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Exact observability of infinite dimensional port-Hamiltonian systems via the Hautus test

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Preface

With the completion of this thesis comes also an end to my student time: a wonderful 6.5 years during which I have grown and learnt a lot and during which I have found lifelong friendships and love. Working on this topic technically started with me learning all about it during my internship in France and I have enjoyed every aspect of this work, even though it has been incredibly challenging. The joy I have found in pursuing this project, however, is in great deal to owe to the many people who have been there to support me.

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Speaking of my move to the Netherlands, I am forever grateful to Carolien and Bertil and their three daughters, Nella, Elze and Henrike, who have welcomed me into their family and hosted me for an entire year. They have truly become a second family to me and I am glad to still be able to spend time with them every year, be it through Sinterklaas, a cooking workshop or our book club.

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Abstract

Observability, and in particular exact observability, is an indispensable tool for observer based controller design for infinite dimensional systems. However, determining exact observability for infinite dimensional systems is often very complex. Finite dimensional systems, on the other hand, know many different tools that easily achieve determination of observability. As such, in this work the possibility is explored to extend the utilisation of tools such as the Hautus test, the Crank-Nicolson scheme and the Lyapunov equation to establish exact controllability for a class of infinite dimensional port-Hamiltonian systems. In particular, the goal of this work is to extend the use of the infinite dimensional Hautus test as a sufficient condition for exact observability. This is done by relating the considered continuous-time system to its discrete-time counterpart: this allows, namely, to link the exact observability of the continuous-time system on the one hand to the observability of the discrete-time system on the other. This work finds that, for the considered class of port-Hamiltonian systems (PHS), exact observability may be determined if the discrete-time counterpart of the considered PHS is observable.

Keywords: port-Hamiltonian systems, exact observability, infinite dimensional system, continuous-time systems, discrete-time systems, time discretisation, Hautus test, Crank-Nicolson, Lyapunov equation

Contents

Chapter 1

Introduction

Observability, the ability to measure the internal state of a system by examining its output, plays a role of particular importance within the domain of observer based controller design. Proving this characteristic within a given system is, however, generally rather challenging, with additional difficulty arising when one regards infinite dimensional systems and, in particular, infinite dimensional port-Hamiltonian systems (PHS), where, in addition, a particular form of observability (exact) is usually required for application. Indeed, while many easy to apply results exist which allow for the establishing of observability in the case of finite-dimensional systems, the situation is very different for infinite dimensional systems: usually, demanding constraints are posed on the system and/or the results constitute and yield necessary conditions rather than sufficient ones. In Tucsnak and Weiss (2009), for example, ample definitions, theorems and results may be found detailing what (exact) observability is and how to ensure an infinite dimensional system is (exactly) observable, but many are not directly applicable or, when directly applied, yield approximate or final state observability (weaker and, therefore, limited forms) rather than exact observability. For example, as will be seen in Chapter 2, the Hautus test for infinite dimensional systems is only a necessary condition for exact observability, while it constitutes a sufficient condition for approximate observability.
These restrictions on known results ensure that much of the work conducted within the field of port-Hamiltonian systems is carried out on a case by case basis. For observer based controller design, for example, research is carried out under the assumption of exact observability or by considering systems which are already known to be observable (see Toledo et al. (2019), Toledo et al. (2020), Toledo et al. (2022)). This limits to some extent research in applications. Time must then be invested in understanding characteristics of the system such as observability or controllability before being able to build upon that knowledge.

Finite dimensional systems, on the other hand, know a variety of useful tools that easily determine whether a system is observable or not, such as the Hautus test, the Kalman condition, or equations such as the Lyapunov equation. These results may not directly be applied to infinite dimensional. However, they may be applied to the discretised version of considered systems. Of interest is then to understand whether exact observability of continuous-time infinite dimensional systems is equivalent to the observability of their discrete-time counterpart or if the observability of the one may in some way be linked to the exact observability of the other (and/or vice versa).

In the following chapters, a specific class of PHS is studied: in particular, the considered class of PHS is time-discretised through different methods and the observability of the discrete-time system is then studied in relation to the exact observability of the continuous-time infinite dimensional system. In order to do so, in Chapter 2 an overview is given of the theory available on (exact) observability for various categories of systems, after which the considered system class is introduced in Chapter 3. The work is then divided into three distinct parts:

- in Chapter 4, the theory from Jacob et al. (2015) and Hastir et al. (2024) is utilised to discretise the PHS after which the Hautus test is applied as a tool to determine observability;
- in Chapter 5, the Crank-Nicholson scheme is utilised to discretise the system and a connection is sought with the theory presented in Hastir et al. (2024);
- in Chapter 6 the Lyapunov equation is used to link the observability of a discrete-time system to the exact observability of a continuous-time system.

In each of the chapters, an example is provided which motivates and articulates the usefulness of the considered theory and results to be used, with the final aim to be able to extend the infinite dimensional Hautus test to a sufficient condition for the considered class of PHS.
Finally, in Chapter 7 a discussion is provided on the limitations of the finding of this work as well as suggestions for future work, while in Chapter 8 a conclusion is given including a recapitulation of the results achieved in this work.

Chapter 2

Theoretical Background

In order to be able to talk about observability for port-Hamiltonian systems (PHS), it is important to understand what exactly is meant by each of these concepts. In particular, in this chapter definitions of observability are introduced for both finite and infinite dimensional systems. PHS and their characteristics are then also detailed. In addition, results on the conversion from infinite-dimensional continuous-time systems into infinite-dimensional discrete-time systems for which the operators dynamics are matrices as reported in Jacob et al. (2015) and Hastir et al. (2024) are discussed.

2.1 Observability

As previously mentioned in Chapter 1, observability is understood as the ability to measure the internal states of a system by examining its outputs. In particular, consider a system determined by the pair (A, C) of which the initial state x_0 is not known, while the output $y(t)$ is observed and the input $u(t)$ is known. If, after some time t_1, the initial state x_0 of the system can always be uniquely determined, then the system is called observable. This should hold for every input function $u(t)$.
In the literature, ample definitions, theorems and results may be found detailing what (exact) observability is and under which circumstance a(n) (in)finite dimensional system is observable. Below the most relevant ones to the avail of this paper are discussed for both finite and infinite dimensional systems.
Unless differently stated, Tucsnak and Weiss (2009) is utilised as the main source for Subsections 2.1.1 and 2.1.2.

2.1.1 Finite dimensional systems

Let U, Y and X be finite-dimensional inner product spaces with $n = \dim(X)$. A finite-dimensional linear time-invariant (LTI) system, denoted by Σ, with U and Y the input and output spaces, respectively, and X as the state space, is described by

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t), \end{cases} \tag{2.1}$$

where $A : X \to X$, $B : U \to X$, $C : X \to Y$ and $D : U \to Y$ are linear operators, $u(t) \in U$ is the *input function*, $y(t) \in Y$ is the *output function* and $x(t) \in X$ is the *state at time* t (see Willems and Polderman (1997)). For any continuous u and any intial state $x(0)$, this system has the unique solution

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\sigma)}Bu(\sigma)d\sigma. \tag{2.2}$$

Consider also the family of operators $\Psi_\tau \in \mathcal{L}(X, L^2([0,\infty); Y))$ defined as

$$(\Psi_\tau x)(t) = \begin{cases} Ce^{At}x & \text{for } t \in [0,\tau], \\ 0 & \text{for } t > \tau. \end{cases} \tag{2.3}$$

Through this family of operators a first definition of observability may be found (see Section 1.4 and, in particular, Proposition 1.4.7 on p. 23 of Tucsnak and Weiss (2009)).

Definition 2.1.1. The system Σ (or the pair (A, C)) is *observable* if for some $\tau > 0$ we have $\ker(\Psi_\tau) = \{0\}$.

As for every $\tau > 0$ it holds that

$$\ker(\Psi_\tau) = \ker \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}, \tag{2.4}$$

from definition 2.1.1 follows the corollary also known as the *Kalman rank condition* (see Corollary 1.4.8 on p. 23 of Tucsnak and Weiss (2009)).

Corollary 2.1.1. *The pair* (A, C) *is observable if and only if*

$$rank \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} = n. \tag{2.5}$$

The matrix utilised in equation (2.5) has n columns such that its rank equating to n ensures it is a full-rank matrix. Indeed then the null-space is equal to $\{0\}$. In a similar fashion, observability may also be defined through the *Hautus test* for observability (Tucsnak and Weiss, 2009; Willems and Polderman, 1997).

Proposition 2.1.1. *The pair* (A, C) *is observable if and only if*

$$rank \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n \qquad \forall \lambda \in \sigma(A) \text{ or } \forall \lambda \in \mathbb{C}. \tag{2.6}$$

From proposition 2.1.1 follows that the pair (A, C) is observable if and only if $Cx_\lambda \neq 0$ for every eigenvector x_λ of A. Furthermore, proposition 2.1.1 may be rewritten as: the pair (A, C) is observable if and only if there exists $k > 0$ such that for every $s \in \mathbb{C}$

$$\|(sI - A)x\|^2 + \|Cx\|^2 \geqslant k^2\|x\|^2 \qquad \forall x \in X. \tag{2.7}$$

2.1.2 Infinite dimensional systems

As previously seen, finite dimensional LTI systems know only one type of observability, although this may be defined and determined in a variety of ways. For infinite-dimensional systems, however, the picture is quite different as one may define entirely different, although related, types of observability. Before those notions are discussed, the necessary concepts of a *strongly continuous semigroup* and of an *infinitesimal generator* are introduced (see Chapter 5 of Jacob and Zwart (2012)).

Definition 2.1.2. Let X be a Hilbert space. $(\mathbb{T}(t))_{t\geqslant 0}$ is called a strongly continuous semigroup (or short C_0-semigroup) if the following holds:

1. For all $t \geqslant 0$, $\mathbb{T}(t)$ is a bounded linear operator on X, i.e., $\mathbb{T}(t) \in \mathcal{L}(X)$;
2. $\mathbb{T}(0) = I$;
3. $\mathbb{T}(t+\tau) = \mathbb{T}(t)\mathbb{T}(\tau)$ for all $t, \tau \geqslant 0$;
4. For all $x_0 \in X$, there holds that $\|\mathbb{T}(t)x_0 - x_0\|_X$ coverges to zero when $t \downarrow 0$, i.e., $t \to \mathbb{T}(t)$ is strongly continuous at zero.

Definition 2.1.3. Let $(\mathbb{T}(t))_{t\geqslant 0}$ be a C_0-semigroup on the Hilbert space X. If, for $x_0 \in X$, the following limit exists

$$\lim_{t\downarrow 0} \frac{\mathbb{T}(t)x_0 - x_0}{t}, \tag{2.8}$$

then it is said that x_0 is an element of the domain of A, i.e., $x_0 \in \mathcal{D}(A)$, and Ax_0 is defined as

$$Ax_0 = \lim_{t\downarrow 0} \frac{\mathbb{T}(t)x_0 - x_0}{t}. \tag{2.9}$$

The operator A is called the *infinitesimal generator* of the strongly continuous semigroup $(\mathbb{T}(t))_{t\geqslant 0}$.

In addition, the following definition is needed (see Section 2.2 and, in particular, Definition 2.2.3 on p. 34 of Tucsnak and Weiss (2009)).

Definition 2.1.4. If $A : \mathcal{D}(A) \to X$, where $\mathcal{D}(A) \subset X$, then the *resolvent set* of A, denoted $\rho(A)$, is the set of points $s \in \mathbb{C}$ for which the operator $sI - A : \mathcal{D}(A) \to X$ is invertible and $(sI - A)^{-1} \in \mathcal{L}(X)$. The spectrum of A in $\mathbb{C}$, denoted $\sigma(A)$, is the complement of the resolvent set.

Now, let U, Y and X be complex Hilbert spaces and let $\mathbb{T}(t)$ be a strongly continuous semigroup on X with generator $A : \mathcal{D}(A) \to X$. Furthermore, let X_1 be $\mathcal{D}(A)$ equipped with the norm $\|x\|_1 = \|(\beta I - A)x\|$, where $\beta \in \rho(A)$ is fixed. Consider the system described by

$$\begin{cases} \dot{x}(t) = Ax(t), & x(0) = x_0 \\ y(t) = Cx(t), \end{cases} \tag{2.10}$$

where $C \in \mathcal{L}(X_1, Y)$ and $x_0 \in X_1$. The above differential equation has the unique solution

$$x(t) = \mathbb{T}(t)x_0. \tag{2.11}$$

Consider the operator $\Psi_\tau \in \mathcal{L}(X_1, L^2([0,\infty); Y))$ (see eq. (2.3)), defined as

$$(\Psi_\tau x_0)(t) = \begin{cases} C\mathbb{T}(t)x_0 & \text{for } t \in [0,\tau], \\ 0 & \text{for } t > \tau. \end{cases} \tag{2.12}$$

The different concepts and definitions of *observability* on an infinite-dimensional system may now be introduced (see Chapter 6 and, in particular, Definition 6.1.1 on p. 183 of Tucsnak and Weiss (2009)).

Definition 2.1.5. Let $\tau > 0$.

- The pair (A, C) is *exactly observable in time* τ if Ψ_τ is bounded from below, i.e., there exists $k_\tau > 0$ such that

$$\|\Psi_\tau x_0\|^2 = \int_0^\tau \|C\mathbb{T}(t)x_0\|^2 dt \geqslant k_\tau^2 \|x_0\|_X^2 \qquad \forall x_0 \in \mathcal{D}(A). \tag{2.13}$$

- (A, C) is *approximately observable in time* τ if $\ker \Psi_\tau = \{0\}$.
- The pair (A, C) is *final state observable in time* τ if there exists a $k_\tau > 0$ such $\|\Psi_\tau x_0\| \geqslant k_\tau \|\mathbb{T}(\tau)x_0\|$ for all $x_0 \in X$.

From this definition it follows that exact observability implies the other observability concepts. In addition, it is easy to see that (A, C) is approximately observable in time τ if

$$\int_0^\tau \|C\mathbb{T}(t)x_0\|^2 dt > 0 \qquad \forall x_0 \neq 0. \tag{2.14}$$

Furthermore, if $\ker(\mathbb{T}(\tau)) = \{0\}$ and if (A, C) is final state observable in time τ, then (A, C) is approximately observable in time τ. A related definition is also introduced.

Definition 2.1.6. The pair (A, C) is *exactly observable in infinite time* if $\Psi_\tau \in \mathcal{L}(X, L^2([0,\infty); Y))$, defined as $\Psi := C\mathbb{T}(t)x_0$, is bounded from below, i.e., there exists $k > 0$ such that

$$\int_0^\infty \|C\mathbb{T}(t)x\|^2 dt \geqslant k^2 \|x\|^2 \qquad \forall x \in \mathcal{D}(A), \tag{2.15}$$

given that $\int_0^\infty \|C\mathbb{T}(t)x\|^2 dt$ exists.
The pair (A, C) is *approximately observable in infinite time* if $\ker(\Psi) = \{0\}$.

The definition of *exponential stability* is introduced next for the purpose of later discussing the infinite dimensional variant of the *Hautus test.*

Definition 2.1.7. The C_0-semigroup $(T(t))_{t\geqslant 0}$ on the Hilbert space X is exponentially stable if there exist positive constants M and α such that

$$\|T(t)\| \leqslant Me^{-at} \text{ for } t \geqslant 0. \tag{2.16}$$

The infinite-dimensional *Hautus test* may now be stated.

Theorem 2.1.1. *If (A, C) is exactly observable in infinite time and the semigroup generated by A is exponentially stable, then there is an $m > 0$ such that for every $s \in \mathbb{C}_-$, $\mathbb{C}_- = \{s \in \mathbb{C} \mid \Re(s) < 0\}$, and every $x \in \mathcal{D}(A)$*

$$\|(sI - A)x\|^2 + |\Re(s)|\|Cx\|^2 \geqslant m|\Re(s)|^2\|x\|^2. \tag{2.17}$$

Furthermore, if this estimate holds, the pair (A, C) is approximately observable in infinite time. If $m \geqslant 1$ and this estimate holds for all $s \in (-\infty, \alpha)$ for some $\alpha \leqslant 0$, then (A, C) is exactly observable, i.e, it is exactly observable in some finite time $\tau > 0$ (see Proposition 6.5.7 on p. 206 of Tucsnak and Weiss (2009)). Additionally, the *Hautus test* (eq. (2.17) in the context of Theorem 2.1.1) is a sufficient condition for certain exponentially stable systems generated by a C_0-group (see Jacob and Zwart (2009a)).
However, eq. (2.17) is in general not a sufficient condition for exact observability. Indeed, while the *Hautus test* is a sufficient condition for approximate observability, it is only a necessary condition for exact observability. In Russell and Weiss (1994) it was initially conjectured that the Hautus test is a sufficient condition for exact observability as well. However, this was later proven false in Jacob et al. (2000). In fact, in Jacob and Zwart (2009b) a different conjecture is proposed as a revised version of the one introduced in Russell and Weiss (1994). It is from this revised conjecture that the condition on the semigroup generated by A posed in Theorem 2.1.1 originates. However, one should also remark that Theorem 2.1.1 is not formulated as an 'if and only if' statement, which is the case for the conjecture in Jacob and Zwart (2009b).

To be noted is that the Hautus test knows multiple variants. In particular, the following equivalent results hold.

Theorem 2.1.2. *There exists $m > 0$ such that for every $s \in \mathbb{C}_-$ and every $x \in \mathcal{D}(A)$*

$$\|(sI - A)x\|^2 + |\Re(s)|\|Cx\|^2 \geqslant m|\Re(s)|^2\|x\|^2$$

if and only if, given $\alpha > 0$, there exists $m_\alpha > 0$ such that for every $s \in \mathbb{C}_-$ and every $x \in \mathcal{D}(A)$

$$\|(sI - A)x\|^2 + \alpha|Re(s)|\|Cx\|^2 \geqslant m_\alpha|Re(s)|^2\|x\|^2. \tag{2.18}$$

Proof. If there exists $m > 0$ such that for every $s \in \mathbb{C}_-$ and every $x \in \mathcal{D}(A)$

$$\|(sI - A)x\|^2 + |\Re(s)|\|Cx\|^2 \geqslant m|\Re(s)|^2\|x\|^2$$

holds, it is then clear that

$$\|(sI - A)x\|^2 + \alpha|\Re(s)|\|Cx\|^2 \geqslant m|\Re(s)|^2\|x\|^2$$

too holds for $\alpha \geqslant 1$, given that $\alpha|\Re(s)|\|Cx\|^2 \geqslant |\Re(s)|\|Cx\|^2$, so that

$$\|(sI - A)x\|^2 + \alpha|\Re(s)|\|Cx\|^2 \geqslant \|(sI - A)x\|^2 + |\Re(s)|\|Cx\|^2 \geqslant m|\Re(s)|^2\|x\|^2.$$

If $1 > \alpha > 0$, it is easy to see that

$$\|(sI - A)x\|^2 + |\Re(s)|\|Cx\|^2 \geqslant m|\Re(s)|^2\|x\|^2$$

is equivalent to

$$\alpha\|(sI - A)x\|^2 + \alpha|\Re(s)|\|Cx\|^2 \geqslant \alpha m|\Re(s)|^2\|x\|^2.$$

Given that, for $1 > \alpha > 0$, $\|(sI - A)x\|^2 > \alpha\|(sI - A)x\|^2$ and by renaming αm as m_α, it follows that the inequality

$$\|(sI - A)x\|^2 + \alpha|\Re(s)|\|Cx\|^2 \geqslant m_\alpha|\Re(s)|^2\|x\|^2$$

also holds.
Suppose now that, given $\alpha > 0$, there exists $m_\alpha > 0$ such that for every $s \in \mathbb{C}_-$ and every $x \in \mathcal{D}(A)$

$$\|(sI - A)x\|^2 + \alpha|Re(s)|\|Cx\|^2 \geqslant m_\alpha|Re(s)|^2\|x\|^2.$$

If $1 > \alpha > 0$, then clearly $\|Re(s)|\|Cx\|^2 > \alpha|Re(s)|\|Cx\|^2$, so that

$$\|(sI - A)x\|^2 + |Re(s)|\|Cx\|^2 \geqslant m_\alpha|Re(s)|^2\|x\|^2$$

must hold. If, instead, $\alpha \geqslant 1$, then

$$\|(sI - A)x\|^2 + \alpha|Re(s)|\|Cx\|^2 \geqslant m_\alpha|Re(s)|^2\|x\|^2$$
$$\Longleftrightarrow$$
$$\frac{1}{\alpha}\|(sI - A)x\|^2 + |Re(s)|\|Cx\|^2 \geqslant \frac{1}{\alpha}m_\alpha|Re(s)|^2\|x\|^2.$$

