obtained relations (2.19) (2.20) to the definition of $\mathcal{F}$ (2.3) it is easy to see that:

$$(e,u) \in \mathcal{F} \iff x \in \mathcal{S} := \mathcal{S}^+ \cup -\mathcal{S}^+, \text{ with } \mathcal{S}^+ := \left\{x \in X_\Sigma \mid \langle x, f_1\rangle \geq 0 \ \wedge \ \langle x, f_2\rangle \geq 0\right\}, \tag{2.21}$$

where the vectors f_1 and f_2 in X_Σ are given by:

$$f_1 := \begin{pmatrix} k_h c_g \\ -1 \end{pmatrix}, \quad f_2 := \begin{pmatrix} 0_{X_\mathcal{G}} \\ 1 \end{pmatrix}. \tag{2.22}$$

Sometimes, it is easier to work with the linear functionals F_1 and F_2 on $X_\mathcal{G}$ given by $F_1 = \langle \cdot, f_1\rangle$ and $F_2 = \langle \cdot, f_2\rangle$.

The next step is to construct sets $\mathcal{S}_1$ and $\mathcal{S}_2$ such that $(x(t), w(t)) \in \mathcal{S}_1 \iff (e(t), u(t), \dot{e}(t)) \in \mathcal{F}_1$ and $(x(t), w(t)) \in \mathcal{S}_2 \iff (e(t), u(t), \dot{e}(t)) \in \mathcal{F}_2$. The dependence on w is needed because this input can directly influence the derivative of the output of the plant, $\dot{e}$. Using (2.16)-(2.18),(2.22) and (2.3), the following relation can be obtained:

$$\begin{aligned}\langle A_1 x(t) + B_1 w(t), f_1\rangle &= k_h C_g \underbrace{(A_g x_g(t) - B_{gv} x_h(t) + B_{gw} w(t))}_{=\dot{x}_g(t)} - \omega_h C_g x_g(t), \\ &= k_h \dot{e}(t) - \omega_h e(t), \\ &= \left\langle \begin{pmatrix} \dot{e}(t) \\ \omega_h e(t)\end{pmatrix}, v_1 \right\rangle.\end{aligned}$$

From this equality, the equalities (2.19) and (2.20) and the definition of $\mathcal{F}_2$ (2.5), it can be seen that the P-mode is needed when $(x(t), w(t)) \in \mathcal{S}_2$:

$$\mathcal{S}_2 := \mathcal{S}_2^+ \cup -\mathcal{S}_2^+ \text{ with } \mathcal{S}_2^+ := \left\{(x,w) \in X_\Sigma \times X_w \mid \langle x, f_2\rangle > 0 \wedge \langle x, f_1\rangle = 0 \wedge \langle (A_1 x + B_1 w), f_1\rangle < 0\right\}. \tag{2.23}$$

The I-mode is thus active for $(x, w) \in \mathcal{S}_1$ with $\mathcal{S}_1$ defined as $\mathcal{S}_1 := \{(x,w) \in X_\Sigma \times X_w \mid x \in \mathcal{S}\} \setminus \mathcal{S}_2$. Using (2.21) and (2.23), $\mathcal{S}_1$ can be written as $\mathcal{S}_1 = \mathcal{S}_1^+ \cup -\mathcal{S}_1^+$, with

$$\mathcal{S}_1^+ := \left\{(x,w) \in X_\Sigma \times X_w \mid \langle x, f_2\rangle \geq 0 \wedge \langle x, f_1\rangle \geq 0 \wedge \left(\langle x, f_1\rangle > 0 \vee \langle (A_1 x + B_1 w), f_1\rangle \geq 0\right)\right\}. \tag{2.24}$$

With A_1, A_2, B_1, B_2 and C from (2.16)-(2.18) and $\mathcal{S}_1$ and $\mathcal{S}_2$ from (2.24) and (2.23), the closed loop system Σ can thus be written as:

$$\Sigma : \begin{cases} \dot{x}(t) = A_1 x(t) + B_1 w(t) & \text{if } (x(t), w(t)) \in \mathcal{S}_1, \\ \dot{x}(t) = A_2 x(t) + B_2 w(t) & \text{if } (x(t), w(t)) \in \mathcal{S}_2, \\ x(0) = x_0 \in \mathcal{S}, \\ e(t) = Cx(t) & \text{and } x(t) \in \mathcal{S}. \end{cases} \tag{2.25}$$

Remark 2.6. *The formulation of $\mathcal{S}_2$ in* (2.23) *is (slightly) different than in earlier literature. In the work by Deenen et al. [9] as well as in earlier work by Sharif et al. [12], the first inequality in the formulation of $\mathcal{S}_2$ is not strict. In this earlier work, the $\mathcal{S}_2$ set is defined as:*

$$\tilde{\mathcal{S}}_2 = \tilde{\mathcal{S}}_2^+ \cup -\tilde{\mathcal{S}}_2^+ \textit{ with } \tilde{\mathcal{S}}_2^+ := \left\{(x,w) \in X_\Sigma \times X_w \mid \langle x, f_2\rangle \geq 0 \wedge \langle x, f_1\rangle = 0 \wedge \langle (A_1 x + B_1 w), f_1\rangle < 0\right\}.$$

This is problematic, because with the formulation of $\mathcal{S}_1$ as in (2.24) *(which is also used in the cited work), there is overlap between the sets:*

$$\mathcal{S}_1 \cap \tilde{\mathcal{S}}_2 = \left\{(x,w) \in X_\Sigma \times X_w \mid \langle x, f_2\rangle = 0 \wedge \langle x, f_1\rangle = 0 \wedge \langle (A_1 x + B_1 w), f_1\rangle \neq 0\right\}.$$

This set is in general not the empty set. As these sets are used to determine which mode is being used in the system description (2.25)*, this means that there are points where two modes are active, which is problematic because then it could happen that Σ as in* (2.25) *prescribes two different equations for $\dot{x}(t)$ simultaneously.*

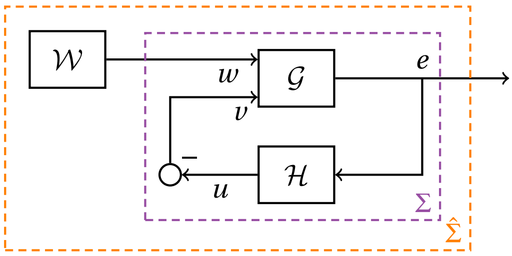


Figure 2.2: By writing w as the output of another linear system, it is possible to describe the output using a system $\hat{\Sigma}$ that has no external inputs.

2.2.1 Incorporating the input in the closed loop system

Under an additional assumption, it is possible to incorporate the input w into the system [9]. This is useful, because the system can then be represented as a system without inputs, which is needed for the proofs in the following chapters. The idea is that if w is the output of a linear system itself, the system including w can be written as a system $\hat{\Sigma}$ that has no inputs. See Figure 2.2.

The following assumption on w is needed to take this approach:

Definition 2.7. *Let X_w and X_{x_w} be Hilbert spaces, with possibly infinite dimensions. Then $w : \mathbb{R}_{\geq 0} \to X_w$ is called a Bohl function if there are bounded linear operators $H_w : X_{x_w} \to X_w$ and $F_w : X_{x_w} \to X_{x_w}$ and a vector $x_{w,0} \in X_{x_w}$ such that w is the output of the linear system given by:*

$$\mathcal{W} : \begin{cases} \dot{x}_w(t) = F_w x_w(t), \\ w(t) = H_w x_w(t), \\ x_w(0) = x_{w,0}. \end{cases} \tag{2.26}$$

If F_w and H_w are matrices, this means that:

$$w(t) = H_w e^{F_w t} x_{w,0}. \tag{2.27}$$

Here, $e^{F_w t}$ is the matrix exponential of $F_w t$, see Equation (3.1) for the definition. The same expression can be used for bounded operators, as will be discussed in the next chapter (Section 3.1).

Remark 2.8. *The definition of a Bohl function in Definition 2.7 is more general than the definition commonly used in literature. Usually, the X_w and X_{x_w} spaces are finite dimensional [28, Sec. 3.5], such that F_w and H_w are represented by matrices and thus the matrix exponential as in (2.27) can be used as an alternative definition.*

The systems from (2.25) and (2.26) can be written as a combined system that switches between two linear modes. The state of this system is $\hat{x} := (x, x_w) \in \hat{X}_\Sigma := (X_\Sigma \times X_{x_w})$.

$$\hat{\Sigma} : \begin{cases} \dot{\hat{x}}(t) = \hat{A}_1 \hat{x}(t) & \text{if } \hat{x}(t) \in \hat{\mathcal{S}}_1, \\ \dot{\hat{x}}(t) = \hat{A}_2 \hat{x}(t) & \text{if } \hat{x}(t) \in \hat{\mathcal{S}}_2, \\ e(t) = \hat{C} \hat{x}(t) & \text{and } \hat{x}(t) \in \hat{\mathcal{S}}. \end{cases} \tag{2.28}$$

The operators $\hat{A}_1 : \hat{X}_\Sigma \to \hat{X}_\Sigma$, $\hat{A}_2 : \hat{X}_\Sigma \to \hat{X}_\Sigma$ and $\hat{C} : \hat{X}_\Sigma \to \mathbb{R}$ are given by:

$$\hat{A}_i = \begin{pmatrix} A_i & B_i H_w \\ 0 & F_w \end{pmatrix}, \quad \forall i \in \{1, 2\}, \qquad \hat{C} = \begin{pmatrix} C & 0 \end{pmatrix}$$

Choose $\hat{f}_i := (f_i, 0)$, such that $\langle x, f_i \rangle = \langle \hat{x}, \hat{f}_i \rangle = F_i(x)$ for $i \in \{1,2\}$. The sets $\hat{\mathcal{S}}_1$ and $\hat{\mathcal{S}}_2$ can then describe the conditions for the integrator and gain modes only in terms of the (expanded) states $\hat{x}$:

$$\begin{aligned}
\hat{\mathcal{S}} &:= \hat{\mathcal{S}}^+ \cup -\hat{\mathcal{S}}^+ \text{ with } \hat{\mathcal{S}}^+ := \left\{\hat{x} \in \hat{X}_\Sigma \mid \langle \hat{x}, \hat{f}_1 \rangle \geq 0 \wedge \langle \hat{x}, \hat{f}_2 \rangle \geq 0\right\}, \\
\hat{\mathcal{S}}_2 &:= \hat{\mathcal{S}}_2^+ \cup -\hat{\mathcal{S}}_2^+ \text{ with } \hat{\mathcal{S}}_2^+ := \left\{\hat{x} \in \hat{X}_\Sigma \mid \langle \hat{x}, \hat{f}_2 \rangle > 0 \wedge \langle \hat{x}, \hat{f}_1 \rangle = 0 \wedge \langle \hat{A}_1 \hat{x}, \hat{f}_1 \rangle < 0\right\}, \\
\hat{\mathcal{S}}_1 &:= \hat{\mathcal{S}} \setminus \hat{\mathcal{S}}_2 = \hat{\mathcal{S}}_1^+ \cup -\hat{\mathcal{S}}_1^+ \text{ with } \hat{\mathcal{S}}_1^+ := \left\{\hat{x} \in \hat{X}_\Sigma \mid \langle \hat{x}, \hat{f}_2 \rangle \geq 0 \wedge \langle \hat{x}, \hat{f}_1 \rangle \geq 0 \wedge \left(\langle \hat{x}, \hat{f}_1 \rangle > 0 \vee \langle \hat{A}_1 \hat{x}, \hat{f}_1 \rangle \geq 0\right)\right\}.
\end{aligned}$$

Because the system is already a switching system, it is relatively easy to allow switches in the $\mathcal{W}$ system too. This means that it is possible to only require w to be piecewise Bohl in the sense of the following definition.

Definition 2.9. *A function $w : \mathbb{R}_{\geq 0} \to X_w$ is called a piecewise Bohl function if there exists $0 < t_1 < t_2 \ldots$ such that each compact interval of $\mathbb{R}_{\geq 0}$ only contains finitely many t_j's and for each interval $[t_j, t_{j+1})$ there is a Bohl function $\tilde{w}_j$ such that $w(t) = \tilde{w}_j(t)$ for all $t \in [t_j, t_{j+1})$.*

2.2.2 Closing the $\mathcal{S}_1$ and $\mathcal{S}_2$ sets

Theorem 2.5 showed that $\mathcal{H}$ has the same Carathéodory solutions as $\overline{\mathcal{H}}$, if e is absolutely continuous. As the output of the plant will be absolutely continuous, it is clear that also the closed loop system Σ can be equivalently written as:

$$\overline{\Sigma} : \begin{cases} \left(\dot{x}(t) = A_1 x(t) + B_1 w(t) \wedge (x(t), w(t)) \in \overline{\mathcal{S}_1}\right) \vee \left(\dot{x}(t) = A_1 x(t) + B_1 w(t) \wedge (x(t), w(t)) \in \overline{\mathcal{S}_2}\right), \\ x(0) = x_0, \\ e(t) = Cx(t). \end{cases} \tag{2.29}$$

2.3 Hybrid System framework

The discontinuous representation of the closed loop system in the previous section fits into a more general class of systems: hybrid systems. These systems describe a combination of continuous and discrete dynamics. Hybrid systems have been studied since the 1960's [29] and have evolved in a broadly studied class of systems [30]. The model structure is useful to describe systems with continuous dynamics and discrete elements, such as switches or digital controller components. In this section, the framework is introduced and it is shown that HIGS fits into this framework. This also clarifies the name of the Hybrid Integrator Gain System (HIGS).

Solutions to hybrid systems are of the form $\phi(t, j) : E \to \mathbb{R}$ where $E \subseteq \mathbb{R}_{\geq 0} \times \mathbb{N}$ is the hybrid domain. A hybrid domain is a union of intervals $[t_j, t_{j+1}] \times \{j\}$, where $0 = t_1 \leq t_2 \leq t_3 \ldots$. The last interval (if it exists) can also be of the form $[t_j, t_{j+1})$ or $[t_j, \infty)$ [31]. A general definition of a hybrid systems is as follows.

Definition 2.10 (Adapted from Goebel et al., [31])**.** *A hybrid system Ψ acting on a state space $\mathcal{X}$ is a tuple $(f, g, \mathcal{C}, \mathcal{D}, \phi_0)$ that contains the data of the Hybrid System:*

- *The flow map $f(\phi) : \mathcal{X} \times \mathbb{R}_{\geq 0} \rightrightarrows \mathcal{X}$ is a set valued mapping that maps elements $\mathcal{X}$ to sets in $\mathcal{X}$ and describes the continuous dynamics.*
- *The jump map $g(\phi) : \mathcal{X} \times \mathbb{R}_{\geq 0} \rightrightarrows \mathcal{X}$ is a set valued mapping that describes the discrete dynamics.*
- *The flow set $\mathcal{C} \subseteq \mathcal{X}$ is the set where the continuous dynamics are active.*

- *The jump set* $\mathcal{D} \subseteq \mathcal{X}$ *is the set where the discrete dynamics are active.*
- $\phi_0 \in \mathcal{X}$ *is the initial condition.*

$\phi(t,j) : E \to \mathbb{R}$ *is called a solution to the hybrid system* Ψ *if the following conditions hold:*

- E *is a hybrid domain.*
- $\phi(0,1) = \phi_0$.
- *For each* $j \in \mathbb{N}$*, the function* $t \mapsto \phi(t,j)$ *is absolutely continuous on the interval* $I_j := \{t \mid (t,j) \in E\}$.
- *For all* $j \in \mathbb{N}$*,*

$$\begin{aligned} \dot{\phi}(t,j) &\in f(\phi(t,j)) && \textit{for almost every } t \in I_j\textit{, and} \\ \phi(t,j) &\in \mathcal{C} && \textit{for all } t \textit{ in the interior of } I_j \end{aligned} \tag{2.30}$$

- *For all* $(t,j) \in E$ *such that* $(t,j+1) \in E$*,*

$$\begin{aligned} &\phi(t,j) \in g(\phi(t,j)), \textit{ and} \\ &\phi(t,j) \in \mathcal{D}. \end{aligned} \tag{2.31}$$

The system description of the hybrid system Ψ shows that when the state is in the flow set $\mathcal{C}$, the solution can 'flow' as described by a differential equation as in (2.30). In the jump set, the solution exhibits discrete behavior by jumping when the discrete variable j increases, whilst no time t is passing.

The differential inclusion as in (2.30) generalizes differential equations. If the set $f(\phi)$ is a singleton for every ϕ, then the differential inclusion is equivalent to a regular differential equation. This generalization is quite common in literature on hybrid systems [32]. One reason for this generalization is that it allows to consider alternative solution concepts (Filippov and Krasovskii solutions) that are useful to study discontinuous controllers. Another reason to consider differential inclusions in the study of HIGS, is that theorems from differential inclusions (e.g. from Aubin & Cellina [33]) are useful to study conditions for existence of solutions, as for example in the work of Heemels & Tanwani [23].

The HIGS can also be formulated as an hybrid system, as shown in the following theorem.

Theorem 2.11. *The closed loop interconnection of the HIGS with a linear system as in* (2.25) *with states* $x \in X$ *and initial condition* x_0 *can be written as a hybrid system as in Definition 2.10 for a continuous input* $w(t)$*. This is done by choosing as state space for the hybrid system* $\mathcal{X}$ *as* $X \times \mathbb{R} \times \mathbb{R}$ *with states* $\phi(t,j) = \begin{pmatrix} \tilde{x}(t,j) & q(t,j) & \tau(t,j) \end{pmatrix}^T$*, and the following flow and jump map, flow and jump set and initial condition:*

$$f\begin{pmatrix} \tilde{x}(t,j) \\ q(t,j) \\ \tau(t,j) \end{pmatrix} = \left\{ \begin{pmatrix} \left(A_1\tilde{x}(t,j) + B_1 w(\tau)\right)(2 - q(t,j)) + \left(A_2\tilde{x}(t,j) + B_2 w(\tau)\right)(q(t,j) - 1) \\ 0 \\ 1 \end{pmatrix} \right\},$$

$$g\begin{pmatrix} \tilde{x}(t,j) \\ q(t,j) \\ \tau(t,j) \end{pmatrix} = \left\{ \begin{pmatrix} \tilde{x}(t,j) \\ 3 - q(t,j) \\ \tau(t,j) \end{pmatrix} \right\},$$

$$\mathcal{C} = \left(\overline{\mathcal{S}}_1 \times \{1\} \times \mathbb{R}_{\geq 0}\right) \cup \left(\overline{\mathcal{S}}_2 \times \{2\} \times \mathbb{R}_{\geq 0}\right),$$

$$\mathcal{D} = \left(\overline{\mathcal{S}}_1 \times \{2\} \times \mathbb{R}_{\geq 0}\right) \cup \left(\overline{\mathcal{S}}_2 \times \{1\} \times \mathbb{R}_{\geq 0}\right),$$

$$\phi_0 = \begin{pmatrix} x_0 \\ 1 \\ 0 \end{pmatrix}.$$

Then for a function $x : \mathbb{R}_{\geq 0} \supseteq \mathbb{T} \to X$*, the following statements are equivalent:*

i) x *is a Carathéodory solution to the system* $\overline{\Sigma}$ *for* $t \in \mathbb{T}$ *that does not switch infinitely many times on a compact time interval.*

ii) E *is a hybrid time domain with* $\{t \in \mathbb{R} \mid (t,j) \in E\} = \mathbb{R}$ *and the function* $\phi : E \to \mathcal{X}$ *given by* $\phi(t,j) = \begin{pmatrix} \tilde{x}(t,j) & q(t,j) & \tau(t,j) \end{pmatrix}^T$ *is a solution to the hybrid system* $(f, g, \mathcal{C}, \mathcal{D}, \phi_0)$ *for some* $q : E \to \mathbb{R}$ *and* $\tau(t,j) : E \to \mathbb{R}$.

Proof. For *ii)* $\Longrightarrow$ *i)*, consider a solution ϕ to the hybrid system. First note, that by looking at the third component of g and f, it is clear that $\tau(t,j) = t$. From the second component, it is clear that for fixed j, $q(t,j)$ is constant and that it switches between 1 and 2. Furthermore, because the jump map does not cause jumps in the state $\tilde{x}$, the set $\{\tilde{x}(t,j) \mid j \in \mathbb{N}$ is such that $(t,j) \in E\}$ only has a single element for each time t. This means that the function $x(t)$ is continuous. It is even absolutely continuous, since it is differentiable everywhere, except on time instances t such that $(t,j) \in E$ and $(t, j+1) \in E$ of which there are only finitely many on each compact time interval.

For almost every $t \in [0, t_{\max})$, the flow map is satisfied and $\phi(t,j) \in \mathcal{C}$. This means that:

$$\left(\dot{\tilde{x}}(t,j) = A_1 \tilde{x}(t,j) + B_1 w(\tau(t,j)) \wedge \phi \in \left(\overline{\mathcal{S}}_1 \times \{1\} \times \mathbb{R}_{\geq 0}\right)\right) \vee \tag{2.32}$$
$$\left(\dot{\tilde{x}}(t,j) = A_2 \tilde{x}(t,j) + B_2 w(\tau(t,j)) \wedge \phi \in \left(\overline{\mathcal{S}}_2 \times \{2\} \times \mathbb{R}_{\geq 0}\right)\right)$$

As $\tilde{x}(t,j) = x(t)$ and $\tau = t$, this means that $x(t)$ is a Carathéodory solution to $\overline{\Sigma}$ as in (2.29).

For *i)* $\Longrightarrow$ *ii)*, consider the Carathéodory solution x to the system Σ. Denote by $\mathcal{T}_1$ the set where $\dot{x}(t) = A_1 x(t) + B_1 w(t)$ is satisfied and by $\mathcal{T}_2$ the set where $\dot{x}(t) = A_2 x(t) + B_2 w(t) \wedge (x, w) \in \overline{\mathcal{S}_2}$ is satisfied. As x is a Carathéodory solution $\overline{\mathcal{T}_1 \cup \mathcal{T}_2} = [0, t_{\max}]$. Define $\tilde{q} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ as:

$$\tilde{q}(t) = \begin{cases} 1 & \text{if } t \in \overline{\mathcal{T}_1}, \\ 2 & \text{if } t \in \overline{\mathcal{T}_2} \setminus \overline{\mathcal{T}_1}. \end{cases}$$

And let j be:

$$j(t) := 1 + \text{the number of switches of } \tilde{q} \text{ on } [0, t].$$

This is well defined, because there are only finitely many switches. Now define the elements of the function $\phi(t,j) = \begin{pmatrix} \tilde{x}(t,j) & q(t,j) & \tau(t,j) \end{pmatrix}$ by:

$$\tilde{x}(t,j) = x(t,j), \quad q(t,j) = q(t_j), \quad \tau(t,j) = t, \quad \forall (t,j) \in E.$$

By construction, the flow map is satisfied. This can be seen from the fact that for almost every t such that $q(t,j) = 1$, it holds that $\tilde{x}(t,j) \in \overline{\mathcal{S}}_1$ and

$$\dot{\tilde{x}}(t,j) = A_1 \tilde{x}(t,j) + B_1 w(\tau(t,j)).$$

Similarly, for almost every t such that $q(t,j) = 2$, it holds that $\tilde{x}(t,j) \in \overline{\mathcal{S}}_2$ and

$$\dot{\tilde{x}}(t,j) = A_2 \tilde{x}(t,j) + B_2 w(\tau(t,j)).$$

To see that the jump map is also satisfied notice that by the construction of j, $q(t,j) \neq q(t, j+1)$. Combining this with the fact that $q(t,j) \in \{1,2\}$ for all $(t,j) \in \mathbb{R}$, it is clear that the jump map is satisfied, since:

$$q(t,j) + q(t,j+1) = 3.$$

Furthermore, by continuity of x and w and the closedness of $\overline{\mathcal{S}}_1$ and $\overline{\mathcal{S}}_2$, it holds that $(x(t), w(t)) \in \mathcal{S}_{q(t_j)}$ for $t \in I_j$. It can thus be concluded that:

$$\begin{aligned} &\phi(t,j) \in \mathcal{C} \quad \text{for all } (t,j) \in E, \text{ such that } t \text{ lies in the interior of } I_j, \quad \text{and} \\ &\phi(t,j) \in \mathcal{D} \quad \text{for all } (t,j) \in E \text{ such that } (t, j+1) \in E. \end{aligned}$$ □

In the above, the flow and jump sets $\mathcal{C}$ and $\mathcal{D}$ are closed. This was possible because $\mathcal{H}$ and $\overline{\mathcal{H}}$ were shown to be equivalent formulations of HIGS in Theorem 2.5. The closedness is important, because existing theorems often rely on the following assumptions [30].

Assumption 2.12 ('Basic Assumptions' on hybrid systems, from [30])**.** *The Basic Assumptions are the following conditions on a hybrid system $(f, g, \mathcal{C}, \mathcal{D}, \phi_0)$:*

1. *The sets $\mathcal{C}$ and $\mathcal{D}$ are closed in $\mathcal{X}$.*
2. *$f : \mathcal{X} \to \mathcal{X}$ is an outer semiconinuous set-valued mapping that is locally bounded on $\mathcal{C}$ and such that $f(\phi)$ is non-empty and convex for every $\phi \in \mathcal{C}$.*
3. *$g : \mathcal{X} \to \mathcal{X}$ is outer semicontinuous that is locally bounded on $\mathcal{D}$ and $g(\phi)$ is nonempty for every $\phi \in \mathcal{D}$.*

The hybrid system as in Theorem 2.11 satisfies these Basic Assumptions.

