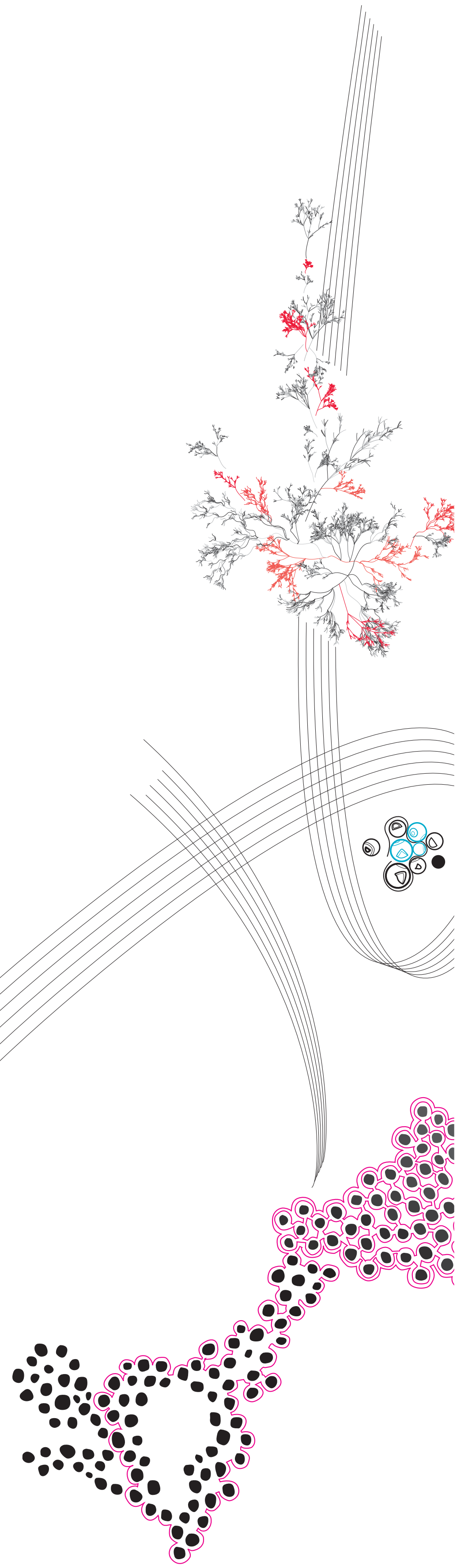
MSc Thesis Applied Mathematics



Resource Sharing Games with Player-Specific Values

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Preface

With this thesis I complete my Masters degree in Applied Mathematics at the University of Twente. I will now use this space to thank everyone who in some capacity helped me write this thesis.

First, I would like to thank my supervisors Marc and Alexander for coming up with the problem and meeting me (almost) every week, to exchange new ideas and give feedback on my work. Furthermore I like to thank Eyal for being the third graduation committee member.

Next, I want to thank my support group, currently consisting of Pelle, Sanne, Wander and Wouter for meeting every morning and hearing about my personal struggles while writing my thesis. And by extend I want to thank Lilian for meeting me in the beginning of the process and steering me in their direction and Eric & Ellen for organising the EEMCS Graduation Support group.

I also want to thank my housemates Gergely and Mara, for showing interest in what I was doing on the white board in our kitchen.

Lastly, I want to thank "Bibliotheek Twente; Hengelo Stad" for serving as inspiration for Example 1.1.

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Abstract

In resource sharing games players compete for a set of resources. Each resource is assigned a strictly positive value. We discuss games where this value is player-specific. The values of the resources contribute to the players' payoff. In an effort to maximize their own payoff players choose sets of resources. When multiple players choose the same resource, the value is affected by a sharing rule. We discuss three types of sharing rules; "no-sharing", "uniform sharing" and "proportional sharing". Each sharing rule forms a different class of resource sharing games with player-specific values. For each class, we discuss the existence of pure Nash equilibria. Then we discuss the quality of these equilibria. The quality is measured with the price of anarchy as introduced by Koutsoupias and Papadimidtriou[1].

With no-sharing, when multiple player choose the same resource they receive a payoff of $-\infty$. We show that pure Nash equilibria always exist. Let θ be the maximum ratio of any resource. We show that the price of anarchy is $\theta + 1$.

With uniform sharing, when multiple player choose the same resource they receive a fraction of the resources' value equal to the number of players choosing it. We show that pure Nash equilibria are not guaranteed. For the subset of instances of n-player games that do have pure Nash equilibria we show that the price of anarchy is n.

With proportional sharing, when multiple player choose the same resource they receive a fraction of the resources' value proportional to the total value of the players choosing it. For 2-player games we show that they are exact potential games and therefore show that pure Nash equilibria always exist. We show for 2-player games that the price of anarchy is $\varphi = \frac{1+\sqrt{5}}{2}$, the golden ratio. Next, for games with more than 2 players, we show that pure Nash equilibria are no longer guaranteed to exist. We end this thesis with a conjecture of $\frac{1+\sqrt{4(n-1)+1}}{2}$ for the price of anarchy of the set of n-player games with pure Nash equilibria.

Keywords: Pure Nash equilibria, Price of anarchy, Resource allocation, Sharing rules

Contents

1 Introduction

First, let us introduce the following situation that can be viewed as a resource sharing game with player-specific values.