For $\alpha \geqslant 1$ it holds that $\|(sI - A)x\|^2 \geqslant \frac{1}{\alpha}\|(sI - A)x\|^2$ so that the inequality

$$\|(sI - A)x\|^2 + |Re(s)|\|Cx\|^2 \geqslant \frac{1}{\alpha}m_\alpha|Re(s)|^2\|x\|^2,$$

with $\frac{1}{\alpha}m_\alpha = m$, holds. □

Theorem 2.1.3. *There exists $m > 0$ such that for every $s \in \mathbb{C}_-$ and every $x \in \mathcal{D}(A)$*

$$\|(sI - A)x\|^2 + |\Re(s)|\|Cx\|^2 \geqslant m|\Re(s)|^2\|x\|^2$$

if and only if, given that $\beta > 0$, there exists $m_\beta > 0$ such that for every $s \in \mathbb{C}_-$ and every $x \in \mathcal{D}(A)$

$$\beta\|(sI - A)x\|^2 + |\Re(s)|\|Cx\|^2 \geqslant m_\beta|\Re(s)|^2\|x\|^2. \tag{2.19}$$

Proof. This result follows directly from Theorem 2.1.2. By defining $\beta = \frac{1}{\alpha}$ and $m_\beta = \frac{m_\alpha}{\alpha}$, it follows that

$$\|(sI - A)x\|^2 + \alpha\|Cx\|^2 \geqslant m_\alpha\|x\|^2$$
$$\Longleftrightarrow$$
$$\frac{1}{\alpha}\|(sI - A)x\|^2 + \|Cx\|^2 \geqslant \frac{m_\alpha}{\alpha}\|x\|^2$$
$$\Longleftrightarrow$$
$$\beta\|(sI - A)x\|^2 + \|Cx\|^2 \geqslant m_\beta\|x\|^2.$$

Given $\alpha > 0$, there exists then $m_\alpha > 0$ such that for every $s \in \mathbb{C}_-$ and every $x \in \mathcal{D}(A)$

$$\|(sI - A)x\|^2 + \alpha\|Cx\|^2 \geqslant m_\alpha\|x\|^2.$$

if and only if, given $\beta > 0$, there exists $m_\beta > 0$ such that for every $s \in \mathbb{C}_-$ and every $x \in \mathcal{D}(A)$

$$\beta\|(sI - A)x\|^2 + \|Cx\|^2 \geqslant m_\beta\|x\|^2,$$

and hence, by Theorem 2.1.2 there exists $m > 0$ such that for every $s \in \mathbb{C}_-$ and every $x \in \mathcal{D}(A)$

$$\|(sI - A)x\|^2 + |\Re(s)|\|Cx\|^2 \geqslant m|\Re(s)|^2\|x\|^2$$

if and only if, given $\beta > 0$, there exists $m_\beta > 0$ such that for every $s \in \mathbb{C}_-$ and every $x \in \mathcal{D}(A)$

$$\beta\|(sI - A)x\|^2 + \|Cx\|^2 \geqslant m_\beta\|x\|^2.$$

□

2.2 Lyapunov equations

Before detailing what port-Hamiltonian systems are, it is necessary to also understand the relation between the Lyapunov equation and observability of a system. In particular, below the Lyapunov equation for global asymptotic stability is introduced for discrete-time systems (Parks, 1992).
Global asymptotic stability indicates the behaviour by which all trajectories of the system converge over time to an equilibrium point x_{eq} (here $x_{\text{eq}} = 0$).

Theorem 2.2.1. *Consider the discrete-time system*

$$\begin{aligned} x_{k+1} &= Ax_k \\ y &= Cx_k, \end{aligned}$$

where A and C are both matrices with complex entries. Given that $C^*C > 0$, there exists a unique $L > 0$ satisfying $A^*LA - L = -C^*C$ if and only if the linear system $x_{k+1} = Ax_k$ is globally asymptotically stable.

A very similar result exists, which links the Lyapunov equation to observability for discrete-time systems given stability. For this purpose, the definition of stability if first introduced.

Definition 2.2.1. An equilibrium point x_{eq} of a differential equation $\dot{x}(t) = f(x(t))$, $x(0) = x_0$ is called stable if $\forall \epsilon > 0\ \exists \delta > 0$ such that $\|x_0 - x_{\text{eq}}\| < \delta$ implies $\|x(t; x_0) - x_{\text{eq}}\| < \epsilon\ \forall t \geqslant 0$.

Theorem 2.2.2. *Consider a stable discrete-time system*

$$\begin{aligned} x_{k+1} &= Ax_k \\ y &= Cx_k. \end{aligned}$$

Given that $C^*C \geqslant 0$, there exists a unique $L > 0$ satisfying $A^*LA - L = -C^*C$ if and only if the pair (A, C) is observable.

Remark 2.2.1. Note that the system need only to be stable and not globally stable. Additionally, now $C^*C \geqslant 0$ rather than $C^*C > 0$. The condition $C^*C \geqslant 0$ always holds.

Remark 2.2.2. If instead of stability, exponential stability of the system is required in Theorem 2.2.2, then, assuming that the Lyapunov equation has a solution, this solution L is always unique.

Very similar theory exists for continuous-time systems. In particular, the following holds (Parks, 1992).

Theorem 2.2.3. *Consider a continuous-time system*

$$\begin{aligned} \dot{x}(t) &= Ax(t) \\ y &= Cx(t), \end{aligned}$$

where A generates a strongly continuous semigroup on X so that $A : \mathcal{D}(A) \to X$, with X a complex Hilbert space, and $A : \mathcal{D}(A) \to Y$, with Y a complex Hilbert space. Given that $C^*C > 0$, there exists a unique $L > 0$ satisfying $A^*L - LA = -C^*C$ if and only if the linear system $\dot{x}(t) = Ax(t)$ is globally asymptotically stable.

Once again, it is then possible to link the Lyapunov equation with exact observability rather then global stability.

Theorem 2.2.4. *Consider an exponentially stable continuous-time system*

$$\begin{aligned} \dot{x}(t) &= Ax(t) \\ y &= Cx(t). \end{aligned}$$

There exists a solution $L > 0$ satisfying $A^*L - LA = -C^*C$ if and only if the pair (A, C) is exact observability.

2.3 Port-Hamiltonian systems

For many physical systems and systems considered in applications, modelling the system as a general LTI is not preferable. Specifically, it is of preference to work with the energy (or Hamiltonian) as the norm rather than a different one. Utilising the energy of the system as the norm provides a direct link with physics. This is, however, not the only advantage, as utilising this norm ensures that certain properties of the system become simpler to prove or are better formulated to suit applications. By utilising the energy of the system as the (squared) norm, one may then formulate and model the system as a port-Hamiltonian system (PHS). In particular, the following definition portrays what is meant by a finite dimensional PHS. Here the choice is made to restrict this to the linear case, in which the Hamiltonian is quadratic, i.e., equals $\mathcal{H}$ is $\frac{1}{2}x^*\mathcal{H}x$, but other Hamiltonians are, however, certainly also possible (see Jacob and Zwart (2012)).

Definition 2.3.1. Let $\mathcal{H}$ be a positive-definite and self-adjoint matrix, i.e., $x^*\mathcal{H}x > 0$ for all vectors $x \neq 0$ and $\mathcal{H}^* = \mathcal{H}$, and let J be a skew-adjoint matrix, i.e., $J^* = -J$. Then the system

$$\dot{x}(t) = J\mathcal{H}x(t) + Bu(t) \tag{2.20}$$

$$y(t) = B^*\mathcal{H}x(t) \tag{2.21}$$

is called a port-Hamiltonian system (PHS) associated to $\mathcal{H}$ and J. J is called the structure matrix and $\mathcal{H}$ is the Hamiltonian density. The Hamiltonian associated to $\mathcal{H}$ is $\frac{1}{2}x^*\mathcal{H}x$.

This definition is a special case of a much more general definition.
PHS possess the following properties.

Lemma 2.3.1. *Let $\mathcal{H}$ be a positive-definite matrix and let the norm $\|\cdot\|_{\mathcal{H}}$ on $\mathbb{C}^n$ be defined as $\|x\|_{\mathcal{H}} := \sqrt{\frac{1}{2}x^*\mathcal{H}x}$. Let x be a (classical) solution of (2.20)–(2.21). Then the following equality holds:*

$$\frac{d\|x(t)\|^2_{\mathcal{H}}}{dt} = \Re(y(t)^*u(t)). \tag{2.22}$$

Proof. As x is a classical solution, the statement follows by differentiating the expression $\|x\|^2_{\mathcal{H}}$ and using the fact that $J^* = -J$ and $\mathcal{H}^* = \mathcal{H}$. □

To describe classes of infinite dimensional PHS, consider partial differential equations of the form

$$\frac{\partial}{\partial t}x(\zeta, t) = (P_1\frac{\partial}{\partial\zeta} + P_0)(\mathcal{H}x)(\zeta, t) \tag{2.23}$$

$$x(\zeta, 0) = x_0(\zeta) \tag{2.24}$$

$$0 = W_B\begin{bmatrix}(\mathcal{H}x)(1,t)\\(\mathcal{H}x)(0,t)\end{bmatrix}, \tag{2.25}$$

$$y(t) = W_C\begin{bmatrix}(\mathcal{H}x)(1,t)\\(\mathcal{H}x)(0,t)\end{bmatrix}. \tag{2.26}$$

Here $\zeta \in [0, 1]$ and $t \geqslant 0$, the $n \times n$ Hermitian matrix P_1 is invertible, P_0 is a $n \times n$ matrix, W_B and W_C are full row rank $n \times 2n$ matrices and $\mathcal{H}$ is a positive Hermitian matrix which may depend on ζ. Here

$$Ax(\zeta, t) := (P_1\frac{\partial}{\partial\zeta})(\mathcal{H}x)(\zeta, t) \tag{2.27}$$

with

$$\mathcal{D}(A) := \left\{x \in L^2((0,1);\mathbb{C}^n) \mid \mathcal{H}x \in H^1((0,1);\mathbb{C}^n), 0 = W_B\begin{bmatrix}(\mathcal{H}x)(1,t)\\(\mathcal{H}x)(0,t)\end{bmatrix}\right\}, \tag{2.28}$$

so that $A : \mathcal{D}(A) \to L^2((0,1);\mathbb{C}^n)$, $\mathcal{D}(A) \subset L^2((0,1);\mathbb{C}^n)$. Additionally,

$$Cx(\zeta, t) = W_C\begin{bmatrix}(\mathcal{H}x)(1,t)\\(\mathcal{H}x)(0,t)\end{bmatrix}, \tag{2.29}$$

so that $C : \mathcal{D}(A) \to \mathbb{C}^n$.
A more complete definition is detailed in Le Gorrec et al. (2005), van der Schaft (2006) and Villegas (2007), where the complete derivation of a PHS formulation is shown.

2.4 Jacob et al. (2015) and Hastir et al. (2024)

Now that the concepts of observability and PHS have been introduced, the theory on the conversion from infinite-dimensional continuous-time systems into infinite-dimensional discrete-time systems for which the operators dynamics are matrices, as introduced in Jacob et al. (2015) and Hastir et al. (2024), may be briefly reported and discussed. In order to do so, the two papers and their relevant contents are discussed individually.

2.4.1 Jacob et al. (2015)

The theory presented in Jacob et al. (2015) is reported below.

Consider the PHS of form

$$\dot{x}(\zeta,t) = (P_1 \frac{\partial}{\partial \zeta})(\mathcal{H}x)(\zeta,t), \tag{2.30}$$

$$x(\zeta,0) = x_0(\zeta), \quad \zeta \in (0,b), \; t \geqslant 0 \tag{2.31}$$

$$u(t) = W_{B,1} \begin{bmatrix} (\mathcal{H}x)(b,t) \\ (\mathcal{H}x)(0,t) \end{bmatrix}, \tag{2.32}$$

$$0 = W_{B,2} \begin{bmatrix} (\mathcal{H}x)(b,t) \\ (\mathcal{H}x)(0,t) \end{bmatrix}, \tag{2.33}$$

$$y(t) = W_C \begin{bmatrix} (\mathcal{H}x)(b,t) \\ (\mathcal{H}x)(0,t) \end{bmatrix}, \tag{2.34}$$

where P_1 is an Hermitian invertible $n \times n$-matrix, $\mathcal{H}$ is a positive $n \times n$-matrix, and $W_B := \begin{bmatrix} W_{B,1} \\ W_{B,2} \end{bmatrix}$ is a $n \times 2n$-matrix of rank n (hence full rank).
The matrix $P_1\mathcal{H}$ is assumed to be constant. It is further diagonalisable and possesses only real non-zero eigenvalues. There exists, therefore, an invertible matrix S such that

$$P_1\mathcal{H} = S^{-1}\text{diag}(p_1, p_2, ..., p_k, n_1, n_2, ..., n_l)S \tag{2.35}$$

$$= S^{-1} \begin{bmatrix} \Lambda & 0 \\ 0 & \Theta \end{bmatrix} S, \tag{2.36}$$

where $p_1, ..., p_k > 0$ and $n_1, ..., n_l < 0$. Hence Λ is a positive definite diagonal $k \times k$-matrix and Θ is a negative definite diagonal $l \times l$-matrix.
By utilising the invertible matrix S, a new state vector is introduced

$$\begin{bmatrix} x_+(\zeta,t) \\ x_-(\zeta,t) \end{bmatrix} = Sx(\zeta,t), \quad \zeta \in [0,b], \tag{2.37}$$

where $x_+(\zeta, t) \in \mathbb{C}^k$ and $x_-(\zeta, t) \in \mathbb{C}^l$. The PHS (2.30)–(2.34) may then be rewritten as

$$\frac{\partial}{\partial t}\begin{bmatrix} x_+(\zeta,t) \\ x_-(\zeta,t) \end{bmatrix} = \frac{\partial}{\partial \zeta}\left(\begin{bmatrix} \Lambda & 0 \\ 0 & \Theta \end{bmatrix}\begin{bmatrix} x_+(\zeta,t) \\ x_-(\zeta,t) \end{bmatrix}\right), \tag{2.38}$$

$$\begin{bmatrix} 0 \\ u(t) \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}\begin{bmatrix} (\Lambda x_+)(b,t) \\ (\Theta x_-)(0,t) \end{bmatrix} + \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}\begin{bmatrix} (\Lambda x_+)(0,t) \\ (\Theta x_-)(b,t) \end{bmatrix}, \tag{2.39}$$

$$y(t) = \begin{bmatrix} O_{21} & O_{22} \end{bmatrix}\begin{bmatrix} (\Lambda x_+)(b,t) \\ (\Theta x_-)(0,t) \end{bmatrix} + \begin{bmatrix} R_{21} & R_{22} \end{bmatrix}\begin{bmatrix} (\Lambda x_+)(0,t) \\ (\Theta x_-)(b,t) \end{bmatrix}, \tag{2.40}$$

where $t \geqslant 0$ and $\zeta \in (0, b)$. The matrix $\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$ is named K and the matrix $\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$ is named Q.

The matrix K of this reformulated PHS informs whether or not the system (2.38)–(2.40) is well-posed (see Jacob and Zwart (2012) and Zwart et al. (2010)), where well-posedness means that for every initial condition $x_0 \in L^2([0, b]; \mathbb{C}^n)$ and every input $u \in L^2_{loc}((0, \infty); \mathbb{C}^p)$, the mild solution $\begin{bmatrix} x_+(\zeta,t) \\ x_-(\zeta,t) \end{bmatrix}$ of the PHS (2.38)–(2.40) is well-defined in the state space $X := L^2([0, b]; \mathbb{C}^n)$ and the output is well-defined in $L^2_{loc}((0, \infty); \mathbb{C}^n)$.

Theorem 2.4.1. *The PHS (2.38)–(2.40) is well-posed on $L^2([0, b]; \mathbb{C}^{n\times n})$ if and only if the matrix K is invertible.*

In Jacob et al. (2015), the diagonalisation of the operator $P_1\mathcal{H}$ is utilised to characterise the zero dynamics of the PHS (2.30)–(2.34).

2.4.2 Hastir et al. (2024)

The theory presented in Hastir et al. (2024) is reported below. To be noted is that the theory developed in Hastir et al. (2024) may be applied after relying on the initial steps presented in Jacob et al. (2015) (and discussed above).

Remark 2.4.1. In Hastir et al. (2024), the initial assumption is made that $P_1\mathcal{H} = -I$, in which case $K = W_0$.

The continuous-time PHS (2.38)–(2.40) may be equivalently rewritten as a discrete-time system of form

$$x_d(j+1)(\zeta) = A_d x_d(j)(\zeta) + B_d u_d(j)(\zeta) \tag{2.41}$$

$$x_d(0)(k(\zeta)) = -\lambda I x_0(\zeta) \tag{2.42}$$

$$y_d(j)(\zeta) = C_d x_d(j)(\zeta) + D_d u_d(j)(\zeta), \tag{2.43}$$

where $j \in \mathbb{N}$, $\zeta \in [0, 1]$, $x_d(j)(\zeta) \in \mathfrak{X}$, $u_d(j) \in \mathcal{U}$ and $y_d \in \mathcal{Y}$ with $\mathfrak{X} := L^2((0, b); \mathbb{C}^n)$, $\mathcal{U} := L^2((0, b); \mathbb{C}^p)$ and $\mathcal{Y} := L^2((0, b); \mathbb{C}^m)$ being the state, input and output spaces, respectively. Hereby $k : [0, b] \to [0, b]$ is defined as $k(\zeta) = b - p(\zeta)p(b)^{-1}$ with $p : [0, b] \to [0, \infty)$ expressed as $p(\zeta) := -\int_0^\zeta \frac{1}{\lambda} d\eta$.

The relationship between the functions x_d, u_d and y_d and the state, the input and the output of the continuous-time system (2.38)–(2.40) is given by the relations

$$x_d(0)(k(\zeta)) = -\lambda I x_0(\zeta), \tag{2.44}$$
$$x_d(j)(\zeta) = f(j+\zeta), \quad j \geqslant 1, \tag{2.45}$$
$$u_d(j)(\zeta) = u((j+\zeta)p(1)), \quad j \in \mathbb{N}, \tag{2.46}$$
$$y_d(j)(\zeta) = y((j+\zeta)p(1)), \quad j \in \mathbb{N}, \tag{2.47}$$

where $x_+(\zeta,t) = -\lambda^{-1} I f(k(\zeta) + p(b)^{-1}t)$, $-\lambda I x_0(\zeta) = f(k(\zeta))$.
The matrices A_d, B_d, C_d and D_d are given by

$$A_d = -K^{-1}Q, \tag{2.48}$$
$$B_d = K^{-1}\begin{bmatrix} 0 \\ I \end{bmatrix}, \tag{2.49}$$
$$C_d = -W_{C,0}K^{-1}Q + W_{C,1}, \tag{2.50}$$
$$D_d = -W_{C,0}K^{-1}\begin{bmatrix} 0 \\ I \end{bmatrix}. \tag{2.51}$$

Below a sketch of how this rewriting may take place is given. A more detailed version may be found later on in Chapter 4.