2.4 (Extended) Projected Dynamical Systems framework

As an alternative to explicitly prescribing the two modes of operation of the HIGS, the HIGS can also be represented using a projection-like operator. In literature, this interpretation of the HIGS is used for both intuitive understanding and establishing mathematical properties such as a well-posedness [23].

In short, a projected dynamical system projects dynamics onto a tangent cone.

2.4.1 The tangent cone

Definition 2.13. *The tangent cone of a point x in a normed space X to a set $A \subseteq X$ denoted $T_A(x)$ is the following set:*

$$T_A(x) := \left\{ w \in X \mid \exists (x_i)_{i\in\mathbb{N}} \text{ in } A,\ (\tau_i)_{i\in\mathbb{N}} \text{ in } \mathbb{R}_+ \text{ with } x_i \to x \text{ and } \tau_i \to 0 \text{ as } i \to \infty \text{ s.t. } w = \lim_{i\to\infty} \frac{x_i - x}{\tau_i} \right\}.$$

The following proposition establishes some properties of the tangent cone.

Proposition 2.14. *Let A and B be sets in a vector space X. Then,*

a) *$T_A(x)$ is a cone. That is, if $w \in T_A(x)$ and $\lambda > 0$, then $\lambda w \in T_A(x)$.*

b) *If A is convex, then $T_A(x)$ is a convex cone.*

c) *$T_{(A\cup B)}(x) = T_A(x) \cup T_B(x) \quad \forall x \in X$.*

Proof. To prove a), fix any $w \in T_A(x)$ and $\lambda > 0$. Then by definition of the tangent cone, there is a sequence $(x_i)_{i\in\mathbb{N}}$ in A and a sequence $(\tau_i)_{i\in\mathbb{N}}$ in $\mathbb{R}_+$ with $x_i \to x$ and $\tau_i \to 0$ as $i \to \infty$ such that $w = \lim_{i\to\infty} \frac{x_i - x}{\tau_i}$. Now let $\tilde{\tau}_i = \tau_i / \lambda$ for any $i \in \mathbb{N}$. Then $\tilde{\tau}_i \to 0$ as $i \to \infty$ and

$$\lim_{i\to\infty} \frac{x_i - x}{\tilde{\tau}_i} = \lim_{i\to\infty} \lambda \frac{x_i - x}{\tau_i} = \lambda w.$$

And thus, $\lambda w \in T_A(x)$.

b): Since by a) $T_A(x)$ is a cone, it suffices to show that for all $w, \tilde{w} \in T_A(x)$, we have that $w + \tilde{w} \in T_A(x)$. To this extent, fix $w, \tilde{w} \in T_A(x)$. Then there exist sequences $(x_i)_{i\in\mathbb{N}}, (\tilde{x}_i)_{i\in\mathbb{N}}$ in A and $(\tau_i)_{i\in\mathbb{N}}, (\tilde{\tau}_i)_{i\in\mathbb{N}}$ in $\mathbb{R}_+$ with $x_i, \tilde{x}_i \to x$ and $\tau_i, \tilde{\tau}_i \to 0$ as $i \to \infty$ such that:

$$w = \lim_{i\to\infty} \frac{x_i - x}{\tau_i} \text{ and } \tilde{w} = \lim_{i\to\infty} \frac{\tilde{x}_i - \tilde{x}}{\tilde{\tau}_i}.$$

Adding these two equalities gives:

$$\begin{aligned} w+\tilde{w} &= \lim_{i\to\infty} \frac{x_i - x}{\tau_i} + \frac{\tilde{x}_i - x}{\tilde{\tau}_i}, \\ &= \lim_{i\to\infty} \frac{1}{\tau_i} x_i + \frac{1}{\tilde{\tau}_i}\tilde{x}_i - \left(\frac{1}{\tau_i} + \frac{1}{\tilde{\tau}_i}\right) x, \\ &= \lim_{i\to\infty} \frac{\tau_i + \tilde{\tau}_i}{\tau_i \tilde{\tau}_i} \left(\frac{\tilde{\tau}_i}{\tau_i + \tilde{\tau}_i} x_i + \frac{\tau_i}{\tau_i + \tilde{\tau}_i}\tilde{x}_i - x\right). \end{aligned} \tag{2.33}$$

Now define for $i \in \mathbb{N}$, $\sigma_i := \frac{\tau_i \tilde{\tau}_i}{\tau_i + \tilde{\tau}_i} > 0$ and $y_i := \frac{\tilde{\tau}_i}{\tau_i+\tilde{\tau}_i} x_i + \frac{\tau_i}{\tau_i+\tilde{\tau}_i}\tilde{x}_i$. By convexity of A, $y_i \in A$. Also, $y_i \to x$ and $\sigma_i \to 0$ as $i \to \infty$. This leads to (2.33) becoming:

$$w + \tilde{w} = \lim_{i\to\infty} \frac{y_i - x}{\sigma_i}.$$

Thus, by definition of the tangent cone, $w + \tilde{w} \in T_A(x)$.

For c): To show that $T_{(A\cup B)}(x) \subseteq T_A(x) \cup T_B(x)$, fix any $w \in T_{(A\cup B)}(x)$. There exists a sequence $(x_i)_{i\in\mathbb{N}}$ in $A\cup B$ that converges to x and a sequence of positive real numbers $(\tau_i)_{i\in\mathbb{N}}$ that converges to 0 such that $w = \lim_{i\to\infty}(x_i - x)/\tau_i$. Now define the sets of indices for which x_i lies in either of the sets:

$$\begin{aligned} N_A &= \{i \in \mathbb{N} \mid x_i \in A\}, \\ N_B &= \{i \in \mathbb{N} \mid x_i \in B\}. \end{aligned}$$

Then, at least one of the subsequences $(x_i)_{i\in N_A}$ and $(x_i)_{i\in N_B}$ has infinitely many element and still converges to x. By selecting the corresponding subsequence of $(\tau_i)_{i\in\mathbb{N}}$, it follows that $w \in T_A(x)$ or $w \in T_B(x)$. As this holds for any $w \in T_{(A\cup B)}(x)$, it can be concluded that indeed $T_{(A\cup B)}(x) \subseteq T_A(x) \cup T_B(x)$

From the definition of the tangent cone and the fact that $A \subseteq A\cup B$ and $B \subseteq A\cup B$, it follows that also $T_{(A\cup B)}(x) \supseteq T_A(x) \cup T_B(x)$. □

As the main idea of the HIGS is to keep the input and output in a sector, the tangent cone of this sector is of special interest. In this view, the following proposition is useful.

Proposition 2.15 (Sharif et al. [12]). *Let S be of the form $S = K \cup -K$ with K the closed cone formed using n_f constraints (defined through f_1 to $f_{n_f} \in X$):*

$$K := \left\{x \in X \mid \langle x, f_j\rangle \geq 0 \ \forall j \in \{1,2,\ldots,n_f\}\right\}. \tag{2.34}$$

Then, the tangent cone to K at x is empty if $x \notin K$ and otherwise given by:

$$T_K(x) = \{w \in X \mid \langle w, f_j\rangle \geq 0 \ \forall j \in J(x)\}$$

With $J(x)$ denoting the set of active constraints $\left\{j \in \{1,2,\ldots,n_f\} \mid \langle x, f_j\rangle = 0\right\}$.
Furthermore, the tangent cone to S at x is given by:

$$T_S(x) = \begin{cases} T_K(x) & \text{for } x \in K \setminus -K, \\ -T_K(x) & \text{for } x \in -K \setminus K, \\ K \cup -K & \text{for } x \in K \cap -K, \\ \emptyset & \text{for } x \notin S. \end{cases}$$

Proof. To prove the expression for $T_K(x)$, notice first that if x lies outside K, there is no sequence in K that has x as its limit, so the tangent cone at x is empty.

For any $x \in K$ and $w \in T_K(x)$ there exists a sequence $(x_i)_{i\in\mathbb{N}}$ in K and a sequence $(\tau_i)_{i\in\mathbb{N}}$ in $\mathbb{R}_+$ with $x_i \to x$ and $\tau_i \to 0$ as $i \to \infty$ such that $w = \lim_{i\to\infty} \frac{x_i - x}{\tau_i}$. For each j in the set of active constraints, it holds that $\langle x, f_j \rangle = 0$. As by assumption the x_i's lie in K and thus also $\langle x_i, f_j \rangle \geq 0$ for any i. Then by linearity of the inner product,

$$\left\langle \frac{x_i - x}{\tau_i}, f_j \right\rangle \geq 0 \quad \forall i \in \mathbb{N}.$$

And thus also the limit w lies in $\{w \in X \mid \langle w, f_j \rangle \geq 0\ \forall j \in J(x)\}$.
This shows that $T_K(x) \subseteq \{w \in X \mid \langle w, f_j \rangle \geq 0\ \forall j \in J(x)\}$.

To show "$\supseteq$", fix any $x \in K$ and any w such that $\langle w, f_j \rangle \geq 0\ \forall j \in J(x)$. To show that w lies in $T_K(x)$, define $\tau_i = \epsilon / i$ and $x_i = x + w\tau_i$ with $\epsilon > 0$ to be determined later. For active constraints, $\langle x, f_j \rangle = 0$ and thus:

$$\langle x_i, f_j \rangle = \tau_i \langle w, f_j \rangle \geq 0 \quad \forall j \in J(x).$$

For non-active constraints, $\langle x, f_j \rangle > 0$ because $x \in K$, this means:

$$\langle x_i, f_j \rangle = \langle x, f_j \rangle + \frac{\epsilon}{i} \langle w, f_j \rangle \quad \forall j \notin J(x).$$

As there are finitely many constraints, ϵ can be chosen such that $\langle x, f_j \rangle + \epsilon \langle w, f_j \rangle$ is positive for all $j \notin J(x)$ and thus x_i lies in K for all $i \in \mathbb{N}$. Also, $\lim_{i\to\infty} x_i = x$ and $\lim_{i\to\infty} \tau_i = 0$. Furthermore, by construction:

$$\frac{x_i - x}{\tau_i} = w \quad \forall i \in \mathbb{N}.$$

And thus by construction $w = \lim_{i\to\infty}(x_i - x)/\tau_i$. From this it can be concluded that w lies in $T_K(x)$.

The tangent cone $T_S(x)$ is obtained immediately from the derived expression of $T_K(x)$ by application of Proposition 2.14 c. □

2.4.2 Projected dynamical systems

Research into Projected Dynamical Systems (PDS) originates from equilibrium problems that arise in various applications, such as economics and operations research [34, 35]. These problems are described by variational inequalities.

Definition 2.16. *Consider a Hilbert space X and let K be a closed, convex and non-empty subset of X, and let $f : K \to X$. Then the variational inequality problem associated with K and f (denoted as VI(F, K)) is:*

$$\text{Find } x \in K \text{ such that } \langle -f(x), y - x \rangle \geq 0 \quad \forall y \in K. \tag{2.35}$$

In the analysis of the problem, it turned out to be useful to consider the following dynamical system.

Definition 2.17. *Consider a Hilbert space X and let K be a closed, convex and non-empty subset of X, and let $f : K \to X$. Then the projected dynamical system associate K and f (denoted as PDS(F, K)) is the following differential equation:*

$$\dot{x}(t) = \Pi_K(x(t), f(x(t))); \quad x(0) = x_0 \in K \tag{2.36}$$

where the projection onto the tangent cone is defined trough

$$\Pi_K(x, v) := \underset{w \in T_K(x)}{\arg\min} \| w - v \|. \tag{2.37}$$

The following result clarifies why the dynamical result was interesting for the analysis of variational inequality problems:

Theorem 2.18. *Let X be a Hilbert space and $K \subset X$ be closed, convex and non-empty, and let $f : K \to X$. Then:*

$$x^* \text{ is a solution to VI(F,K)} \quad \Longleftrightarrow \quad \Pi_K(x^*, f(x^*)) = 0.$$

Thus, x^* is a critical point, or equilibrium of the dynamical system.

For a proof, see [36]. This work by Cojocaru & Jonker also proves the following result, showing existence of local solutions in a possibly infinite dimensional setting.

Theorem 2.19. *Consider a Hilbert space X and let $K \subset X$ be closed, convex and non-empty. Let $f : K \to X$ be a Lipschitz continuous mapping with Lipschitz constant b. Take $x_0 \in K, L > 0$ such that $\|x_0\| \leq L$. Then the system:*

$$\begin{cases} \dot{x}(t) = \Pi_K(x(t), f(x(t))), \\ x(0) = x_0. \end{cases}$$

has a unique solution on the interval $[0, l]$ where $l = \frac{L}{\|f(x_0)\| + bL}$.

2.4.3 Extended projected dynamical systems

The HIGS constrains the dynamics to a sector. This is similar to a projection. However, to study the dynamics of systems with projections in controllers, the framework of PDS needs to be adjusted. Because only the controller performs some kind of projection, the plant states should not be affected by the projection of the controller dynamics. Thus only partial projection of the states is desired in such a way that only the controller states are changed. The following extension to the PDS framework was first proposed by Sharif et al. [12] to be able to describe the dynamics of controllers such as the HIGS, called an extended Projected Dynamical System (ePDS):

$$\dot{x}(t) = \Pi_{S,E}(x(t), f(x(t))), \tag{2.38}$$

where the extended projection onto the tangent cone is defined trough

$$\Pi_{S,E}(x, v) := \underset{\substack{w \in T_S(x) \\ w - v \in E}}{\arg\min} \|w - v\|. \tag{2.39}$$

Here, E is a subspace of X which defines in which directions the dynamics may be changed by the projection operator. For controlled systems, such a constraint is needed to ensure that the projection operator does not influence plant states, only states that correspond to the controller.

Proposition 2.20 (Similar to Lemma 3.1 in [12])**.** *Assume that:*

- *K is a cone in Hilbert space X of the form given by* (2.34) *with $n_f = 2$.*
- *S is the double cone $K \cup -K$.*
- *E is a one dimensional subspace of X.*
- *$E \cap S = \{0\}$ and $S + E = X$.*

Then the operator $\Pi_{S,E}(x, v)$ has a unique outcome for every $x \in S$ and $v \in X$.

Proof. Let E be spanned by the vector $h \in X$ with $\|h\| = 1$. As $w - v \in E$, there exists an $\eta \in \mathbb{R}$ such that $w - v = \eta h$. The projection operator in (2.39) can be equivalently written as:

$$\Pi_{S,E}(x, v) = \underset{\eta \in \Lambda_{T_S(x)}(v)}{\operatorname{argmin}} \left\|\eta h\right\|^2. \tag{2.40}$$

The set $\Lambda_A(v)$ is defined for sets $A \subseteq X$, and $v \in X$ as

$$\Lambda_A(v) := \left\{\eta \in \mathbb{R} \mid \eta h + v \in A\right\}.$$

If $\Lambda_{T_S(x)}(v)$ is non-empty and convex, the minimization problem in (2.40) has a unique outcome. This follows from the observation that $\left\|\eta h\right\|^2 = \left|\eta\right|_2^2$ is strictly convex.

It thus remains to show that $\Lambda_{T_S(x)}(v)$ is non-empty, closed and convex for every $x \in S$, $v \in X$. First note that if $A \subseteq X$ is closed and convex, then so is $\Lambda_A(v)$. For convexity consider:

$$\eta, \tilde{\eta} \in \Lambda_A(v), \alpha \in (0,1) \implies v + \left(\alpha\eta + (1-\alpha)\tilde{\eta}\right) h = \alpha\left(v + \eta h\right) + (1-\alpha)\left(v + \tilde{\eta} h\right) \in \Lambda_A(v). \tag{2.41}$$

For closedness, consider a sequence $(\eta_i)_{i\in\mathbb{N}}$ with $\eta_i \in \Lambda_A$ that converges to $\tilde{\eta} \in \mathbb{R}$. Then:

$$\lim_{i\to\infty} \left\|\eta_i h + v - (\tilde{\eta} h + v)\right\| = \lim_{i\to\infty} \left|\eta_i - \tilde{\eta}\right| = 0$$

And thus, $(\eta_i h + v)_{i\in\mathbb{N}}$ converges to $(\tilde{\eta} h + v) \in X$. Furthermore, as for every i, $(\eta_i h + v) \in A$ and A is closed, also the limit $(\tilde{\eta} h + v) \in A$. This means $\tilde{\eta} \in \Lambda_A$. Because the limit is in Λ_A for an arbitrary converging sequence that lies in the set, Λ_A is closed.

To show that $\Lambda_{T_S(x)}(v)$ is non-empty, closed and convex, consider the following cases separately:

Case 1: $x \in K \cap -K$

Recall from Proposition 2.15 that $T_S(x) = S$ in this case. From the assumption that $S + E = X$, it then follows that $\Lambda_{T_S(x)}(v)$ is nonempty.

$\Lambda_{T_S(x)}(v)$ can be rewritten as:

$$\Lambda_S(v) = \left\{\eta \mid \eta h + v \in S\right\} = \underbrace{\left\{\eta \mid \eta h + v \in K\right\}}_{\Lambda_K(v)} \cup \underbrace{\left\{\eta \mid \eta h + v \in -K\right\}}_{\Lambda_{-K\setminus K}(v)}.$$

In fact, $\Lambda_S(v) = \Lambda_K(v)$ or $\Lambda_S(v) = \Lambda_{-K\setminus K}(v)$. To prove this, assume towards a contradiction that there exist elements:

$$\eta^+ \in \Lambda_K(v) \quad \text{and} \quad \eta^- \in \Lambda_{-K\setminus K}(v). \tag{2.42}$$

Then,

$$\eta^+ h + v \in K \quad \text{and} \quad \eta^- h + v \in -K \implies -\eta^- h - v \in K.$$

By convexity of K, it then follows that:

$$\frac{1}{2}(\eta^+ - \eta^-) h \in K.$$

But $\frac{1}{2}(\eta^+ - \eta^-) h$ is also clearly an element of E. Then by the assumption that $E \cap K = \{0\}$ it follows that $\eta^+ - \eta^- = 0$, contradicting (2.42). It must then hold that indeed $\Lambda_S(v) = \Lambda_K(v)$ or $\Lambda_S(v) = \Lambda_{-K\setminus K}(v)$. As both K and $-K \setminus K$ are closed and convex, it follows from (2.41) that $\Lambda_{T_S(x)}(v)$ is also convex.

Case 2: $x \in K \setminus -K$

Because $x \in K \setminus -K$, there must be $i \in \{1,2\}$ such that $\langle x, f_i > 0 \rangle$. So constraint i is not an active constraint.

As $h \notin S$, the inner products $\langle h, f_1 \rangle$ and $\langle h, f_2 \rangle$ are nonzero and have opposite sign. Thus for any $v \in X$, η can be chosen to be:

$$\eta = -\frac{\langle v, f_j \rangle}{\langle h, f_j \rangle},$$

where $j \neq i$ is the other constraint that may be active or not. Then, by this choice of η:

$$\langle v + \eta h, f_j \rangle = \langle v, f_j \rangle + \eta \langle h, f_j \rangle = 0.$$

And thus, $v + \eta h \in T_K(x)$ as in Proposition 2.15 and thus $\eta \in \Lambda_{T_S(x)}(v)$, which is thus non-empty.

By Proposition 2.14 and closedness and convexity of K, $T_S(v) = T_K(v)$ is convex. The closedness and convexity of $\Lambda_{T_S(x)}(v)$ then follows from (2.41).

Case 3: $x \in -K \setminus K$ The same reasoning as in Case 2 can be applied to conclude that $\Lambda_{T_S(x)}(v)$ is non-empty, closed and convex.

□

In the proposition, it is assumed that the subspace E is one dimensional, which is enough for controllers with only one state. As HIGS has only one state, this result is sufficient for the moment. It is very likely that this result can be extended to general finite dimensional subspaces E: In the work by Sharif et al. [12] where a similar result is shown for the case where $X = \mathbb{R}^n$ no assumption is needed on the dimension of the subspace E.

2.4.4 The HIGS in the ePDS framework

The HIGS can be written in the ePDS framework for suitable S, E and $f(x)$. This is established in the following theorem that is inspired by the work of Sharif et al. [12].

Theorem 2.21. *The HIGS defined trough the discontinuous representation of* (2.25) *can equivalently be written as an ePDS, that is, of the form of* (2.38). *This is done by choosing:*

- $E := \mathrm{span}\{f_2\}$, *where* $f_2 = (0_{X_G}, 1)$ *is as before the unit vector in the component corresponding to the state of the HIGS element.*
- $S := \mathcal{S}$, *with* $\mathcal{S}$ *being as defined before through* (2.21).
- $f(x) := A_1 x$.

Proof. First note that with this choice of S and E, the assumptions from Proposition 2.20 are satisfied and thus $\Pi_{\mathcal{S},E}(x, A_1 x)$ is well posed.

To show the equivalence to the discontinuous representation consider the following cases for $x \in \mathcal{S} \cap \mathrm{D}(A_1)$:

Case 1: There are no active constraints: $\langle x, f_1 \rangle \neq 0$ and $\langle x, f_2 \rangle \neq 0$.
Then, $T_{\mathcal{S}}(x) = X \ni A_1 x$, so $\eta = 0$ is feasible and clearly optimal.

Case 2: Both constraints are active: $\langle x, f_1 \rangle = 0$ and $\langle x, f_2 \rangle = 0$.
In this case, $T_{\mathcal{S}}(x) = \mathcal{S}$ (by Proposition 2.15). The constraints imply that $C_g x_g = 0$ and $x_h = 0$. Thus $A_1 x = \begin{pmatrix} A_g x_g & 0 \end{pmatrix}^T$ which implies that $\langle A_1 x, f_2 \rangle = 0$, and thus $A_1 x \in \mathcal{S}$ and again $\eta = 0$ is feasible and optimal.

Case 3: Only the second constraint is active $\langle x, f_1 \rangle \neq 0$ and $\langle x, f_2 \rangle = 0$.
Then, by Proposition 2.15 the tangent cone is given by the set where $\langle v, f_2 \rangle$ is zero or has the same sign as $\langle x, f_1 \rangle$:

$$T_{\mathcal{S}}(x) = \left\{ v \mid \langle v, f_1 \rangle \langle x, f_2 \rangle \geq 0 \right\}.$$

Now notice that as $\langle x, f_2 \rangle = x_h = 0$, the value of $\langle A_1 x, f_2 \rangle = \omega_h C_g x_g$. As $\langle x, f_1 \rangle = k_h C_g x_g$ and ω_h and k_h are positive, it follows that $A_1 x \in T_{\mathcal{S}}(x)$. Thus $\eta = 0$ is again feasible.

Case 4: Only the first constraint is active $\langle x, f_1 \rangle = 0$ and $\langle x, f_2 \rangle \neq 0$.
In this case the tangent cone is given by:

$$T_{\mathcal{S}}(x) = \left\{ v \mid \langle v, f_2 \rangle \langle x, f_1 \rangle \geq 0 \right\}.$$

In this case it can happen that $\langle A_1 x, f_1 \rangle$ and $\langle x, f_2 \rangle$ have opposite sign and thus that $A_1 x \notin T_{\mathcal{S}}(x)$. Then, from Equation (2.40) it follows that η should be chosen such that $|\eta|$ is smallest and

$$\langle x, f_2 \rangle \left(\langle A_1 x, f_1 \rangle + \eta \langle h, f_1 \rangle \right) \geq 0.$$

If $x \in \left\{ \pm x \mid \langle x, f_1 \rangle = 0 \wedge \langle x, f_2 \rangle > 0 \wedge \langle A_1 x, f_1 \rangle < 0 \right\}$, then the above is not satisfied for $\eta = 0$. The set above is exactly how $\mathcal{S}_2$ was defined in the discontinuous representation (2.23). The optimal η to the optimization problem (2.40) for these x is:

$$\eta = -\frac{\langle A_1 x, f_1 \rangle}{\langle h, f_1 \rangle} = \langle A_1 x, f_1 \rangle.$$

So in conclusion:

$$\dot{x}(t) = \Pi_{\mathcal{S},E}(x, v) = \begin{cases} A_1 x + \langle A_1 x, f_1 \rangle f_2 & \text{if } x \in \mathcal{S}_2, \\ A_1 x & \text{otherwise.} \end{cases} \tag{2.43}$$

By substituting the definition of F_1, it can be seen that:

$$\begin{aligned} A_1 x + \langle A_1 x, f_1 \rangle f_2 &= \begin{pmatrix} A_g & -B_{gv} \\ \omega_h C_g & 0 \end{pmatrix} x + \left(\begin{pmatrix} k_h C_g & -1 \end{pmatrix} \begin{pmatrix} A_g & -B_{gv} \\ \omega_h C_g & 0 \end{pmatrix} x \right) \begin{pmatrix} 0_{X_G} \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} A_g & -B_{gv} \\ k_h C_g A_g & -k_h C_g B_{gv} \end{pmatrix} x = A_2 x. \end{aligned} \tag{2.44}$$

Hence, it is clear that (2.43) describes the same dynamics as (2.25). □

Besides providing an interesting new perspective, the formulation of the HIGS as an ePDS also makes it clearer how A_1 and A_2 are related. From (2.44) it becomes clear that A_2 is a rank-1 perturbation of A_1, furthermore the perturbation can be expressed in terms of A_1 as :

$$A_2 x - A_1 x = \langle A_1 x, f_1 \rangle f_2. \tag{2.45}$$

It is also possible to express the perturbation in terms of A_2 as:

$$A_1 x - A_2 x = -\langle A_2 x, f_2 \rangle f_2 + \frac{\omega_h}{k_h} \langle x, f_1 + f_2 \rangle f_2. \tag{2.46}$$

This relation can be verified by substituting the definitions of the A_1, A_2, F_1 and F_2.

2.5 Conclusion

This chapter shows that HIGS can be represented in multiple ways. In Theorem 2.5 it was shown that in the open loop definition can be written such that the modes are active on closed sets $\overline{\mathcal{F}_1}$ and $\overline{\mathcal{F}_2}$. In the closed loop setting, it is possible to write the plant and HIGS in a combined system that switches between two linear modes of operation. This representation fits into the Hybrid Systems framework. It was shown that this feedback connection can also be written as an extended Projected Dynamical System (ePDS), where only the controller state of the integrator dynamics is projected to ensure that the dynamics stay inside the $\mathcal{S}$ sector. These different representations provide insights for stability and well-posedness results, as becomes clear in the following chapters.

3 Semigroup theory

This chapter introduces the theory of semigroups. The definitions and most of the results are from the text books by Engel & Nagel [37] (Chapter I, II and IV) and Curtain & Zwart [38]. The aim of this chapter is to give a brief introduction and state the results that will be needed to study the well posedness of the closed loop system in the following two chapters.

A semigroup generalizes the exponential and matrix exponential. It is well-known that these play a central role in the solutions to linear differential equations. For example in the scalar case, $x(t) = \mathrm{e}^{at}x_0, a \in \mathbb{R}$ is the solution to the ordinary differential equation (ODE):

$$\dot{x}(t) = ax(t); \quad x(0) = x_0.$$

For the square matrix $M \in \mathbb{R}^{n\times n}$, the matrix exponential is defined as:

$$\mathrm{e}^M = \sum_{i=0}^{\infty} \frac{1}{i!} M^i. \tag{3.1}$$

A linear system of ODEs of the form $\dot{x}(t) = Mx(t);\ x(0) = x_0$ has the solution $x(t) = \mathrm{e}^{Mt}x_0$. As the two modes of the controlled system in the discontinuous representation (2.28) are of a similar form: $\dot{x(t)} = Ax(t)$ where $A : \hat{X}_\Sigma \to \hat{X}_\Sigma$ is an operator. To be able to construct solutions to these linear systems, a generalization of the matrix exponential is needed.

Definition 3.1. *Let X be a Hilbert space. Consider a family of operators $(T(t))_{t\geq 0}$, where $T(t)$ is a linear bounded operator from X to X. $(T(t))_{t\geq 0}$ is called a strongly continuous semigroup, often denoted as C_0-semigroup, if it satisfies:*

i) $T(t+s) = T(t)T(s) \qquad \forall t,s \geq 0 \qquad$ *(semigroup law).*

ii) $T(0) = I$.

iii) $\lim_{t\downarrow 0} \|T(t)x - x\| = 0 \qquad \forall x \in X \qquad$ *(strong continuity).*

The infinitesimal generator A of the semigroup is defined whenever the following limit exits

$$Ax = \lim_{t\downarrow 0} \frac{1}{t}\left(T(t) - I\right)x.$$

The domain $\mathrm{D}(A)$ is the set of $x \in X$ where the above limit exists.

Example 3.2. The simplest example of a C_0-semigroup is the exponential function on $\mathbb{R}$: $T(t) = \mathrm{e}^{at}\cdot$ with $a \in \mathbb{R}$. It indeed satisfies the properties of Definition 3.1, property *i)* and *ii)* follow from the well-known properties of the exponential:

i) $\mathrm{e}^{a(t+s)} = \mathrm{e}^{at}\mathrm{e}^{as}$.

ii) $\mathrm{e}^0 = 1$.

The third property follows from the continuity of e^{at}:

iii) $0 \leq \left\|\mathrm{e}^{at}x - x\right\| \leq \left|\mathrm{e}^{at} - 1\right| |x| \to 0$ as $t \downarrow 0$ for every $x \in \mathbb{R}$.

The generator of this semigroup is the number a:

$$\lim_{t\downarrow 0} \frac{1}{t}\left(\mathrm{e}^{at} - 1\right) = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{at}|_{t=0} = a.$$

Also the matrix exponential e^{Mt} that is defined through (3.1) is a C_0-semigroup and M is its generator. It is in fact a specific case of the semigroups considered in Section 3.1.

Proposition 3.3. *Let $T(t)$ be a strongly continuous semigroup on Hilbert space X. Then there exist constants $M \geq 1$ and $w \in \mathbb{R}$ such that*

$$\|T(t)\| \leq M e^{wt}. \tag{3.2}$$

For a proof of this statement, see [37, Sec. I-5].

The following proposition clarifies why semigroups play an important role in the analysis of differential equations: $T(t)x(0)$ is the solution to the (abstract) differential equation $\dot{x}(t) = Ax(t)$.

Proposition 3.4. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on Hilbert space X. Then for all $x_0 \in \mathrm{D}(A)$ the following statements hold:*

i) $T(t)x_0 \in \mathrm{D}(A) \qquad \forall t \geq 0.$

ii) $T(t)x_0$ *is differentiable with respect to t for* $t > 0$ *and right differentiable at* $t = 0$.

iii) $\dfrac{\mathrm{d}}{\mathrm{d}t}(T(t)x_0) = AT(t)x_0 \qquad \forall t > 0.$

iv) $AT(t)x_0 = T(t)Ax_0 \qquad \forall t \geq 0.$

Proof. To prove *i)* and *iv)*, first consider the right limit in X:

$$\lim_{s\downarrow 0} \frac{T(t+s)x_0 - T(t)x_0}{s} = T(t)\lim_{s\downarrow 0} \frac{1}{s}(T(s)-I)x_0.$$

By the definition of the generator of the semigroup in Definition 3.1, the right hand side equals $T(t)Ax_0$. From Definition 3.1, it follows that $T(t)T(s) = T(s)T(t)$, and thus the limit can also be written as:

$$T(t)\lim_{s\downarrow 0} \frac{1}{s}(T(s)-I)x_0 = \lim_{s\downarrow 0} \frac{T(t+s)x_0 - T(t)x_0}{s} = \lim_{s\downarrow 0} \frac{1}{s}(T(s)-I)T(t)x_0.$$

As the left limit exist, also the right limit exists and thus $T(t)x_0 \in \mathrm{D}(A)$ and:

$$AT(t)x_0 = T(t)Ax_0. \tag{3.3}$$

This proves *i)* and *iv)* and the right differentiability at $t = 0$. To prove *ii)* and *iii)* it remains to show that also the left limit exists and equals the above right limit. For $0 < s < t$, it holds that

$$\frac{T(t-s)x_0 - T(t)x_0}{-s} = T(t-s)\frac{(T(s)-I)x_0}{s}. \tag{3.4}$$

Taking the limit $s \downarrow 0$ gives that also the left derivative $\lim_{s\downarrow 0} \frac{T(t-s)x_0 - T(t)x_0}{-s}$ exists and equals $T(t)Ax_0$. Then, by the equality in (3.3) the left and right derivative are equal for $t > 0$, proving differentiability. □

3.1 Uniformly continuous semigroups

For a bounded operator $A : X \to X$, the following operator can be defined:

$$T(t) := e^{At} := \sum_{i=0}^{\infty} \frac{t^i}{i!}A^i. \tag{3.5}$$

For bounded A, this series converges in the operator norm and is indeed bounded for fixed t, as can be seen by taking in the norms:

$$\left\| \sum_{i=0}^{\infty} \frac{t^i}{i!} A^i \right\| \leq \sum_{i=0}^{\infty} \frac{t^i}{i!} \left\| A^i \right\| = e^{t\|A\|}.$$

The series in (3.5) indeed defines a C_0-semigroup and A is the infinitesimal generator. See [38, Example 2.1.3 and 2.1.12] for a proof of this. For property *iii)* of Definition 3.1, it can be shown that the convergence of $T(t)$ to I as $t \to 0$ is uniform in X [38]:

$$0 \leq \|T(t)x - x\| \leq \|T(t) - I\| \, \|x\| \leq \sum_{i=1}^{\infty} \frac{t^i}{i!} \|A\|^i \, \|x\| = \left(e^{\|A\|t} - 1\right) \|x\| .$$

Clearly, $\left(e^{\|A\|t} - 1\right) \to 0$, as this is independent on x, the convergence is uniform. Such a semigroup is called uniformly continuous.

Definition 3.5. *A C_0-semigroup is called uniformly continuous if:*

$$\lim_{t \downarrow 0} \|T(t) - I\| = 0.$$

Here, $\|\cdot\|$ denotes the operator norm.

As the above analysis can be done for any bounded operator A on the Hilbert space X, it is clear that A generates a uniformly continuous semigroup if it is bounded (and thus also a C_0-semigroup). Conversely, any uniformly continuous semigroup also has a generator that is bounded. For a proof of this last result, see [37, Th. I-3.7].

3.2 Resolvents and their properties

The series that was used to describe the semigroup generated by a bounded operator in (3.5) is not suitable to use for unbounded operators, because the domain of an unbounded operator is not the whole space X. Before another formulation of a semigroup is given in Section 3.3, the resolvents of an operator need to be defined. For the following definition, a complex Hilbert space is needed. Up to this point, this thesis worked with real operators on real Hilbert spaces. These spaces and operators can be complexified to continue with the following theory, see for example [39, Sec. 12.1].

Definition 3.6. *For an operator $A : \mathrm{D}(A) \subset X \to X$ that generates a semigroup on a complex Hilbert space X, the resolvent set is defined as:*

$$\rho(A) := \{\lambda \in \mathbb{C} \mid \lambda I - A : \mathrm{D}(A) \to X \textit{ has a bounded inverse}\}.$$

The spectrum of A, denoted as $\sigma(A)$ is the complement of the resolvent set.

For $\lambda \in \rho(A)$, the resolvent is defined as the following bounded operator from $X \to X$:

$$R(\lambda, A) = (\lambda - A)^{-1}. \tag{3.6}$$

Proposition 3.7. *For the resolvents of generators A and B on a Hilbert space X, the following properties hold:*

a) $AR(\lambda, A) = R(\lambda, A)A = \lambda R(\lambda, A) - I \quad \forall \lambda \in \rho(A)$.

b) $R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A) \quad \forall \lambda, \mu \in \rho(A)$ *(Resolvent equation).*

c) $R(\lambda, B) - R(\lambda, A) = R(\lambda, A)(B - A)R(\lambda, B) \quad \forall \lambda \in \rho(A) \cup \rho(B)$ *(Second resolvent equation).*

For a proof, see [37, Sec. IV.1].

The above theory holds for any linear operator A on a Hilbert space. If A generates a C_0-semigroup, another characterization of the reolvent can be given.

Proposition 3.8. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Hilbert space X and let $M, w \in \mathbb{R}$ be such that* (3.2) *is satisfied. Then:*

i) if $\lambda \in \mathbb{C}$ is such that the following integral exists for every $x \in X$

$$\int_0^\infty \mathrm{e}^{-\lambda t} T(t) x \,\mathrm{d}t.$$

Then $\lambda \in \rho(A)$ and $R(\lambda, A)x$ is equal to the above integral.

ii) If $\Re(\lambda) > w$, then $\lambda \in \rho(A)$ and the resolvent is given by the integral in i).

iii) $\|R(\lambda, A)\| \leq \frac{M}{\Re(\lambda - w)}$ for all λ such that $\Re(\lambda) > w$.

3.3 Analytic semigroups

In this section, the resolvents introduced in the previous section will be used to show that the following (possibly unbounded) operators generate a semigroup with nice properties.

Definition 3.9. *An operator $A : \mathrm{D}(A) \subset X \to X$ on a Hilbert space X is called sectorial of angle $0 < \delta \leq \pi/2$ if the resolvent set $\rho(A)$ contains the sector given by:*

$$S_{\pi/2+\delta} := \left\{\lambda \in \mathbb{C} \mid \left|\arg(\lambda)\right| < \pi/2 + \delta\right\} \setminus \{0\},$$

and if for $0 < \epsilon < \delta$ there is an $M_\epsilon \geq 0$ such that:

$$\|R(\lambda, A)\| \leq \frac{M_\epsilon}{|\lambda|} \quad \forall \lambda \in \overline{S_{\pi/2+\delta}} \setminus \{0\}.$$

For these sectorial operators, a family of operators $(T(t))_{t\geq 0}$ can be defined trough:

$$\begin{cases} T(z) := I & \text{for } z = 0, \\ T(z) := \frac{1}{2\pi i} \int_\gamma e^{\lambda z} R(\lambda, A) \,\mathrm{d}\lambda & \text{for } z > 0. \end{cases} \tag{3.7}$$

Here, γ is a piecewise smooth curve in the sector $S_{\pi/2+\delta}$ going from $\infty \mathrm{e}^{-i(\pi/2+\delta')}$ to $\infty \mathrm{e}^{+i(\pi/2+\delta')}$ with $\left|\arg(z)\right| < \delta' < \delta$. An example of such a γ curve sketched in Figure 3.1. As the integrand is analytic, the integral does not depend on the path that is chosen between these endpoints.

The family of operators as defined in (3.7) has the following properties.

Proposition 3.10. *Let A be a sectorial operator of angle δ on a Hilbert space X. Then:*

i) $T(z)$ is a bounded operator from X to X for any for $z \in S_\delta$ $\|T(z)\|$ is uniformly bounded for $z \in S_{\delta'}$ with $0 < \delta' < \delta$.

ii) $z \mapsto T(z)$ is an analytic map in S_δ.

iii) $T(z_1 + z_2) = T(z_1)T(z_2) \quad \forall z_1, z_2 \in S_\delta$.

iv) $z \mapsto T(z)$ is strongly continuous in $S_{\delta'} \cup \{0\}$ with $0 < \delta' < \delta$.

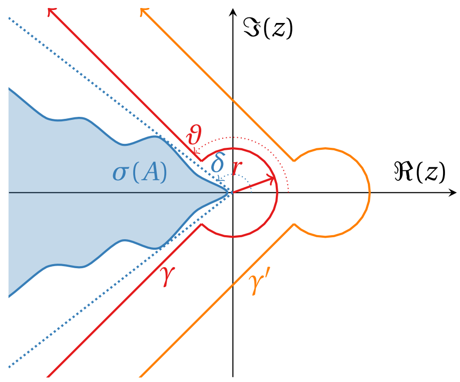


Figure 3.1: Sketch of the integration paths γ and γ'.

Proof. Here, only the proof of *iii)* is given, as it gives an impression of how the structure of $T(z)$ as in (3.7) can be used. For a proof of the other statements, see [37, Sec. II-4a].
By the definition of $T(z)$ as in (3.7),

$$T(z_1)T(z_2) = \frac{1}{(2\pi i)^2}\int_\gamma\int_{\gamma'} \mathrm{e}^{\mu z_1}\mathrm{e}^{\lambda z_2} R(\mu, A)R(\lambda, A)\,\mathrm{d}\lambda\ \mathrm{d}\mu.$$

Here, γ is the curve sketched in Figure 3.1 that consists of three parts:

$$\gamma_1 : \left\{-\rho \mathrm{e}^{-i\vartheta} | -\infty \le \rho \le r\right\},$$
$$\gamma_2 : \left\{r\mathrm{e}^{-i\alpha} | -\vartheta \le \alpha \le \vartheta\right\},$$
$$\gamma_3 : \left\{\rho \mathrm{e}^{i\vartheta} | r \le \rho \le \infty\right\},$$

where $\delta < \vartheta < \max\left\{\left|\arg(z_1)\right|, \left|\arg(z_1)\right|\right\}$ such that z_1 and z_2 lie to the left of the γ curve. The γ' curve is the same curve, but shifted to the right by $\epsilon > 0$ with ϵ small enough such that z_1 and z_2 also lie to the right of the γ' curve. The radius r is chosen small enough such that the curves do not intersect.

Using the resolvent identity (Proposition 3.7 *b)*,

$$T(z_1)T(z_2) = \frac{1}{(2\pi i)^2}\int_\gamma\int_{\gamma'} \mathrm{e}^{\mu z_1}\mathrm{e}^{\lambda z_2} \frac{1}{\lambda-\mu}\big(R(\mu, A) - R(\lambda, A)\big)\,\mathrm{d}\lambda\ \mathrm{d}\mu.$$

Splitting the integral and applying Fubini's theorem in the second part gives

$$T(z_1)T(z_2) = \frac{1}{(2\pi i)^2}\int_\gamma \mathrm{e}^{\mu z_1}\int_{\gamma'} \frac{\mathrm{e}^{\lambda z_2}}{\lambda-\mu}\,\mathrm{d}\lambda\ R(\mu, A)\,\mathrm{d}\mu - \frac{1}{(2\pi i)^2}\int_{\gamma'} \mathrm{e}^{\lambda z_2}\int_{\gamma} \frac{\mathrm{e}^{\mu z_1}}{\lambda-\mu}\,\mathrm{d}\mu\ R(\lambda, A)\,\mathrm{d}\lambda.$$

By closing the curve γ on the left at infinite radius, which can be done because the exponential will be zero there, Cauchy's integral theorem can be applied to the second term. From which it can be concluded that this term is zero. Similarly, by closing the γ' to the left Cauchy's integral formula can be used in the first term to conclude that $\frac{1}{2\pi i}\int_\gamma \frac{\mathrm{e}^{\lambda z_2}}{\lambda-\mu} = \mathrm{e}^{\mu z_2}$. Substitution then gives:

$$T(z_1)T(z_2) = \frac{1}{2\pi i}\int_\gamma \mathrm{e}^{\mu z_1}\mathrm{e}^{\mu z_2}\ R(\mu, A)\,\mathrm{d}\mu.$$

By definition of the semigroup as in (3.7), this is just $T(z_1 + z_2)$. $\square$

From Proposition 3.10 *iii)* and *iv)*, it is clear that $T(z)$ as in (3.7) is a C_0-semigroup. The generator of this semigroup is the sectorial operator A [37, Prop. II-4.4]. The additional properties established in Proposition 3.10 make it a special C_0-semigroup, which is called a bounded analytic semigroup.

Definition 3.11. *A family of bounded linear operators* $(T(z))_{z\in S_{\delta\cup\{0\}}}$ *on a Hilbert space X is called an analytic semigroup if ii)-iv) from Proposition 3.10 hold. If in addition also i) holds, then the semigroup is called bounded analytic.*

It follows directly from Proposition 3.10 that every sectorial operator generates a bounded analytic semigroup. The converse also holds: The generator of a bounded analytic semigroup is sectorial [37, Th. II-4.6].

Although bounded analytic semigroups are always generated by a sectorial operator, this is not true for general analytic semigroups: Also operators that are not sectorial can generate analytic semigroups. The following perturbation result gives such a result.

Theorem 3.12. *Let A generate an analytic semigroup on a Hilbert space X. Then there exists a constant* $\alpha > 0$ *such that:*

$$\exists a < \alpha, b > 0 \textit{such that} \|Fx\| \leq a\,\|Ax\| + b\,\|x\| \quad \implies \quad (A+F) \textit{ generates an analytic semigroup.}$$

For a proof, see [37, Sec. III-2].

Remark 3.13. *In particular, Theorem 3.12 shows that if A is sectorial and thus the generator of an analytic semigroup, then* $(A+cI)$ *also generates a semigroup for every* $c \in \mathbb{R}$*. This means that if there is a c such that* $(A+cI)$ *is sectorial, then it generates an analytic semigroup.*

3.4 Conclusion

In this chapter, semigroup theory was introduced. It was shown that a semigroup is a generalization of the exponential and matrix exponential and that the semigroup can be seen as the solution to the abstract differential equation $\dot{x}(t) = Ax(t)$ as $\frac{\mathrm{d}}{\mathrm{d}t}T(t)x_0 = AT(t)x_0$.

In the following chapters, the theory introduced here will be used to study the well-posedness of the infinite dimensional systems that are formed by interconnecting HIGS with an infinite dimensional plant, that are described by such abstract differential equations.

4 Well-posedness of HIGS as a controller for plants with a bounded generator

In this chapter, the well-posedness of HIGS as a controller for a bounded linear plant is established. The system that is studied is the combined closed-loop system Σ of a linear plant $\mathcal{G}$ with a HIGS element $\mathcal{H}$ as controller. This is depicted in Figure 2.1. Different descriptions of this system were studied in Chapter 2. In earlier research, two different proofs were established for the existence of solutions to this interconnection in the finite-dimensional case. One proof by Deenen et al. [9] explicitly constructs local solutions to the system, and provides an argument why these local solutions can be extended to global solutions. This proof uses the discontinuous representation of HIGS as in Section 2.2. A later proof by Heemels & Tanwani [23] uses the projection-based representation to apply theorems from differential inclusions and hybrid systems to establish existence of solutions. This latter proof needs less assumptions on the plant structure and the input $w(t)$ and is thus more general.

As the theory from differential inclusions and hybrid systems underlying the later proof by Heemels & Tanwani [23] are not readily available in the infinite dimensional setting, the explicit construction of solutions as in the proof by Deenen et al. [9] seemed to us to be the most straightforward approach to establish a first well-posedness result for the interconnection of HIGS with an infinite dimensional plant.

This chapter generalizes the result by Deenen et al. [9] The structure of the proof is strongly motivated by their proof. The result of this chapter includes the result of Deenen et al. [9] and generalizes it: Here, it is not assumed that $CB_{gv} = CB_{gw} = 0$, in contrast to the work by Deenen et al. Furthermore, the proof by Deenen et al. assumes a finite dimensional state and input space, so that the generators of the semigroups describing the I and P-mode (A_1 and A_2) can be represented by matrices. In this chapter it is only assumed that the semigroup generators are bounded operators that act on a Hilbert space X that may be infinite dimensional.

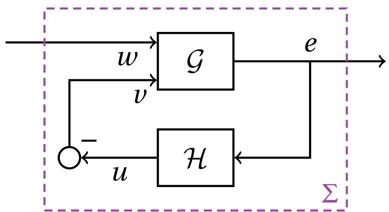


Figure 4.1: The closed-loop interconnection of HIGS with a plant $\mathcal{G}$.

4.1 Aim of this chapter

The aim of this chapter is to show that solutions to the feedback connection as in Figure 4.1 exist. As in Assumption 2.1, the plant is linear and has no direct feed-trough. In this chapter, it is also assumed that the linear operators in the plant (A_g, B_{gv}, B_{gw} and C_g) are bounded.

In the existence proof presented in this chapter, existence is shown by explicitly constructing local solutions and showing that these can be extended. For this it is useful to consider the discontinuous system representation of the HIGS controlled interconnection (Σ) that was introduced in Section 2.2. It makes it explicit that HIGS is switching between two linear systems that have similarities:

$$\Sigma : \begin{cases} \dot{x}(t) = \underbrace{\begin{pmatrix} A_g & -B_{gv} \\ \omega_h C_g & 0 \end{pmatrix}}_{A_1} x(t) + \underbrace{\begin{pmatrix} B_{gw} \\ 0 \end{pmatrix}}_{B_1} w(t) & \text{for } (x(t), w(t)) \in \mathcal{S}_1, \\ \dot{x}(t) = \underbrace{\begin{pmatrix} A_g & -B_{gv} \\ k_h C_g A_g & -k_h C_g B_{gv} \end{pmatrix}}_{A_2} x(t) + \underbrace{\begin{pmatrix} B_{gw} \\ k_h C_g B_{gw} \end{pmatrix}}_{B_2} w(t) & \text{for } (x(t), w(t)) \in \mathcal{S}_2, \\ e(t) = \underbrace{\begin{pmatrix} C_g & 0 \end{pmatrix}}_{C} x(t) & \text{and } x(t) \in \mathcal{S}. \end{cases} \tag{4.1}$$

Here, $k_h, \omega_h > 0$ are parameters that come from the HIGS element ($\mathcal{H}$). The operators that originate from the plant are $A_g \colon X_{\mathcal{G}} \to X_{\mathcal{G}}$, $B_{gv} \colon \mathbb{R} \to X_{\mathcal{G}}$, $B_{gw} \colon X_w \to X_{\mathcal{G}}$ and $C_g \colon X_{\mathcal{G}} \to \mathbb{R}$. Recall that in this chapter, these operators are all assumed to be bounded and thus also A_1 and A_2 are bounded operators. From Section 3.1 it is therefore clear that A_1 and A_2 generate a C_0-semigroup. If the system would not switch and the inputs are zero, it would thus be easy to state what the solution would be, it would simply be given by $\mathrm{e}^{At} x_0$.