Consider the functions $k : [0,b] \to [0,b]$, $p : [0,b] \to [0,\infty)$ expressed as $k(\zeta) = b - p(\zeta)p(b)^{-1}$, $p(\zeta) := -\int_0^\zeta \frac{1}{\lambda} d\eta = -\frac{1}{\lambda}\zeta$. Here p is monotonic with $p(0) = 0$ and k satisfies that $k(0) = b$ and $k(b) = 0$. In particular, there holds that $p(\zeta)p(b)^{-1} \in [0,b]$ for every $\zeta \in [0,b]$.
Observe that $x(\zeta,t) = -\lambda^{-1} f(k(\zeta) + p(b)^{-1}t)$ is the solution to (2.38) (Hastir et al., 2024) for some scalar function f and that, in particular, $f(k(\zeta)) = -\lambda x_0(\zeta)$, $\zeta \in [0,b]$.
The substitution of the expression $x(\zeta,t) = -\lambda^{-1} f(k(\zeta) + p(b)^{-1}t)$ into the system (2.38)–(2.40) yields

$$f(k(\zeta)) = -\lambda x_0(\zeta), \quad \zeta \in [0,b], \tag{2.52}$$
$$\begin{bmatrix} 0 \\ u(t) \end{bmatrix} = K f(1 + p(b)^{-1}t) + Q f(p(b)^{-1}t), \tag{2.53}$$
$$y(t) = W_{C,0} f(1 + p(b)^{-1}t) + W_{C,1} f(p(b)^{-1}t). \tag{2.54}$$

By utilising the invertibility of K one finds then that

$$f(k(\zeta)) = -\lambda I x_0(\zeta), \quad \zeta \in [0,b] \tag{2.55}$$
$$f(1 + p(b)^{-1}t) = -K^{-1}Q f(p(b)^{-1}t) + K^{-1}\begin{bmatrix} 0 \\ u(t) \end{bmatrix}, \quad t \geqslant 0, \tag{2.56}$$
$$\begin{aligned} y(t) =& (-W_{C,0}K^{-1}Q + W_{C,1}) f(p(b)^{-1}t) \\ &- W_{C,0}K^{-1}\begin{bmatrix} 0 \\ u(t) \end{bmatrix}, \end{aligned} \tag{2.57}$$

where eq. (2.57) is obtained by substituting eq. (2.56) into eq. (2.54).
The matrices A_d, B_d, C_d and D_d may now be defined as

$$A_d = -K^{-1}Q, \tag{2.58}$$

$$B_d = K^{-1}\begin{bmatrix} 0 \\ I \end{bmatrix}, \tag{2.59}$$

$$C_d = -W_{C,0}K^{-1}Q + W_{C,1}, \tag{2.60}$$

$$D_d = -W_{C,0}K^{-1}\begin{bmatrix} 0 \\ I \end{bmatrix} \tag{2.61}$$

so that the system (2.56)–(2.57) may be rewritten as

$$f(1 + p(b)^{-1}t) = A_d f(p(b)^{-1}t) + B_d u(t), \quad t \geqslant 0, \tag{2.62}$$

$$y(t) = C_d f(p(b)^{-1}t) + D_d u(t). \tag{2.63}$$

In Hastir et al. (2024), rewriting and transforming the system as seen above is utilised to link the solution of a LQ-optimal control problem to the solution of an equivalent LQ-optimal control problem in discrete-time.

Chapter 3

Problem statement

For linear, time-invariant finite-dimensional systems there are several theorems known giving necessary and sufficient conditions for observability (see Subsection 2.1.1). These tests, such as the Kalman rank condition test or the Hautus test, are normally used to check observability of the systems. Characterising observability for system described by linear time invariant partial differential equations has been, however, an active field of research since the birth of mathematical systems theory. Several non-equivalent definitions have been introduced and tests characterising these definitions have been derived (see Subsection 2.1.2), culminating in the introduction of a variant of the Hautus test by David Russell and George Weiss (Russell and Weiss, 1994), who showed that this test is necessary for the system to be exactly observable.

Consider the class of PHS of form

$$\frac{\partial}{\partial t} w(\zeta, t) = P_1 \mathcal{H} \frac{\partial}{\partial \zeta} w(\zeta, t) \tag{3.1}$$

$$0 = W_B \begin{bmatrix} w(1,t) \\ w(0,t) \end{bmatrix} \tag{3.2}$$

$$y(t) = W_C \begin{bmatrix} w(1,t) \\ w(0,t) \end{bmatrix}, \tag{3.3}$$

where $\zeta \in [0,1]$ and $P_1\mathcal{H}$ is a $n \times n$ diagonal matrix with λ or $-\lambda$, $\lambda > 0$, real and constant, on the diagonal: $P_1\mathcal{H}$ is, hence, symmetric. The matrices W_B and W_C encode the boundary conditions and which boundary values are outputs, respectively. W_B is a $n \times 2n$ full rank matrix. In similar fashion to eq. (2.27) and eq. (2.28), here

$$Ax(\zeta, t) := P_1 \mathcal{H} \frac{\partial}{\partial \zeta} w(\zeta, t) \tag{3.4}$$

with

$$\mathcal{D}(A) := \left\{ x \in L^2((0,1);\mathbb{C}^n) \mid \mathcal{H}x \in H^1((0,1);\mathbb{C}^n), 0 = W_B \begin{bmatrix} (\mathcal{H}x)(1,t) \\ (\mathcal{H}x)(0,t) \end{bmatrix} \right\}, \tag{3.5}$$

so that $A : \mathcal{D}(A) \to L^2((0,1);\mathbb{C}^n)$, $\mathcal{D}(A) \subset L^2((0,1);\mathbb{C}^n)$. Additionally, in accordance with eq. (2.29),

$$Cx(\zeta, t) = W_C \begin{bmatrix} w(1,t) \\ w(0,t) \end{bmatrix}, \tag{3.6}$$

so that $C : \mathcal{D}(A) \to \mathbb{C}^n$.
For infinite dimensional systems, and in particular PHS, determining exact observability is not easy feat. In many cases it is not possible to determine exact observability with a result which implements simple matrix or operator conditions. This hinders applications, which is why the work carried out in the coming chapter aims to extend results for the purpose of being able to more easily attain exact observability for specific classes of infinite dimensional systems.
As mentioned in Subsection 2.1.2, the variant of the Hautus test introduced by David Russell and George Weiss is not sufficient to characterise observability. At the same time, the condition is sufficient for many practical cases. For this reason, this work aims to research whether the infinite dimensional Hautus test may be generalised as a sufficient condition for the class of PHS introduced here above. In particular, the relation between continuous-time PHS and their respective discrete-time versions is studied with the intention of understanding whether exact observability of continuous-time PHS may be concluded from observability of discrete-time systems. This problem is tackled in a variety of ways.

As mentioned in Chapter 1, in Chapters 4, 5 and 6 the class of PHS of form as the PHS (3.1)–(3.3) is studied. In further detail, in Chapter 4 this class of systems will be rewritten as a time discrete system expressed through matrices upon which will be studied whether the Hautus test may serve as sufficient condition to establish observability. In Chapter 5, the Crank-Nicholson scheme (Crank and Nicolson, 1947) is utilised to try and bring the theory of Lyapunov equation together with the theory presented in Jacob et al., 2015 and Hastir et al. (2024) so as to prove exact observability for PHS (3.1)–(3.3). In Chapter 6, the Lyapunov equation is used a final, individual tool, to determine the circumstances under which PHS (3.1)–(3.3) may be exactly observable.

Chapter 4

Hautus test for exact observability of infinite dimensional PHS

As mentioned in Subsection 2.1.2, the infinite dimensional Hautus test (Theorem 2.1.1), is a necessary, rather than a sufficient, condition for exact observability for infinite dimensional PHS. In particular, it is in general unclear whether exact observability may be determined for infinite dimensional PHS based on the fact that equation (2.17) holds and there are, in fact, counterexamples to this statement (see Jacob et al. (2000)). In finite dimensions, however, the Hautus test is a powerful tool that grants the ability to, in a few easy steps, indicate whether or not a system is observable or not. This is due to fact that, in finite dimensions, approximate and exact observability are one and the same.

In this chapter, the theory detailed in Chapter 2 by Jacob et al. (2015) and Hastir et al. (2024) is utilised in order to rewrite the considered continuous time class of systems (PHS (3.1)–(3.3)) as a discrete time one. Through this, exact observability results aimed to be expanded through the use of the Hautus test and the study of its sufficiency.

4.1 Motivating example

Before delving into the problem statement, a motivating example for the use of the Hautus test, as well as for the use of the proposed theory is given.
To this avail, consider the PHS

$$\frac{\partial x}{\partial t}(\zeta, t) = \frac{\partial x}{\partial \zeta}(\zeta, t), \tag{4.1}$$

$$0 = x(1, t), \tag{4.2}$$

$$y(t) = x(0, t) \tag{4.3}$$

modelling a shift. If the above PHS is exactly observable, Theorem 2.1.1 must hold. In particular, the inequality

$$\|(sI - A)x\|^2 + |Re(s)|\|Cx\|^2 \geqslant m|Re(s)|^2\|x\|^2 \qquad \forall s \in \mathbb{C}_-,\ x \in \mathcal{D}(A) \tag{4.4}$$

must hold for some constants $m > 0$. In particular, the left-hand side must be bounded from below for any $s \in \mathbb{C}_-$. Here A and C are defined in similar fashion to eq. (2.27) and eq. (2.29). In particular, $Ax(\zeta,t) := \frac{\partial x}{\partial \zeta}(\zeta,t)$ so that $A : \mathcal{D}(A) \to L^2((0,1);\mathbb{C})$ and $Cx(\zeta,t) := x(0,t)$ so that $C : \mathcal{D}(A) \to \mathbb{C}$.
The Hautus test may, for this system, imply exact observability and be a sufficient condition. In order to see that this is true, take $s = -r + i\omega$, with $r > 0$ and $\omega \in \mathbb{R}$. Then

$$
\begin{aligned}
&\|(sI - A)x\|^2 + |Re(s)|\|Cx\|^2 \\
=&\| - rx + (i\omega - A)x\|^2 + |Re(s)|\|Cx\|^2 \\
=&\langle -rx + (i\omega - A)x, -rx + (i\omega - A)x\rangle + r\|x(0)\|^2 \\
=&r^2\|x\|^2 + \|(i\omega - A)x\|^2 - r\langle x, (i\omega - A)x\rangle - r\langle (i\omega - A)x, x\rangle + r\|x(0)\|^2 \\
=&r^2\|x\|^2 + \|(i\omega - A)x\|^2 - r\langle x, i\omega x\rangle + r\langle x, Ax\rangle - r\langle i\omega x, x\rangle + r\langle Ax, x\rangle \\
&+ r\|x(0)\|^2 \\
=&r^2\|x\|^2 + \|(i\omega - A)x\|^2 + i\omega r\|x\|^2 + r\int_0^1 x\overline{\frac{\partial x}{\partial \zeta}}d\zeta - i\omega r\|x\|^2 + r\int_0^1 \frac{\partial x}{\partial \zeta}\overline{x}d\zeta \\
&+ r\|x(0)\|^2 \\
=&r^2\|x\|^2 + \|(i\omega - A)x\|^2 + r[x(\zeta)\overline{x(\zeta)}]_0^1 - r\int_0^1 \frac{\partial x}{\partial \zeta}\overline{x}d\zeta + r\int_0^1 \frac{\partial x}{\partial \zeta}\overline{x}d\zeta \\
&+ r\|x(0)\|^2 \\
=&r^2\|x\|^2 + \|(i\omega - A)x\|^2 - r\|x(0)\|^2 + r\|x(0)\|^2 \\
=&r^2\|x\|^2 + \|(i\omega - A)x\|^2 \\
\geqslant&r^2\|x\|^2 = |Re(s)|^2 m\|x\|^2
\end{aligned}
$$

for $m = 1$. Through the Hautus test for infinite dimensional systems it is in this case possible to at least conclude that this PHS is approximately observable in finite time. Additionally, because $m = 1$, it is also possible to conclude exact observability (see Proposition 6.5.7 on p. 197 of Tucsnak and Weiss (2009)).

The Hautus test is then clearly a powerful tool that may give necessary information about the PHS at hand. It remains to research whether or not this result may be extended to ensure that the Hautus test also proves exact observability for classes of PHS of form such as system (3.1)–(3.3).

4.2 From a continuous time to a discrete time system

In the considered class of systems, the matrix $P_1\mathcal{H}$ is already diagonal, meaning that diagonalising the system as shown in Jacob et al. (2015) can be done in a few steps. In particular

$$
P_1\mathcal{H} = S^{-1}\text{diag}(\lambda, ..., \lambda, -\lambda, ..., -\lambda)S = S^{-1}\begin{bmatrix}\lambda I & 0 \\ 0 & -\lambda I\end{bmatrix}S, \tag{4.5}
$$

where S is a unitary permutation matrix which simply switches around the rows of the vector or matrix it is multiplied with so as group together the diagonal elements λ and the diagonal elements $-\lambda$ of the diagonal matrix $P_1\mathcal{H}$.

Remark 4.2.1. The steps taken below in Section 4.2 as well as later on in Section 4.3 may in very similar fashion be carried out for a matrix $P_1\mathcal{H}$ which is not diagonal, but is diagonalisable. In Remark 4.3.1 further insight is given into the most important differences that go along with $P_1\mathcal{H}$ being diagonalisable rather than diagonal.

As done in Section 2.4, by rewriting $w(\zeta,t)$ as $\begin{bmatrix} w_+(\zeta,t) \\ w_-(\zeta,t) \end{bmatrix} = Sw(\zeta,t)$, the PHS (3.1)–(3.3) may be rewritten, with $\zeta \in [0,1]$, as

$$\frac{\partial}{\partial t}\begin{bmatrix} w_+(\zeta,t) \\ w_-(\zeta,t) \end{bmatrix} = \frac{\partial}{\partial \zeta}\left(\begin{bmatrix} \lambda I & 0 \\ 0 & -\lambda I \end{bmatrix}\begin{bmatrix} w_+(\zeta,t) \\ w_-(\zeta,t) \end{bmatrix}\right), \tag{4.6}$$

$$0 = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}\begin{bmatrix} (\lambda w_+)(1,t) \\ (-\lambda w_-)(0,t) \end{bmatrix} + \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}\begin{bmatrix} (\lambda w_+)(0,t) \\ (-\lambda w_-)(1,t) \end{bmatrix}, \tag{4.7}$$

$$y(t) = \begin{bmatrix} O_{21} & O_{22} \end{bmatrix}\begin{bmatrix} (\lambda w_+)(1,t) \\ (-\lambda w_-)(0,t) \end{bmatrix} + \begin{bmatrix} R_{21} & R_{22} \end{bmatrix}\begin{bmatrix} (\lambda w_+)(0,t) \\ (-\lambda w_-)(1,t) \end{bmatrix}, \tag{4.8}$$

with $\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} = K$ and $\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} = Q$.

Here it must be noted that equations (4.7) and (4.8) follow by construction. For equation (4.7) it holds that

$$\begin{aligned} 0 = W_B\begin{bmatrix} w(1,t) \\ w(0,t) \end{bmatrix} &= W_B\begin{bmatrix} S^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix}\begin{bmatrix} w_+(1,t) \\ w_-(1,t) \\ w_+(0,t) \\ w_-(0,t) \end{bmatrix} \\ &= \tilde{W_B}\begin{bmatrix} S^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix}\begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix}\begin{bmatrix} P_1\mathcal{H} & 0 \\ 0 & P_1\mathcal{H} \end{bmatrix}\begin{bmatrix} S^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix}\begin{bmatrix} w_+(1,t) \\ w_-(1,t) \\ w_+(0,t) \\ w_-(0,t) \end{bmatrix}, \end{aligned}$$

where $\tilde{W_B}\begin{bmatrix} P_1\mathcal{H} & 0 \\ 0 & P_1\mathcal{H} \end{bmatrix} = W_B$. So

$$\begin{aligned} 0 &= \tilde{W_B}\begin{bmatrix} S^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix}\begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix}\begin{bmatrix} P_1\mathcal{H} & 0 \\ 0 & P_1\mathcal{H} \end{bmatrix}\begin{bmatrix} S^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix}\begin{bmatrix} w_+(1,t) \\ w_-(1,t) \\ w_+(0,t) \\ w_-(0,t) \end{bmatrix} \\ &= \tilde{W_B}\begin{bmatrix} S^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix}\begin{bmatrix} \lambda I & 0 & 0 & 0 \\ 0 & -\lambda I & 0 & 0 \\ 0 & 0 & \lambda I & 0 \\ 0 & 0 & 0 & -\lambda I \end{bmatrix}\begin{bmatrix} w_+(1,t) \\ w_-(1,t) \\ w_+(0,t) \\ w_-(0,t) \end{bmatrix} \\ &= \tilde{W_B}\begin{bmatrix} S^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix}\begin{bmatrix} \lambda w_+(1,t) \\ -\lambda w_-(1,t) \\ \lambda w_+(0,t) \\ -\lambda w_-(0,t) \end{bmatrix}. \end{aligned}$$

The matrix $\tilde{W}_B \begin{bmatrix} S^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix}$ is written as the block matrix $\begin{bmatrix} K_{11} & Q_{11} & Q_{12} & K_{12} \\ K_{21} & Q_{21} & Q_{22} & K_{22} \end{bmatrix}$ such that

$$\begin{aligned} 0 &= \tilde{W}_B \begin{bmatrix} S^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix} \begin{bmatrix} \lambda w_+(1,t) \\ -\lambda w_-(1,t) \\ \lambda w_+(0,t) \\ -\lambda w_-(0,t) \end{bmatrix} \\ &= \begin{bmatrix} K_{11} & Q_{12} & Q_{11} & K_{12} \\ K_{21} & Q_{22} & Q_{21} & K_{22} \end{bmatrix} \begin{bmatrix} \lambda w_+(1,t) \\ -\lambda w_-(1,t) \\ \lambda w_+(0,t) \\ -\lambda w_-(0,t) \end{bmatrix}. \end{aligned}$$

Finally this may be rewritten as

$$0 = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} \lambda w_+(1,t) \\ -\lambda w_-(0,t) \end{bmatrix} + \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} \lambda w_+(0,t) \\ -\lambda w_-(1,t) \end{bmatrix}.$$

For equation (4.8) a similar result holds. Similarly,

$$\begin{aligned} y(t) = W_C \begin{bmatrix} w(1,t) \\ w(0,t) \end{bmatrix} &= W_C \begin{bmatrix} S^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix} \begin{bmatrix} w_+(1,t) \\ w_-(1,t) \\ w_+(0,t) \\ w_-(0,t) \end{bmatrix} \\ &= \tilde{W}_C \begin{bmatrix} S^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} P_1\mathcal{H} & 0 \\ 0 & P_1\mathcal{H} \end{bmatrix} \begin{bmatrix} S^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix} \begin{bmatrix} w_+(1,t) \\ w_-(1,t) \\ w_+(0,t) \\ w_-(0,t) \end{bmatrix}, \end{aligned}$$

where $\tilde{W}_C \begin{bmatrix} P_1\mathcal{H} & 0 \\ 0 & P_1\mathcal{H} \end{bmatrix} = W_C$. Then

$$\begin{aligned} y(t) &= \tilde{W}_C \begin{bmatrix} S^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} P_1\mathcal{H} & 0 \\ 0 & P_1\mathcal{H} \end{bmatrix} \begin{bmatrix} S^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix} \begin{bmatrix} w_+(1,t) \\ w_-(1,t) \\ w_+(0,t) \\ w_-(0,t) \end{bmatrix} \\ &= \tilde{W}_C \begin{bmatrix} S^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix} \begin{bmatrix} \lambda I & 0 & 0 & 0 \\ 0 & -\lambda I & 0 & 0 \\ 0 & 0 & \lambda I & 0 \\ 0 & 0 & 0 & -\lambda I \end{bmatrix} \begin{bmatrix} w_+(1,t) \\ w_-(1,t) \\ w_+(0,t) \\ w_-(0,t) \end{bmatrix} \\ &= \tilde{W}_C \begin{bmatrix} S^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix} \begin{bmatrix} \lambda w_+(1,t) \\ -\lambda w_-(1,t) \\ \lambda w_+(0,t) \\ -\lambda w_-(0,t) \end{bmatrix}. \end{aligned}$$

The matrix $\tilde{W}_C \begin{bmatrix} S^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix}$ is written as the block matrix $\begin{bmatrix} O_{21} & R_{22} & R_{21} & O_{22} \end{bmatrix}$

such that

$$\tilde{W}_C \begin{bmatrix} S^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix} \begin{bmatrix} \lambda w_+(1,t) \\ -\lambda w_-(1,t) \\ \lambda w_+(0,t) \\ -\lambda w_-(0,t) \end{bmatrix} = \begin{bmatrix} O_{21} & R_{22} & R_{21} & O_{22} \end{bmatrix} \begin{bmatrix} \lambda w_+(1,t) \\ -\lambda w_-(1,t) \\ \lambda w_+(0,t) \\ -\lambda w_-(0,t) \end{bmatrix}. \tag{4.9}$$

Finally this may be rewritten as

$$y(t) = \begin{bmatrix} O_{21} & O_{22} \end{bmatrix} \begin{bmatrix} \lambda w_+(1,t) \\ -\lambda w_-(0,t) \end{bmatrix} + \begin{bmatrix} R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} \lambda w_+(0,t) \\ -\lambda w_-(1,t) \end{bmatrix}.$$

To further rewrite the system, the matrix K must be invertible (see Section 2.4). For system (4.6)–(4.8), this is, however, guaranteed and may be assumed as the PHS (3.1)–(3.3) must be well-posed and always have a solution.