Due to the switching behavior of the Σ system it is not clear whether solutions exist. One could imagine that it would be possible to end up in a situation where HIGS switches infinitely many times and therefore would end up getting 'stuck'.

Recall from Section 2.2 that the sets $\mathcal{S}_1$ and $\mathcal{S}_2$ that determine which mode is active are given by $\mathcal{S}_i = \mathcal{S}_i^+ \cup -\mathcal{S}_i^+$ for $i \in \{1,2\}$, with:

$$\begin{aligned} \mathcal{S}_1^+ &:= \left\{(x, w) \in X_\Sigma \times X_w \mid \langle x, f_2 \rangle \geq 0 \wedge \langle x, f_1 \rangle \geq 0 \wedge \left(\langle x, f_1 \rangle > 0 \vee \langle (A_1 x + B_1 w), f_1 \rangle \geq 0\right)\right\}, \\ \mathcal{S}_2^+ &:= \left\{(x, w) \in X_\Sigma \times X_w \mid \langle x, f_2 \rangle > 0 \wedge \langle x, f_1 \rangle = 0 \wedge \langle (A_1 x + B_1 w), f_1 \rangle < 0\right\}, \\ \mathcal{S} &:= \mathcal{S}_1 \cup \mathcal{S}_2, \qquad S_1 \cap S_2 = \emptyset. \end{aligned}$$

To simplify the notation of $\mathcal{S}_1$, the following definition will be introduced, which also plays a key role in the equivalences established in the next section.

Definition 4.1. *With the (strict) lexicographic inequality, denoted by ($>_l$) $\geq_l$ for elements $a_1, a_2, \dots \in \mathbb{R}$ is defined as follows.*

- *The strict lexicographic inequality $(a_1, a_2, \dots) >_l 0$ holds if $\exists j \in \mathbb{N}$ such that $a_j > 0$ and $a_i = 0\ \forall i < j$. In other words, the first non-zero element of the sequence $(a_i)_{i \in \mathbb{N}}$ is positive.*
- *The non-strict lexicographic inequality $(a_1, a_2, \dots) \geq_l 0$ holds if $(a_1, a_2, \dots) >_l 0$ or $(a_1, a_2, \dots) = \mathbf{0}$.*

Using a lexicographic inequality, the set $\mathcal{S}_1^+$ can also be written as:

$$\mathcal{S}_1^+ = \left\{(x, w) \in X_\Sigma \times X_w \mid \langle x, f_2 \rangle \geq 0 \wedge \left(\langle x, f_1 \rangle, \langle (A_1 x + B_1 w), f_1 \rangle\right) \geq_l 0\right\}. \tag{4.2}$$

As the system Σ is discontinuous, any solution can have a discontinuous derivative at time instances at which the mode of operation changes. Solutions in the classical sense are required to be continuous and satisfying the differential equations everywhere (in the usual definition, see for example [25]). The HIGS controlled system cannot be expected to have a continuous derivative. Thus, the aim is to show existence of Carathéodory solutions as in Definition 2.3, satisfying the system equations almost everywhere.

In this chapter, it is shown that Σ has a Carathéodory solution for all $x_0 \in S$ and all inputs w that are piecewise Bohl functions (see Definition 2.9). This assumption on w is needed to be able to write w as an exo-system locally, and thus to be able to locally write a system $\hat{\Sigma}$ as in (2.28).

4.2 Local behavior of the semigroup evaluation

The following equivalences between local behavior of the semigroup evaluation $\langle e^{At}x, f\rangle$ and lexicographic inequalities involving the generator A are important in the proof of local existence of solutions.

Lemma 4.2. *Consider a Hilbert space $(X;\langle\cdot,\cdot\rangle)$. For vectors $f, x \in X$ and a bounded linear operator $A \in \mathcal{L}(X;X)$, the following equivalences hold:*

a) $\exists \epsilon > 0$ *s.t.* $\forall t \in [0,\epsilon], \langle e^{At}x, f\rangle = 0 \iff (\langle x, f\rangle, \langle Ax, f\rangle, \langle A^2x, f\rangle, \ldots) = \mathbf{0}$.

b) $\exists \epsilon > 0$ *s.t.* $\forall t \in (0,\epsilon], \langle e^{At}x, f\rangle > 0 \iff (\langle x, f\rangle, \langle Ax, f\rangle, \langle A^2x, f\rangle, \ldots) >_l 0$.

c) $\exists \epsilon > 0$ *s.t.* $\forall t \in [0,\epsilon], \langle e^{At}x, f\rangle \geq 0 \iff (\langle x, f\rangle, \langle Ax, f\rangle, \langle A^2x, f\rangle, \ldots) \geq_l 0$.

Proof.
$\Longleftarrow$ *in (a).* Using the definition of the semigroup generated by A as in (3.5) and the assumption that $\left\langle A^i x, f\right\rangle = 0$ for any i in $\mathbb{N}$, the following can be obtained:

$$\langle e^{At}x, f\rangle = \sum_{i=0}^{\infty} \frac{t^i}{i!}\left\langle A^i x, f\right\rangle = 0 \quad \forall t \geq 0.$$

$\Longrightarrow$ *in (a).* By assumption, there exists $\epsilon > 0$ such that for all $t \in [0,\epsilon], \langle e^{At}x, f\rangle = 0$. In particular, evaluation at $t = 0$ gives $\langle x, f\rangle = 0$. Furthermore, function $\langle e^{At}x, f\rangle$ is right-differentiable with respect to t at $t = 0$ infinitely many times. Taking the i-th right derivative with $i \in \mathbb{N}$ gives that:

$$\left\langle A^i x, f\right\rangle = 0 \qquad \forall i \in \mathbb{N}.$$

$\Longleftarrow$ *in (b).* By assumption:

$$(\underbrace{\langle x, f\rangle}_{=:a_0}, \underbrace{\langle Ax, f\rangle}_{=:a_1}, \underbrace{\langle A^2x, f\rangle}_{=:a_2}, \ldots) >_l 0.$$

By definition of the lexicographic inequality, there is at least one non-zero term in the sequence $(a_i)_{i\in\mathbb{N}_0}$, and the first non-zero term is positive, i.e.:

$$\exists j \in \mathbb{N}_0 \text{ s.t. } a_j > 0 \text{ and } a_k = 0 \ \forall \ k < j.$$

The following bound can be established:

$$\sum_{i=j+1}^{\infty}\left|\frac{1}{i!}\left\langle A^i x, f\right\rangle\right| \leq \sum_{i=0}^{\infty}\left|\frac{1}{i!}\left\langle A^i x, f\right\rangle\right| \leq \|f\| \sum_{i=0}^{\infty}\left(\frac{1}{i!}\|A\|^i\right)\|x\| = \|f\| e^{\|A\|}\|x\| =: K < \infty. \quad (4.3)$$

This bound can be used to obtain the following inequality for $t \leq 1$:

$$\begin{aligned}
\langle e^{At}x, f\rangle &= \frac{t^j}{j!}\left\langle A^j x, f\right\rangle + \sum_{i=j+1}^{\infty}\frac{t^i}{i!}\left\langle A^i x, f\right\rangle, \\
&\geq \frac{t^j}{j!}\left\langle A^j x, f\right\rangle - t^{j+1}\sum_{i=j+1}^{\infty}\frac{t^{i-j-1}}{i!}\left|\left\langle A^i x, f\right\rangle\right|, \\
&\geq \frac{t^j}{j!}\left\langle A^j x, f\right\rangle - t^{j+1}K && \text{(Using (4.3) and } i-j-1 \geq 0\text{)}, \\
&= t^j\left(\frac{a_j}{j!} - tK\right). && (4.4)
\end{aligned}$$

As $a_j > 0$ and $K \geq 0$, it follows that $\epsilon := \min\left\{\frac{a_j}{j!(K+1)}, 1\right\} > 0$. Then for $t \in [0,\epsilon]$, it holds that $a_j / j! > tK \geq 0$. Hence, by (4.4), $\langle e^{At}x, f\rangle > 0$ for all $t \in (0,\epsilon]$.

$\Longleftarrow$ *in (c).* For $t \in (0,\epsilon]$, this follows directly from ($\Longleftarrow$) of *(a)* or *(b)*. For $t = 0$, it is clear that $\langle e^{At}x, f\rangle = \langle x, f\rangle \geq 0$, as $\langle x, f\rangle$ is the first term of the assumed lexicographic inequality.

$\Longrightarrow$ *in (b).* To prove this direction of the equivalence, it is easier to prove the contrapositive. That is to show:

$$\left(\langle x,\tilde{f}\rangle, \langle Ax,\tilde{f}\rangle, \langle A^2x,\tilde{f}\rangle, \dots\right) \leq_l 0 \implies \forall \epsilon > 0, \exists t \in (0,\epsilon], \langle e^{At}x, \tilde{f}\rangle \leq 0. \tag{4.5}$$

Note that $\Longleftarrow$ of *(c)* with $f = -\tilde{f}$ already gives that:

$$\left(\langle x,-\tilde{f}\rangle, \langle Ax,-\tilde{f}\rangle, \langle A^2x,-\tilde{f}\rangle, \dots\right) \geq_l 0 \implies \exists \epsilon > 0 \text{ s.t. } \forall t \in [0,\epsilon], \langle e^{At}x, -\tilde{f}\rangle \geq 0.$$

This is already a stronger statement than (4.5).

$\Longrightarrow$ *in (c).* Likewise, the contrapositive of this statement follows directly from $\Longleftarrow$ of *(b)* with $f = -\tilde{f}$. □

In particular, this lemma shows that there is an interval near zero where $\langle e^{At}x, f\rangle$ has a unique sign.

Corollary 4.3. *Consider a Hilbert space $(X; \langle\cdot,\cdot\rangle)$. For vectors $f, x \in X$ and a bounded linear operator $A \in \mathcal{L}(X;X)$, exactly one of the following statement holds:*

i) $\exists \epsilon > 0$ *such that* $\forall t \in [0,\epsilon], \langle e^{At}x, f\rangle = 0$.

ii) $\exists \epsilon > 0$ *such that* $\forall t \in (0,\epsilon], \langle e^{At}x, f\rangle > 0$.

iii) $\exists \epsilon > 0$ *such that* $\forall t \in (0,\epsilon], \langle e^{At}x, f\rangle < 0$.

Proof. From the definition of the lexicographic inequality, it follows that exactly one of the following statements is true:

i) $\left(\langle x,f\rangle, \langle Ax,f\rangle, \langle A^2x,f\rangle, \dots\right) = \mathbf{0}$.

ii) $\left(\langle x,f\rangle, \langle Ax,f\rangle, \langle A^2x,f\rangle, \dots\right) >_l 0$.

iii) $\left(\langle x,f\rangle, \langle Ax,f\rangle, \langle A^2x,f\rangle, \dots\right) <_l 0$.

The proof then follows directly from the equivalences established in Lemma 4.2. □

4.3 Proving local existence of solutions

In this section, local existence of solutions is established for linear plants of the form (2.1) with a bounded A_g. The approach in this proof is that first the input w is incorporated in the system, by assuming that the input is a piecewise Bohl function. Next, the region $\mathcal{S}$ is split into two parts, corresponding to the two modes of operation. For initial conditions in each of these regions, candidate solutions are proposed that correspond to the two modes of operation of the HIGS element.

Theorem 4.4. *Assume that A_g, B_{gv} and C_g are bounded operators. Then, the HIGS-controlled system Σ as in Equation (4.1) is locally well-posed in the sense that for any $x_0 \in \mathcal{S}$ and any piecewise Bohl input w it admits a Carathéodory solution on the time interval $[0,\epsilon]$ for some $\epsilon > 0$.*

Proof. Because w is a piecewise Bohl function, it is Bohl on an interval $[0,\tilde{\epsilon}]$ for some $\tilde{\epsilon} > 0$ sufficiently small. Thus, $w(t)$ can be written as the output of an exosystem $\mathcal{W}$ of the form as in (2.26).

The systems Σ and $\mathcal{W}$ can be written combined on the interval $[0,\tilde{\epsilon}]$, with states $\hat{x} := (x, x_w) \in \hat{X}_\Sigma := (X_\Sigma \times X_{x_w})$, as before in (2.28):

$$\hat{\Sigma}: \begin{cases} \dot{\hat{x}}(t) = \underbrace{\begin{pmatrix} A_1 & B_1 H_w \\ 0 & F_w \end{pmatrix}}_{\hat{A}_1} \hat{x}(t) & \text{if } \hat{x}(t) \in \hat{\mathcal{S}}_1, \\ \dot{\hat{x}}(t) = \underbrace{\begin{pmatrix} A_2 & B_2 H_w \\ 0 & F_w \end{pmatrix}}_{\hat{A}_2} \hat{x}(t) & \text{if } \hat{x}(t) \in \hat{\mathcal{S}}_2, \\ e(t) = \underbrace{\begin{pmatrix} C & 0 \end{pmatrix}}_{\hat{C}} \hat{x}(t) & \text{and } \hat{x}(t) \in \hat{\mathcal{S}}. \end{cases} \tag{4.6}$$

By the boundedness of A_g, $B_g v$ and C_g, also the operators $\hat{A}_1$ and $\hat{A}_2$ are bounded. The sets $\hat{\mathcal{S}}_1$ and $\hat{\mathcal{S}}_2$ describe the conditions for the integrator and gain modes in terms of the (expanded) states ($\hat{x}$) and are given by:

$$\hat{\mathcal{S}}_1 = \hat{\mathcal{S}}_1^+ \cup -\hat{\mathcal{S}}_1^+ \;\text{ with } \hat{\mathcal{S}}_1^+ := \left\{\hat{x} \in \hat{X}_\Sigma \mid \langle \hat{x}, \hat{f}_2 \rangle \geq 0 \wedge \left(\langle \hat{x}, \hat{f}_1 \rangle, \langle \hat{A}_1 \hat{x}, \hat{f}_1 \rangle\right) \geq_l 0\right\}, \tag{4.7}$$

$$\hat{\mathcal{S}}_2 := \hat{\mathcal{S}}_2^+ \cup -\hat{\mathcal{S}}_2^+ \;\text{ with } \hat{\mathcal{S}}_2^+ := \left\{\hat{x} \in \hat{X}_\Sigma \mid \langle \hat{x}, \hat{f}_2 \rangle > 0 \wedge \langle \hat{x}, \hat{f}_1 \rangle = 0 \wedge \langle \hat{A}_1 \hat{x}, \hat{f}_1 \rangle < 0\right\}, \tag{4.8}$$

$$\hat{\mathcal{S}} := \hat{\mathcal{S}}^+ \cup -\hat{\mathcal{S}}^+ \;\text{ with } \hat{\mathcal{S}}^+ := \hat{\mathcal{S}}_1^+ \cup \hat{\mathcal{S}}_2^+ = \left\{\hat{x} \in \hat{X}_\Sigma \mid \langle \hat{x}, \hat{f}_1 \rangle \geq 0 \wedge \langle \hat{x}, \hat{f}_2 \rangle \geq 0\right\}. \tag{4.9}$$

In these definitions, $\hat{f}_i := (f_i, 0)$, such that $\langle x, f_i \rangle = \langle \hat{x}, \hat{f}_i \rangle = F_i(x)$ for $i \in \{1,2\}$.

Now, the region $\hat{\mathcal{S}}_{\text{int}}$ is defined as:

$$\hat{\mathcal{S}}_{\text{int}} := \left\{\hat{x}_0 \in \hat{\mathcal{S}} \mid \exists \epsilon > 0, \forall t \in [0,\epsilon], \mathrm{e}^{\hat{A}_1 t} \hat{x}_0 \in \hat{\mathcal{S}}\right\} \tag{4.10}$$

Using Lemma 4.2, it can be seen that $\hat{\mathcal{S}}_{\text{int}}$ can also be written as $\hat{\mathcal{S}}_{\text{int}} = \hat{\mathcal{S}}_{\text{int}}^+ \cup -\hat{\mathcal{S}}_{\text{int}}^+$ with:

$$\hat{\mathcal{S}}_{\text{int}}^+ := \left\{\hat{x} \in \hat{X}_\Sigma \mid \left(\langle \hat{x}, \hat{f}_1 \rangle, \langle \hat{A}_1 \hat{x}, \hat{f}_1 \rangle, \ldots\right) \geq_l 0 \wedge \left(\langle \hat{x}, \hat{f}_2 \rangle, \langle \hat{A}_1 \hat{x}, \hat{f}_2 \rangle, \ldots\right) \geq_l 0\right\}. \tag{4.11}$$

The cases that the initial condition $\hat{x}_0$ lies in $\hat{\mathcal{S}}_{\text{int}}$ or in $\hat{\mathcal{S}} \setminus \hat{\mathcal{S}}_{\text{int}}$ are considered separately.

Case 1: $\hat{x}_0 \in \hat{\mathcal{S}}_{\text{int}}$:

From (4.11) it is clear that $\hat{\mathcal{S}}_{\text{int}} \subseteq \hat{\mathcal{S}}_1$. Even more, it will be shown that $\hat{x}(t) := \mathrm{e}^{\hat{A}_1 t} \hat{x}_0$ is in $\hat{\mathcal{S}}_1$ for $t \in [0,\varepsilon]$ for some $\varepsilon > 0$ and is thus a solution on this interval.

By definition of $\hat{\mathcal{S}}_{\text{int}}$, there is an $\epsilon > 0$ such that $\mathrm{e}^{\hat{A}_1 s} \hat{x}_0 \in \mathcal{S}$ for all $s \in [0,\epsilon]$. Let $s = t + \delta$ and $t, \delta \in [0, \epsilon/2]$, then by the semigroup law,

$$\mathrm{e}^{\hat{A}_1 s} \hat{x}_0 = \mathrm{e}^{\hat{A}_1 \delta} \mathrm{e}^{\hat{A}_1 t} \hat{x}_0 \in \mathcal{S}, \quad \forall t, \delta \in [0, \epsilon/2]$$

It then follows from the definition of $\hat{\mathcal{S}}_{\text{int}}$ in (4.10) that even $\mathrm{e}^{\hat{A}_1 t} \hat{x}_0 \in \hat{\mathcal{S}}_{\text{int}}$ for $t \in [0, \epsilon/2]$. Since $\hat{\mathcal{S}}_{\text{int}} \subseteq \hat{\mathcal{S}}_1$, it follows that $\mathrm{e}^{\hat{A}_1 t}$ is a solution on this interval.

Case 2: $\hat{x}_0 \in \hat{\mathcal{S}} \setminus \hat{\mathcal{S}}_{\text{int}}$:

In the case that $\hat{x}_0 \in \hat{\mathcal{S}} \setminus \hat{\mathcal{S}}_{\text{int}}$, it will be shown that the function $\hat{x}_{\mathrm{P}}(t)$ defined by $\hat{x}_{\mathrm{P}}(t) := \mathrm{e}^{\hat{A}_2 t} \hat{x}_0$ takes values in $\hat{\mathcal{S}}_2$ for $t \in (0,\epsilon]$ and is thus a Carathéodory solution on $[0,\epsilon]$ for some $\epsilon > 0$.

For $\hat{x}_0 \in \hat{\mathcal{S}} \setminus \hat{\mathcal{S}}_{\text{int}}$, it will first be shown that:

$$\hat{x}_0 \in \hat{\mathcal{S}} \setminus \hat{\mathcal{S}}_{\text{int}} \implies \langle \hat{x}_0, \hat{f}_1 \rangle = 0 \wedge \langle \hat{x}_0, \hat{f}_2 \rangle \neq 0. \tag{4.12}$$

The contrapositive is proven in cases. Notice that the right side of (4.12) being false can be written as $\big(\langle \hat{x}_0, \hat{f}_1 \rangle < 0 \vee \langle \hat{x}_0, \hat{f}_1 \rangle > 0 \vee \langle \hat{x}_0, \hat{f}_2 \rangle = 0\big)$. Now three cases are considered:

Case a: $\big(\langle \hat{x}_0, \hat{f}_1 \rangle > 0 \wedge \langle \hat{x}_0, \hat{f}_2 \rangle < 0\big) \vee \big(\langle \hat{x}_0, \hat{f}_1 \rangle < 0 \wedge \langle \hat{x}_0, \hat{f}_2 \rangle > 0\big)$:
In this case $\hat{x}_0 \notin \hat{\mathcal{S}}$, hence also $\hat{x}_0 \notin \hat{\mathcal{S}} \setminus \hat{\mathcal{S}}_{\text{int}}$.

Case b: $\big(\langle \hat{x}_0, \hat{f}_1 \rangle > 0 \wedge \langle \hat{x}_0, \hat{f}_2 \rangle > 0\big) \vee \big(\langle \hat{x}_0, \hat{f}_1 \rangle < 0 \wedge \langle \hat{x}_0, \hat{f}_2 \rangle < 0\big)$:
In this case, $\hat{x}_0$ lies in the interior of $\hat{\mathcal{S}}_1$. In other words, there is some $\delta > 0$ such that the open ball $\mathcal{B}(\hat{x}_0, \delta) \subseteq \hat{\mathcal{S}}_1$. Then by continuity of $\mathrm{e}^{\hat{A}_1 t} \hat{x}_0$, there is some $\epsilon > 0$ such that this function lies inside of $\hat{\mathcal{S}}_1$ for all $t \in [0, \epsilon]$. Hence, by definition of $\hat{\mathcal{S}}_{\text{int}}$ in (4.10), $\hat{x}_0 \in \hat{\mathcal{S}}_{\text{int}}$, implying that $\hat{x}_0 \notin \hat{\mathcal{S}} \setminus \hat{\mathcal{S}}_{\text{int}}$.

Case c: $\langle \hat{x}_0, \hat{f}_2 \rangle = 0$:
First, it can be observed that the following equality holds (by substitution):

$$\hat{F}_2 \hat{A}_1 = \frac{\omega_h}{k_h} (\hat{F}_1 + \hat{F}_2).$$

From which it can also be concluded that:

$$\langle \hat{A}_1 \hat{x}, \hat{f}_2 \rangle = \frac{\omega_h}{k_h} \langle \hat{x}, \hat{f}_1 + \hat{f}_2 \rangle \quad \forall \hat{x}. \tag{4.13}$$

Now, consider one of the sequences of the lexicographic inequalities in the alternative definition of $\hat{\mathcal{S}}_{\text{int}}$ as in (4.11): $\big(\langle \hat{x}, \hat{f}_1 \rangle, \langle \hat{A}_1 \hat{x}, \hat{f}_1 \rangle, \ldots\big)$, which is denoted here as a. It must hold that $a \geq_l 0$ or $-a \geq_l 0$. If it can be shown that the other sequence $b = \big(\langle \hat{x}, \hat{f}_2 \rangle, \langle \hat{A}_1 \hat{x}, \hat{f}_2 \rangle, \ldots\big)$ satisfies a lexicographic inequality with the same sign (i.e. $a \geq_l 0 \implies b \geq_l 0$ and $a \leq_l 0 \implies b \leq_l 0$), it is clear that $\hat{x} \in \hat{\mathcal{S}}_{\text{int}}$ from (4.11).

Notice that if all elements of a are zero, and by assumption $\langle \hat{x}_0, \hat{f}_2 \rangle = 0$, then (4.13) implies that all elements of b are zero.

On the other hand, if a_ρ is the first nonzero element of a, then by similar reasoning $b_i = 0$ for $i \leq \rho$. For the $\rho+1$-th element, substitution in Equation (4.13) gives:

$$\left\langle \hat{A}_1^{\rho+1} \hat{x}_0, \hat{f}_2 \right\rangle = \frac{\omega_h}{k_h} \langle \hat{A}_1^{\rho} \hat{x}_0, \hat{f}_1 \rangle.$$

Hence, the first nonzero element of b, $b_{\rho+1}$, has the same sign as the first nonzero element of a. Therefore, a lexicographic inequality on a implies the same direction of lexicographic inequality on b. It can thus be concluded that in this case $\hat{x}_0 \in \hat{\mathcal{S}}_{\text{int}}$.

Thus in all cases (a-c), the conclusion is drawn that $\hat{x}_0 \notin \hat{\mathcal{S}} \setminus \hat{\mathcal{S}}_{\text{int}}$, and thus (4.12) is proven.

Define $\hat{x}_{\mathrm{P}}^{+}(t) := \mathrm{e}^{\hat{A}_2 t} \hat{x}_0^{+}$, with:

$$\hat{x}_0^{+} := \begin{cases} \hat{x}_0 & \text{if } \langle \hat{x}_0, \hat{f}_2 \rangle > 0, \\ -\hat{x}_0 & \text{otherwise,} \end{cases} \tag{4.14}$$

If it is shown that this is in $\hat{\mathcal{S}}_2$ for $t \in (0, \epsilon]$, then by the symmetry of $\hat{\mathcal{S}}_2$, also $\hat{x}_{\mathrm{P}}(t)$ is in $\hat{\mathcal{S}}_2$. From (4.12) it follows that for $\hat{x}_0 \in \hat{\mathcal{S}} \setminus \hat{\mathcal{S}}_{\text{int}}$ it must hold that either $\langle \hat{x}_0, \hat{f}_2 \rangle > 0$ or

$\langle \hat{x}_0, \hat{f}_2 \rangle < 0$. For both cases, by definition of $\hat{x}_0^+$ in (4.14), $\langle \hat{x}_0^+, \hat{f}_2 \rangle > 0$. By continuity, there is some $\epsilon_1 > 0$ such that $\langle \hat{x}_{\mathrm{P}}^+(t), \hat{f}_2 \rangle > 0$ for all $t \in [0, \epsilon_1]$.

As A_2 describes proportional feedback it is not surprising that $k_h e(t) = u(t)$ and thus $\langle \hat{x}_{\mathrm{P}}^+(t), \hat{f}_1 \rangle = 0$ for $t \in (0, \epsilon_1]$. To show this, notice that by (4.12), it holds that in this case $\langle \hat{x}_0^+, \hat{f}_1 \rangle = 0$. Also, from the definitons of $\hat{F}_1$ and $\hat{A}_2$:

$$\hat{F}_1 \hat{A}_2 = \begin{pmatrix} k_h C_g & -1 \end{pmatrix} \begin{pmatrix} A_g & -B_{gv} \\ k_h C_g A_g & -k_h C_g B_{gv} \end{pmatrix} = 0.$$

And thus,

$$\begin{aligned} &\left(\langle \hat{x}_0^+, \hat{f}_1 \rangle, \langle \hat{A}_2 \hat{x}_0^+, \hat{f}_1 \rangle, \langle \hat{A}_2^2 \hat{x}_0^+, \hat{f}_1 \rangle, \ldots \right) = \\ &\qquad \left(\langle \hat{x}_0^+, \hat{f}_1 \rangle, \hat{F}_1 \hat{A}_2 \hat{x}_0^+, \hat{F}_1 \hat{A}_2^2 \hat{x}_0^+, \ldots \right) = 0. \end{aligned}$$

By Lemma 4.2 and the definition of $\hat{x}_{\mathrm{P}}^+(t)$, it can then be concluded that indeed $\langle \hat{x}_{\mathrm{P}}^+(t), \hat{f}_1 \rangle = 0$.

To show that $\hat{x}_{\mathrm{P}}^+(t)$ is in $\hat{\mathcal{S}}_2$ for a.e. t in $[0, \epsilon]$ it remains to show that $\langle \hat{A}_1 \hat{x}_{\mathrm{P}}^+(t), \hat{f}_1 \rangle < 0$ for a.e. t in $[0, \epsilon]$. For this, notice that by (4.11):

$$\hat{x}_0^+ \in \hat{\mathcal{S}} \setminus \hat{\mathcal{S}}_{\mathrm{int}} \implies \left(\langle \hat{x}_0^+, \hat{f}_1 \rangle, \langle \hat{A}_1 \hat{x}_0^+, \hat{f}_1 \rangle, \ldots \right) <_l 0. \tag{4.15}$$

The following relation holds, as can be verified by substitution. It also follows from (2.44):

$$\hat{A}_2 x = \hat{A}_1 \hat{x} + \langle \hat{A}_1 \hat{x}, \hat{f}_1 \rangle \hat{f}_2, \qquad \forall \hat{x} \in \hat{X}_\Sigma. \tag{4.16}$$

With this, the following claim can be made:
Claim: If for some $N \in \mathbb{N}$, $\langle \hat{A}_1^k \hat{x}, \hat{f}_1 \rangle = 0 \ \forall k \in \{0, 1, \ldots, N\}$, then it will be shown that:

$$\left\langle \hat{A}_1 \hat{A}_2^i \hat{x}, \hat{f}_1 \right\rangle = \left\langle \hat{A}_1^{i+1} \hat{x}, \hat{f}_1 \right\rangle \qquad \text{(for } i \in \{0, 1, \ldots, N\}\text{)}.$$

In particular:

$$\left\langle \hat{A}_1 \hat{A}_2^i \hat{x}, \hat{f}_1 \right\rangle = 0 \qquad \text{(for } i \in \{0, 1, \ldots, (N-1)\}\text{)}.$$

Proof of the claim. By induction. As induction basis, the statement trivially holds that for $i = 0$. For the induction step, assume that for some $0 < h < N$, it holds that:

$$\left\langle \hat{A}_1 \hat{A}_2^i \hat{x}, \hat{f}_1 \right\rangle = \left\langle \hat{A}_1^{i+1} \hat{x}, \hat{f}_1 \right\rangle \qquad \forall 0 \le i \le h. \tag{4.17}$$

Then, using (4.16) with $\hat{x} = \hat{A}_2^j \hat{x}_0^+$, the following can be obtained:

$$\hat{A}_2^{j+1} \hat{x}_0^+ = \hat{A}_1 \hat{A}_2^j \hat{x}_0^+ + \left\langle \hat{A}_1 \hat{A}_2^j \hat{x}_0^+, \hat{f}_1 \right\rangle \hat{f}_2 \qquad \forall 0 \le j \le h. \tag{4.18}$$

By substitution of the induction assumption (4.17), and using the assumption in the claim this can be reduced to:

$$\hat{A}_2^{j+1} \hat{x}_0^+ = \hat{A}_1 \hat{A}_2^j \hat{x}_0^+ \qquad \forall 0 \le j < h. \tag{4.19}$$

Substitution of (4.19) for $j = h-1, h-2, \ldots, 1, 0$ gives:

$$\hat{A}_2^h \hat{x}_0^+ = \hat{A}_1^h \hat{x}_0^+.$$

With this the induction step can be completed:

$$\left\langle \hat{A}_1 \hat{A}_2^{h+1} \hat{x}_0^+, \hat{f}_1 \right\rangle = \left\langle \hat{A}_1 \hat{A}_2 \hat{A}_1^h \hat{x}_0^+, \hat{f}_1 \right\rangle.$$

Substitution of (4.16) gives:

$$\begin{aligned}\left\langle \hat{A}_1 \hat{A}_2^{h+1} \hat{x}_0^+, \hat{f}_1 \right\rangle &= \left\langle \hat{A}_1^{h+2} \hat{x}_0^+, \hat{f}_1 \right\rangle + \left\langle \hat{A}_1^{h+1} \hat{x}_0^+, \hat{f}_1 \right\rangle \langle \hat{A}_1 \hat{f}_2, \hat{f}_1 \rangle, \\ &= \left\langle \hat{A}_1^{h+2} \hat{x}_0^+, \hat{f}_1 \right\rangle.\end{aligned}$$

Thus, for any $0 \le h < N$, the induction assumption holds also for $h+1$. *This proves the claim.*

Combining (4.15) with the claim gives:

$$\hat{x}_0^+ \in \hat{\mathcal{S}} \setminus \hat{\mathcal{S}}_{\text{int}} \implies \left(\langle \hat{A}_1 \hat{x}_0^+, \hat{f}_1 \rangle, \langle \hat{A}_1 \hat{A}_2 \hat{x}_0^+, \hat{f}_1 \rangle, \langle \hat{A}_1 \hat{A}_2^2 \hat{x}_0^+, \hat{f}_1 \rangle, \ldots \right) <_l 0. \tag{4.20}$$

From Lemma 4.2, it can then be concluded that there is some ϵ_2 for which $t \in (0, \epsilon_2] \implies \left\langle \hat{A}_1 e^{\hat{A}_2 t} \hat{x}_0^+, \hat{f}_1 \right\rangle < 0$. In conclusion, by taking $\epsilon = \min\{\epsilon_1, \epsilon_2\}$, it holds that $\hat{x}_P(t) \in \hat{\mathcal{S}}_2$ for all $t \in (0, \epsilon]$. Thus $\hat{x}_P(t)$ is a local Cartheodory solution of the HIGS controlled system Σ.

In the two cases, a local Carathéodory solution to the system $\hat{\Sigma}$ could be constructed based on the semigroup generated by A_1 and A_2 respectively. This proves that a local solution exists for all $\hat{x}_0 \in \hat{\mathcal{S}}$, and thus for all $x_0 \in \mathcal{S}$ and piecewice Bohl inputs w. □

Remark 4.5. *In the proof of Theorem 4.4 that is given above, the steps taken are very similar to the proof by Deenen et al. [9] where A is assumed to be finite dimensional (a matrix). Because of the boundedness of A, the equivalences as in Lemma 4.2 are available on which the proof heavily relies. In contrast to the result by Deenen et al. [9], it is not assumed in Theorem 4.4 that* $C_g B_{gv} = 0$. *Omitting this assumption leads to* A_2 *and* B_2 *having more non-zero elements in the second row. This results in* w *appearing in the definition of the* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *sets. However, this dependence quickly drops when the combined system* $\hat{\Sigma}$ *is constructed: The sets* $\hat{\mathcal{S}}_1$ *and* $\hat{\mathcal{S}}_2$ *are then again of the same form as in the work by Deenen et al. [9].*

4.4 Global existence of solutions

Now that local existence of solutions is proven, it can be shown that these solutions can be extended. In other words, that the solutions are forward complete in the sense of the following definition.

Definition 4.6. *Let* $\mathbb{T} \subseteq \mathbb{R}_{\ge 0}$ *be of the form* $[0, T]$, $[0, T)$ *or* $\mathbb{R}_{\ge 0}$ *with* $T \in \mathbb{R}$. *A solution* $x : \mathbb{T} \to X_\Sigma$ *is called* <u>*maximal*</u> *if there does not exist another solution* $x' : \mathbb{T}' \to X_\Sigma$ *with* $\mathbb{T}' = [0, T') \supsetneq \mathbb{T}$ *that satisfies* $x(t) = x'(t)\ \forall t \in \mathbb{T}$.
$x : \mathbb{T} \to X_\Sigma$ *is called* <u>*forward complete*</u> *if it is maximal and* $\mathbb{T}$ *is* $\mathbb{R}_{\ge 0}$.

Theorem 4.7. *All maximal solutions to the HIGS controlled system* Σ *are forward complete.*

Proof. It will be shown that a solution $x : \mathbb{T} \to X_\Sigma$ to the HIGS controlled system with $\mathbb{T} = [0, T]$ or $\mathbb{T} = [0, T)$ for any $T > 0$ is not maximal (i.e. it cannot be extended), by showing that the left limit $\lim_{t \uparrow T} x(t)$ exists and is in $\mathcal{S}$. Then the local-existence result of Theorem 4.4 can be used to extend the solution to $[0, T + \epsilon)$.

Case 1: $\mathbb{T} = [0, T]$:
As the solutions are assumed to be absolutely continuous, the left limit exists.

Case 2: $\mathbb{T} = [0, T)$:

Again, the assumption that $w(t)$ is a piecewise Bohl function is used. By this assumption, there is a t_i such that w is a Bohl function on $[t_i, T)$. Then by the reasoning as in Section 2.2.1, a system like $\hat{\Sigma}$ in (2.28) can be obtained (possibly with different $\hat{A}_1$ and $\hat{A}_2$). Therefore, it holds that:

$$\dot{\hat{x}} = \begin{cases} \hat{A}_1\hat{x} & \text{if } \hat{x} \in \hat{\mathcal{S}}_1, \\ \hat{A}_2\hat{x} & \text{if } \hat{x} \in \hat{\mathcal{S}}_2. \end{cases}$$

As $\hat{A}_1$ and $\hat{A}_2$ are bounded, there is some M such that $\left\|\dot{\hat{x}}(t)\right\| \leq M\left\|\hat{x}(t)\right\|$ for all $t \in [t_i, T)$.

By the Fundamental Theorem of Calculus:

$$\begin{aligned} \hat{x}(t) &= \hat{x}(t_i) + \int_{t_i}^{T} \dot{\hat{x}}(\tau)\,\mathrm{d}\tau, \\ \|\hat{x}(t)\| &\leq \|\hat{x}(t_i)\| + \int_{t_i}^{T} \left\|\dot{\hat{x}}(\tau)\right\|\mathrm{d}\tau, \\ \|\hat{x}(t)\| &\leq \|\hat{x}(t_i)\| + M\int_{t_i}^{T} \|\hat{x}(\tau)\|\,\mathrm{d}\tau. \end{aligned}$$

Then by Gronwalls Lemma, it can be concluded that there exists an $L > 0$ such that $\|\hat{x}(t)\| \leq L$ for all $t \in [t_i, T)$. Furthermore, for $s, t \in [t_i, T)$:

$$\begin{aligned} \|x(t) - x(s)\| \leq \|\hat{x}(t) - \hat{x}(s)\| &\leq \int_{s}^{t} \left\|\dot{\hat{x}}(\tau)\right\|\mathrm{d}\tau, \\ &\leq M\int_{s}^{t} \|\hat{x}(\tau)\|\,\mathrm{d}\tau, \\ &\leq ML(t-s). \end{aligned}$$

So x is Lipschitz continuous on $[t_i, T)$, this implies that the limit $\lim_{t\uparrow T} x(t)$ exists. As the set $\mathcal{S}$ is closed in X_Σ, this limit also lies in $\mathcal{S}$. □

4.5 Conclusion

In this chapter, existence of Carathéodory solutions to the closed loop system of HIGS with a plant $\mathcal{G}$ of the form (2.1) with bounded A_g, B_{gv} and C_g was established. First, local existence of such solutions was established in Theorem 4.4, where the equivalences from Lemma 4.2 played a central role. In the proof of Theorem 4.4, the following steps were taken:

1. Use the assumption that w is a piecewise Bohl function to write w as the output of an exosystem $\mathcal{W}$ on an interval $[0, \tilde{\epsilon}]$.

2. Combine the systems Σ and $\mathcal{W}$ to obtain a single system $\hat{\Sigma}$ that has no external inputs.

3. Define the set $\hat{\mathcal{S}}_{\text{int}}$ as in (4.10) and show that for $\hat{x}_0 \in \hat{\mathcal{S}}_{\text{int}}$, a local solution is given by $\hat{x}_{\mathrm{I}}(t) = \mathrm{e}^{\hat{A}_1 t}\hat{x}_0$ by showing that $\hat{x}_{\mathrm{I}}(t) \in \hat{\mathcal{S}}_1$ for $t \in [0, \epsilon]$.

4. For $\hat{x}_0 \in \hat{\mathcal{S}} \backslash \hat{\mathcal{S}}_{\text{int}}$, show that a local solution is given by $\hat{x}_{\mathrm{P}}(t) = \mathrm{e}^{\hat{A}_2 t}\hat{x}_0$ by showing that $\hat{x}_{\mathrm{P}}(t) \in \hat{\mathcal{S}}_2$ for $t \in [0, \epsilon]$ by proving the three conditions from the definition of $\hat{\mathcal{S}}_2$ as in (4.8):

   a. $\left\langle \hat{x}_{\mathrm{P}}^{+}(t), \hat{f}_2 \right\rangle > 0$.

   b. $\left\langle \hat{x}_{\mathrm{P}}^{+}(t), \hat{f}_1 \right\rangle = 0$.

   c. $\left\langle A_1\hat{x}_{\mathrm{P}}^{+}(t), \hat{f}_1 \right\rangle < 0$.

To show that the local solutions can be extended to global solutions that are define for all $t \in \mathbb{R}_{\geq 0}$ in Theorem 4.7, the following steps were made:

1. A solution that is defined on $[0, T]$ can be extended using the local result from Theorem 4.4 can be used with $x_0 = x(T)$.
2. Similarly, a solution defined on $[0, T)$ can also be extended to a larger interval as by the boundedness of $\hat{A}_1$ and $\hat{A}_2$, x is Lipschitz continuous and thus $\lim_{t \uparrow T} x(t)$ exists. Thus again Theorem 4.4 can be used to extend the solution.
3. As solutions can be extended if they are defined on the interval $[0, T]$ or $[0, T)$, it can be concluded that solutions cannot be maximal if they are not forward complete.

In the next chapter, an attempt is made to further generalize these existence results by trying to relax the assumption that A_g is bounded.

5 An attempt to establish well-posedness of HIGS as a controller for plants with an unbounded generator

In the previous chapter, it was shown that global solutions exist for the HIGS in feedback with a linear system of the form

$$\mathcal{G}\colon \begin{cases} \dot{x}_g(t) = A_g x_g(t) + B_{gv} v(t) + B_{gw} w(t), \\ e(t) \;\;= C_g x_g(t), \end{cases}$$

when $A_g\colon X_{\mathcal{G}} \to X_{\mathcal{G}}$, $B_{gv}\colon \mathbb{R} \to X_{\mathcal{G}}$, $B_{gw}\colon X_w \to X_{\mathcal{G}}$ and $C_g\colon X_{\mathcal{G}} \to \mathbb{R}$ are bounded operators.

The proof of local existence of solutions as in Chapter 4 relies heavily on the assumption that these operators are bounded, and thus also A_1 and A_2 are bounded. As many interesting examples, such as partial differential equations, have an A_g that is not bounded, it would be nice if an existence result could be obtained for a more general set of generators. In this section, an attempt is made to generalize the well-posedness results from the previous chapter to analytic semigroups.

For simplicity, the focus is on a system for which the input w is zero. Once that question would be settled, it is likely possible to generalize this result by assuming that that the input is piecewise Bohl. The system without inputs that is considered in this chapter is given by:

$$\Sigma\colon \begin{cases} \dot{x}(t) = \underbrace{\begin{pmatrix} A_g & -B_{gv} \\ \omega_h C_g & 0 \end{pmatrix}}_{A_1} x(t) & \text{if } x \in \mathcal{S}_1, \\ \dot{x}(t) = \underbrace{\begin{pmatrix} A_g & -B_{gv} \\ k_h C_g A_g & -k_h C_g B_{gv} \end{pmatrix}}_{A_2} x(t) & \text{if } x \in \mathcal{S}_2, \\ e(t) = \underbrace{\begin{pmatrix} C_g & 0 \end{pmatrix}}_{C} x(t) & \text{and } x \in \mathcal{S} \end{cases} \tag{5.1}$$

The definitions of $\mathcal{S}$, $\mathcal{S}_1$ and $\mathcal{S}_2$ from (2.21), (2.23) and (2.24) reduce to:

$$\mathcal{S}_1 := \mathcal{S}_1^+ \cup -\mathcal{S}_1^+ \text{ with } \mathcal{S}_1^+ := \left\{x \in X_\Sigma \mid \langle x, f_2\rangle \geq 0 \wedge \left(\langle x, f_1\rangle, \langle A_1 x, f_1\rangle\right) \geq_l 0\right\}, \tag{5.2}$$

$$\mathcal{S}_2 := \mathcal{S}_2^+ \cup -\mathcal{S}_2^+ \text{ with } \mathcal{S}_2^+ := \left\{x \in X_\Sigma \mid \langle x, f_2\rangle > 0 \wedge \langle x, f_1\rangle = 0 \wedge \langle A_1 x, f_1\rangle < 0\right\}, \tag{5.3}$$

$$\mathcal{S} := \mathcal{S}_1 \cup \mathcal{S}_2. \tag{5.4}$$

Notice that this system is very similar to the $\hat{\Sigma}$ system that is used for the well-posedness proof in Chapter 4 after the input had been incorporated in the system.

In the bounded case, it was quite clear that A_1 and A_2 generate semigroups. This is not obvious in the unbounded case. Conditions for A_1 and A_2 to be generators of semigroups are given in Section 5.1.

The proof of Theorem 4.4 relies on the characterization of the behavior of $\langle e^{At} x, f\rangle$ near $t = 0$ from Lemma 4.2. The equivalences from this lemma cannot hold in the case where A is unbounded, even more, the lexicographic inequalities can not even be written down in a meaningful way if $x \notin \mathrm{D}(A^\infty) = \bigcap_{i\in\mathbb{N}} \mathrm{D}(A^i)$. Therefore, an alternative characterization is proposed in Lemma 5.4. This new characterization is shown to be equivalent for bounded A using Lemma 4.2.

In Section 5.3 an attempt is made to show local existence of solutions using the alternative characterization.

To prove the characterization, it is proposed to make an assumption on the initial condition in Section 5.4.

5.1 Generation of semigroups

In the previous chapter, explicit solutions were proposed and shown to satisfy the combined system $(\hat{\Sigma})$ to achieve local existence of solutions to this system. These solutions were of the form $\mathrm{e}^{\hat{A}_1 t}x_0$ or $\mathrm{e}^{\hat{A}_2 t}x_0$. Here, $\mathrm{e}^{\hat{A}_i t}$ denotes the semigroup generated by $\hat{A}_i$. In the previous chapter, $\hat{A}_1$ and $\hat{A}_2$ were bounded. Therefore it was clear that these operators indeed generate a semigroup, by (3.5). In this chapter, A_1 and A_2 can be unbounded. The semigroups are essential to be able to construct local solutions. Therefore, the first question to answer is under which conditions A_1 and A_2 generate a semigroup. Application of Theorem 3.12 already gives a partial answer:

Proposition 5.1. *Consider A_1 as in* (2.25)*. Then A_1 generates an analytic semigroup on X if A_g generates an analytic semigroup on X_G and B_{gv} and C_g are bounded.*

Proof. The operator A_1 can be written as:

$$A_1 = \underbrace{\begin{pmatrix} A_g & 0 \\ 0 & 0 \end{pmatrix}}_{\tilde{A}_g} + \underbrace{\begin{pmatrix} 0 & -B_{gv} \\ \omega_h C_g & 0 \end{pmatrix}}_{\tilde{F}}.$$

In the above, $\tilde{A}_g$ clearly generates an analyitic semigroup on $X = X_g \times \mathbb{R}$, as A_g generates an analytic semigroup on X_g. The second operator is bounded. Thus, A_1 is a bounded perturbation of $\tilde{A}_g$. Then, it follows from Theorem 3.12 that also $A_1 = \tilde{A}_g + \tilde{F}$ generates an analytic semigroup. □

The perturbation $A_2 - A_1$ is not necessarily bounded, so a different argument is needed to show that also A_2 generates a semigroup. The special structure of A_1 and A_2 results in the fact that $A_2 - A_1$ is a rank-1 perturbation (as established in Equation (2.45)):

$$A_2 - A_1 = \langle A_1 x, f_1 \rangle f_2.$$

The following theorem from Desch and Schappacher [40, Thm. 1] guarantees that A_2 also generates an analytic semigroup:

Theorem 5.2. *Let A be the generator of an analytic semigroup and let $F : (\mathrm{D}(A), \|A\cdot\| + \|\cdot\|) \to (X, \|\cdot\|)$ be a compact linear operator. Then $(A + F)$ also generates an analytic semigroup.*

As an operator from $(\mathrm{D}(A_1), \|A_1\cdot\| + \|\cdot\|)$ to $(X, \|\cdot\|)$, $\langle A_1 x, f_1 \rangle f_2$ is bounded. As its range is one dimensional, and thus it is a compact operator. Hence, Theorem 5.2 gives the result that indeed A_2 generates an analytic semigroup if A_1 does, also the domains of A_1 and A_2 are the same.

5.2 An alternative characterization of behavior of semigroups for small t

For an unbounded operator A, the terms involving A in the lexicographic inequalities from Lemma 4.2 cannot be defined when $x \notin \mathrm{D}(A^i)$ for $i \geq 1$. To be able to adapt the proof to a more general situation, another characterization of the behavior of $\langle \mathrm{e}^{At}x, f \rangle$ near zero is needed. In this section, the following alternative characterization is proposed and proven under certain conditions:

$$\exists \epsilon > 0 \text{ s.t. } \forall t \in [0,\epsilon], \left\langle e^{At}x, f\right\rangle = 0 \iff \exists \Lambda > 0 \text{ s.t. } \forall \lambda \geq \Lambda, \int_0^\infty \lambda e^{-\lambda t}\left\langle e^{At}x, f\right\rangle \mathrm{d}t = 0, \tag{5.5a}$$

$$\exists \epsilon > 0 \text{ s.t. } \forall t \in (0,\epsilon], \left\langle e^{At}x, f\right\rangle > 0 \iff \exists \Lambda > 0 \text{ s.t. } \forall \lambda \geq \Lambda, \int_0^\infty \lambda e^{-\lambda t}\left\langle e^{At}x, f\right\rangle \mathrm{d}t > 0, \tag{5.5b}$$

$$\exists \epsilon > 0 \text{ s.t. } \forall t \in [0,\epsilon], \left\langle e^{At}x, f\right\rangle \geq 0 \iff \exists \Lambda > 0 \text{ s.t. } \forall \lambda \geq \Lambda, \int_0^\infty \lambda e^{-\lambda t}\left\langle e^{At}x, f\right\rangle \mathrm{d}t \geq 0. \tag{5.5c}$$

In Lemma 5.4 a generalized version of the proposed equivalences is established, under a condition of sign uniqueness in an interval near zero. Corollary 4.3 shows that this condition is satisfied for bounded semigroups. It could be that this property also holds for more general semigroups, but we were not able to provide a proof.

The following counterexample already shows that the above result does not generalize to non-analytic semigroups.

Remark 5.3. *The equivalences proposed in Equation* (5.5) *cannot hold in general for semigroups that are not analytic. To see this, consider the left shift on the Hilbert space $L_2(\mathbb{R}_{\geq 0})$, with the usual inner product, that is:*

$$(T(t)g)(\eta) := g(\eta + t), \quad \forall g \in L_2(\mathbb{R}_{\geq 0}), t, \eta \in \mathbb{R}_{\geq 0}.$$

As shown e.g. in [41, Example 2.1.4], this is indeed a strongly continuous semigroup on $L_2(\mathbb{R}_{\geq 0})$. It is not an analytic semigroup.