Once the matrix K has been determined to be invertible, the system (4.6)–(4.8) may be further rewritten. In particular, functions $k : [0,1] \to [0,1]$ and $p : [0,1] \to \mathbb{R}^+$ are defined by $k(\zeta) = 1 - p(\zeta)p(1)^{-1}$ and $p(\zeta) := \int_0^\zeta \frac{1}{\lambda} d\eta$, respectively. The function p is monotonic and satisfies $p(0) = 0$, while the function k is such that $k(1) = 0$ and $k(0) = 1$.

The solution of equation (4.6) may be expressed as $\begin{bmatrix} \lambda w_{+0}(\zeta) \\ -\lambda w_{-0}(\zeta) \end{bmatrix} = \begin{bmatrix} f_+(k(\zeta)) \\ f_-(k(\zeta)) \end{bmatrix}$ for some functions f_+ and f_- with $\begin{bmatrix} w_+(\zeta,t) \\ w_-(\zeta,t) \end{bmatrix} = \begin{bmatrix} \lambda^{-1} f_+(k(\zeta) + p(1)^{-1}t) \\ -\lambda^{-1} f_-(k(\zeta) + p(1)^{-1}t) \end{bmatrix}$ (see Hastir et al. (2024)). By substituting this expression into equations (4.7) and (4.8), one finds that

$$\begin{aligned}
\begin{bmatrix} f_+(k(\zeta)) \\ f_-(k(\zeta)) \end{bmatrix} &= \begin{bmatrix} \lambda w_{+0}(\zeta) \\ -\lambda w_{-0}(\zeta) \end{bmatrix} \quad \zeta \in [0,1] \\
0 &= \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} f_+(p(1)^{-1}t) \\ f_-(1 + p(1)^{-1}t) \end{bmatrix} + \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} f_+(1 + p(1)^{-1}t) \\ f_-(p(1)^{-1}t) \end{bmatrix} \\
y(t) &= \begin{bmatrix} O_{21} & O_{22} \end{bmatrix} \begin{bmatrix} f_+(p(1)^{-1}t) \\ f_-(1 + p(1)^{-1}t) \end{bmatrix} + \begin{bmatrix} R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} f_+(1 + p(1)^{-1}t) \\ f_-(p(1)^{-1}t) \end{bmatrix},
\end{aligned}$$

which may be rewritten as

$$\begin{aligned}
0 &= K \begin{bmatrix} f_+(p(1)^{-1}t) \\ f_-(1 + p(1)^{-1}t) \end{bmatrix} + Q \begin{bmatrix} f_+(1 + p(1)^{-1}t) \\ f_-(p(1)^{-1}t) \end{bmatrix} \\
y(t) &= \begin{bmatrix} O_{21} & O_{22} \end{bmatrix} \begin{bmatrix} f_+(p(1)^{-1}t) \\ f_-(1 + p(1)^{-1}t) \end{bmatrix} + \begin{bmatrix} R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} f_+(1 + p(1)^{-1}t) \\ f_-(p(1)^{-1}t) \end{bmatrix}.
\end{aligned}$$

Using the invertibility of K, it can be found that

$$\begin{aligned}
0 = K \begin{bmatrix} f_+(p(1)^{-1}t) \\ f_-(1 + p(1)^{-1}t) \end{bmatrix} + Q \begin{bmatrix} f_+(1 + p(1)^{-1}t) \\ f_-(p(1)^{-1}t) \end{bmatrix} & \iff \\
K \begin{bmatrix} f_+(p(1)^{-1}t) \\ f_-(1 + p(1)^{-1}t) \end{bmatrix} = -Q \begin{bmatrix} f_+(1 + p(1)^{-1}t) \\ f_-(p(1)^{-1}t) \end{bmatrix} & \iff \\
\begin{bmatrix} f_+(p(1)^{-1}t) \\ f_-(1 + p(1)^{-1}t) \end{bmatrix} = -K^{-1} Q \begin{bmatrix} f_+(1 + p(1)^{-1}t) \\ f_-(p(1)^{-1}t) \end{bmatrix} &
\end{aligned}$$

and, therefore, that

$$y(t) = \begin{bmatrix} O_{21} & O_{22} \end{bmatrix} \begin{bmatrix} f_+(p(1)^{-1}t) \\ f_-(1+p(1)^{-1}t) \end{bmatrix} + \begin{bmatrix} R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} f_+(1+p(1)^{-1}t) \\ f_-(p(1)^{-1}t) \end{bmatrix} \iff$$

$$y(t) = -\begin{bmatrix} O_{21} & O_{22} \end{bmatrix} K^{-1}Q \begin{bmatrix} f_+(1+p(1)^{-1}t) \\ f_-(p(1)^{-1}t) \end{bmatrix} + \begin{bmatrix} R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} f_+(1+p(1)^{-1}t) \\ f_-(p(1)^{-1}t) \end{bmatrix} \iff$$

$$y(t) = (-\begin{bmatrix} O_{21} & O_{22} \end{bmatrix} K^{-1}Q + \begin{bmatrix} R_{21} & R_{22} \end{bmatrix}) \begin{bmatrix} f_+(1+p(1)^{-1}t) \\ f_-(p(1)^{-1}t) \end{bmatrix}.$$

For any $t \geqslant 0$, $j \in \mathbb{N}$ and $\zeta \in [0,1]$ may be found such that $j + \zeta = p(1)^{-1}t$. With this definition it is obtained that

$$\begin{bmatrix} f_+(1+p(1)^{-1}t) \\ f_-(p(1)^{-1}t) \end{bmatrix} = \begin{bmatrix} f_+(j+1+\zeta) \\ f_-(j+\zeta) \end{bmatrix} =: \begin{bmatrix} w_{+_d}(j+1)(\zeta) \\ w_{-_d}(j)(\zeta) \end{bmatrix}.$$

Finally, for $j \in \mathbb{N}$, $\zeta \in [0,1]$

$$\begin{bmatrix} w_{+_d}(j+1)(\zeta) \\ w_{-_d}(j)(\zeta) \end{bmatrix} \in L^2(0,1;\mathbb{C}^n), \quad y_d(j) \in L^2(0,1;\mathbb{C}^m)$$

are defined by

$$\begin{bmatrix} w_{+_d}(0)(k(\zeta)) \\ w_{-_d}(0)(k(\zeta)) \end{bmatrix} = \begin{bmatrix} \lambda w_{+_0}(\zeta) \\ -\lambda w_{-_0}(\zeta) \end{bmatrix} \tag{4.10}$$

$$\begin{bmatrix} w_{+_d}(j+1)(\zeta) \\ w_{-_d}(j)(\zeta) \end{bmatrix} = \begin{bmatrix} f_+(j+1+\zeta) \\ f_-(j+\zeta) \end{bmatrix} = \begin{bmatrix} f_+(1+p(1)^{-1}t) \\ f_-(p(1)^{-1}t) \end{bmatrix}, \; j \geqslant 1 \tag{4.11}$$

$$y_d(j)(\zeta) = y((j+\zeta)p(1)) = y(t), \; j \in \mathbb{N} \tag{4.12}$$

so that the following system is obtained

$$\begin{bmatrix} w_{+_d}(j)(\zeta) \\ w_{-_d}(j+1)(\zeta) \end{bmatrix} = A_d \begin{bmatrix} w_{+_d}(j+1)(\zeta) \\ w_{-_d}(j)(\zeta) \end{bmatrix} \tag{4.13}$$

$$\begin{bmatrix} w_{+_d}(0)(k(\zeta)) \\ w_{-_d}(0)(k(\zeta)) \end{bmatrix} = \begin{bmatrix} \lambda I & 0 \\ 0 & -\lambda I \end{bmatrix} \begin{bmatrix} w_{+_0}(\zeta) \\ w_{-_0}(\zeta) \end{bmatrix} \tag{4.14}$$

$$y_d(j)(\zeta) = C_d \begin{bmatrix} w_{+_d}(j+1)(\zeta) \\ w_{-_d}(j)(\zeta) \end{bmatrix}, \tag{4.15}$$

with

$$A_d = -K^{-1}Q, \tag{4.16}$$

$$C_d = -\begin{bmatrix} O_{21} & O_{22} \end{bmatrix} K^{-1}Q + \begin{bmatrix} R_{21} & R_{22} \end{bmatrix}. \tag{4.17}$$

Of interest is now to understand whether the Hautus test characterising the observability of system (4.13)–(4.15) implies the characterisation of the exact observability for system (3.1)–(3.3). First, however, the concept of observability for the system (4.13)–(4.15) mujst be understood. The system (4.13)–(4.15) is observable if there exists $N > 0$ such that, for some constant $\tilde{k}$, it holds that

$$\sum_{j=0}^{N-1} \|y_d(j)\|^2 \geqslant \tilde{k}\|w_{d_0}\|^2. \tag{4.18}$$

It is now necessary to understand whether observability of system (4.13)–(4.15) and exact observability of system (3.1)–(3.3) are equivalent. To this avail, the following proposition is introduced.

Proposition 4.2.1. *Consider the discrete time system output $y_d(j)(\zeta)$ as introduced in system (4.13)–(4.15). For all $N \in \mathbb{N}$, it holds that*

$$\sum_{j=0}^{N-1} \|y_d(j)\|^2 = p(1)^{-1} \int_0^{Np(1)} \|y(t)\|^2 dt. \tag{4.19}$$

Proof. The norm $\|y_d(j)\|^2$ is equal to $\int_0^1 \|y_d(j)(\zeta)\|^2 d\zeta$, so that it is possible to rewrite $\sum_{j=1}^{N} \|y_d(j)\|^2$ as

$$\sum_{j=0}^{N-1} \|y_d(j)\|^2 = \sum_{j=0}^{N-1} \int_0^1 \|y_d(j)(\zeta)\|^2 d\zeta. \tag{4.20}$$

Given, as seen in Section 2.4 and as seen above, that $j + \zeta = p(1)^{-1}t$ such that $d\zeta = p(1)^{-1}dt$, it follows that

$$\sum_{j=0}^{N-1} \int_0^1 \|y_d(j)(\zeta)\|^2 d\zeta = \sum_{j=0}^{N-1} \int_{jp(1)}^{(j+1)p(1)} \|y(t)\|^2 p(1)^{-1} dt. \tag{4.21}$$

The equation may now be further rewritten as

$$\begin{aligned} \sum_{j=0}^{N-1} \int_{jp(1)}^{(j+1)p(1)} \|y(t)\|^2 p(1)^{-1} dt &= \int_0^{Np(1)} \|y(t)\|^2 p(1)^{-1} dt \\ &= p(1)^{-1} \int_0^{Np(1)} \|y(t)\|^2 dt. \end{aligned} \tag{4.22}$$

Combining equations (4.21) and (4.22), it follows that

$$\sum_{j=0}^{N-1} \|y_d(j)\|^2 = p(1)^{-1} \int_0^{Np(1)} \|y(t)\|^2 dt. \tag{4.23}$$

□

Remark 4.2.2. A similar result holds for $\begin{bmatrix} w_{+_d}(j)(\zeta) \\ w_{-_d}(j)(\zeta) \end{bmatrix}$ by which some equality is also established between $\begin{bmatrix} w_{+_d}(j)(\zeta) \\ w_{-_d}(j)(\zeta) \end{bmatrix}$ and $\begin{bmatrix} w_{+}(\zeta,t) \\ w_{-}(\zeta,t) \end{bmatrix}$. In fact

$$\begin{aligned} \|w_0\|^2 = \left\| S^{-1} \begin{bmatrix} w_{+_0} \\ w_{-_0} \end{bmatrix} \right\|^2 &= \left\| S^{-1} \begin{bmatrix} w_{+_0}(\zeta) \\ w_{-_0}(\zeta) \end{bmatrix} \right\|^2 \\ &= \left\| S^{-1} \begin{bmatrix} \lambda w_{d_{+_0}}(k(\zeta)) \\ -\lambda w_{d_{-_0}}(k(\zeta)) \end{bmatrix} \right\|^2 \\ &= \left\| \begin{bmatrix} \lambda w_{d_{+_0}}(k(\zeta)) \\ -\lambda w_{d_{-_0}}(k(\zeta)) \end{bmatrix} \right\|^2 = |\lambda|^2 \left\| \begin{bmatrix} w_{d_{+_0}} \\ w_{d_{-_0}} \end{bmatrix} \right\|^2 \end{aligned}$$

by use of the fact that S^{-1} is unitary and that $\begin{bmatrix} w_{+0}(\zeta) \\ w_{-0}(\zeta) \end{bmatrix} = \begin{bmatrix} \lambda w_{d_{+0}}(k(\zeta)) \\ -\lambda w_{d_{-0}}(k(\zeta)) \end{bmatrix}$ holds (see eq. (4.10)). This result is below formulated as Lemma 4.2.1.

Lemma 4.2.1. *Consider the discrete time system output $y_d(j)(\zeta)$ as introduced in equation (4.15). For all $N \in \mathbb{N}$, it holds that*

$$\|w_0\|^2 = |\lambda|^2 \left\| \begin{bmatrix} w_{d_{+0}} \\ w_{d_{-0}} \end{bmatrix} \right\|^2. \tag{4.24}$$

From definition 2.1.5, follows that the PHS (3.1)–(3.3) is exactly observable in time $t > 0$ if there exists some constant $k_\tau > 0$ such that

$$\int_0^{Np(1)} \|y(t)\|^2 dt \geqslant k_\tau^2 \|w_0\|^2. \tag{4.25}$$

Similarly, system (4.13)–(4.15) is observable if for some constant $\tilde{k}$ it holds that

$$\sum_{j=1}^{N} \|y_d(j)\|^2 \geqslant \tilde{k} \|w_{d_0}\|. \tag{4.26}$$

Proposition 4.2.1 and remark 4.2.2 show the equivalence of the definitions across both systems, so that the following may be stated.

Proposition 4.2.2. *The PHS (3.1)–(3.3) is exactly observable if and only if the system (4.13)–(4.15) is observable.*

It follows from this proposition that observability is preserved when rewriting the PHS (3.1)–(3.3) as system (4.13)–(4.15). As such, a relationship between the formulation of the Hautus test for both systems may be investigated.
It remains then to first formulate the Hautus test such that it may characterise the observability of the system (4.13)–(4.15)

4.3 Formulation of the Hautus test

System (4.13)–(4.15) is a discrete time infinite dimensional system. In other words, it is formulated so that Proposition 2.1.1 and equation (2.7) must hold in the following manner: if A_d is exponentially stable, then the pair (A_d, C_d) is (exactly) observable if and only if there exists $k > 0$ such that for every $z \in \mathbb{C}$

$$\left\| (zI - A_d) \begin{bmatrix} w_{+_d}(\zeta) \\ w_{-_d}(\zeta) \end{bmatrix} \right\|^2 + \left\| C_d \begin{bmatrix} w_{+_d}(\zeta) \\ w_{-_d}(\zeta) \end{bmatrix} \right\|^2 \geqslant k^2 \left\| \begin{bmatrix} w_{+_d}(\zeta) \\ w_{-_d}(\zeta) \end{bmatrix} \right\|^2. \tag{4.27}$$

Alternatively, the pair (A_d, C_d) is (exactly) observable if and only if there exists $k > 0$ such that eq. (4.27) holds for every $z \in \mathbb{D}_1 := \{z : |z| < 1\}$.
At the same time, if the initial infinite dimensional system (3.1)–(3.3) is exactly

observable, then Theorem 2.1.1 must hold. In particular, there must exist in $m > 0$ such that eq. (2.17), formulated as

$$\left\| sw(\zeta) - P_1\mathcal{H}\frac{\partial}{\partial\zeta}w(\zeta)\right\|^2 + |\Re(s)| \left\| W_C \begin{bmatrix} w(1) \\ w(0) \end{bmatrix}\right\|^2 \geqslant m|\Re(s)|^2\|w(\zeta)\|^2, \qquad (4.28)$$

must hold for every $s \in \mathbb{C}_-$ and every $w \in \mathcal{D}(A)$. Alternatively, if system (4.6)–(4.8) is exactly observable, then there must exist $m > 0$ such that

$$\begin{aligned} &\left\| s \begin{bmatrix} w_+(\zeta) \\ w_-(\zeta) \end{bmatrix} - \frac{\partial}{\partial\zeta} \begin{bmatrix} \lambda I & 0 \\ 0 & -\lambda I \end{bmatrix} \begin{bmatrix} w_+(\zeta) \\ w_-(\zeta) \end{bmatrix} \right\|^2 \\ +&|\Re(s)| \left\| \begin{bmatrix} O_{21} & O_{22} \end{bmatrix} \begin{bmatrix} (\lambda w_+)(1) \\ (-\lambda w_-)(0) \end{bmatrix} + \begin{bmatrix} R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} (\lambda w_+)(0) \\ (-\lambda w_-)(1) \end{bmatrix} \right\|^2 \\ \geqslant& m|\Re(s)|^2 \left\| \begin{bmatrix} w_+(\zeta) \\ w_-(\zeta) \end{bmatrix} \right\|^2 . \end{aligned} \qquad (4.29)$$

for every $s \in \mathbb{C}_-$ and every $\begin{bmatrix} w_+ \\ w_- \end{bmatrix} \in \mathcal{D}(A)$.

If eq. (4.27) holding can guarantee that eq. (4.28) holds and vice versa, the Hautus test may be found to be a sufficient condition for exact observability of system (3.1)–(3.3).

4.3.1 Sufficiency of the Hautus test

Before tackling the issue of whether the Hautus test may be proven to be a sufficient condition for the exact observability of PHS (3.1)–(3.3), it is necessary to consider and prove the following proposition.

Proposition 4.3.1. *Consider PHS (3.1)–(3.3) and (4.6)–(4.8). There exists $m > 0$ such that eq. (4.28)*

$$\left\| sw(\zeta) - P_1\mathcal{H}\frac{\partial}{\partial\zeta}w(\zeta)\right\|^2 + |\Re(s)| \left\| W_C \begin{bmatrix} w(1) \\ w(0) \end{bmatrix}\right\|^2 \geqslant m|\Re(s)|^2\|w(\zeta)\|^2$$

holds for every $s \in \mathbb{C}_-$ and every $w \in \mathcal{D}(A)$ if and only if there exists $m > 0$ such that eq. (4.29)

$$\begin{aligned} &\left\| s \begin{bmatrix} w_+(\zeta) \\ w_-(\zeta) \end{bmatrix} - \frac{\partial}{\partial\zeta} \begin{bmatrix} \lambda I & 0 \\ 0 & -\lambda I \end{bmatrix} \begin{bmatrix} w_+(\zeta) \\ w_-(\zeta) \end{bmatrix} \right\|^2 \\ +&|\Re(s)| \left\| \begin{bmatrix} O_{21} & O_{22} \end{bmatrix} \begin{bmatrix} (\lambda w_+)(1) \\ (-\lambda w_-)(0) \end{bmatrix} + \begin{bmatrix} R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} (\lambda w_+)(0) \\ (-\lambda w_-)(1) \end{bmatrix} \right\|^2 \\ \geqslant& m|\Re(s)|^2 \left\| \begin{bmatrix} w_+(\zeta) \\ w_-(\zeta) \end{bmatrix} \right\|^2 . \end{aligned}$$

holds for every $s \in \mathbb{C}_-$ and every $w \in \mathcal{D}(\tilde{A})$.

Proof. In order to prove that the two inequalities are equivalent, one may use the fact that parts of the two inequalities are equal to each other. In particular, both inequalities are considered as

$$\underbrace{\left\| sw(\zeta) - P_1\mathcal{H}\frac{\partial}{\partial\zeta}w(\zeta)\right\|^2}_{(a)} + \underbrace{|\Re(s)|\left\|W_C\begin{bmatrix} w(1) \\ w(0)\end{bmatrix}\right\|^2}_{(b)} \geqslant \underbrace{m|\Re(s)|^2\|w(\zeta)\|^2}_{(c)}$$

and

$$\underbrace{\left\| s\begin{bmatrix} w_+(\zeta) \\ w_-(\zeta)\end{bmatrix} - \frac{\partial}{\partial\zeta}\begin{bmatrix}\lambda I & 0 \\ 0 & -\lambda I\end{bmatrix}\begin{bmatrix} w_+(\zeta) \\ w_-(\zeta)\end{bmatrix}\right\|^2}_{(a^*)}$$

$$+\underbrace{|\Re(s)|\left\|\begin{bmatrix} O_{21} & O_{22}\end{bmatrix}\begin{bmatrix}(\lambda w_+)(1) \\ (-\lambda w_-)(0)\end{bmatrix} + \begin{bmatrix} R_{21} & R_{22}\end{bmatrix}\begin{bmatrix}(\lambda w_+)(0) \\ (-\lambda w_-)(1)\end{bmatrix}\right\|^2}_{(b^*)}$$

$$\geqslant \underbrace{m|\Re(s)|^2\left\|\begin{bmatrix} w_+(\zeta) \\ w_-(\zeta)\end{bmatrix}\right\|^2}_{(c^*)}.$$

It will be shown that (a) and (a^*), (b) and (b^*) and (c) and (c^*) are equal pairwise. All subsequent steps follow then from how the PHS (3.1)–(3.3) is rewritten as the PHS (4.6)–(4.8) in Section 4.2. Indeed, use is made once again of the fact that $w(\zeta,t) = S^{-1}\begin{bmatrix} w_+(\zeta,t) \\ w_-(\zeta,t)\end{bmatrix}$ and that $P_1\mathcal{H} = S^{-1}\begin{bmatrix}\lambda I & 0 \\ 0 & -\lambda I\end{bmatrix}S$ (as by eq. (4.5)). Equality between (a) and (a^*) can then be easily seen as

$$\begin{aligned}
\left\| sw(\zeta) - P_1\mathcal{H}\frac{\partial}{\partial\zeta}w(\zeta)\right\|^2 &= \left\| sS^{-1}\begin{bmatrix} w_+(\zeta) \\ w_-(\zeta)\end{bmatrix} - \frac{\partial}{\partial\zeta}S^{-1}\begin{bmatrix}\lambda I & 0 \\ 0 & -\lambda I\end{bmatrix}SS^{-1}\begin{bmatrix} w_+(\zeta) \\ w_-(\zeta)\end{bmatrix}\right\|^2 \\
&= \left\| S^{-1}s\begin{bmatrix} w_+(\zeta) \\ w_-(\zeta)\end{bmatrix} - S^{-1}\frac{\partial}{\partial\zeta}\begin{bmatrix}\lambda I & 0 \\ 0 & -\lambda I\end{bmatrix}\begin{bmatrix} w_+(\zeta) \\ w_-(\zeta)\end{bmatrix}\right\|^2 \\
&= \left\| S^{-1}\left(s\begin{bmatrix} w_+(\zeta) \\ w_-(\zeta)\end{bmatrix} - \frac{\partial}{\partial\zeta}\begin{bmatrix}\lambda I & 0 \\ 0 & -\lambda I\end{bmatrix}\begin{bmatrix} w_+(\zeta) \\ w_-(\zeta)\end{bmatrix}\right)\right\|^2 \\
&= \left\| s\begin{bmatrix} w_+(\zeta) \\ w_-(\zeta)\end{bmatrix} - \frac{\partial}{\partial\zeta}\begin{bmatrix}\lambda & 0 \\ 0 & -\lambda I\end{bmatrix}\begin{bmatrix} w_+(\zeta) \\ w_-(\zeta)\end{bmatrix}\right\|^2,
\end{aligned}$$

where the last equality holds due to S^{-1} being unitary. It follows that

$$\left\| sw(\zeta) - \frac{\partial}{\partial\zeta}P_1\mathcal{H}w(\zeta)\right\|^2 = \left\| s\begin{bmatrix} w_+(\zeta) \\ w_-(\zeta)\end{bmatrix} - \frac{\partial}{\partial\zeta}\begin{bmatrix}\lambda I & 0 \\ 0 & -\lambda I\end{bmatrix}\begin{bmatrix} w_+(\zeta) \\ w_-(\zeta)\end{bmatrix}\right\|^2.$$

Similarly, equality between (c) and (c^*) follows as

$$\begin{aligned}
\|w(\zeta)\|^2 &= \left\| S^{-1}\begin{bmatrix} w_+(\zeta) \\ w_-(\zeta)\end{bmatrix}\right\|^2 \\
&= \left\|\begin{bmatrix} w_+(\zeta) \\ w_-(\zeta)\end{bmatrix}\right\|^2.
\end{aligned}$$

It follows that

$$m|Re(s)|^2\|w(\zeta)\|^2 = m|Re(s)|^2 \left\| \begin{bmatrix} w_+(\zeta) \\ w_-(\zeta) \end{bmatrix} \right\|^2 .$$

Finally, equality must be established between (b) and (b^*). It follows that

$$\begin{aligned}
&\left\| \begin{bmatrix} O_{21} & O_{22} \end{bmatrix} \begin{bmatrix} (\lambda w_+)(1) \\ (-\lambda w_-)(0) \end{bmatrix} + \begin{bmatrix} R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} (\lambda w_+)(0) \\ (-\lambda I w_-)(1) \end{bmatrix} \right\|^2 \\
&= \|O_{21}(\lambda w_+)(1) + O_{22}(-\lambda w_-)(0) + R_{21}(\lambda w_+)(0) + R_{22}(-\lambda w_-)(1)\|^2 \\
&= \|O_{21}(\lambda w_+)(1) + R_{22}(-\lambda w_-)(1) + R_{21}(\lambda w_+)(0) + O_{22}(-\lambda w_-)(0)\|^2 \\
&= \left\| \begin{bmatrix} O_{21} & R_{22} \end{bmatrix} \begin{bmatrix} (\lambda w_+)(1) \\ (-\lambda w_-)(1) \end{bmatrix} + \begin{bmatrix} R_{21} & O_{22} \end{bmatrix} \begin{bmatrix} (\lambda w_+)(0) \\ (-\lambda w_-)(0) \end{bmatrix} \right\|^2 \\
&= \left\| \begin{bmatrix} O_{21} & R_{22} & R_{21} & O_{22} \end{bmatrix} \begin{bmatrix} (\lambda w_+)(1) \\ (-\lambda w_-)(1) \\ (\lambda w_+)(0) \\ (-\lambda w_-)(0) \end{bmatrix} \right\|^2 \\
&= \left\| \begin{bmatrix} O_{21} & R_{22} & R_{21} & O_{22} \end{bmatrix} \begin{bmatrix} \lambda I & 0 & 0 & 0 \\ 0 & -\lambda I & 0 & 0 \\ 0 & 0 & \lambda I & 0 \\ 0 & 0 & 0 & -\lambda I \end{bmatrix} \begin{bmatrix} w_+(1) \\ w_-(1) \\ w_+(0) \\ w_-(0) \end{bmatrix} \right\|^2 .
\end{aligned}$$

Given that $\begin{bmatrix} w_+(\zeta, t) \\ w_-(\zeta, t) \end{bmatrix} = Sw(\zeta, t)$, it follows that

$$\begin{aligned}
&\left\| \begin{bmatrix} O_{21} & R_{22} & R_{21} & O_{22} \end{bmatrix} \begin{bmatrix} \lambda I & 0 & 0 & 0 \\ 0 & -\lambda I & 0 & 0 \\ 0 & 0 & \lambda I & 0 \\ 0 & 0 & 0 & -\lambda I \end{bmatrix} \begin{bmatrix} w_+(1) \\ w_-(1) \\ w_+(0) \\ w_-(0) \end{bmatrix} \right\|^2 \\
&= \left\| \begin{bmatrix} O_{21} & R_{22} & R_{21} & O_{22} \end{bmatrix} \begin{bmatrix} \lambda I & 0 & 0 & 0 \\ 0 & -\lambda I & 0 & 0 \\ 0 & 0 & \lambda I & 0 \\ 0 & 0 & 0 & -\lambda I \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} w(1) \\ w(0) \end{bmatrix} \right\|^2 = \\
&= \left\| W_C \begin{bmatrix} w(1) \\ w(0) \end{bmatrix} \right\|^2 ,
\end{aligned}$$

where the final equality holds true based on equation (4.9). In particular,

$$\begin{aligned}
&|Re(s)| \left\| \begin{bmatrix} O_{21} & O_{22} \end{bmatrix} \begin{bmatrix} (\lambda w_+)(1) \\ (-\lambda w_-)(0) \end{bmatrix} + \begin{bmatrix} R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} (\lambda w_+)(0) \\ (-\lambda I w_-)(1) \end{bmatrix} \right\|^2 \\
&= |Re(s)| \left\| W_C \begin{bmatrix} w(1) \\ w(0) \end{bmatrix} \right\|^2 .
\end{aligned}$$

In conclusion, the inequality

$$\begin{aligned}&\left\| s\begin{bmatrix} w_+(\zeta) \\ w_-(\zeta)\end{bmatrix} - \frac{\partial}{\partial\zeta}\begin{bmatrix}\lambda I & 0 \\ 0 & -\lambda I\end{bmatrix}\begin{bmatrix} w_+(\zeta) \\ w_-(\zeta)\end{bmatrix}\right\|^2 \\ &+|\Re(s)|\left\|\begin{bmatrix}O_{21} & O_{22}\end{bmatrix}\begin{bmatrix}(\lambda w_+)(1) \\ (-\lambda w_-)(0)\end{bmatrix} + \begin{bmatrix}R_{21} & R_{22}\end{bmatrix}\begin{bmatrix}(\lambda w_+)(0) \\ (-\lambda w_-)(1)\end{bmatrix}\right\|^2 \\ &\geqslant m|\Re(s)|^2\left\|\begin{bmatrix} w_+(\zeta) \\ w_-(\zeta)\end{bmatrix}\right\|^2\end{aligned}$$

holds if and only if the inequality

$$\left\| sw(\zeta) - P_1\mathcal{H}\frac{\partial}{\partial\zeta}w(\zeta)\right\|^2 + |\Re(s)|\left\|W_C\begin{bmatrix}w(1)\\ w(0)\end{bmatrix}\right\|^2 \geqslant m|\Re(s)|^2\|w(\zeta)\|^2$$

holds as they are equivalent (and in fact equal) to one another. □

Remark 4.3.1. If a diagonalisable $P_1\mathcal{H}$ is considered rather than a diagonal $P_1\mathcal{H}$, then the proof of Proposition 4.3.1 varies slightly. In particular, eq. (4.28) and eq. (4.29) are then not equal, but can be proven to be equivalent through the use of Theorem 2.1.2 and Theorem 2.1.3.

Conjecture 4.3.1 may now be introduced. Indeed, as also mentioned in Subsection 2.1.2, the infinite dimensional formulation of the Hautus test relative to the PHS (3.1)–(3.3) (eq. (4.28)) can only be proven to be a sufficient condition for exact observability if the finite dimensional formulation of the Hautus test relative to the system (4.13)–(4.15) (eq. (4.27)) holds whenever eq. (4.28). In particular, in order to prove the sufficiency of eq. (4.28), the following proposition must hold true.

Conjecture 4.3.1. *Suppose that the operator* $A := P_1\mathcal{H}\frac{\partial}{\partial\zeta}$, *with domain* $\mathcal{D}(A) = \left\{x \in L^2((0,1);\mathbb{C}^n) \mid \mathcal{H}x \in H^1((0,1);\mathbb{C}^n), 0 = W_B\begin{bmatrix}(\mathcal{H}x)(1,t)\\ (\mathcal{H}x)(0,t)\end{bmatrix}\right\}$ *(see eq. (3.5)), is exponentially stable. There exists* $m > 0$ *such that*

$$\left\| sw(\zeta) - P_1\mathcal{H}\frac{\partial}{\partial\zeta}w(\zeta)\right\|^2 + |\Re(s)|\left\|W_C\begin{bmatrix}w(1)\\ w(0)\end{bmatrix}\right\|^2 \geqslant m|\Re(s)|^2\|w(\zeta)\|^2$$

holds for every $s \in \mathbb{C}_-$ *and every* $w \in \mathcal{D}(A)$ *if and only if* A_d *is exponentially stable and there exists a* $k > 0$ *such that*

$$\left\|(zI - A_d)\begin{bmatrix}w_{+_d}(\zeta)\\ w_{-_d}(\zeta)\end{bmatrix}\right\|^2 + \left\|C_d\begin{bmatrix}w_{+_d}(\zeta)\\ w_{-_d}(\zeta)\end{bmatrix}\right\|^2 \geqslant k^2\left\|\begin{bmatrix}w_{+_d}(\zeta)\\ w_{-_d}(\zeta)\end{bmatrix}\right\|^2$$

holds for every $z \in \mathbb{D}_1$.

Remark 4.3.2. By Proposition 4.3.1 an equivalent result to Conjecture 4.3.1 may be formulated where systems (4.6)–(4.8) and (4.13)–(4.15) and related Hautus test formulations are utilised.

Remark 4.3.3. The operator $A := P_1\mathcal{H}\frac{\partial}{\partial\zeta}$ (and, in particular, PHS (3.1)–(3.3)) is exponentially stable if and only if the matrix A_d is exponentially stable (Jacob et al., 2019).

In order to prove Conjecture 4.3.1 the following proposition is introduced, which may be proven by use of Proposition 4.2.1.

Proposition 4.3.2. *Suppose that A_d is exponentially stable. If there exists a $k > 0$ such that*

$$\left\|(zI - A_d)\begin{bmatrix} w_{+_d}(\zeta) \\ w_{-_d}(\zeta) \end{bmatrix}\right\|^2 + \left\|C_d\begin{bmatrix} w_{+_d}(\zeta) \\ w_{-_d}(\zeta) \end{bmatrix}\right\|^2 \geqslant k^2\left\|\begin{bmatrix} w_{+_d}(\zeta) \\ w_{-_d}(\zeta) \end{bmatrix}\right\|^2$$

holds for every $z \in \mathbb{D}_1$, then there exists $m > 0$ such that

$$\left\|sw(\zeta) - P_1\mathcal{H}\frac{\partial}{\partial\zeta}w(\zeta)\right\|^2 + |\Re(s)|\left\|W_C\begin{bmatrix} w(1) \\ w(0) \end{bmatrix}\right\|^2 \geqslant m|\Re(s)|^2\|w(\zeta)\|^2$$

holds for every $s \in \mathbb{C}_-$ and every $w \in \mathcal{D}(A)$, with A exponentially stable.

Proof of Proposition 4.3.2. Suppose that A_d is exponentially stable. If there exists $k > 0$ such that

$$\left\|(zI - A_d)\begin{bmatrix} w_{+_d}(\zeta) \\ w_{-_d}(\zeta) \end{bmatrix}\right\|^2 + \left\|C_d\begin{bmatrix} w_{+_d}(\zeta) \\ w_{-_d}(\zeta) \end{bmatrix}\right\|^2 \geqslant k^2\left\|\begin{bmatrix} w_{+_d}(\zeta) \\ w_{-_d}(\zeta) \end{bmatrix}\right\|^2$$

holds for every $z \in \mathbb{D}_1$, then, by Theorem 2.1.1, system (4.13)–(4.15) is observable. If system (4.13)–(4.15) is observable, then, by Proposition 4.2.2, the PHS (3.1)–(3.3) is exactly observable in finite time. If PHS (3.1)–(3.3) is exactly observable in finite time, then, by Remark 4.3.3, A is exponentially stable and, by Theorem 2.1.1, there exists some $m > 0$ such that

$$\left\|sw(\zeta) - P_1\mathcal{H}\frac{\partial}{\partial\zeta}w(\zeta)\right\|^2 + |\Re(s)|\left\|W_C\begin{bmatrix} w(1) \\ w(0) \end{bmatrix}\right\|^2 \geqslant m|\Re(s)|^2\|w(\zeta)\|^2$$

holds true for every $s \in \mathbb{C}_-$ and every $w \in \mathcal{D}(A)$. □

Remark 4.3.4. By Proposition 4.3.1 an equivalent result holds between eq. (4.27) and eq. (4.29).

By Proposition 4.3.2, Conjecture 4.3.1 may now be rewritten as the following conjecture.

Conjecture 4.3.2. *Suppose that A is exponentially stable. If there exists $m > 0$ such that*

$$\left\|sw(\zeta) - P_1\mathcal{H}\frac{\partial}{\partial\zeta}w(\zeta)\right\|^2 + |\Re(s)|\left\|W_C\begin{bmatrix} w(1) \\ w(0) \end{bmatrix}\right\|^2 \geqslant m|\Re(s)|^2\|w(\zeta)\|^2$$

holds for every $s \in \mathbb{C}_-$ and every $w \in \mathcal{D}(A)$, then there exists a $k > 0$ such that

$$\left\|(zI - A_d)\begin{bmatrix} w_{+_d}(\zeta) \\ w_{-_d}(\zeta) \end{bmatrix}\right\|^2 + \left\|C_d\begin{bmatrix} w_{+_d}(\zeta) \\ w_{-_d}(\zeta) \end{bmatrix}\right\|^2 \geqslant k^2\left\|\begin{bmatrix} w_{+_d}(\zeta) \\ w_{-_d}(\zeta) \end{bmatrix}\right\|^2$$

holds for every $z \in \mathbb{D}_1$, with A_d exponentially stable.

Proving Conjecture 4.3.2, however, is very complicated. No proof has currently been attained for Conjecture 4.3.2.

Before declaring that the methodology yields no results whatsoever, however, it may be of use, in accordance with the work presented in Hastir et al. (2024) and referenced in Remark 2.4.1, to attempt to prove conjecture 4.3.2 for a system where $P_1\mathcal{H} = -\lambda I$.

4.3.2 System with $P_1\mathcal{H} = -\lambda I$

Consider the class of PHS of form

$$\frac{\partial}{\partial t}w(\zeta,t) = -\lambda\frac{\partial}{\partial\zeta}w(\zeta,t) \tag{4.30}$$

$$0 = W_B\begin{bmatrix} w(1,t) \\ w(0,t)\end{bmatrix} \tag{4.31}$$

$$y(t) = W_C\begin{bmatrix} w(1,t) \\ w(0,t)\end{bmatrix}, \tag{4.32}$$

where $\zeta \in [0,1]$ and $\lambda > 0$ and real so that $P_1\mathcal{H}$ is symmetric. The matrices W_B and W_C encode the boundary conditions and which boundary values are outputs, respectively. W_B is a $n \times 2n$ full rank matrix.
Since the system only presents negative eigenvalues, there is no necessity to consider a transformation matrix S as previously done. Instead, it follows that this system may be rewritten as

$$\frac{\partial}{\partial t}w(\zeta,t) = \frac{\partial}{\partial\zeta}\left(-\lambda I w(\zeta,t)\right), \tag{4.33}$$

$$0 = \frac{1}{\lambda}W_{B,1}\lambda w(1,t) + \frac{1}{\lambda}W_{B,0}\lambda w(0,t), \tag{4.34}$$

$$y(t) = \frac{1}{\lambda}W_{C,1}\lambda w(1,t) + \frac{1}{\lambda}W_{C,0}\lambda w(0,t), \tag{4.35}$$

with, by Remark 2.4.1, $-\frac{1}{\lambda}W_{B,0} = K$ and $-\frac{1}{\lambda}W_{B,1} = Q$, where K must be invertible for the system to have a solution. Once the matrix K has been determined to be invertible, the system (4.33)–(4.35) may be further rewritten. In particular, as seen in Subsection 2.4.2, functions $k : [0,1] \to [0,1]$ and $p : [0,1] \to \mathbb{R}^+$ are defined by $k(\zeta) = 1 - p(\zeta)p(1)^{-1}$ and $p(\zeta) := -\int_0^\zeta \frac{1}{\lambda}d\eta = -\frac{1}{\lambda}\zeta$, respectively.
By substituting the expression $w(\zeta,t) = -\lambda^{-1}f_-(k(\zeta) + p(1)^{-1}t)$ (see Subsection 2.4.2) into equation (4.35) and using the invertibility of $K = -\frac{1}{\lambda}W_{B,0}$, one finds that

$$\begin{aligned} 0 &= -\frac{1}{\lambda}W_{B,1}f(p(1)^{-1}t) - \frac{1}{\lambda}W_{B,0}f(1 + p(1)^{-1}t) && \iff \\ \frac{1}{\lambda}W_{B,0}f(1 + p(1)^{-1}t) &= -\frac{1}{\lambda}W_{B,1}f(p(1)^{-1}t) && \iff \\ f(1 + p(1)^{-1}t) &= -W_{B,0}^{-1}W_{B,1}f(p(1)^{-1}t) \end{aligned}$$

and, therefore, that

$$y(t) = (\frac{1}{\lambda}W_{C,0}W_{B,0}^{-1}W_{B,1} - \frac{1}{\lambda}W_{C,1})f(p(1)^{-1}t).$$

For any $t \geqslant 0$, $j \in \mathbb{N}$ and $\zeta \in [0,1]$ may be found such that $j + \zeta = p(1)^{-1}t$. With this definition it is obtained that

$$f(p(1)^{-1}t) = f(j+\zeta) =: w_d(j)(\zeta).$$

Finally, for $j \in \mathbb{N}$, $\zeta \in [0,1]$

$$w_d(j)(\zeta) \in L^2(0,1;\mathbb{C}^n), \quad y_d(j) \in L^2(0,1;\mathbb{C}^m)$$

are defined by

$$w_d(0)(k(\zeta)) = -\lambda w_{-0}(\zeta) \tag{4.36}$$
$$w_d(j)(\zeta) = f(j+\zeta) = f(p(1)^{-1}t),\ j \geqslant 1 \tag{4.37}$$
$$y_d(j)(\zeta) = y((j+\zeta)p(1)) = y(t),\ j \in \mathbb{N} \tag{4.38}$$

so that the system

$$w_d(j+1)(\zeta) = A_d w_d(j)(\zeta) \tag{4.39}$$
$$w_d(0)(k(\zeta)) = -\lambda I w_0(\zeta) \tag{4.40}$$
$$y_d(j)(\zeta) = C_d w_d(j)(\zeta), \tag{4.41}$$

is obtained, where

$$A_d = -W_{B,0}^{-1}W_{B,1}, \tag{4.42}$$
$$C_d = \frac{1}{\lambda}W_{C,0}W_{B,0}^{-1}W_{B,1} - \frac{1}{\lambda}W_{C,1}. \tag{4.43}$$

Now, if A_d is exponentially stable, the pair (A_d, C_d) is (exactly) observable if and only if there exists $k > 0$ such that for every $z \in \mathbb{C}$

$$\|(zI + A_d)w_d(\zeta)\|^2 + \|C_d w_d(\zeta)\|^2 \geqslant k^2 \|w_d(\zeta)\|^2 . \tag{4.44}$$

Alternatively, if A_d is exponentially stable, the pair (A_d, C_d) is (exactly) observable if and only if there exists $k > 0$ such that eq. (4.44) holds for every $z \in \mathbb{D}_1 := \{z : |z| < 1\}$.