Now take f to be $\mathbb{1}_{[0,1]}$, so one on the interval $[0,1]$ *and zero elsewhere. And let g be $-\mathbb{1}_{[2,3]}$. Then for $t \in \mathbb{R}_{\geq 0} \setminus (1,3)$, $\left\langle T(t)g, f\right\rangle = 0$, however for $t \in (1,3)$, $\left\langle T(t)g, f\right\rangle < 0$, and hence:*

$$\int_0^\infty \lambda e^{-\lambda t}\left\langle T(t)g, f\right\rangle \mathrm{d}t < 0 \qquad \forall \lambda > 0.$$

Therefore the '$\Longrightarrow$'-direction of (5.5a) *and* (5.5c) *fails. Furthermore, the '$\Longleftarrow$'-direction of* (5.5b) *fails when considering $-f$.*

In the following lemma, a sufficient condition that gives the equivalences for special analytic semigroups.

Lemma 5.4. *Consider an analytic function $g : \mathbb{C} \supseteq \mathrm{D}(g) \to \mathbb{C}$ with $\mathrm{D}(g) \supset \mathbb{R}_{\geq 0}$ that is exponentially bounded and for which it holds that exactly one of the following statements is true:*

i) $\exists \epsilon > 0$ *s.t.* $\forall t \in [0,\epsilon], g(t) = 0.$

ii) $\exists \epsilon > 0$ *s.t.* $\forall t \in (0,\epsilon], g(t) > 0.$

iii) $\exists \epsilon > 0$ *s.t.* $\forall t \in (0,\epsilon], g(t) < 0.$

Then, the following equivalences hold:

a) $\exists \epsilon > 0$ *s.t.* $\forall t \in [0,\epsilon], g(t) = 0 \iff \exists \Lambda > 0$ *s.t.* $\forall \lambda \geq \Lambda, \int_0^\infty \lambda e^{-\lambda t} g(t)\,\mathrm{d}t = 0.$

b) $\exists \epsilon > 0$ *s.t.* $\forall t \in (0,\epsilon], g(t) > 0 \iff \exists \Lambda > 0$ *s.t.* $\forall \lambda \geq \Lambda, \int_0^\infty \lambda e^{-\lambda t} g(t)\,\mathrm{d}t > 0.$

c) $\exists \epsilon > 0$ *s.t.* $\forall t \in [0,\epsilon], g(t) \geq 0 \iff \exists \Lambda > 0$ *s.t.* $\forall \lambda \geq \Lambda, \int_0^\infty \lambda e^{-\lambda t} g(t)\,\mathrm{d}t \geq 0.$

Proof.
$\implies$ *in (a).* It is assumed that the function g is zero in an interval $[0,\epsilon]$. As g is analytic on an open set in $\mathbb{C}$ containing $(0,\infty)$, it follows from the identity theorem that the function is zero for all $t \in (0,\infty)$ and thus the integral is zero too.

$\implies$ *in (b).* This direction follows from continuity of g. First, the integral is split into three parts:

$$\int_0^\infty \lambda e^{-\lambda t} g(t)\,dt = \int_0^{\epsilon/2} \lambda e^{-\lambda t} g(t)\,dt + \int_{\epsilon/2}^{\epsilon} \lambda e^{-\lambda t} g(t)\,dt + \int_{\epsilon}^{\infty} \lambda e^{-\lambda t} g(t)\,dt.$$

By the assumption that $g(t) > 0$ for $t \in (0,\epsilon]$, the first integral is non-negative. Furthermore, note that there exists $m_\epsilon > 0$ such that for $t \in [\epsilon/2,\epsilon]$, $g(t) > m_\epsilon$. The expression can then be bounded from below by:

$$\int_0^\infty \lambda e^{-\lambda t} g(t)\,dt \geq \int_{\epsilon/2}^{\epsilon} \lambda e^{-\lambda t} m_\epsilon\,dt - \int_{\epsilon}^{\infty} \lambda e^{-\lambda t} \left|g(t)\right|\,dt.$$

As g is exponentially bounded, there exist M and ω such that $\left|g(t)\right| < Me^{\omega t}$ for $t > 0$:

$$\int_0^\infty \lambda e^{-\lambda t} g(t)\,dt \geq \int_{\epsilon/2}^{\epsilon} \lambda e^{-\lambda t} m_\epsilon\,dt - \int_{\epsilon}^{\infty} \lambda e^{-\lambda t} M e^{\omega t}\,dt.$$

For $\lambda > \omega$ both integrals on the right exist and equal:

$$\int_0^\infty \lambda e^{-\lambda t} g(t)\,dt \geq m_\epsilon e^{-\lambda\epsilon/2} - m_\epsilon e^{-\lambda\epsilon} - M\frac{\lambda}{\lambda-\omega} e^{\omega\epsilon-\lambda\epsilon}.$$

For $\lambda > 2\omega$, this can be bounded from below by $\Big(\underbrace{m_\epsilon}_{>0} - \underbrace{(m_\epsilon + M/2\, e^{\omega\epsilon})e^{-\lambda\epsilon/2}}_{\to 0 \text{ for } \lambda\to\infty}\Big)e^{-\lambda\epsilon/2}$.

Thus, for sufficiently large λ, the result is positive.

$\impliedby$ *in (a).* From having that:

$$\exists \Lambda > 0 \text{ s.t. } \forall \lambda \geq \Lambda, \int_0^\infty \lambda e^{-\lambda t} g(t)\,dt = 0,$$

it follows that:

$$\nexists \Lambda > 0 \text{ s.t. } \forall \lambda \geq \Lambda, \int_0^\infty \lambda e^{-\lambda t} g(t)\,dt > 0,$$

and:

$$\nexists \Lambda > 0 \text{ s.t. } \forall \lambda \geq \Lambda, \int_0^\infty \lambda e^{-\lambda t} g(t)\,dt < 0.$$

From the contrapositive of $\implies$ in (b) it then follows that *ii)* and *iii)* from the statement of the lemma are false. And thus, by assumption:

$$\exists \epsilon > 0 \text{ s.t. } \forall t \in [0,\epsilon], g(t) = 0.$$

$\impliedby$ *in (b).* From having that:

$$\exists \Lambda > 0 \text{ s.t. } \forall \lambda \geq \Lambda, \int_0^\infty \lambda e^{-\lambda t} g(t)\,dt > 0,$$

it follows that:

$$\nexists \Lambda > 0 \text{ s.t. } \forall \lambda \geq \Lambda, \int_0^\infty \lambda e^{-\lambda t} g(t)\,dt = 0,$$

and:

$$\nexists \Lambda > 0 \text{ s.t. } \forall \lambda \geq \Lambda, \int_0^\infty \lambda e^{-\lambda t} \langle e^{At}x, -f\rangle\,dt > 0.$$

From the contrapositive of $\implies$ in (b) (with $-f$) and $\implies$ in (a) it then follows that statements *i)* and *iii)* are false and that thus *ii)* must hold.

The equivalence in (c) follows directly from the established equivalences from (a) and (b), combined with the assumption. □

5.3 First steps towards well-posedness using the alternative characterization

The alternative characterization for the behavior of the semigroup near $t = 0$ proposed in Equation (5.5) has been established for semigroups generated by bounded A (following from Lemma 5.4 and Corollary 4.3). For these bounded operators also the characterization from Lemma 4.2 is available. Despite this, there is hope that the proposed equivalences can also be established for more general, i.e. analytic, semigroups. Under the following assumption, many steps of the proof for the bounded case (as in Section 4.5) can also be done in the unbounded case.

Assumption 5.5. *For the HIGS-controlled system Σ as in Equation* (4.1)*, it is assumed that x_0, A_1 and A_2 satisfy the following statements:*

1. *A_g generates an analytic semigroup.*
2. *$x_0 \in \mathrm{D}(A_1)$ $(= \mathrm{D}(A_2))$.*
3. *For all $i, j \in \{1,2\}$, exactly one of the following holds:*
   i) $\exists \epsilon > 0$ *s.t.* $\forall t \in [0,\epsilon], \left\langle e^{A_i t} x_0, f_j \right\rangle = 0.$
   ii) $\exists \epsilon > 0$ *s.t.* $\forall t \in (0,\epsilon], \left\langle e^{A_i t} x_0, f_j \right\rangle > 0.$
   iii) $\exists \epsilon > 0$ *s.t.* $\forall t \in (0,\epsilon], \left\langle e^{A_i t} x_0, f_j \right\rangle < 0.$
4. *For $i \in \{1,2\}$, exactly one of the following holds:*
   i) $\exists \epsilon > 0$ *s.t.* $\forall t \in [0,\epsilon], \left\langle A_1 e^{A_i t} x_0, f_1 \right\rangle = 0.$
   ii) $\exists \epsilon > 0$ *s.t.* $\forall t \in (0,\epsilon], \left\langle A_1 e^{A_i t} x_0, f_1 \right\rangle > 0.$
   iii) $\exists \epsilon > 0$ *s.t.* $\forall t \in (0,\epsilon], \left\langle A_1 e^{A_i t} x_0, f_1 \right\rangle < 0.$

Recall that the proof for local existence of solutions as presented in Section 4.3 in the case where A_1 and A_2 are bounded consisted of the following steps (For the situation in this chapter the first two steps are not relevant as it was assumed that $w = 0$):

1. Use the assumption that w is a piecewise Bohl function to write w as the output of an exosystem $\mathcal{W}$ on an interval $[0,\tilde{\epsilon}]$.
2. Combine the systems Σ and $\mathcal{W}$ to obtain a single system $\hat{\Sigma}$ that has no external inputs.
3. Define the set $\hat{\mathcal{S}}_{\text{int}}$ as in (4.10) and show that for $\hat{x}_0 \in \hat{\mathcal{S}}_{\text{int}}$, a local solution is given by $\hat{x}_{\text{I}}(t) = e^{\hat{A}_1 t}\hat{x}_0$ by showing that $\hat{x}_{\text{I}}(t) \in \hat{\mathcal{S}}_1$ for $t \in [0,\epsilon]$.
4. For $\hat{x}_0 \in \hat{\mathcal{S}} \setminus \hat{\mathcal{S}}_{\text{int}}$, show that a local solution is given by $\hat{x}_{\text{P}}(t) = e^{\hat{A}_2 t}\hat{x}_0$ by showing that $\hat{x}_{\text{P}}(t) \in \hat{\mathcal{S}}_2$ for $t \in [0,\epsilon]$ by proving the three conditions from the definition of $\hat{\mathcal{S}}_2$ as in (4.8):
   a. $\left\langle \hat{x}_{\text{P}}^+(t), \hat{f}_2 \right\rangle > 0.$
   b. $\left\langle \hat{x}_{\text{P}}^+(t), \hat{f}_1 \right\rangle = 0.$
   c. $\left\langle A_1 \hat{x}_{\text{P}}^+(t), \hat{f}_1 \right\rangle < 0.$

Steps 1 and 2 are not relevant in this chapter, as the simplification $w = 0$ was made. The steps to focus on are thus 3 and 4. These steps involve the $\hat{\mathcal{S}}_{\text{int}}$ region. It is defined as:

$$\mathcal{S}_{\text{int}} := \left\{ x_0 \in \mathcal{S} \mid \exists \epsilon > 0, \forall t \in [0,\epsilon], e^{\hat{A}_1 t} x_0 \in \mathcal{S} \right\}. \tag{5.6}$$

Now using the new characterization from Lemma 5.4 (with $g(t) = \langle e^{A_1 t}x, f_j\rangle$) that holds by Assumption 3, it can be seen that $\hat{\mathcal{S}}_{\text{int}}$ can also be written as:

$$\mathcal{S}_{\text{int}} = \mathcal{S}^+_{\text{int}} \cup -\mathcal{S}^+_{\text{int}} \text{ with } \mathcal{S}^+_{\text{int}} = \Big\{x \in X_\Sigma \mid \exists \Lambda > 0 \text{ s.t. } \forall \lambda \geq \Lambda, \int_0^\infty e^{-\lambda t}\langle e^{A_1 t}x_0, f_1\rangle \, dt \geq 0 \wedge \int_0^\infty e^{-\lambda t}\langle e^{A_1 t}x_0, f_2\rangle \, dt \geq 0\Big\}. \tag{5.7}$$

5.3.1 An alternative for Step 3 of proving local existence of solutions without Lemma 4.2

Proposition 5.6. *Assume Assumption 5.5 and assume that $x_0 \in \mathcal{S}_{int}$. Then $x_I(t) := e^{A_1 t}x_0 \in \mathcal{S}_1$ for $t \in (0,\epsilon]$ and some $\epsilon > 0$. And thus $x_I(t)$ is a solution to Σ on this interval.*

Proof. It is clear that if $x_0 \in \mathcal{S}_{\text{int}}$, then $x_0 \in \mathcal{S}^+_{\text{int}}$ or $-x_0 \in \mathcal{S}^+_{\text{int}}$. As in the first proof, define

$$x_0^+ := \begin{cases} x_0 & \text{if } x_0 \in \mathcal{S}^+_{\text{int}} \vee \langle x_0, f_2\rangle > 0, \\ -x_0 & \text{otherwise,} \end{cases} \tag{5.8}$$

such that $x_0 \in \mathcal{S}_{\text{int}} \implies x_0^+ \in \mathcal{S}^+_{\text{int}}$. It will be shown that this implies that for some $\varepsilon > 0$, $x_I^+(t) := e^{A_1 t}x_0^+ \in \mathcal{S}_1^+, \forall t \in [0,\varepsilon]$.

From the definition of $\mathcal{S}^+_{\text{int}}$, it can be concluded that:

$$\exists \varepsilon_1 \text{ such that } t \in [0,\varepsilon_1] \implies \langle e^{A_1 t}x_0^+, f_1\rangle = \langle x_I^+(t), f_1\rangle \geq 0, \tag{5.9}$$

and:

$$\exists \varepsilon_2 \text{ such that } t \in [0,\varepsilon_2] \implies \langle e^{A_1 t}x_0^+, f_2\rangle = \langle x_I^+(t), f_2\rangle \geq 0. \tag{5.10}$$

From (5.9), it is clear that statement *iii)* from Assumption 5.5.3 cannot hold. Now consider the other two cases separately:

Case a: Assumption 5.5.3.*ii)*: $\exists \varepsilon_3 > 0$ s.t. $\forall t \in (0,\varepsilon_3], \langle e^{A_1 t}x_0, f_1\rangle > 0$.
Then it also follows that for $t \in (0,\varepsilon_3]$, $\big(\langle x_I^+(t), f_1\rangle, \langle A_1 x_I^+(t), f_1\rangle\big) \geq_l 0$.

Case b: Assumption 5.5.3.*i)*: $\exists \varepsilon_4 > 0$ s.t. $\forall t \in [0,\varepsilon_4], \langle e^{A_1 t}x_0, f_1\rangle = 0$.
Apply integration by parts on the first integral of the alternative definition of $\mathcal{S}^+_{\text{int}}$ as in (5.7) to obtain:

$$\int_0^\infty \lambda e^{-\lambda t}\langle e^{A_1 t}x_0, f_1\rangle \, dt \geq 0 \quad \forall \lambda > \Lambda,$$
$$\Big[-e^{-\lambda t}\langle e^{A_1 t}x_0, f_1\rangle\Big]_0^\infty + \int_0^\infty e^{-\lambda t}\langle A_1 e^{A_1 t}x_0, f_1\rangle \geq 0 \quad \forall \lambda > \Lambda.$$

The evaluation of the first term at $t = 0$ gives 0 by assumption. Using (3.2), it can be concluded that also the evaluation at $t = \infty$ gives 0 for $\lambda > \omega$. Hence:

$$\int_0^\infty \lambda e^{-\lambda t}\langle A_1 e^{A_1 t}x_0, f_1\rangle \geq 0 \quad \forall \lambda > \max\{\Lambda, \omega\}.$$

Then, using the equivalence from Lemma 5.4 (which holds by Assumption 5.5.4i), there is some ε_3 such that for each $t \in [0,\varepsilon_3]$, it holds that $\langle A_1 x_I^+(t), f_1\rangle \geq 0$.

Combined with the assumption of this case it therefore also holds that in this interval:

$$\big(\langle A_1 x_0^+, f_1\rangle, \langle A_1^2 x_0^+, f_1\rangle, \dots\big) \geq_l 0.$$

In case a and b, it is concluded that there is some ε_3 such that for $t \in (0,\varepsilon_3]$, $\big(\langle x_I^+(t), f_1\rangle, \langle A_1 x_I^+(t), f_1\rangle\big) \geq_l 0$. Combined with (5.10), it can be concluded that for $\varepsilon = \min\{\varepsilon_2, \varepsilon_3\}$, $x_I^+ \in \mathcal{S}_1 \; \forall t \in [0,\varepsilon]$. As by definition of $\mathcal{S}_1$, $x \in \mathcal{S}_1 \implies -x \in \mathcal{S}_1$ it can be concluded that also $x_I(t) \in \mathcal{S}_1 \; \forall t \in [0,\varepsilon]$. As $x_I(t)$ is a solution to the integral mode by construction, $x_I(t)$ is a local solution for initial values $x_0 \in \mathcal{S}_{\text{int}}$. □

5.3.2 An alternative for Step 4 of proving local existence of solutions without Lemma 4.2

For $x_0 \in \mathcal{S} \setminus \mathcal{S}_{\text{int}}$, the goal is to show that $x_P(t) = \mathrm{e}^{A_2 t} x_0$ by showing that $x_\text{P}(t) \in \mathcal{S}_2$ for $t \in [0,\epsilon]$ by proving the three conditions from the definition of $\mathcal{S}_2$ as in (5.3):

a. $\langle x_\text{P}^+(t), f_2 \rangle > 0.$

b. $\langle x_\text{P}^+(t), f_1 \rangle = 0.$

c. $\langle A_1 x_\text{P}^+(t), f_1 \rangle < 0.$

For a and b, the following proposition provides a proof.

Proposition 5.7. *Assume Assumption 5.5 and assume that $x_0 \in \mathcal{S} \setminus \mathcal{S}_{int}$. Then for $x_P(t) := \mathrm{e}^{A_2 t} x_0$ there exists $\epsilon > 0$ such that for all $t \in (0,\epsilon]$ it holds that:*

a. $\langle x_P^+(t), f_2 \rangle > 0.$

b. $\langle x_P^+(t), f_1 \rangle = 0.$

Proof. For $x_0 \in \mathcal{S} \setminus \mathcal{S}_{\text{int}}$, it will first be shown that:

$$x_0 \in \mathcal{S} \setminus \mathcal{S}_{\text{int}} \implies \langle x_0, f_1 \rangle = 0 \wedge \langle x_0, f_2 \rangle \neq 0. \tag{5.11}$$

Like in the proof of Theorem 4.4, the contrapositive is proven in cases:

Case a & b: $\langle x_0, f_1 \rangle < 0 \vee \langle x_0, f_1 \rangle > 0$:
The proof of these cases is identical to the proof of Theorem 4.4: It follows that either $x_0 \notin \mathcal{S}$ or $x_0 \in \mathcal{S}_{\text{int}}$.

Case c: $\langle x_0, f_2 \rangle = 0$:
Recall the following equality from the proof of Theorem 4.4 (that can be verified by substitution):

$$\langle A_1 x, f_2 \rangle = \frac{\omega_h}{k_h} \langle x, f_1 + f_2 \rangle \quad \forall x \in X. \tag{5.12}$$

From the combination of the statement of assumption 3 and Lemma 5.4, it follows that at least one of the following two statements is true:

$$\exists \epsilon > 0 \text{ s.t. } \exists \Lambda > 0 \text{ s.t. } \forall \lambda \geq \Lambda, \int_0^\infty \lambda \mathrm{e}^{-\lambda t} \langle \mathrm{e}^{A_1 t} x_0, f_1 \rangle \,\mathrm{d}t \geq 0, \tag{5.13a}$$

$$\exists \epsilon > 0 \text{ s.t. } \exists \Lambda > 0 \text{ s.t. } \forall \lambda \geq \Lambda, \int_0^\infty \lambda \mathrm{e}^{-\lambda t} \langle \mathrm{e}^{A_1 t} x_0, f_1 \rangle \,\mathrm{d}t \leq 0. \tag{5.13b}$$

To show that x_0 is in $\mathcal{S}_{\text{int}}$, the aim is to show that such a statement also holds with the same sign for f_2 instead of f_1. To do this, integration by parts is used:

$$\int_0^\infty \lambda \mathrm{e}^{-\lambda t} \langle \mathrm{e}^{A_1 t} x_0, f_2 \rangle \,\mathrm{d}t = \left[-\mathrm{e}^{-\lambda t} \langle \mathrm{e}^{A_1 t} x_0, f_2 \rangle \right]_0^\infty + \int_0^\infty \mathrm{e}^{-\lambda t} \langle A_1 \mathrm{e}^{A_1 t} x_0, f_2 \rangle \,\mathrm{d}t.$$

Again, by the assumption that $\langle x_0, f_2 \rangle = 0$ and the growth bound of the C_0-semigroup (3.2), the first term on the r.h.s. is zero. Then using (5.12) gives:

$$\int_0^\infty \lambda \mathrm{e}^{-\lambda t} \langle \mathrm{e}^{A_1 t} x_0, f_2 \rangle \,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-\lambda t} \frac{\omega_h}{k_h} \langle \mathrm{e}^{A_1 t} x_0, f_2 \rangle + \mathrm{e}^{-\lambda t} \frac{\omega_h}{k_h} \langle \mathrm{e}^{A_1 t} x_0, f_1 \rangle \,\mathrm{d}t.$$

Taking the first term on the right to the left hand side gives:

$$\left(\lambda - \frac{\omega_h}{k_h} \right) \int_0^\infty \mathrm{e}^{-\lambda t} \langle \mathrm{e}^{A_1 t} x_0, f_2 \rangle \,\mathrm{d}t, = \frac{\omega_h}{k_h} \int_0^\infty \mathrm{e}^{-\lambda t} \langle \mathrm{e}^{A_1 t} x_0, f_1 \rangle \,\mathrm{d}t.$$

For $\lambda \geq \max\{\Lambda, \omega_h / k_h, \omega\}$, it is thus clear that indeed the same sign is obtained as in (5.13a) or (5.13b). Thus, by the alternative definition of $\mathcal{S}_{\text{int}}$ as in (5.7), x_0 is in $\mathcal{S}_{\text{int}}$.

Thus in all cases (a-c), the conclusion is drawn that $x_0 \notin \mathcal{S} \setminus \mathcal{S}_{\text{int}}$, and thus (5.11) *is proven.*

In particular, it then follows that for x_0 in $\mathcal{S} \setminus \mathcal{S}_{\text{int}}$ it must hold that $\langle x_0^+, f_2 \rangle > 0$. By continuity, there is some $\epsilon_1 > 0$ such that $\langle x_{\text{P}}^+(t), f_2 \rangle > 0$ for all $t \in [0, \epsilon_1]$.