At the same time, if the infinite dimensional system (4.30)–(4.32) is exactly observable, then Theorem 2.1.1 must hold. In particular, there must exist $m > 0$ such that eq. (2.17), formulated as

$$\left\| sw(\zeta) + \lambda\frac{\partial}{\partial\zeta}w(\zeta) \right\|^2 + |\Re(s)| \left\| W_C \begin{bmatrix} w(1) \\ w(0) \end{bmatrix} \right\|^2 \geqslant m|\Re(s)|^2 \|w(\zeta)\|^2, \tag{4.45}$$

must hold for every $s \in \mathbb{C}_-$ and every $w \in \mathcal{D}(A)$. By Proposition 4.3.1, this inequality is equal to the inequality

$$\begin{aligned} &\left\| sw(\zeta) + \lambda\frac{\partial}{\partial\zeta}w(\zeta) \right\|^2 \\ +&|\Re(s)| \left\| \frac{1}{\lambda}W_{C,0}\lambda w(0) + \frac{1}{\lambda}W_{C,1}\lambda w(1) \right\|^2 \\ \geqslant& m|\Re(s)|^2 \|w(\zeta)\|^2. \end{aligned} \tag{4.46}$$

It remains to investigate whether it is possible that equation (4.44), with all relative conditions, holds if and only if equation (4.45) holds with all relative conditions (or, alternatively, whether it holds if and only if equation (4.46) holds). Proving this is, however, still not simple.

Let A_d be exponentially stable. If there exists $k > 0$ such that for every $z \in \mathbb{D}_1$

$$\|(zI + A_d)w_d(\zeta)\|^2 + \|C_d w_d(\zeta)\|^2 \geqslant k^2 \|w_d(\zeta)\|^2$$

holds, then clearly the pair (A_d, C_d) is observable. By proposition 4.2.2, then the pair (A, C) is exactly observable, so that, by theorem 2.1.1, there exists $m > 0$ such that

$$\left\|sw(\zeta) + \lambda\frac{\partial}{\partial\zeta}w(\zeta)\right\|^2 + |\Re(s)| \left\|W_C \begin{bmatrix} w(1) \\ w(0) \end{bmatrix}\right\|^2 \geqslant m|\Re(s)|^2\|w(\zeta)\|^2,$$

must hold for every $s \in \mathbb{C}_-$ and every $w \in \mathcal{D}(A)$. This also follows from Proposition 4.3.2.

Let now A be exponentially stable. If there exists $m > 0$ such that

$$\left\|sw(\zeta) + \lambda\frac{\partial}{\partial\zeta}w(\zeta)\right\|^2 + |\Re(s)| \left\|W_C \begin{bmatrix} w(1) \\ w(0) \end{bmatrix}\right\|^2 \geqslant m|\Re(s)|^2\|w(\zeta)\|^2,$$

holds for every $s \in \mathbb{C}_-$ and every $w \in \mathcal{D}(A)$, then, by Proposition 4.3.1, it necessarily also holds that

$$\begin{aligned} &\left\|sw(\zeta) + \lambda\frac{\partial}{\partial\zeta}w(\zeta)\right\|^2 \\ +&|\Re(s)| \left\|\frac{1}{\lambda}W_{C,0}\lambda w(0) + \frac{1}{\lambda}W_{C,1}\lambda w(1)\right\|^2 \\ \geqslant& m|\Re(s)|^2\|w(\zeta)\|^2. \end{aligned}$$

However, it remains unclear, even in this case, whether this implies that the inequality

$$\begin{aligned} &\|(sI - A_d)\, w_d(0)(k(\zeta)\|^2 \\ +&|\Re(s)| \left\|(K^{-1}Q)W_{C,0} - W_{C,1})w_d(t)(\zeta)\right\|^2 \\ \geqslant& m|\Re(s)|^2\|w_d(0)(k(\zeta)\|^2 \end{aligned}$$

too must hold for $z \in \mathbb{D}_1$. This methodology does then not quite yield the desired theory. A different path or different manipulation are necessary in order to be able to prove conjecture 4.3.2 or a similar result.

Remark 4.3.5. Very similar steps may be taken for $P_1\mathcal{H} = \lambda I$. It should be, however, noted that for this case it is Q, rather than K, that should be invertible.

Chapter 5

Crank-Nicolson scheme and Lyapunov equations

By use of discrete-time and continuous-time Lyapunov equations, rather than the infinite-dimensional and the finite-dimensional Hautus test, it is also possible to establish a relationship between the observability of a continuous-time and a discrete-time control system. In particular, this relationship can be established through the use of the Crank-Nicolson scheme, a finite difference method introduced in Crank and Nicolson (1947), where it's main application was evaluating numerical solutions of the non-linear partial differential equation $\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial \zeta^2} - \frac{\partial w}{\partial t}$.

5.1 Motivating example

Consider continuous-time system

$$\frac{d}{dt}x(t) = Ax(t) \tag{5.1}$$

$$y(t) = Cx(t) \tag{5.2}$$

with A an exponentially stable operator, $A : \mathcal{D}(A) \to X$, and $C : \mathcal{D}(A) \to Y$, where X and Y are Hilbert spaces and $\mathcal{D}(A) \subseteq X$. By Theorem 2.2.4, the pair (A, C) is exactly observable if and only is the Lyapunov equation

$$A^*L + LA = -C^*C \tag{5.3}$$

has a unique solution L which is positive definite.
A related discrete-time system is found by applying the Crank-Nicolson scheme to eq. (5.1). In particular, the time derivative is approached through the term $\frac{x(t+h)-x(t)}{h}$ for some $h > 0$, while the right-hand side of eq. (5.1) is approached with its average. This yields

$$\begin{aligned} \frac{d}{dt}x(t) &= Ax(t) \\ \implies \quad \frac{x(t+h) - x(t)}{h} &= A\frac{x(t+h) + x(t)}{2} \\ \iff \quad x(t+h) - x(t) &= \frac{h}{2}A(x(t+h) + x(t)). \end{aligned} \tag{5.4}$$

Without loss of generality, h is taken to be equal to 2, so that, on $\mathcal{D}(A)$,

$$\begin{aligned}
x(t+h) - x(t) &= \frac{h}{2}A(x(t+h) + x(t)) && \Longleftrightarrow \\
x(t+2) - x(t) &= A(x(t+2) + x(t)) && \Longleftrightarrow \\
x(t+2) - Ax(t+2) &= x(t) + Ax(t) && \Longleftrightarrow \\
(I-A)x(t+2) &= (I+A)x(t) && \Longleftrightarrow \\
x(t+2) &= (I-A)^{-1}(I+A)x(t) && \Longleftrightarrow \\
x(t+2) &= (I+A)(I-A)^{-1}x(t),
\end{aligned}$$

where the last equality follows from the fact that

$$\begin{aligned}
(I-A)^{-1}(I+A) &= (I-A)^{-1}(I+A)(I-A)(I-A)^{-1} \\
&= (I-A)^{-1}(I-A^2)(I-A)^{-1} \\
&= (I-A)^{-1}(I-A)(I+A)(I-A)^{-1} \\
&= (I+A)(I-A)^{-1}.
\end{aligned}$$

The scheme is now also applied to eq. (5.2) by rewriting the right-hand side as an average so as to yield

$$\begin{aligned}
y(t) &= Cx(t) \\
&= C\frac{x(t+2) + x(t)}{2} \\
&= C\frac{(I+A)(I-A)^{-1}x(t) + x(t)}{2} \\
&= C\frac{(I-A)^{-1}(I+A)x(t) + (I-A)^{-1}(I-A)x(t)}{2} \\
&= C(I-A)^{-1}\frac{(I+A)x(t) + (I-A)x(t)}{2} \\
&= C(I-A)^{-1}x(t).
\end{aligned} \tag{5.5}$$

The discrete-time system

$$x(t+2) = A_{CN}x(t) = (I+A)(I-A)^{-1}x(t) \tag{5.6}$$

$$y(t) = C_{CN}x(t) = C(I-A)^{-1}x(t) \tag{5.7}$$

is the obtained, where A_{CN} is stable (see Proposition 5.1.1 as discussed later on). For this system, given that A_{CN} is stable, the pair (A_{CN}, C_{CN}) is observable if and only if the Lyapunov equation

$$A_{CN}^* L_{CN} A_{CN} - L_{CN} = -C_{CN}^* C_{CN} \tag{5.8}$$

has a unique solution L_{CN} which is positive definite (see Theorem 2.2.2).

It is now possible to show that the pair (A, C) is exactly observable if and only if the pair (A_{CN}, C_{CN}) is observable. In particular, suppose that A_{CN} is stable (so that so is A) and that the Lyapunov equation

$$A_{CN}^* L_{CN} A_{CN} - L_{CN} = -C_{CN}^* C_{CN},$$

relative to system (5.6)–(5.7), has the solution L_{CN}, so that the pair (A_{CN}, C_{CN}) is observable. It follows that

$$
\begin{aligned}
((I+A)(I-A)^{-1})^* L_{CN}(I+A)(I-A)^{-1} - L_{CN} &= -(C(I-A)^{-1})^* C(I-A)^{-1} \\
(I-A)^{-*}(I+A)^* L_{CN}(I+A)(I-A)^{-1} - L_{CN} &= -(I-A)^{-*} C^* C(I-A)^{-1}.
\end{aligned}
$$

By multiplying both sides of the equality by $(I-A)^*$, one finds that

$$
\begin{aligned}
(I+A)^* L_{CN}(I+A) - (I-A)^* L_{CN}(I-A) &= -C^*C \\
(I+A^*) L_{CN}(I+A) - (I-A^*) L_{CN}(I-A) &= -C^*C \\
2L_{CN}A + 2A^* L_{CN} &= -C^*C.
\end{aligned}
$$

Finally it is concluded that

$$
\begin{aligned}
A_{CN}^* L_{CN} A_{CN} - L_{CN} &= -C_{CN}^* C_{CN} \\
&\Longleftrightarrow \\
2L_{CN}A + 2A^* L_{CN} &= -C^*C.
\end{aligned} \tag{5.9}
$$

This means that for A stable, the Lyapunov equation

$$LA + A^*L = -C^*C$$

has a unique positive definite solution $L = 2L_{CN}$ so that the pair (A, C) is exactly observable. By reversing the steps, one can prove that if the pair (A, C) is exactly observable, then so is the pair (A_{CN}, C_{CN}) (Power, 1967).
Showing that the pair (A, C) is exactly observable if and only if (A_{CN}, C_{CN}) is observable was done here under the assumption that A is stable if and only if A_{CN} is stable. Below is shown why this is the case.

Proposition 5.1.1. *Consider the continuous-time system (5.1)–(5.2) and the discrete-time system (5.6)–(5.7). The operator A is globally asymptotically stable if and only if A_{CN} is globally asymptotically stable.*

Proof. By Theorem 2.2.3, A is globally asymptotically stable if and only if, given that $C^*C \geqslant \epsilon I$ with $\epsilon > 0$, there exists a unique $L > 0$ satisfying

$$LA + A^*L = -C^*C.$$

By eq. (5.9), if there exists a unique $L > 0$ satisfying

$$LA + A^*L = -C^*C,$$

then there exists a unique $L > 0$ satisfying

$$A_{CN}^* L_{CN} A_{CN} - L_{CN} = -C_{CN}^* C_{CN}.$$

By Theorem 2.2.1, A_{CN} is globally asymptotically stable if and only if, given that $C_{CN}^* C_{CN} > 0$, there exists a unique $L > 0$ satisfying

$$A_{CN}^* L_{CN} A_{CN} - L_{CN} = -C_{CN}^* C_{CN},$$

hence A_{CN} too is globally asymptotically stable.
By reversing these steps it also proven that if A_{CN} is globally asymptotically stable, then so is A. □

The observability of the continuous-time system (5.1)–(5.2) and the discrete-time system (5.6)–(5.7) may also be compared by use of the Hautus test. Indeed, suppose, with A being stable, that there exists $m > 0$ such that for every $s \in \mathbb{C}_-$ and every $x \in \mathcal{D}(A)$ the inequality

$$\|(sI - A)x\|^2 + |\Re(s)|\|Cx\|^2 \geqslant m|\Re(s)|^2\|x\|^2$$

holds. By applying the substitution $x = \frac{1}{1-s}(I - A)^{-1}\tilde{x}$, where $s \neq 1$ as $s \in \mathbb{C}_-$ so that $\Re(s) < 0$, one finds that

$$\|(sI - A)x\|^2 + |\Re(s)|\|Cx\|^2 \geqslant m|\Re(s)|^2\|x\|^2$$
$$\Longleftrightarrow$$
$$\begin{aligned}&\|\frac{1}{1-s}(sI - A)(I - A)^{-1}\tilde{x}\|^2 \\ &+|\Re(s)|\|\frac{1}{1-s}C(I - A)^{-1}\tilde{x}\|^2 \geqslant m|\Re(s)|^2\|\frac{1}{1-s}(I - A)^{-1}\tilde{x}\|^2\end{aligned}$$
$$\Longleftrightarrow$$
$$\|\frac{1}{1-s}(sI - A)(I - A)^{-1}\tilde{x}\|^2 + |\Re(s)|\|\frac{1}{1-s}C_{CN}\tilde{x}\|^2 \geqslant m|\Re(s)|^2\|\frac{1}{1-s}(I - A)^{-1}\tilde{x}\|^2.$$

By rewriting $\frac{1}{1-s}(sI - A)(I - A)^{-1}\tilde{x}$ as

$$\begin{aligned}\frac{1}{1-s}(sI - A)(I - A)^{-1}\tilde{x} &= \frac{1}{1-s}[sI - A](I - A)^{-1}\tilde{x} \\ &= \frac{1}{1-s}[sI - \frac{1}{2}I + \frac{1}{2}I - \frac{1}{2}sA + \frac{1}{2}sA - A](I - A)^{-1}\tilde{x} \\ &= \frac{1}{1-s}[\frac{1}{2}(1+s)(I - A) - \frac{1}{2}(1-s)(I + A)](I - A)^{-1}\tilde{x} \\ &= \frac{1}{2}\frac{1}{1-s}[(1+s)(I - A) - (1-s)(I + A)](I - A)^{-1}\tilde{x} \\ &= \frac{1}{2}\frac{1}{1-s}[(1+s)I - (1-s)(I + A)(I - A)^{-1}]\tilde{x} \\ &= \frac{1}{2}\frac{1}{1-s}[(1+s)I - (1-s)A_{CN}]\tilde{x} \\ &= \frac{1}{2}[\frac{1+s}{1-s}I - A_{CN}]\tilde{x}\end{aligned}$$

and by defining $\frac{1+s}{1-s} = z$, one finds hence that

$$\|(sI - A)x\|^2 + |\Re(s)|\|Cx\|^2 \geqslant m|\Re(s)|^2\|x\|^2$$
$$\Longleftrightarrow$$
$$\begin{aligned}\|\frac{1}{2}[\frac{1+s}{1-s}I - A_{CN}]\tilde{x}\|^2 + |\Re(s)|\|\frac{1}{1-s}C_{CN}\tilde{x}\|^2 &\geqslant m|\Re(s)|^2\|\frac{1}{1-s}(I - A)^{-1}\tilde{x}\|^2 \\ \|\frac{1}{2}[zI - A_{CN}]\tilde{x}\|^2 + |\Re(s)|\|\frac{1}{1-s}C_{CN}\tilde{x}\|^2 &\geqslant m|\Re(s)|^2\|\frac{1}{1-s}(I - A)^{-1}\tilde{x}\|^2 \\ \|[zI - A_{CN}]\tilde{x}\|^2 + 4|\Re(s)|\frac{1}{1-s}\frac{1}{1-\overline{s}}\|C_{CN}\tilde{x}\|^2 &\geqslant 4m|\Re(s)|^2\frac{1}{1-s}\frac{1}{1-\overline{s}}\|(I - A)^{-1}\tilde{x}\|^2.\end{aligned}$$

Here it follows that, since $\Re(s) < 0$

$$\begin{aligned}\frac{1}{1-s}\frac{1}{1-\overline{s}} &= \frac{1}{1+|\Re(s)|-Im(s)i}\frac{1}{1+|\Re(s)|+Im(s)i}\\ &= \frac{1}{(1+|\Re(s)|)^2+(Im(s))^2}\\ &= \frac{1}{|s|^2+2|\Re(s)|+1}.\end{aligned}$$

The inequality $\|(sI-A)x\|^2+|\Re(s)|\|Cx\|^2 \geqslant m|\Re(s)|^2\|x\|^2$ may then be rewritten as

$$\|[zI-A_{CN}]\tilde{x}\|^2 + \frac{4|\Re(s)|}{|s|^2+2|\Re(s)|+1}\|C_{CN}\tilde{x}\|^2 \geqslant \frac{4m|\Re(s)|^2}{|s|^2+2|\Re(s)|+1}\|(I-A)^{-1}\tilde{x}\|^2.$$

Finally, all terms containing s must be rewritten so as to contain z instead. This is done via the following equivalences:

$$\begin{aligned}z &= \frac{1+s}{1-s}\\ (1-s)z &= 1+s\\ z-sz &= 1+s\\ sz+s &= z-1\\ (z+1)s &= z-1\\ s &= \frac{z-1}{z+1}\\ s &= \frac{z-1}{z+1}\frac{\overline{z}+1}{\overline{z}+1}\\ s &= \frac{|z|^2-1+2\Im(z)i}{|z+1|^2},\end{aligned}$$

so that $\Re(s) = \frac{|z|^2-1}{|z+1|^2}$. It follows then that for a constant $m > 0$, the inequality

$$\|[zI-A_{CN}]\tilde{x}\|^2 + \frac{4\left|\frac{|z|^2-1}{|z+1|^2}\right|}{\left|\frac{z-1}{z+1}\right|^2+2\left|\frac{|z|^2-1}{|z+1|^2}\right|+1}\|C_{CN}\tilde{x}\|^2 \geqslant \frac{4m\left|\frac{|z|^2-1}{|z+1|^2}\right|^2}{\left|\frac{z-1}{z+1}\right|^2+2\left|\frac{|z|^2-1}{|z+1|^2}\right|+1}\|(I-A)^{-1}\tilde{x}\|^2$$

holds for every $x \in \mathcal{D}(A)$. Here $z \in \mathbb{D}_1$ given that $s \in \mathbb{C}_-$ and, hence, that $\Re(s) < 0$. In fact, from $\Re(s) = \frac{|z|^2-1}{|z+1|^2}$, it follows that

$$\Re(s) < 0 \iff |z| < 1,$$

so that the pair (A_{CN}, C_{CN}) is observable by the finite dimensional Hautus test. By the Lyapunov equations, (A, C) will then also be observable. Also by use of the Lyapunov equations it is further possible to state that the inverse is true: if the Hautus test holds for the pair (A_{CN}, C_{CN}), then the pair (A_{CN}, C_{CN}) will be observable, meaning that the pair (A, C) will be exactly observable so that its relative Hautus test also holds.

5.2 Crank-Nicolson and PHS

It remains now to investigate whether a similar equivalence through the Crank-Nicolson scheme may also be obtained for the considered system (3.1) – (3.3). In particular, the relation between A_{CN} and A_d is investigated so as to be able to utilise this relation to prove proposition 4.3.1.

Consider equation (3.1), which has form similar to equation (5.1) with the operator A such that $Ax := \left(P_1\mathcal{H}\frac{\partial}{\partial\zeta}\right)(x)$ for $x \in \mathcal{D}(A)$. Recall that $P_1\mathcal{H}$ is constant and diagonal. Once again the Crank-Nicolson scheme is implemented on the left-hand side, while the right-hand side is replaced by its average, obtaining

$$\begin{aligned}\frac{\partial}{\partial t}w(\zeta,t) &= P_1\mathcal{H}\frac{\partial}{\partial\zeta}w(\zeta,t)\\ w_{i,j+1} - w_{i,j} &= \left(\frac{h}{2}P_1\mathcal{H}\frac{\partial}{\partial\zeta}\right)(w_{i,j+1} + w_{i,j})\end{aligned}$$

through the same steps as seen in Section 5.1. This finally yields the operator $\left(I + P_1\mathcal{H}\frac{\partial}{\partial\zeta}\right)\left(I - P_1\mathcal{H}\frac{\partial}{\partial\zeta}\right)^{-1} = A_{CN}$.
The inverse $\left(I - P_1\mathcal{H}\frac{\partial}{\partial\zeta}\right)^{-1}$ must be determined such that

$$\left(I - P_1\mathcal{H}\frac{\partial}{\partial\zeta}\right)^{-1} f = g \tag{5.10}$$

for some functions f and g, $g \in \mathcal{D}(A)$. In other words, $\left(I - P_1\mathcal{H}\frac{\partial}{\partial\zeta}\right)^{-1}$ must be defined so that

$$\begin{aligned}f &= \left(I - P_1\mathcal{H}\frac{\partial}{\partial\zeta}\right)g\\ &= g - P_1\mathcal{H}\frac{\partial g}{\partial\zeta}.\end{aligned} \tag{5.11}$$

In particular, equation (5.11) must be solved. In order to do so, equation (5.11) is rewritten as follows:

$$\frac{\partial}{\partial\zeta}g = (P_1\mathcal{H})^{-1}g - (P_1\mathcal{H})^{-1}f. \tag{5.12}$$

Eq. (5.12) has the solution (see Haberman (2014))

$$g(\zeta) = e^{(P_1\mathcal{H})^{-1}\zeta}g(0) - \int_0^\zeta e^{(P_1\mathcal{H})^{-1}(\zeta-\tau)}(P_1\mathcal{H})^{-1}f(\tau)d\tau. \tag{5.13}$$

Within eq. (5.13), $g(0)$ may be determined by utilising the boundary conditions. In particular it must hold that

$$\begin{aligned}0 &= W_B\begin{bmatrix}g(1)\\ g(0)\end{bmatrix}\\ 0 &= \begin{bmatrix}W_{B,1} & W_{B,0}\end{bmatrix}\begin{bmatrix}g(1)\\ g(0)\end{bmatrix}.\end{aligned}$$

Given that

$$g(1) = e^{(P_1\mathcal{H})^{-1}}g(0) - \int_0^1 e^{(P_1\mathcal{H})^{-1}(1-\tau)}(P_1\mathcal{H})^{-1}f(\tau)d\tau, \tag{5.14}$$

the boundary condition yields that

$$\begin{aligned} 0 =& W_{B,1}g(1) + W_{B,0}g(0) \\ 0 =& W_{B,1}\left(e^{(P_1\mathcal{H})^{-1}}g(0) - \int_0^1 e^{(P_1\mathcal{H})^{-1}(1-\tau)}(P_1\mathcal{H})^{-1}f(\tau)d\tau\right) \\ &+ W_{B,0}g(0) \\ 0 =& W_{B,1}e^{(P_1\mathcal{H})^{-1}}g(0) - W_{B,1}\int_0^1 e^{(P_1\mathcal{H})^{-1}(1-\tau)}(P_1\mathcal{H})^{-1}f(\tau)d\tau \\ &+ W_{B,0}g(0) \\ W_{B,0}g(0) + W_{B,1}e^{(P_1\mathcal{H})^{-1}}g(0) =& W_{B,1}\int_0^1 e^{(P_1\mathcal{H})^{-1}(1-\tau)}(P_1\mathcal{H})^{-1}f(\tau)d\tau \\ \left(W_{B,0} + W_{B,1}e^{(P_1\mathcal{H})^{-1}}\right)g(0) =& W_{B,1}\int_0^1 e^{(P_1\mathcal{H})^{-1}(1-\tau)}(P_1\mathcal{H})^{-1}f(\tau)d\tau, \end{aligned}$$

so that

$$\begin{aligned} g(0) = &\left(W_{B,0} + W_{B,1}e^{(P_1\mathcal{H})^{-1}}\right)^{-1} \cdot \\ &W_{B,1}\int_0^1 e^{(P_1\mathcal{H})^{-1}(1-\tau)}(P_1\mathcal{H})^{-1}f(\tau)d\tau. \end{aligned} \tag{5.15}$$

Here $W_0 + W_1e^{(P_1\mathcal{H})^{-1}}$ is invertible due to the stability of the system. Hence the solution of eq. (5.12) equals

$$\begin{aligned} g(\zeta) =& e^{(P_1\mathcal{H})^{-1}\zeta}\left(W_{B,0} + W_{B,1}e^{(P_1\mathcal{H})^{-1}}\right)^{-1}W_{B,1}\int_0^1 e^{(P_1\mathcal{H})^{-1}(1-\tau)}(P_1\mathcal{H})^{-1}f(\tau)d\tau \\ &- \int_0^\zeta e^{(P_1\mathcal{H})^{-1}(\zeta-\tau)}(P_1\mathcal{H})^{-1}f(\tau)d\tau. \end{aligned}$$

The inverse operator $\left(I - P_1\mathcal{H}\frac{\partial}{\partial\zeta}\right)^{-1}$, based on equation (5.10), equals then

$$\begin{aligned} \left(\left(I - P_1\mathcal{H}\frac{\partial}{\partial\zeta}\right)^{-1}w\right)(\zeta) :=& e^{(P_1\mathcal{H})^{-1}\zeta}\left(W_{B,0} + W_{B,1}e^{(P_1\mathcal{H})^{-1}}\right)^{-1} \cdot \\ &W_{B,1}\int_0^1 e^{(P_1\mathcal{H})^{-1}(1-\tau)}(P_1\mathcal{H})^{-1}w(\tau)d\tau \\ &- \int_0^\zeta e^{(P_1\mathcal{H})^{-1}(\zeta-\tau)}(P_1\mathcal{H})^{-1}w(\tau)d\tau. \end{aligned} \tag{5.16}$$

The operator A_{CN} has the form

$$\begin{aligned}
A_{CN}w(\zeta) &= \left(I + P_1\mathcal{H}\frac{\partial}{\partial\zeta}\right)\left(I - P_1\mathcal{H}\frac{\partial}{\partial\zeta}\right)^{-1} w(\zeta) \\
&= \left[2I - (I - P_1\mathcal{H}\frac{\partial}{\partial\zeta})\right]\left(I - P_1\mathcal{H}\frac{\partial}{\partial\zeta}\right)^{-1} w(\zeta) \\
&= \left[2\left(I - P_1\mathcal{H}\frac{\partial}{\partial\zeta}\right)^{-1} - I\right] w(\zeta) \\
&=2e^{(P_1\mathcal{H})^{-1}\zeta}\left(W_{B,0} + W_{B,1}e^{(P_1\mathcal{H})^{-1}}\right)^{-1}. \qquad (5.17) \\
&\quad W_{B,1}\int_0^1 e^{(P_1\mathcal{H})^{-1}(1-\tau)}(P_1\mathcal{H})^{-1}w(\tau)d\tau \\
&\quad - 2\int_0^\zeta e^{(P_1\mathcal{H})^{-1}(\zeta-\tau)}(P_1\mathcal{H})^{-1}w(\tau)d\tau - w(\zeta) \\
&=2e^{(P_1\mathcal{H})^{-1}\zeta}w(0) - 2\int_0^\zeta e^{(P_1\mathcal{H})^{-1}(\zeta-\tau)}(P_1\mathcal{H})^{-1}w(\tau)d\tau - w(\zeta),
\end{aligned}$$

with

$$w(0) = \left(W_{B,0} + W_{B,1}e^{(P_1\mathcal{H})^{-1}}\right)^{-1} W_{B,1}\int_0^1 e^{(P_1\mathcal{H})^{-1}(1-\tau)}(P_1\mathcal{H})^{-1}w(\tau)d\tau. \qquad (5.18)$$

Similarly, by combining eq. (5.5) and eq. (5.16), the operator C_{CN} is given by

$$C_{CN}w(\zeta) = W_C\begin{bmatrix}\left(\left(I - P_1\mathcal{H}\frac{\partial}{\partial\zeta}\right)^{-1} w\right)(1) \\ \left(\left(I - P_1\mathcal{H}\frac{\partial}{\partial\zeta}\right)^{-1} w\right)(0)\end{bmatrix}.$$

Working this expression out and introducing eq. (5.14) and eq. (5.15) into it, yields

$$\begin{aligned}
C_{CN}w(\zeta) &=W_{C,1}\left(I - P_1\mathcal{H}\frac{\partial}{\partial\zeta}\right)^{-1} w(1) + W_{C,0}\left(I - P_1\mathcal{H}\frac{\partial}{\partial\zeta}\right)^{-1} w(0) \\
&=W_{C,1}e^{(P_1\mathcal{H})^{-1}}\left(W_{B,0} + W_{B,1}e^{(P_1\mathcal{H})^{-1}}\right)^{-1}. \\
&\quad W_{B,1}\int_0^1 e^{(P_1\mathcal{H})^{-1}(1-\tau)}(P_1\mathcal{H})^{-1}w(\tau)d\tau \\
&\quad - W_{C,1}\int_0^1 e^{(P_1\mathcal{H})^{-1}(1-\tau)}(P_1\mathcal{H})^{-1}w(\tau)d\tau \\
&\quad + W_{C,0}\left(W_{B,0} + W_{B,1}e^{(P_1\mathcal{H})^{-1}}\right)^{-1}. \\
&\quad W_{B,1}\int_0^1 e^{(P_1\mathcal{H})^{-1}(1-\tau)}(P_1\mathcal{H})^{-1}w(\tau)d\tau.
\end{aligned}$$

Finally, the operator C_{CN} may be defined as

$$\begin{aligned}
C_{CN}w(\zeta) =& W_{C,1}\left(e^{(P_1\mathcal{H})^{-1}}\left(W_{B,0}+W_{B,1}e^{(P_1\mathcal{H})^{-1}}\right)^{-1}W_{B,1}-1\right)\cdot\\
&\int_0^1 e^{(P_1\mathcal{H})^{-1}(1-\tau)}(P_1\mathcal{H})^{-1}w(\tau)d\tau\\
&+W_{C,0}\left(W_{B,0}+W_{B,1}e^{(P_1\mathcal{H})^{-1}}\right)^{-1}\cdot\\
&W_{B,1}\int_0^1 e^{(P_1\mathcal{H})^{-1}(1-\tau)}(P_1\mathcal{H})^{-1}w(\tau)d\tau\\
=& W_{C,1}g(1)+W_{C,0}g(0).
\end{aligned} \tag{5.19}$$

It remains now to compare these two operators to the previously obtained matrices A_d and C_d. If this is possible, then the exact observability of the pair (A, C) may be linked, through the pair (A_{CN}, C_{CN}), to the observability of the pair (A_d, C_d). In this fashion, by the use of the Lyapunov equation, it would then be possible to extend the infinite dimensional Hautus test to a sufficient condition.

5.2.1 Comparison with A_d and C_d

In Section 4.2, matrices A_d and C_d had been found of form

$$\begin{aligned}
A_d &= -K^{-1}Q,\\
C_d &= -\begin{bmatrix}R_{21} & O_{22}\end{bmatrix}K^{-1}Q+\begin{bmatrix}O_{21} & R_{22}\end{bmatrix}.
\end{aligned}$$

In order to compare these two matrices to the operators A_{CN} and C_{CN} found above, assume first that $P_1\mathcal{H} = -I$. Where there no tangible result to be found for this simpler version of $P_1\mathcal{H}$, it would certainly not be fruitful to attempt the same result for more convoluted versions of the matrix.

If $P_1\mathcal{H} = -I$, then, as seen in Subsection 4.3.2,

$$\begin{aligned}
A_d &= -W_{B,0}^{-1}W_{B,1},\\
C_d &= -W_{C_0}W_{B,0}^{-1}W_{B,1}+W_{C,1}
\end{aligned}$$

so that

$$\begin{aligned}
w(0,t) &= -W_{B,0}^{-1}W_{B,1}w(1,t),\\
y(t) &= (-W_{C_0}W_{B,0}^{-1}W_{B,1}+W_{C,1})w(1,t).
\end{aligned}$$

Recall eq. (5.17) and eq. (5.19)

$$\begin{aligned}
A_{CN}w(\zeta,t) &= 2e^{(P_1\mathcal{H})^{-1}\zeta}w(0,t)-2\int_0^\zeta e^{(P_1\mathcal{H})^{-1}(\zeta-\tau)}(P_1\mathcal{H})^{-1}w(\tau,t)d\tau - w(\zeta,t),\\
C_{CN}w(\zeta,t) &= W_{C,1}w(1,t)+W_{C,0}w(0,t).
\end{aligned}$$

Comparison between C_{CN} and C_d could be possible. In particular, $C_{CN}w(\zeta,t) = C_dw(\zeta,t)$ if $w(0,t) = -W_{B,0}^{-1}W_{B,1}w(1,t)$. However, the same cannot be said about a comparison between A_{CN} and A_d. In fact, these two operators do not map to the same space, indicating a substantial difference between the two.

Remark 5.2.1. Comparison between the pairs (A_{CN}, C_{CN}) and (A_d, C_d) can also be attempted by rewriting $w(0)$, defined by eq. (5.15), in terms of $w(1)$, defined as eq. (5.14), and by subsequently comparing the obtained result with

$$w(0) = -W_{B,0}^{-1} W_{B,1} w(t).$$

However, this methodology too, does not yield a satisfactory comparison for the operator A_{CN} and requires, additionally, many assumptions to be made about $W_{B,0}$, $W_{B,1}$ or combinations of the two.

Comparison between the pairs (A_{CN}, C_{CN}) and (A_d, C_d) is then not directly possible, meaning that Conjecture 4.3.2 remains unproven. However, the exact observability of PHS of form (3.1)–(3.3) may still potentially be proven. The Lyapunov equation itself may provide valuable insights to this avail.

Chapter 6

The Lyapunov equation

The Lyapunov equation is a useful tool in determining exact observability. Recall, once again, Theorem 2.2.4: for a continuous-time system of form

$$\dot{x}(t) = Ax(t) \tag{6.1}$$

$$y(t) = Cx(t) \tag{6.2}$$

with A a stable operator, the pair (A, C) is exactly observable if the Lyapunov equation

$$A^*L + LA = -C^*C \tag{6.3}$$

has a unique solution L which is positive definite, which means that necessarily $L = L^*$ (Parks, 1992).
As seen in Theorem 2.2.3, the Lyapunov equation is also stongly associated with stability as, given any $C^*C > 0$, there exists a unique $L > 0$

$$A^*L + LA = -C^*C$$

if and only if the linear system $\dot{x} = Ax$ is globally asymptotically stable. The quadratic function $V(x) = \langle x, Lx\rangle$ is then the Lyapunov function that can be used to verify stability (see Theorem 2.2.3).

In the following section, the Lyapunov equation is utilised to eventually try and find a solution when considering the PHS (3.1)–(3.3).

6.1 Motivating example

Normally speaking, solving a Lyapunov equation need not be straightforward. Consider, however, the exponentially stable system

$$\begin{aligned}\frac{\partial}{\partial t}x(\zeta, t) &= \frac{\partial}{\partial \zeta}x(\zeta, t)\\ \beta x(0, t) &= x(1, t)\\ y(t) &= Cx(\zeta, t) = x(0, t),\end{aligned}$$

where $A := \frac{\partial}{\partial\zeta}$, with $\mathcal{D}(A) := \left\{x \in L^2((0,1);\mathbb{C}) \mid \mathcal{H}x \in H^1((0,1);\mathbb{C}), 0 = W_B \begin{bmatrix} (\mathcal{H}x)(1,t) \\ (\mathcal{H}x)(0,t) \end{bmatrix}\right\}$, and $C : \mathcal{D}(A) \to \mathbb{C}$. For L a complex multiplication operator, $Lf = pf$ with $p \in \mathbb{R}$, and $\forall f, g \in \mathcal{D}(A)$

$$\begin{gathered}
A^*L + LA = -C^*C \\
\Longleftrightarrow \\
A^*p + pA = -C^*C \\
\Longleftrightarrow \\
\langle Af, pg\rangle + \langle pf, Ag\rangle = -\langle Cf, Cg\rangle \\
\Longleftrightarrow \\
\int_0^1 \frac{\partial f}{\partial\zeta}(\zeta)p\overline{g(\zeta)}d\zeta + \int_0^1 pf(\zeta)\overline{\frac{\partial g}{\partial\zeta}(\zeta)}d\zeta = -\langle f(0), g(0)\rangle \qquad (6.4) \\
\Longleftrightarrow \\
\int_0^1 p\left[\frac{\partial f}{\partial\zeta}(\zeta)\overline{g(\zeta)} + f(\zeta)\overline{\frac{\partial g}{\partial\zeta}(\zeta)}\right]d\zeta = -f(0)\overline{g(0)} \\
\Longleftrightarrow \\
p\int_0^1 \frac{\partial}{\partial\zeta}[f(\zeta)\overline{g(\zeta)}]d\zeta = -f(0)\overline{g(0)},
\end{gathered}$$

since $\frac{\partial}{\partial\zeta}[f(\zeta)\overline{g(\zeta)}] = \frac{\partial f}{\partial\zeta}(\zeta)\overline{g(\zeta)} + f(\zeta)\overline{\frac{\partial g}{\partial\zeta}(\zeta)}$. It follows then that eq. (6.4) is equivalent to

$$\begin{gathered}
p[f(\zeta)\overline{g(\zeta)}]_0^1 = -f(0)\overline{g(0)} \\
\Longleftrightarrow \\
p(f(1)\overline{g(1)} - f(0)\overline{g(0)}) = -f(0)\overline{g(0)}.
\end{gathered} \qquad (6.5)$$

Since for $f \in \mathcal{D}(A)$ there holds that $\beta f(0) = f(1)$, eq. (6.5) is equivalent to

$$\begin{aligned}
p(|\beta|^2 f(0)\overline{g(0)} - f(0)\overline{g(0)}) &= -f(0)\overline{g(0)} \quad \Longleftrightarrow \\
p(|\beta|^2 - 1)f(0)\overline{g(0)} &= -f(0)\overline{g(0)} \quad \Longleftrightarrow \\
-p(1 - |\beta|^2)f(0)\overline{g(0)} &= -f(0)\overline{g(0)} \quad \Longleftrightarrow \\
pf(0)\overline{g(0)} &= \frac{1}{1 - |\beta|^2}f(0)\overline{g(0)}.
\end{aligned}$$

The Lyapunov equation has then solution $p = \frac{1}{1-|\beta|^2}$. Since the system is exponentially stable, it must hold that $|\beta| < 1$, so that $p = \frac{1}{1-|\beta|^2} > 0$. This solution is then positive and the pair (A, C) is then, in this case, exactly observable.

Remark 6.1.1. The considered system has eigenvalue $\lambda = 1$. However, were a system to be considered where $A = \lambda\frac{\partial}{\partial\zeta}$ or $A = -\lambda\frac{\partial}{\partial\zeta}$, with $\lambda > 0$ and real, the solution to the Lyapunov equation is then $p = \frac{1}{\lambda(1-|\beta|^2)}$. Seeing as the considered system is stable, two cases are then possible:

1. if $\lambda > 0$, then $|\beta| < 1$ and $p > 0$;

2. if $\lambda < 0$, then $|\beta| > 1$ and $p > 0$.

In this case, observability could be determined with ease. It remains then to see whether a similar result may be obtained for L being a matrix as per the considered class of PHS.