To show that $\langle x_{\text{P}}^+(t), f_1 \rangle = 0$ for $t \in (0, \epsilon_1]$, notice that by (5.11), it holds that in this case $\langle x_0^+, f_1 \rangle = 0$. By definition of f_1 and A_2:

$$\langle A_2 x, f_1 \rangle = 0 \quad \forall x \in \hat{X}_\Sigma. \tag{5.14}$$

By integration by parts, using that $\langle x_0^+, f_1 \rangle = 0$ (from (5.11)) and the growth bound from (3.2):

$$\int_0^\infty \lambda \mathrm{e}^{-\lambda t} \langle \mathrm{e}^{A_2 t} x_0, f_1 \rangle \,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-\lambda t} \langle A_2 \mathrm{e}^{A_2 t} x_0, f_1 \rangle \,\mathrm{d}t = 0, \tag{5.15}$$

for $\lambda > \omega$. By assumption (3) combined with the equivalences established in Lemma 5.4 and the definition of $x_{\text{P}}^+(t)$, it can then be concluded that indeed $\langle x_{\text{P}}^+(t), f_1 \rangle = 0$. □

To show that $x_{\text{P}}^+(t)$ is in $\mathcal{S}_2$ for almost every t in $[0, \epsilon]$ it remains to show that $\langle A_1 x_{\text{P}}^+(t), f_1 \rangle < 0$ for a.e. t in $[0, \epsilon]$ (Step 4c). By Assumption 4 and Lemma 5.4, this can be done by showing that for some $\Lambda > 0$:

$$\int_0^\infty \lambda \mathrm{e}^{-\lambda t} \langle A_1 x_{\text{P}}^+(t), f_1 \rangle \,\mathrm{d}t < 0 \quad \forall \lambda > \Lambda.$$

Using (2.46), the integral can be rewritten as:

$$\begin{aligned}\int_0^\infty \lambda \mathrm{e}^{-\lambda t} \langle A_1 x_{\text{P}}^+(t), f_1 \rangle \,\mathrm{d}t = \int_0^\infty \lambda \mathrm{e}^{-\lambda t} \Big[&\langle A_2 x_{\text{P}}^+(t), f_1 \rangle - \langle A_2 x_{\text{P}}^+(t), f_2 \rangle \langle h, f_1 \rangle \\ &+ \frac{\omega_h}{k_h} \langle x_{\text{P}}^+(t), f_1 + f_2 \rangle \langle h, f_1 \rangle \Big] \,\mathrm{d}t.\end{aligned}$$

Using $\langle h, f_1 \rangle = -1$ and (5.15), this can be reduced to:

$$\begin{aligned}\int_0^\infty \lambda \mathrm{e}^{-\lambda t} \langle A_1 x_{\text{P}}^+(t), f_2 \rangle \,\mathrm{d}t &= \int_0^\infty \lambda \mathrm{e}^{-\lambda t} \Big[\langle A_2 \mathrm{e}^{A_2 t} x_0^+, f_2 \rangle - \frac{\omega_h}{k_h} \langle \mathrm{e}^{A_2 t} x_0^+, f_2 \rangle \Big] \,\mathrm{d}t, \\ &= \left\langle A_2 \lambda \int_0^\infty \mathrm{e}^{-\lambda t} \mathrm{e}^{A_2 t} x_0^+ \,\mathrm{d}t, f_2 \right\rangle - \frac{\omega_h}{k_h} \left\langle \lambda \int_0^\infty \mathrm{e}^{-\lambda t} \mathrm{e}^{A_2 t} x_0^+ \,\mathrm{d}t, f_2 \right\rangle.\end{aligned}$$

Using the integral representation of the resolvent (Proposition 3.8):

$$\begin{aligned}&= \langle \lambda(\lambda - A_2)^{-1} A_2 x_0^+, f_2 \rangle - \frac{\omega_h}{k_h} \langle \lambda(\lambda - A_2)^{-1} x_0^+, f_2 \rangle, \\ &= \langle \lambda(\lambda - A_2)^{-1} A_2 x_0^+, f_2 \rangle - \frac{\omega_h}{k_h} \langle (\lambda - \lambda + A_2)(\lambda - A_2)^{-1} x_0^+ + x_0^+, f_2 \rangle, \\ &= \left\langle \left(\lambda - \frac{\omega_h}{k_h} \right) (\lambda - A_2)^{-1} A_2 x_0^+, f_2 \right\rangle - \frac{\omega_h}{k_h} \langle x_0^+, f_2 \rangle.\end{aligned}$$

Thus it was shown that:

$$\int_0^\infty \lambda \mathrm{e}^{-\lambda t} \langle A_1 x_{\text{P}}^+(t), f_1 \rangle \,\mathrm{d}t < 0 \iff \left\langle \left(\lambda - \frac{\omega_h}{k_h} \right) (\lambda - A_2)^{-1} A_2 x_0^+, f_2 \right\rangle - \frac{\omega_h}{k_h} \langle x_0^+, f_2 \rangle < 0. \tag{5.16}$$

In the above, the achievement made is that the semigroup could be replaced by the resolvents and that A_1 is not combined with the semigroup of A_2 anymore. Furthermore, it is known that the term $\langle x_0^+, f_2 \rangle$ is negative. However, unfortunately it is not clear whether the entire expression is also negative for sufficiently large λ.

5.4 How additional conditions on x_0 could help to satisfy the assumptions of Lemma 5.4 and why this is problematic

In order to satisfy the assumptions of Lemma 5.4, it can be of help to impose additional conditions on the initial state x_0. Let $T_1(t)$ denote the analytic semigroup generated by A_1. Then, if it is assumed that x_0 is of the form $T_1(\delta)\tilde{x}_0$ for some $\tilde{x}_0 \in X_\Sigma$, x_0 would be in $\mathrm{D}(A_1^\infty)$. Furthermore, because the map $t \mapsto T_1(t)T_1(\delta)\tilde{x}_0 = T_1(t+\delta)\tilde{x}_0$ is analytic at 0, the property in the assumption of Lemma 5.4 that says that $\langle T_1(t)x_0, f\rangle$ has a unique sign in a small interval $(0,\epsilon]$ would be satisfied, as follows from the following proposition.

Proposition 5.8. *If, given an $\epsilon > 0$, the function $g(t) : \mathbb{C} \supseteq \mathrm{D}(g) \to \mathbb{R}$ with $\mathrm{D}(g) \supset (-\epsilon, \infty)$ is analytic on its domain, then exactly one of the following statements holds:*

i) $\exists\epsilon > 0$ *s.t.* $\forall t \in [0,\epsilon], g(t) = 0$.

ii) $\exists\epsilon > 0$ *s.t.* $\forall t \in (0,\epsilon], g(t) > 0$.

iii) $\exists\epsilon > 0$ *s.t.* $\forall t \in (0,\epsilon], g(t) < 0$.

Proof. Towards a contradiction assume that the above statement is false, then it must be the case that none holds (as they exclude each other). If no statement holds true it means that $\forall\epsilon_n > 0$, there are t_+ and $t_- \in (0,\epsilon]$ such that $g(t_+) \geq 0$ and $g(t_-) \leq 0$). Then, by continuity the interval $[\min\{t_-, t_+\}, \max\{t_-, t_+\}]$ contains at least one zero, say at t_i. As this holds for any ϵ, procede by taking $\epsilon = t_i/2$ to construct a sequence $\{t_i\}_{n\in\mathbb{N}}$ such that for any i, $g(t_i) = 0$. Then by the identity theorem it follows that $g = 0$, contradicting that statement *i)* is false. □

The proposed assumption to let x_0 be of the form $T_1(\delta)\tilde{x}_0$ is however not immediately sufficient to satisfy all statements in Assumption 5.5, as analyticity would be needed both for $t \mapsto T_1(t)T_1(\delta)\tilde{x}_0$ and $t \mapsto T_2(t)T_1(\delta)\tilde{x}_0$. One could hope that the special structure of A_1 and A_2 would still provide the analyticity of both functions. This is however not true, as a result of the following counterexample:

Lemma 5.9. *Let A be an unbounded operator generating the analytic semigroup $(T(t))_{t\geq 0}$. Then, there exists a bounded, rank 1 perturbation P such that $(A+P)$ has an eigenvector g corresponding to a nonzero eigenvalue λ with $g \notin \mathrm{D}(A^\infty)$.*
Moreover, for any $p \in X \setminus \mathrm{D}(A^\infty)$, $\lambda \in \rho(A) \setminus \{0\}$ and $Q : \mathrm{D}(A) \to \mathbb{R}$ such that $Q((\lambda I - A)^{-1}p) = 1$, $Px = Q(x) \cdot p$ such a perturbation.

Proof. The perturbation of the form $Px = Q(x) \cdot p$ with $p \in X$ and $Q : \mathrm{D}(A) \to \mathbb{R}$ has rank 1 by construction and is bounded if Q is bounded. Let λ be in $\rho(A) \setminus \{0\}$. Choose $p \notin \mathrm{D}(A^i)$ for some $i \in \mathbb{N}$, which is possible because A is unbounded. Then define $g := (\lambda I - A)^{-1}p$, as $p \notin \mathrm{D}(A^\infty)$ and $(\lambda I - A)^{-1}$ is linear, $g \neq 0$. Now choose Q such that $Q(g) = 1$, which is possible by the Hahn-Banach theorem [27, Sec. 16.1]. From the definition of g, it follows that:

$$p = (\lambda I - A)g.$$

Using $Q(g) = 1$, the above can be rewritten as:

$$Ag + Q(g) \cdot p = \lambda g.$$

Then, by definition of P, this is:

$$(A+P)g = \lambda g.$$

And thus, g is an eigenvector of $(A+P)$. From the assumption that $p \notin \mathrm{D}(A^i)$, it can be concluded that:

$$g = (\lambda I - A)^{-1} p \notin \mathrm{D}(A^i(\lambda I - A)) \supseteq \mathrm{D}(A^{(i+1)}) \supseteq \mathrm{D}(A^\infty).$$

$\square$

A consequence of this lemma is that if A and $(A+P)$ generate analytic semigroups $(T_1(t))_{t\geq 0}$ and $(T_2(t))_{t\geq 0}$, the mapping $t \mapsto T_1(t)T_2(\delta)x_0$ with $\delta > 0$ is not necessarily analytic at $t=0$ for generic perturbations P. Also not in general for perturbations P that are bounded or only of rank 1: By choosing x_0 as the eigenvector of $(A+P)$ as in Lemma 5.9, the following is the result:

$$T_1(t)T_2(\delta)g = T_1(t)e^{(\delta\lambda)}g.$$

As $g \notin \mathrm{D}(A^\infty)$, the function is not differentiable at $t=0$, and can thus not be analytic at this point.

For the specific case of the system Σ, the generator of the semigroup corresponding to the integrator mode A_1 and the perturbation P that is added for the proportional mode are given by:

$$A = A_1 = \begin{pmatrix} A_g & -B_{gv} \\ \omega_h C_g & 0 \end{pmatrix} \qquad P = A_2 - A_1 = \underbrace{\langle A_1 x, f_1 \rangle}_{Q(x)} \underbrace{f_2}_{p}.$$

The perturbation in case of the HIGS is thus rank 1 and of the form as in Lemma 5.9. The fact that $Af_2 = \begin{pmatrix} -B_{gv} \\ 0 \end{pmatrix}$ means that if $B_{gv} \notin \mathrm{D}(A_g)$, then $f_2 \notin \mathrm{D}(A_1^\infty)$. In view of Lemma 5.9, the question that remains is whether it is possible in the structure of the HIGS to have that $Q(g) = 1$. The function $Q : \mathrm{D}(A_1) \to \mathbb{R}$ can be rewritten as:

$$Q(x) = \langle A_1 x, f_1 \rangle = \begin{pmatrix} C_g(k_h A_g - \omega_h I) & -k_h C_g B_{gv} \end{pmatrix} \begin{pmatrix} x_g \\ x_h \end{pmatrix}. \tag{5.17}$$

Choose $\lambda \in \rho(A_g) \cap \mathbb{R}_+$, then g can be written using the Schur complement of the $A_1 - \lambda I$ block operator [42, p. 650] as:

$$S := -\lambda + C_g(A_g - \lambda I)^{-1} B_{gv} \in \mathbb{R},$$
$$g = \begin{pmatrix} A_g - \lambda I & -B_{gv} \\ \omega_h C_g & -\lambda \end{pmatrix}^{-1} f_2 = \frac{1}{S} \begin{pmatrix} (A_g - \lambda I)^{-1} B_{gv} \\ 1 \end{pmatrix}.$$

Substituting this into (5.17) gives:

$$\begin{aligned} Q(g) &= \frac{1}{S} \begin{pmatrix} C_g(k_h A_g - \omega_h I) & -k_h C_g B_{gv} \end{pmatrix} \begin{pmatrix} (A_g - \lambda I)^{-1} B_{gv} \\ 1 \end{pmatrix}, \\ &= \frac{1}{S} \left[C_g(k_h A_g - \omega_h I)(A_g - \lambda I)^{-1} B_{gv} - k_h C_g B_{gv} \right]. \end{aligned}$$

To have that $Q(g) = 1$, the following should thus be satisfied:

$$\begin{aligned} S &= C_g(k_h A_g - \omega_h I)(A_g - \lambda I)^{-1} B_{gv} - k_h C_g B_{gv}, \\ -\lambda + C_g(A_g - \lambda I)^{-1} B_{gv} &= C_g(k_h A_g - \omega_h I)(A_g - \lambda I)^{-1} B_{gv} - k_h C_g B_{gv}, \\ -\lambda + C_g(A_g - \lambda I)^{-1} B_{gv} &= k_h C_g A_g (A_g - \lambda I)^{-1} B_{gv} - \omega_h C_g (A_g - \lambda I)^{-1} B_{gv} - k_h C_g B_{gv}, \\ -\lambda + C_g(A_g - \lambda I)^{-1} B_{gv} &= k_h C_g B_{gv} - (\omega_h - k_h \lambda) C_g (A_g - \lambda I)^{-1} B_{gv} - k_h C_g B_{gv}, \\ -\lambda + C_g(A_g - \lambda I)^{-1} B_{gv} &= -(\omega_h - k_h \lambda) C_g (A_g - \lambda I)^{-1} B_{gv}. \end{aligned}$$

This can be satisfied by choosing $C_g(\cdot) = -\langle \cdot, (A_g - \lambda I)^{-1} B_{gv} \rangle / \| (A_g - \lambda I)^{-1} B_{gv} \|$ and ω_h and k_h such that $1 + \lambda = \omega_h - k_h \lambda$, which is possible for λ positive and real.

5.5 Conclusion

In this chapter, the goal was to generalize the results on existence of solutions from Chapter 4 to the case where A_g and thus also A_1 and A_2 are unbounded operators.

A new characterization of local behavior was proposed in Section 5.2, that uses the Laplace transform for large λ to conclude something about what the sign is of the semigroup evaluation $\langle T(t)x, f\rangle$ for small t. The proposed equivalences hold in the bounded case, where also the equivalences from Lemma 4.2 are available.

In Section 5.3, it was shown that the new characterization can be used to replace arguments that only hold for bounded A_1 and A_2. The last step of the proof remains an open problem, we were unable to show without arguments that originate from the bounded case that for $x_0 \in \mathcal{S} \setminus \mathcal{S}_{\text{int}}$ and small t, $\langle A_1 x_{\text{p}}^+(t), f_1\rangle < 0$.

An option to establish the equivalences in an unbounded case would be to choose the initial condition x_0 of a special form. This is proposed in Section 5.4. However, because of the switching behavior of the HIGS, it is not reasonable to make this assumption.

6 Simulations of HIGS as a controller

In this chapter, numerical simulations of HIGS as a controller are presented. The focus is on controlling a plant that has two integrators, as this highlights a limitation of LTI control. This example is inspired by the work of Dinther et al. [10], in Remark 6.3 an important difference to their example will be highlighted.

A feedback connection as in Figure 6.1 is studied. In this setting a reference r is tracked. The focus is on the case where $\mathcal{P}$ is a double integrator and r is a unit step. That is:

$$\mathcal{P}: \begin{cases} \dot{x}_p(t) = A_p x_p(t) + B_p u(t), \\ y(t) = C_p x_p(t), \\ x_p(0) = x_{p,0}, \end{cases} \tag{6.1}$$

with $x_p \in \mathbb{R}^2$ and:

$$A_p := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad B_p := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C_p := \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

Alternatively, $\mathcal{P}$ can be described by the transfer function $P(s) := 1/s^2$. First a linear PID controller is simulated in Section 6.1. This example will show overshoot in the step response. In Section 6.2, it is shown that this is no coincidence, because any causal linear controller that stabilizes the system will have overshoot in the step response, which is a known limiatation of LTI control, see e.g. [43, Ch. 1]. However, by using a controller with a HIGS element it is possible to avoid any overshoot in the step response, as shown in the simulations in Section 6.3.

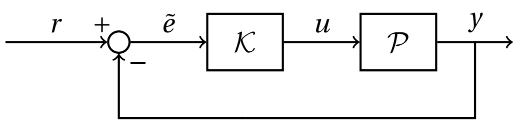


Figure 6.1: Feedback connection of a plant $\mathcal{P}$ and a controller $\mathcal{K}$ tracking a reference r.

6.1 Example of a linear controller acting on the double integrator plant

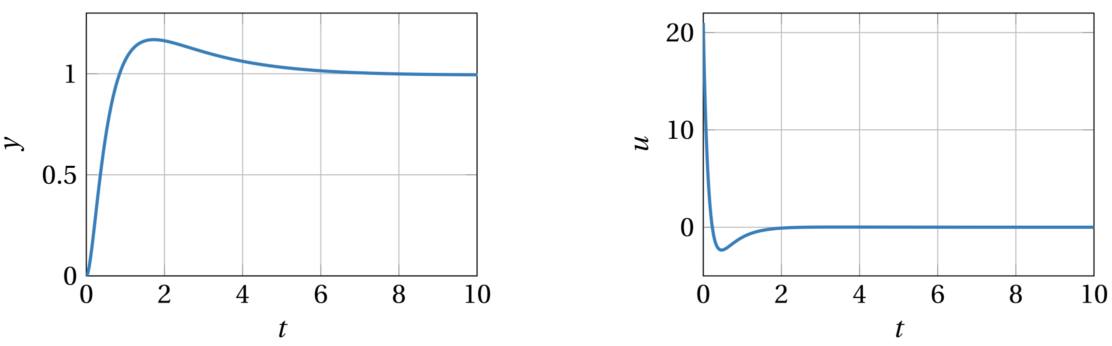
Before using HIGS as a controller, an example is given of the closed loop step response of this system if $\mathcal{K}$ is a PID controller that is described by the transfer function:

$$K(s) = k_p + k_i \frac{1}{s} + \frac{k_d s}{T_f s + 1}.$$

Notice that the derivative part is not a pure derivative, an additional pole is added at $1/T_f$. This ensures that the controller is causal. With $k_p = 1$, $k_i = 0.1$, $k_d = 2$ and $T_f = 0.1$, the system is stable. The step response is shown in Figure 6.2. As is clear from Figure 6.2a, the step response has overshoot.

6.2 Proof that linear controllers always have overshoot in this situation

It is no coincidence that the PID controller considered above leads to overshoot in the step response of the closed loop system. In fact, for any causal LTI controller (that does not have a zero at $s = 0$) the closed loop system would have overshoot in the step response. This is a consequence of the following theorem, which is inspired by [43, Th. 1.3.2]. (In [43], only the case $n = 2$ is treated.)



(a) Plant output y **(b)** Controller output u

Figure 6.2: Step response of the closed loop system where $\mathcal{P}$ is a double integrator and $\mathcal{K}$ a PID controller.

Theorem 6.1. *Consider the closed loop system as in Figure 6.1, where $\mathcal{K}$ and $\mathcal{P}$ are linear systems described by the transfer functions $K(s)$ and $P(s)$. Assume that the closed loop system is stable. Then, if there is an $n \geq 2$ such that:*

$$\lim_{s\to 0} s^n P(s)K(s) = c,$$

for some c that satisfies $0 < |c| < \infty$, then for the error $\tilde{e}$ resulting from a step input it holds that:

$$\lim_{t\to\infty} \tilde{e}(t) = 0, \quad \textit{and} \quad \int_0^\infty \tilde{e}(t)\,\mathrm{d}t = 0.$$

Proof. The Laplace transform of a step input is given by $R(s) = 1/s$ for s in a right half plane. Using this, the Laplace transform of the error can be written as:

$$E(s) = R(s)\frac{1}{1+P(s)K(s)}.$$

Therefore,

$$\lim_{s\to 0} E(s) = \lim_{s\to 0} \frac{1}{s}\frac{1}{1+P(S)K(s)},$$
$$\lim_{s\to 0} E(s) = \lim_{s\to 0} \frac{s^{(n-1)}}{s^n + s^n P(s)K(s)}.$$

As the numerator goes to zero as $s \to 0$ and the denominator goes to c, the limit $\lim_{s\to 0} E(s) = 0$. From the stability of the closed loop system, it follows that $s = 0$ is in the region of convergence of the Laplace transform (see e.g. [43, Lemma 1.3.1]), and thus:

$$\int_0^\infty \tilde{e}(t)\,\mathrm{d}t = \lim_{s\to 0} E(s) = 0.$$

Similarly,

$$\lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{1}{1+P(s)K(s)} = 0.$$

Now apply the final value theorem to conclude that:

$$\lim_{t\to\infty} \tilde{e}(t) = \lim_{s\to 0} sE(s) = 0. \qquad \square$$

Corollary 6.2. *Consider the step response of the closed loop system as in Figure 6.1 with $P(s) = 1/s^2$ (that is $\mathcal{P}$ as in (6.1)), then for any causal stabilizing LTI controller $\mathcal{K}$ that does not have a zero at $s = 0$, the step response has overshoot.*

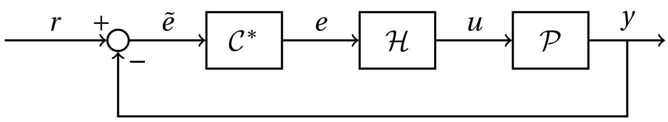


Figure 6.3: Feedback connection of a plant $\mathcal{P}$ and a controller composed of a linear element C^* and a HIGS element $\mathcal{H}$ tracking a reference r.

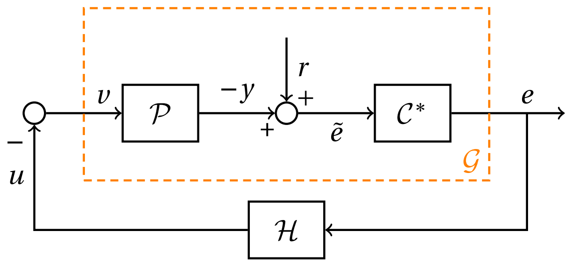


Figure 6.4: Feedback connection of HIGS $\mathcal{H}$ with a linear system $\mathcal{G}$. The linear system consists of a controller element $\mathcal{C}^*$, the plant $\mathcal{P}$ and a reference r that is assumed to be constant.

Proof. From the causality and the assumption that the controller $\mathcal{K}$ does not have a zero at $s = 0$, it follows that there exists a $j \geq 0$ such that:

$$\lim_{s\to 0} s^j K(s) = c, \quad \text{for some } c \text{ that satisfies } 0 < |c| < \infty.$$

Then it follows from Theorem 6.1 that:

$$\int_0^\infty \tilde{e}(t)\,\mathrm{d}t = 0. \tag{6.2}$$

Since $\tilde{e}(0) = 1$ for a step response and $\tilde{e}$ is continuous, there must be an $\epsilon > 0$ such that $\tilde{e}(t) > 1/2$ for $t \in [0, \epsilon]$, therefore:

$$\int_0^\epsilon \tilde{e}(t)\,\mathrm{d}t > \frac{1}{2}\epsilon.$$

Combining (6.2) with (6.2) gives that:

$$\int_\epsilon^\infty \tilde{e}(t)\,\mathrm{d}t < \frac{1}{2}\epsilon.$$

Therefore, there must be a t such that $\tilde{e}(t) < 0$, implying that there is overshoot. □

6.3 Example of HIGS based controller acting on the double integrator plant

Again, the feedback connection of Figure 6.1 is considered with $\mathcal{P}$ being a double integrator as in (6.1). It will be shown that if $\mathcal{K}$ is a series connection of a specific linear controller $\mathcal{C}^*$ with a HIGS element $\mathcal{H}$ as in Figure 6.3 that then the step response does not exhibit overshoot.

Notice that if the reference input is constant, then the linear systems $\mathcal{C}^*$ and $\mathcal{P}$, together with the reference input can be written as a linear system $\mathcal{G}$, that is sketched in Figure 6.4. Because $\mathcal{C}^*$ is linear and causal, it can be written in state space form as:

$$\mathcal{C}^* : \begin{cases} \dot{x}_c(t) = A_c x_c(t) + B_c \tilde{e}(t), \\ e(t) = C_c x_c(t) + D_c \tilde{e}(t), \\ x_c(0) = x_{c,0}. \end{cases} \tag{6.3}$$

Here, A_c, B_c and C_c are matrices of appropriate sizes. Using (6.1), (6.3) and a new state x_r that is intended to be equal to the reference r, the system $\mathcal{G}$ can be written as:

$$\mathcal{G}: \begin{cases} \begin{pmatrix} \dot{x}_p(t) \\ \dot{x}_r(t) \\ \dot{x}_c(t) \end{pmatrix} = \begin{pmatrix} A_p & 0 & 0 \\ 0 & 0 & 0 \\ B_c C_p & B_c & A_c \end{pmatrix} \begin{pmatrix} x_p(t) \\ x_r(t) \\ x_c(t) \end{pmatrix} + \begin{pmatrix} B_p \\ 0 \\ 0 \end{pmatrix} v(t), \\ e(t) = \begin{pmatrix} C_c & D_c & D_c C_p \end{pmatrix} \begin{pmatrix} x_p(t) \\ x_r(t) \\ x_c(t) \end{pmatrix}, \\ \begin{pmatrix} x_p(0) \\ x_r(0) \\ x_c(0) \end{pmatrix} = \begin{pmatrix} x_{p,0} \\ r \\ x_{c,0} \end{pmatrix}. \end{cases}$$

The feedback connection as in (6.4) with $\mathcal{G}$ as above thus fits exactly in the structure that is studied in Chapter 4 with $w = 0$, as can be seen when comparing to the assumed structure (2.1) and Figure 4.1. The operators A_g, $B_g v$ and C_g are represented by a matrix a column and a row vector respectively:

$$A_g = \begin{pmatrix} A_p & 0 & 0 \\ 0 & 0 & 0 \\ B_c C_p & B_c & A_c \end{pmatrix}, \quad B_{gv} = \begin{pmatrix} B_p \\ 0 \\ 0 \end{pmatrix}, \quad C_g = \begin{pmatrix} C_c & D_c & D_c C_p \end{pmatrix}.$$

The structure of $\mathcal{G}$ fits with the setting of Theorems 4.4 and 4.7, which guarantees that the system considered here is well-posed in the sense that a global Carathéodory solution must exist.