6.2 Solution to the Lyapunov equation for a matrix L

Before applying and solving the Lyapunov equation for the PHS (3.1)–(3.3), it is of essence that finding a matrix solution L is possible for an exponentially stable system of form

$$\begin{aligned}\frac{\partial}{\partial t}x(\zeta,t) &= \lambda\frac{\partial}{\partial \zeta}x(\zeta,t)\\ y(t) &= Cx(\zeta,t),\end{aligned}$$

where $A = \lambda\frac{\partial}{\partial\zeta}I$, with $\lambda I = P_1\mathcal{H}$ and with

$$\mathcal{D}(A) := \left\{x \in L^2((0,1);\mathbb{C}^n) \mid \mathcal{H}x \in H^1((0,1);\mathbb{C}^n), 0 = W_B\begin{bmatrix}(\mathcal{H}x)(1,t)\\ (\mathcal{H}x)(0,t)\end{bmatrix}\right\},$$

with $M_0f(0) = M_1f(1)$, where $M_1 = K$ and $M_0 = Q$ are $n \times n$ matrices, and $Cf = C_0f(0)$, $f(\zeta) \in \mathbb{C}^n$ and $C_0 \in \mathbb{C}^{k\times n}$. Additionally, consider that $(Lf)(\zeta) = L_0f(\zeta)$, where, necessarily, $L_0 = L_0^*$. It follows, $\forall g, f \in \mathcal{D}(A)$, that

$$\begin{gathered}A^*L + LA = -C^*C\\ \Longleftrightarrow\\ \langle Af, Lg\rangle + \langle Lf, Ag\rangle = -\langle Cf, Cg\rangle\\ \Longleftrightarrow\\ \int_0^1 \frac{\partial f^*}{\partial\zeta}(\zeta)\lambda L_0 g(\zeta)d\zeta + \int_0^1 f(\zeta)^* L_0\lambda\frac{\partial g}{\partial\zeta}(\zeta)d\zeta = -\langle C_0f(0), C_0g(0)\rangle.\end{gathered} \tag{6.6}$$

Since λI is diagonal, $\lambda L_0 = L_0\lambda$. It follows that eq. (6.6) is equivalent to

$$\begin{gathered}\int_0^1 \frac{\partial}{\partial\zeta}[f^*(\zeta)\lambda L_0 g(\zeta)]d\zeta = -f(0)^*C_0^*C_0g(0)\\ [f^*(\zeta)\lambda L_0g(\zeta)]_0^1 = -f(0)^*C_0^*C_0g(0)\\ f^*(1)\lambda L_0g(1) - f^*(0)\lambda L_0g(0) = -f(0)^*C_0^*C_0g(0)\\ \Longleftrightarrow\\ \langle f(1), \lambda L_0g(1)\rangle - \langle f(0), \lambda L_0g(0)\rangle = -\langle f(0), C_0^*C_0g(0)\rangle.\end{gathered} \tag{6.7}$$

It is now necessary to be able to rewrite $f(1)$ (or $g(1)$) as $f(0)$ (or $g(0)$). For this purpose, $M_1 = K$ must, in line with the theory discussed in Section 2.4, be invertible so that $f(1) = M_1^{-1}M_0 f(0)$. It follows that eq. (6.7) is equivalent to

$$\begin{aligned}
\langle M_1^{-1}M_0 f(0), \lambda L_0 M_1^{-1}M_0 g(0)\rangle - \langle f(0), \lambda L_0 g(0)\rangle &= -\langle f(0), C_0^* C_0 g(0)\rangle \\
\langle f(0), M_0^*(M_1^{-1})^*\lambda L_0 M_1^{-1}M_0 g(0)\rangle - \langle f(0), \lambda L_0 g(0)\rangle &= -\langle f(0), C_0^* C_0 g(0)\rangle \\
\langle f(0), M_0^*(M_1^{-1})^*\lambda L_0 M_1^{-1}M_0 g(0) - \lambda L_0 g(0)\rangle &= -\langle f(0), C_0^* C_0 g(0)\rangle \\
\langle f(0), [M_0^*(M_1^{-1})^*\lambda L_0 M_1^{-1}M_0 - \lambda L_0] g(0)\rangle &= -\langle f(0), C_0^* C_0 g(0)\rangle \\
&\Longleftrightarrow \\
M_0^*(M_1^{-1})^*\lambda L_0 M_1^{-1}M_0 - \lambda L_0 &= -C_0^* C_0.
\end{aligned}$$

The expression $M_0^*(M_1^{-1})^*\lambda L_0 M_1^{-1}M_0 - \lambda L_0 = -C_0^* C_0$ is to be recognised as the discrete-time Lyapunov equation with $M_1^{-1}M_0$ acting as the discrete-time A and C_0 acting as the discrete-time C. In particular,

$$M_0^*(M_1^{-1})^*\lambda L_0 M_1^{-1}M_0 - \lambda L_0 = -C_0^* C_0 \tag{6.8}$$

can be solved for $\lambda L_0 > 0$ if the matrix $M_1^{-1}M_0$ is stable so that the system defined by the pair $(M_1^{-1}M_0, C_0)$ is observable.

Remark 6.2.1. The pair $(M_1^{-1}M_0, C_0)$ is equivalent to the pair (A_d, C_d). In particular, given that $M_1 = -K$ and $M_0 = Q$, that is the case if C_0 can be expressed as $C_0 = -\begin{bmatrix} R_{21} & O_{22} \end{bmatrix} M_1^{-1}M_0 + \begin{bmatrix} O_{21} & R_{22} \end{bmatrix}$.

The following, more general, result then holds.

Proposition 6.2.1. *Consider the exponentially stable continuous-time PHS of form*

$$\begin{aligned}
\frac{\partial}{\partial t}x(\zeta,t) &= \frac{\partial}{\partial \zeta}(\lambda(\zeta)x(\zeta,t)) \\
y(t) &= Cx(\zeta,t),
\end{aligned}$$

with $P_1\mathcal{H}(\zeta) = \lambda(\zeta)I > 0$, real and dependent on $\zeta \in [0,1]$ so that $\lambda(0) = \lambda(1)$, $M_0 f(0) = M_1 f(1)$, where M_1 is invertible, and $Cf = C_0 f(0)$, $f(\zeta) \in \mathbb{C}^n$ and $C_0 \in \mathbb{C}^{k\times n}$. The pair (A, C) is exactly observable if the pair $(M_1^{-1}M_0, C_0)$ is observable.

Proof. This result follows from the steps previously utilised in Section 6.2. Indeed, eq. (6.6) may here be rewritten as

$$\int_0^1 \left(\frac{\partial}{\partial \zeta}(\lambda(\zeta)f(\zeta))\right)^* L_0 g(\zeta)d\zeta + \int_0^1 f^*(\zeta)L_0\frac{\partial}{\partial \zeta}(\lambda(\zeta)g(\zeta))d\zeta = -\langle C_0 f(0), C_0 g(0)\rangle,$$

It follows that eq. (6.7) becomes in this case

$$\langle f(1), \lambda(1)L_0 g(1)\rangle - \langle f(0), \lambda(0)L_0 g(0)\rangle = -\langle f(0), C_0^* C_0 g(0)\rangle.$$

Finally, eq. (6.8) becomes

$$\begin{aligned}
M_0^*(M_1^{-1})^*\lambda(0)L_0 M_1^{-1}M_0 - \lambda(0)L_0 &= -C_0^* C_0 \\
&\Longleftrightarrow \\
M_0^*(M_1^{-1})^*\lambda(1)L_0 M_1^{-1}M_0 - \lambda(1)L_0 &= -C_0^* C_0,
\end{aligned}$$

which has a unique and positive solution $\lambda(0)L_0 = \lambda(1)L_0$. □

The use of the Lyapunov equation, just as the use of the theory provided by Jacob et al. (2015) and Hastir et al. (2024) as seen in section 4.3, proves, for the considered form of continuous-time PHS, that these continuous-time PHS are exactly observable if their discrete-time counterpart is observable. The reverse, however, (if the continuous-time PHS is exactly observable, then its discrete-time counterpart is observable) is not as clear.

Chapter 7

Discussion

While the premise of this work found base in the fact that finite dimensional control systems (and by extension port-Hamiltonian systems) know many powerful theorems to determine observability, the results found also outline limitations of these theorems when trying to extend their infinite dimensional counterpart in the context of exact observability.
In this chapter, a complete overview of the encountered limitations is given.

In Chapter 4, theory from Jacob et al. (2015) and Hastir et al. (2024) was utilised to rewrite the continuous-time infinite dimensional PHS (3.1)–(3.3)

$$\begin{aligned}\frac{\partial}{\partial t}w(\zeta,t) &= P_1\mathcal{H}\frac{\partial}{\partial \zeta}w(\zeta,t)\\ 0 &= W_B\begin{bmatrix} w(1,t)\\ w(0,t)\end{bmatrix}\\ y(t) &= W_C\begin{bmatrix} w(1,t)\\ w(0,t)\end{bmatrix}\end{aligned}$$

into first the continuous-time infinite dimensional PHS (4.6)–(4.8)

$$\begin{aligned}\frac{\partial}{\partial t}\begin{bmatrix} w_+(\zeta,t)\\ w_-(\zeta,t)\end{bmatrix} &= \frac{\partial}{\partial \zeta}\left(\begin{bmatrix}\lambda I & 0\\ 0 & -\lambda I\end{bmatrix}\begin{bmatrix} w_+(\zeta,t)\\ w_-(\zeta,t)\end{bmatrix}\right),\\ 0 &= \begin{bmatrix} K_{11} & K_{12}\\ K_{21} & K_{22}\end{bmatrix}\begin{bmatrix}(\lambda w_+)(1,t)\\ (-\lambda w_-)(0,t)\end{bmatrix} + \begin{bmatrix} Q_{11} & Q_{12}\\ Q_{21} & Q_{22}\end{bmatrix}\begin{bmatrix}(\lambda w_+)(0,t)\\ (-\lambda w_-)(1,t)\end{bmatrix},\\ y(t) &= \begin{bmatrix} O_{21} & O_{22}\end{bmatrix}\begin{bmatrix}(\lambda w_+)(1,t)\\ (-\lambda w_-)(0,t)\end{bmatrix} + \begin{bmatrix} R_{21} & R_{22}\end{bmatrix}\begin{bmatrix}(\lambda w_+)(0,t)\\ (-\lambda w_-)(1,t)\end{bmatrix}\end{aligned}$$

and then into the discrete-time PHS (4.13)–(4.15)

$$\begin{aligned}\begin{bmatrix} w_{+_d}(j)(\zeta)\\ w_{-_d}(j+1)(\zeta)\end{bmatrix} &= A_d\begin{bmatrix} w_{+_d}(j+1)(\zeta)\\ w_{-_d}(j)(\zeta)\end{bmatrix}\\ \begin{bmatrix} w_{+_d}(0)(k(\zeta))\\ w_{-_d}(0)(k(\zeta))\end{bmatrix} &= \begin{bmatrix}\lambda I & 0\\ 0 & -\lambda I\end{bmatrix}\begin{bmatrix} w_{+_0}(\zeta)\\ w_{-_0}(\zeta)\end{bmatrix}\\ y_d(j)(\zeta) &= C_d\begin{bmatrix} w_{+_d}(j+1)(\zeta)\\ w_{-_d}(j)(\zeta)\end{bmatrix}.\end{aligned}$$

The Hautus test was then applied with the intention of finally determining whether eq. (4.28)

$$\left\| sw(\zeta) - P_1\mathcal{H}\frac{\partial}{\partial\zeta}w(\zeta)\right\|^2 + |\Re(s)| \left\| W_C \begin{bmatrix} w(1) \\ w(0) \end{bmatrix}\right\|^2 \geqslant m|\Re(s)|^2\|w(\zeta)\|^2,$$

eq. (4.29)

$$\begin{aligned}
&\left\| s\begin{bmatrix} w_+(\zeta) \\ w_-(\zeta) \end{bmatrix} - \frac{\partial}{\partial\zeta}\begin{bmatrix} \lambda I & 0 \\ 0 & -\lambda I \end{bmatrix}\begin{bmatrix} w_+(\zeta) \\ w_-(\zeta) \end{bmatrix}\right\|^2 \\
+&|\Re(s)| \left\| \begin{bmatrix} O_{21} & O_{22} \end{bmatrix}\begin{bmatrix} (\lambda w_+)(1) \\ (-\lambda w_-)(0) \end{bmatrix} + \begin{bmatrix} R_{21} & R_{22} \end{bmatrix}\begin{bmatrix} (\lambda w_+)(0) \\ (-\lambda w_-)(1) \end{bmatrix}\right\|^2 \\
\geqslant& m|\Re(s)|^2 \left\| \begin{bmatrix} w_+(\zeta) \\ w_-(\zeta) \end{bmatrix}\right\|^2 .
\end{aligned}$$

and eq. (4.27)

$$\left\| (zI - A_d)\begin{bmatrix} w_{+_d}(\zeta) \\ w_{-_d}(\zeta) \end{bmatrix}\right\|^2 + \left\| C_d\begin{bmatrix} w_{+_d}(\zeta) \\ w_{-_d}(\zeta) \end{bmatrix}\right\|^2 \geqslant k^2 \left\| \begin{bmatrix} w_{+_d}(\zeta) \\ w_{-_d}(\zeta) \end{bmatrix}\right\|^2$$

are equivalent and, in fact, imply one another. While this was attainable for equations (4.28) and (4.29), the same was not true for equations (4.28) and (4.27) or for equations (4.29) and (4.27). Indeed, in spite of the fact that proposition 4.2.1 ensures the equivalence of exact observability of system (3.1)–(3.3) and observability of system (4.13)–(4.15), equation (4.27) implies equation (4.28), but the other way around is not true. In particular, system (3.1)–(3.3) is exactly observable if system (4.13)–(4.15) is observable. Conjecture 4.3.2, and therefore conjecture 4.3.1, remains unproven in spite of the many different attempts that have been made to try and rewrite eq. (4.28) as eq. (4.27). However, it is possible that different mathematical manipulation or, alternatively, a different class of PHS, may ensure a more complete and positive result where Theorem 2.1.1 is indeed ensured to be a sufficient condition rather then a necessary one.
In figure 7.1 a schematic representation is given of of the implications achieved and known to be true as presented in Chapter 4.

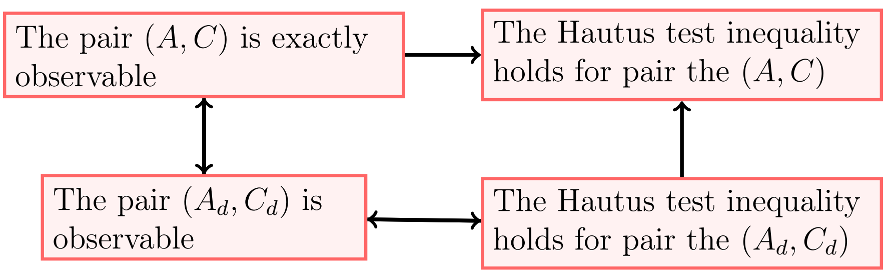

Figure 7.1: Scheme of the results as obtained in Chapter 4

Figure 7.1 explicitly shows what was discussed above: the exact observability of the pair (A, C) can be proven through the observability of the pair (A_d, C_d). On the other hand, it also shows that the Hautus test for the pair (A, C) currently does not yield exact observability nor does it imply the Hautus test for the pair (A_d, C_d).

In Chapter 5, the Crank-Nicolson scheme is utilised to obtain a discrete-time system from PHS (3.1)–(3.3). This yields the pair (A_{CN}, C_{CN}). Through the Lyapunov equation and the Hautus test, the presented motivating example shows, once again, that it is possible to ensure that exact observability of the continuous-time system and observability of the discrete-time system are equivalent. In fact, it would seem as if the Hautus test is a sufficient condition for the motivating example. However, correlating the pair (A_{CN}, C_{CN}) with the pair (A_d, C_d) found in Chapter 4 is not possible without setting strict constraints upon the considered system, even for one of the simpler cases of the system (where $P_1 = -\lambda I$ or $P_1 = \lambda I$).

In Chapter 6, the Lyapunov equation is used as a tool to establish exact observability and, in fact, once again, it becomes clear here that the considered system, which need be stable, is exactly observable if its discrete-time counterpart, described by the pair $(M_1^{-1}M_0, C_0)$, is stable and observable. This, however, still provides no tool that allows for the infinite dimensional Hautus test to be extended to a sufficient condition.

Chapter 8

Conclusion

In the work carried out through Chapters 4, 5 and 6, the relevance of considering the discrete-time counterparts to continuous-time infinite dimensional systems becomes apparent, even when limitations are met in the ability to extend results (as discussed in Chapter 7). Similarly within these limitations, it becomes clear that powerful results in finite dimensions may play a role of importance in elevating results in infinite dimensions. This is, for example, the case with the Hautus test: while for the considered PHS (3.1)–(3.3) the infinite dimensional Hautus test can still not be proven to be a sufficient condition for exact observability, the finite dimensional Hautus test manages to prove exact observability for this same system by use of Proposition 4.2.1, which links observability of the discrete-time system to exact observability of the continuous-time infinite dimensional system. This result is also schematically represented in figure 7.1. The Lyapunov equation also corroborates this, ensuring that, indeed, the stable PHS (3.1)–(3.3) is exactly observable if its discrete-time counterpart is stable and observable. So while the work presented here was not able to extend the infinite dimensional Hautus test to a sufficient condition for exact observability, it provided evidence to the fact that the continuous-time infinite dimensional PHS (3.1)–(3.3), if stable, is itself exactly observable if the discrete-time counterpart (which may be obtained through the theory presented in Jacob et al. (2015) and Hastir et al. (2024), by use of the Crank-Nicolson scheme or by used of the Lyapunov equation) is stable and observable.

The absence of the wished for positive result does not indicate an impossibility to obtain said result. In fact, the current work does not delve into whether it is possible to prove that, were eq. (4.27)

$$\left\|(zI - A_d)\begin{bmatrix} w_{+_d}(\zeta) \\ w_{-_d}(\zeta) \end{bmatrix}\right\|^2 + \left\|C_d\begin{bmatrix} w_{+_d}(\zeta) \\ w_{-_d}(\zeta) \end{bmatrix}\right\|^2 \geqslant k^2 \left\|\begin{bmatrix} w_{+_d}(\zeta) \\ w_{-_d}(\zeta) \end{bmatrix}\right\|^2,$$

not to hold true, then neither would eq. (4.28)

$$\left\|sw(\zeta) - P_1\mathcal{H}\frac{\partial}{\partial\zeta}w(\zeta)\right\|^2 + |\Re(s)|\left\|W_C\begin{bmatrix} w(1) \\ w(0) \end{bmatrix}\right\|^2 \geqslant m|\Re(s)|^2\|w(\zeta)\|^2.$$

In other words, it remains, as of yet, unclear whether eq. (4.27) can hold while eq. (4.28) does not. Similarly, it remains then unclear whether eq. (4.27) can hold true while both the continuous-time infinite dimensional PHS (3.1)–(3.3)

$$
\begin{aligned}
\frac{\partial}{\partial t} w(\zeta, t) &= P_1 \mathcal{H} \frac{\partial}{\partial \zeta} w(\zeta, t) \\
0 &= W_B \begin{bmatrix} w(1,t) \\ w(0,t) \end{bmatrix} \\
y(t) &= W_C \begin{bmatrix} w(1,t) \\ w(0,t) \end{bmatrix}
\end{aligned}
$$

and the discrete-time PHS (4.13)–(4.15)

$$
\begin{aligned}
\begin{bmatrix} w_{+_d}(j)(\zeta) \\ w_{-_d}(j+1)(\zeta) \end{bmatrix} &= A_d \begin{bmatrix} w_{+_d}(j+1)(\zeta) \\ w_{-_d}(j)(\zeta) \end{bmatrix} \\
\begin{bmatrix} w_{+_d}(0)(k(\zeta)) \\ w_{-_d}(0)(k(\zeta)) \end{bmatrix} &= \begin{bmatrix} \lambda I & 0 \\ 0 & -\lambda I \end{bmatrix} \begin{bmatrix} w_{+_0}(\zeta) \\ w_{-_0}(\zeta) \end{bmatrix} \\
y_d(j)(\zeta) &= C_d \begin{bmatrix} w_{+_d}(j+1)(\zeta) \\ w_{-_d}(j)(\zeta) \end{bmatrix}
\end{aligned}
$$

are not observable. Future research should aim to look into whether, indeed, it is possible to prove that eq. (4.27) not holding implies that eq. (4.28) cannot hold true or if a counter example to this statement can be found.

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