To simulate the closed loop system, it is useful to consider the discontinuous representation as in Section 2.2 again. It is then clear that the closed loop system is switching between two linear systems:

$$\Sigma: \begin{cases} \dot{x}(t) = \underbrace{\begin{pmatrix} A_g & -B_{gv} \\ \omega_h C_g & 0 \end{pmatrix}}_{A_1} x(t) & \text{if } x \in \mathcal{S}_1, \\ \dot{x}(t) = \underbrace{\begin{pmatrix} A_g & -B_{gv} \\ k_h C_g A_g & -k_h C_g B_{gv} \end{pmatrix}}_{A_2} x(t) & \text{if } x \in \mathcal{S}_2, \\ e(t) = \underbrace{\begin{pmatrix} C_g & 0 \end{pmatrix}}_{C} x(t) & \text{and } x \in \mathcal{S}. \end{cases} \tag{6.4}$$

With $\mathcal{S}1$, $\mathcal{S}_2$ and $\mathcal{S}$ as in (5.2)-(5.4). The simulation is performed in MATLAB using the following approach:

1. The step response is obtained by choosing x_p and x_c to be the zero vector such that the system is in steady state with $r = 0$, and performing a simulation with $r = 1$.

2. It is checked whether x_0 (with $r = 1$) lies in $\mathcal{S}_1$ or $\mathcal{S}_2$.

3. The corresponding linear system is simulated using `ode45()`, an ODE solver in MATLAB that is based on an explicit Runge-Kutta method [44]. Options of the solver are used to detect when the border of $\mathcal{S}_1$ or $\mathcal{S}_2$ is reached. When this happens the simulation is stopped.

4. When the simulation is stopped because it was detected that the border of $\mathcal{S}_1$ or $\mathcal{S}_2$ might be reached, the initial condition is updated to the final states of the simulation that had ended. Then, we return to step 1.

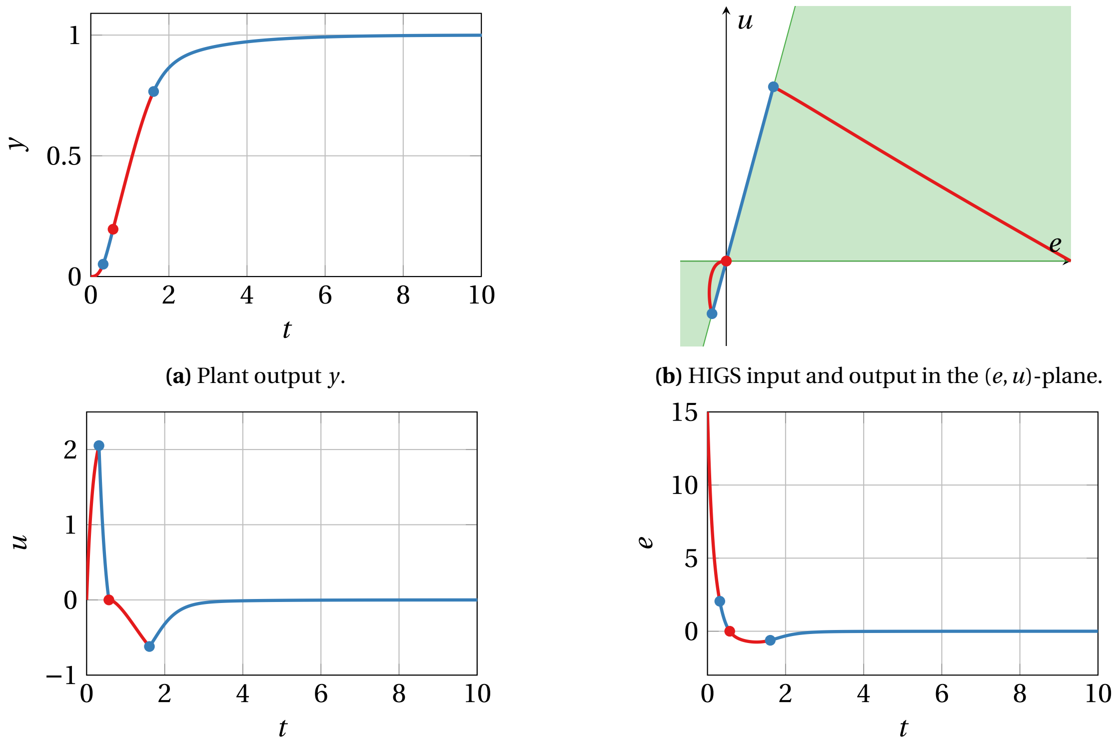
5. If the simulation is stopped because the final simulation time is reached, the simulation is done (and the results are plotted).

The MATLAB-code that was used for the simulation is in Appendix A.

Using $\omega_h = k_h = 1$ for the parameters of the HIGS and using a prefilter of the form:

$$C^*(s) = \frac{k_d s}{T_f s + 1} + k_p.$$

That is, the PID controller from before without integral action. With $k_d = 2$, $k_p = 1$ and $T_f = 0.1$, the results as in Figure 6.5 were obtained for the step response. From Figure 6.5a it can be seen that this controller consisting of a linear part $\mathcal{C}^*$ and a HIGS element $\mathcal{H}$ is able to achieve a step response without overshoot.



(a) Plant output y.

(b) HIGS input and output in the (e, u)-plane.

(c) Controller output u.

(d) HIGS input e.

Figure 6.5: Step response of the closed loop system where $\mathcal{P}$ is a double integrator that is controlled by PID controller. The color of the lines indicates the mode of operation of the HIGS element, red for I-mode and blue for P-mode.

Remark 6.3. *The example shown here is very similar to that in the work of Deenen et al. [9]. However, in their work it is proposed to use the following prefilter:*

$$C^*(s) = k_p\left(\frac{s}{\omega_c} + 1\right).$$

However, as the number of zeros is bigger than the number of poles, this system is not causal. In view of overcoming a limitation of LTI control this is problematic, as with a non-causal LTI controller it would be possible to perfectly track the reference (by choosing the inverse of the plant, $K(s) = s^2$ in this case). The example presented in this section shows that it is also possible to prevent overshoot using a causal prefilter.

6.4 Conclusion

In this section, simulations were performed with a HIGS based controller acting on a linear plant. The focus was on using a double integrator as a plant. It was found that an LTI controller that stabilizes the closed loop and does not have a zero at $s = 0$ always results in overshoot of the step response. On the contrary, a HIGS based controller was shown to have no overshoot in this situation, which motivates why the HIGS can be useful in applications where there is a stringent requirement on the overshoot.

7 Conclusion and Discussion

7.1 Conclusion

The main contributions of this thesis are the following:

1. In Chapter 2, the main representations of a closed loop system with HIGS that are commonly used in literature for the case where the plant is finite dimensional are presented in an infinite dimensional setting. The discontinuous system shows that the closed loop system is switching between two linear systems. The representation as an extended projected dynamical system provides an alternative view, which could be useful especially to extend results to nonlinear plants and more general classes of inputs.

   Furthermore, it was shown that the closed loop system is a specific case of a Hybrid System. To satisfy common assumptions in literature on Hybrid Systems, it was shown that the sets used to define the modes can be closed without changing the (Carathéodory) solutions to the system.

2. The well-posedness result that shows existence of local and global solutions in the closed loop setting presented in Chapter 4 generalizes an earlier result from Deenen et al. [9]. It was possible to use similar arguments as in the cited work to obtain this result for cases where the operators in the plant act on possibly infinite dimensional state spaces, as long as these operators are bounded. Furthermore, an assumption that was made in earlier work could be dropped. That is, it was shown that it is not required that $C_g B_{gv} = 0$.

3. Chapter 5, presents first steps towards an extention of the well-posedness result to plants in which there are unbounded operators (A_g). Unbounded operators are typically present in models represented by partial differential equations. For this generalization, another characterization of local behavior of the semigroup evaluation $\langle e^{At}x, f\rangle$ is needed. In Section 5.2, an alternative characterization is proposed and it is shown that if A is bounded this characterisation is equivalent to Lemma 4.2 that was used in the proof for the bounded case. It is unclear if the assumption holds in general for unbounded operators. In a specific case where an assumption is made on the initial condition, it is possible to show that the alternative characterization holds. However, as shown in Section 5.4 this approach does not work in the case of the feedback connection with HIGS.

Thus, it can be concluded that the feedback interconnection of HIGS with a linear plant has global solutions if the operators in the plant are bounded. In case of unbounded A this thesis does not yet provide an answer to the well-posedness question, but it gives ideas on what approaches might or might not be promising to investigate further.

7.2 Discussion

In this section, a reasoning is provided for some of the choices made in the research described in this thesis. Also, opportunities for further research are identified.

7.2.1 Structure of existing proofs for well-posedness in finite dimensional case and choices made to extend these to an infinite dimensional setting

In the finite dimensional setting, two proofs exist that establish existence of solutions for the closed loop system of HIGS with a finite dimensional plant [9, 23]. The proof by Deenen et al. [9] uses the representation of the closed loop system as a discontinuous system (as in Section 2.2). The proof by Heemels & Tanwani [23] is based on the representation as an extended projected dynamical system (as in Section 2.4). The later proof by Heemels & Tanwani [23] is more general, as it allows for a larger class of input functions and nonlinear plants, in contrast to the proof by Deenen et al. [9] where the assumption is made that the input is a piecewise Bohl function and the plant is linear. The proof by Deenen et al. [9] explicitly constructs local solutions and provides an argument why these can always be extended to global solutions. On the other hand, the proof by Heemels & Tanwani [23] is based more on existing results from differential inclusions and hybrid systems.

In Chapters 4 and 5, where these results are extended to the infinite dimensional setting, it was chosen to base the proof on the earlier work by Deenen et al. [9] and thus to attempt to explicitly construct local solutions and show later that these could be extended to global solutions. The reasoning behind this choice was that in other proving strategies, such as for example using the representation of HIGS as an extended Projected Dynamical System (ePDS), some structure of the HIGS might be hard to use. Most results used in a proof that takes this approach by Heemels & Tanwani are the finite dimensional setting, which limits the applicability in the infinite dimensional setting that is considered here. Furthermore, in infinite dimensional setting it was envisioned that it would be much harder to achieve a general result for any plant than in the finite dimensional case. This can for example be seen from the fact that it is a priori not obvious that A_1 and A_2 generate a semigroup: Because these systems were seen as linear systems, a positive result could be established in Section 5.1. Such a result would come much less natural when one would start from the representation as an ePDS.

The approach to explicitly construct local solutions and thereafter show that global solutions exist was shown to be successful in Chapter 4: Under the assumption that the plant is linear, has no direct feed-trough and that the operators that define the plant are bounded, the existence of solutions could be guaranteed. For the bounded case, many of the arguments from the finite dimensional proof as in the work by Deenen et al. [9] could still be used.

7.2.2 Difficulties in further extending results to plants with unbounded generators

Establishing a well-posedness result in a situation where the generator of the plant is unbounded still remains an open problem. In Chapter 5, first steps are made towards such a result. To be able to use a similar proving strategy as in the case of bounded generators, an alternative characterization of local behavior was need. Although a formulation was found that was equivalent for bounded generators and meaningful also for unbounded ones, the statement could not be proven without stringent assumptions in the unbounded setting.

In Section 5.3, it was shown that the new characterization from Lemma 5.4 can be used in most steps of the proof for local well-posedness to replace arguments that require that the generator is bounded. However, the proof has not yet been completed it remains an open problem, to show assuming that the generator is bounded that for $x_0 \in \mathcal{S} \setminus \mathcal{S}_{\text{int}}$ and small t, $\langle A_1 e^{A_2 t} x_0, f_1 \rangle < 0$. Also, we have not been able to produce a counterexample. In the bounded case, it is known that this statement is true from Theorem 4.4, but also this is not clear immediately in this case from the new characterization. Since the new characterization is equivalent to the characterization used in the bounded case (Lemma 4.2), we expect it to be possible to use the structure of A_1, A_2, f_1 and f_2 to complete the argument. One could hope that these arguments could then also transfer to the unbounded case.

The assumptions made to achieve the equivalences of Lemma 5.4, Assumption 5.5-3 and 4 are hard to achieve in case of semigroups with an unbounded generator. Remark 5.3 already shows that the equivalences cannot hold for a semigroup that is not analytic, so one could only hope to show the equivalences for analytic semigroups. One option to achieve such a result would be to make an assumption on the initial condition of the semigroup, to ensure that it lies in the domain of A^∞. This would satisfy the assumptions to use Lemma 5.4, but it might also even be possible to use Lemma 4.2 if it could be guaranteed that the initial position is in $D(A^\infty)$. Section 5.4 shows that the switching behavior of the HIGS makes such an assumption problematic: At the switching instants it cannot be guaranteed to stay inside $D(A_1^\infty)$ or $D(A_2^\infty)$, since these are not the same. It might therefore be necessary in future research to consider more specific classes or examples of analytic semigroups instead of attempting to prove this statment for general analytic semigroups.

7.2.3 Other possibilities for generalizations of the well-posedness proofs

It might be possible to achieve a more general well-posedness result trough the representation of the HIGS as an extended Projected Dynamical System (ePDS), as Heemels & Tanwani [23] achieved for the finite dimensional case. In this thesis it was shown that the $\Pi_{S,E}(x, v)$ operator has a unique outcome in the setting of HIGS also in infinite dimensional case, so it is possible to at least describe the infinite dimensional system in this framework. The theory from Hybrid Systems and differential inclusions that the proof of Heemels & Tanwani [23] uses is formulated in the finite dimensional setting, but there is a local existence result for regular PDS by Cojocaru & Jonker [36] (in this thesis as Theorem 2.19). By using this as a starting point it might be possible to extend their results to the ePDS setting. In this way it might be possible to generalize the existence result to nonlinear plants and allows to drop the assumption that the input needs to be a piecewise Bohl function. However, in the theorem by Cojocaru & Jonker there is an assumption that the function describing the unprojected dynamics $f(x(t), t)$ should be Lipschitz continuous. In the linear case, where $f(x(t), t) = A_g x(t) + B_{gv} w(t)$, this would mean that A_g is bounded. This is an indication that it is also here difficult to obtain a result that holds for cases where A_g is unbounded.

In Section 2.3, it was shown that the sets used to define the modes can be closed without changing the Carathéodory solutions to the system. This last result has to the best of the authors knowledge not been established before and makes it possible to use theorems from Hybrid Systems, for example to establish uniqueness of solutions or stability results. In this thesis, this property has not been used to its fullest potential as this result was obtained quite late in the project.

7.2.4 Stability results

In this thesis, the focus was on achieving well-posedness results. For applications, it is of course important to also show stability properties of the infinite dimensional closed loop system. This should be a topic of future research. In this work, we did not yet touch upon this topic because it is meaningless to show stability results for a system where existence of solutions has not been proven. As this work shows, achieving a well-posedness result is challenging, especially if the generator of the plant is unbounded, which is typical for applications described by partial differential equations which would be the most interesting for application of HIGS.

There are however situations where intuitively stability is clear: For example, for a has a power bond between input and output and is passive (such as a port-Hamiltonian system, see for example [24]). The fact that the HIGS ensures that the input and output have equal sign already ensures that the energy in the system can only decrease, which would ensure stability. A similar argument is used in the finite dimensional case for example by Heertjes et al. [2]. Also for Negative Imaginary systems, that include for example systems where the input is a force and the output a position (where the poles have negative imaginary part), a generic stability result is achieved by Shi et al. [21], which might be relatively easily transferable to the infinite dimensional case.

A MATLAB code for simulating a HIGS based controller for the double integrator plant

Below is the MATLAB-code that is used to simulate the closed loop system of a HIGS based controller in feedback with a plant that is the double integrator, as discussed in Section 6.3.

```
clear;
close all;
clc;

% This script sets up the linear systems and perform numerical
% simulation for the following system:
%                       | r (const)
%  u  |-----|  y_real   v +  e2   |-----|    e
% --->|  P  |----------->O------->|  C* |-------->
% |   |-----|  Cout*xg -          |-----|    |
% |                                          |
% |                |-----|                   |
% |----------------|  H  |<------------------|
%                  |-----|
%
% The combination of P, C* with the contsant reference is considered
% to be the system G, with which we are back in the feedback
% connection that was studied in the well posedness proof.
%
% For the simulation, we only need the output e of this combined system
% For the result, we are mainly interested in y_real or e2 (as this
% should resemble a step)
%
% To simulate a step response, the initial conditions should be such
% that r=0 would be steady state. Then with r=1 the step response is
% simulated.

%% HIGS parameters
kh = 1;
wh = 1;
tfinal = 15;
xh0 = 0; %initial state

%% reference
r = 1;

%% Plant: Double integrator
Ap = [0 0
      1 0];
Bp = [1;0];
Cp =  [0 1];

xp0 = [0, 0]; %initial position and velocity

%% Prefilter
kd = 2;
N  = 7;
kp = 1;
Cstar = tf([kd*N 0],[1 N]) + kp;
Cstarss = ss(Cstar);
Ac = Cstarss.A;
Bc = Cstarss.B;
Cc = Cstarss.C;
```

```
Dc = Cstarss.D;

xc0 = zeros(size(Bc));


%% Assuming constant reference, combine the systems to the system G
% (that is plant in original feedback connection)

%The state vector will be: [xr, r, xc]
xg0 = [xp0, r,  xc0];
%The combined system:
np = size(Ap,1); nr = 1; nc = size(Ac,1);
ng = np+nr+nc;
Ag  = [Ap   zeros(np,nr+nc);
       zeros(nr,np+nr+nc);
      Bc*Cp Bc  Ac];
Bgv = [Bp; zeros(nr+nc,1)];
Cg = [Dc*Cp  Dc  Cc];

%functions to extract the relevant outputs:
xToe = @(x) Cg*x(1:ng,:);
xToy = @(x) Cg*x(1:ng,:);
Cout = [Cp zeros(1,nr+nc)];
xToyout = @(x) -Cout*x(1:ng,:);
xToeTilde = @(x) 1 - xToyout(x);

%% Combine with the HIGS:
A1 = [Ag -Bgv;
      wh*Cg 0];
A2 = [Ag -Bgv;
      kh*Cg*Ag -kh*Cg*Bgv];

xh0 = 0;
x0 = [xg0,xh0]';

%functions for determining the switching behavior
F1 = [kh*Cg -1];
F2 = [zeros(size(Cg)), 1];

%functions that deterimine the switching, the switchToP and -I functions
%are added at the end of this script
switchToPfun = @(¬,x)  switchToP(0,x,F1,F2,A1);
switchToIfun = @(¬,x)  switchToI(0,x,F1,F2,A1);

%functions for the dynamics in I and P mode
fI = @(t,x) A1*x;
fP = @(t,x) A2*x;

%function to extract HIGS output
xTou = @(x) x(1+ng,:); %u = xh

%% numerical solving
refine = 5;
RelTol = 1e-8;
AbsTol = 1e-9;
%Solver settings
optionsI = odeset('RelTol',RelTol,'AbsTol',AbsTol,'Events',...
    switchToPfun,'OutputSel',1,'Refine',refine);
optionsP = odeset('RelTol',RelTol,'AbsTol',AbsTol,'Events',...
    switchToIfun,'OutputSel',1,'Refine',refine);

%initialization
tstart = 0;
```

```
tout = tstart;
xout = x0.';
teout = [];
yeout = [];
ieout = [];
secout = [];
sec = 1;

%main simulation loop. A ode solver runs until a switch in mode is ,
%needed after which the initial condition is set to the end of the
%simulated segment and simulation can continue in the same or a
%different mode.
%This is repeated till the final simulation time is reached.
%
while tout(end) < tfinal*0.999
    tstart = tout(end);
    %check in which mode we are (with small tollerance)
    if abs(F1*x0) < 1e-10 && (F2*x0)*sign(F1*A1*x0) < -1e-10
        %update initial conditions (x_h) to satisfy u = kh*e exactly
        x0(ng+1) = kh*xToe(x0);
        %we need the P-mode
        fprintf('Simulating with P-mode starting from t = %.2f and ...
            (e,u) = (%.2f, %.2f) \n',tstart, xToe(xout(end,:)'), ...
            xTou(xout(end,:)'))
        [t,x,te,xe,ie] = ode45(fP,[tstart tfinal],x0,optionsP);
        if ¬isempty(xe)
            t = [t; te(end)];
            x = [x; xe(end,:)];
        end
        x(end,ng+1) = kh*xToe(x(end,:)');
        if rem(sec,2) == 1 %is odd -> comes from I-mode
            %only increase section number when we actually switched
            sec = sec + 1;
        end


    else
        %we can use the I mode=
        fprintf('Simulating with I-mode starting from t = %.2f and ...
            (e,u) = (%.2f, %.2f) \n',tstart, xToe(xout(end,:)'), ...
            xTou(xout(end,:)'))
        %simulate using I-mode until event is reached
        [t,x,te,xe,ie] = ode45(fI,[tstart tfinal],x0,optionsI);
        if ¬isempty(xe)
            t = [t; te(end)];
            x = [x; xe(end,:)];
        end
        if rem(sec,2) == 0 %is even -> comes from P-mode
            %only increase section number when we actually switched
            sec = sec + 1;
        end
    end

    %save the results
    tout = [tout; t];
    xout = [xout; x];
    secout = [secout; sec*ones(size(t))];

    %setup initial conditions
    tstart = tout(end);
    x0 = x(end,:)';
end

```

```
%% Plotting
close all
subplot(2,4,1:4)
eout = xToe(xout');
e2out = xToeTilde(xout');
uout = xTou(xout');
yout = xToyout(xout');
hold on
for sec = 1:2:max(secout)
    %odd pieces -> I mode
    plot(eout(secout==sec),uout(secout==sec),'LineWidth',2,'Color','blue')
end
for sec = 2:2:max(secout)
    %even pieces -> P mode
    plot(eout(secout==sec),uout(secout==sec),'LineWidth',2,'Color','red')
end
%add the line
hold on
ref1 = refline(kh);
ref2 = refline(0);
ref1.Color = 'black';
ref2.Color = 'black';
xlabel('HIGS input $e$','Interpreter','latex')
ylabel('HIGS ouput $u$','Interpreter','latex')


subplot(2,4,5)
hold on
for sec = 1:2:max(secout)
    %odd pieces -> I mode
    plot(tout(secout==sec),e2out(secout==sec),'LineWidth',1,'Color','blue')
end
for sec = 2:2:max(secout)
    %even pieces -> P mode
    plot(tout(secout==sec),e2out(secout==sec),'LineWidth',1,'Color','red')
end
ref1 = yline(0);
ref1.Color = 'black';
xlabel('Time $t$','Interpreter','latex')
ylabel('Error $\tilde{e}$','Interpreter','latex')


subplot(2,4,6)
hold on
for sec = 1:2:max(secout)
    %odd pieces -> I mode
    plot(tout(secout==sec),eout(secout==sec),'LineWidth',1,'Color','blue')
end
for sec = 2:2:max(secout)
    %even pieces -> P mode
    plot(tout(secout==sec),eout(secout==sec),'LineWidth',1,'Color','red')
end
ref1 = yline(0);
ref2 = yline(1);
ref1.Color = 'black';
ref2.Color = 'black';
xlabel('Time $t$','Interpreter','latex')
ylabel('HIGS input $e$','Interpreter','latex')

subplot(2,4,7)
hold on
for sec = 1:2:max(secout)
    %odd pieces -> I mode
```

```
    plot(tout(secout==sec),uout(secout==sec),'LineWidth',1,'Color','blue')
end
for sec = 2:2:max(secout)
    %even pieces -> P mode
    plot(tout(secout==sec),uout(secout==sec),'LineWidth',1,'Color','red')
end
ref1 = yline(0);
ref1.Color = 'black';
xlabel('Time $t$','Interpreter','latex')
ylabel('HIGS output $u$','Interpreter','latex')

subplot(2,4,8)
hold on
for sec = 1:2:max(secout)
    %odd pieces -> I mode
    plot(tout(secout==sec),yout(secout==sec),'LineWidth',1,'Color','blue')
end
for sec = 2:2:max(secout)
    %even pieces -> P mode
    plot(tout(secout==sec),yout(secout==sec),'LineWidth',1,'Color','red')
end
ref1 = yline(0);
ref2 = yline(1);
ref1.Color = 'black';
ref2.Color = 'black';
xlabel('Time $t$','Interpreter','latex')
ylabel('Plant output $y$','Interpreter','latex')


%Event functions that stop the simulation, after which it is
%determined in which mode to continue. These functions indicate
%where a switch MIGHT happen, it need not be true that there is
%actually a switch needed.
function [value,isterminal,direction] = switchToP(¬,x,F1,F2,A1)
    %F1*x should be 0 and F2*x and F1*A1*x should have opposite sign (no
    %switch needed if F1*A1*x == 0), if one of these becomes zero we
    %might need to switch
   value = (F1*x) + (max( (F2*x) * (F1*(A1*x)),0));
   isterminal = 1;
   direction  = 0;
end

function [value,isterminal,direction] = switchToI(¬,x,F1,F2,A1)
    %F2*x =0  and F1*A1*x = 0 are the conditions that
    %could lead to leaving S_2
   value = [(F2*x) , (F1*(A1*x))];
   isterminal = [1 1];
   direction  = [0 0];
end
```

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