Example 1.1 (Library book allocation)**.** A number of libraries, each serving a different demographic, are allocating books from a shared depot to the libraries' personal collections. To do this, each library chooses a set of books they like to add to their collection. Not every book in the depot is desired by every library, and each library is limited by the physical space they have for it.

The libraries assign each book a value based on the number of times they expect the book to be borrowed within a year. Because the libraries serve different demographics, this value is expected to be different between libraries. Furthermore, in an effort to promote reading, each library aims to have a personal collection of books that maximizes their total amount of expected borrowed books.

Each book is assumed to be an unique copy, hence multiple libraries cannot not have it in their possession simultaneously. Before the allocation is made, it has been decided by the libraries whether or not they are going to share books, in cases when multiple libraries choose the same book. When they do not want to share, books should only be chosen by a single library. Alternatively, when they do want to share, the libraries are in possession of the books for a fraction of the year and they expect then those books to get checked out with the same fraction of the assigned value.

In resource sharing games players compete for a set of resources, in case of Example 1.1 the resources are books. Each resource is assigned a strictly positive value. We discuss games where this value is player-specific, that is, the value is not necessary the same for all players. The values of the resources will contribute to the payoff of the players. In an effort to maximize their own payoff, players choose resources. The possible choices for each player is given by a strategy set. The strategy set consists of sets of resources a player can choose at once. In Example 1.1 it is natural to assume if the libraries have space for a set of book they should have space for any subset of them, likewise it is assumed that if a player can choose some set of resources they also can choose any subset of it. Resources chosen by a single player add the full value of the resource to that player's payoff. Otherwise, when multiple players choose the same resource, the value added to the payoffs is affected by a sharing rule.

We discuss three flavours of sharing rules; "*no-sharing*", when resources are chosen by multiple players then $-\infty$ is added to the players' payoffs; "*uniform sharing*", when resources are chosen by multiple players then a fraction of the resource's value is added to the players' payoffs equal for each player choosing it; and "*proportional sharing*", when resources are chosen by multiple players then a fraction of the resource's value is added to the players' payoffs proportional to the values of all players choosing it.

Each sharing rule forms a different class of resource sharing games with player specific values. For each class, we discuss the existence of pure Nash equilibria. That is when there exist an allocation of resources, corresponding to a collection of choices by the players (one for each player), where no player can get a higher payoff by choosing a different subset of resources when the other players keep their current choice. The collection of choices by the players is also called a strategy profile and the choice of subset resources is also called a strategy. While the existence of a mixed Nash equilibrium have been shown to always exist by Nash[2], they are not a topic of this thesis. Hence any reference hereafter to Nash equilibria will strictly refer to pure Nash equilibria. Since resource sharing games with player specific values are not guaranteed to have pure Nash equilibria with every sharing

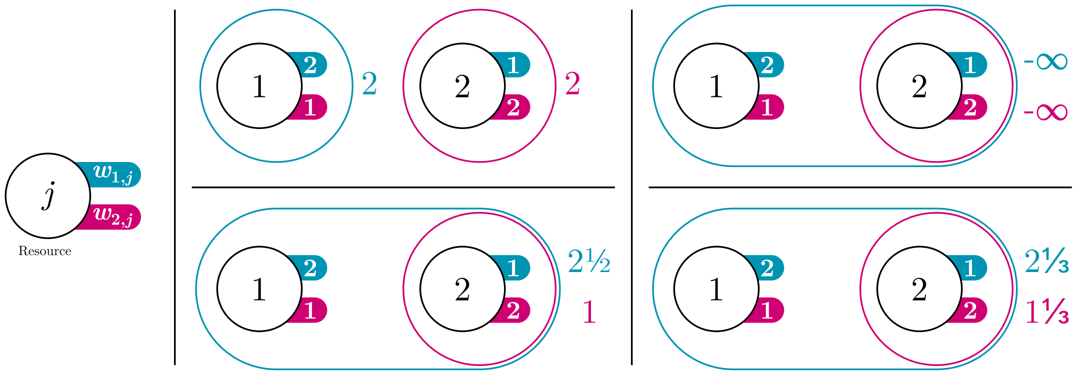


Figure 1: Illustration corresponding to Example 1.2. On the far left we depict how in general we will visualise a resource in figures; a circle with the resource's name and attached to it in colour the values of the various players. To the right of that are depicted, the optimal allocation $\mathbf{OPT} = (\{1\}, \{2\})$ in the top left, and allocation $\mathbf{S} = (\{1, 2\}, \{2\})$, where player 1 chooses both resources and player 2 chooses resource 2. This with no-sharing (top right), uniform sharing (bottom left) and proportional sharing (bottom right). The choices and values of player 1 are associated with the colour blue and strategies and values of player 2 are associated with the colour magenta.

rule. We then restrict ourselves to the class of instances that do have pure Nash equilibria. For these classes we measure the quality of the equilibria.

The quality is measured with the the price of anarchy as introduced by Koutsoupias and Papadimitriou[1]. For an instance of a resource sharing games the price of anarchy is calculated by dividing the payoff obtained from the allocation of the resources that maximizes the total payoff by the total payoff of the worst pure Nash equilibrium. Then the price of anarchy for the entire class, this equals the worst price of anarchy of any instance in the class. Hence this measures how much worse the total payoff can get when the players act selfishly. For example, perhaps in Example 1.1 there is an observing party, the Ministry of Education, who wants the total number of check-outs across all libraries to be maximized. Then the price of anarchy represents how much at its worst the total number of expected check-outs can get when the libraries act selfishly and only care about maximizing their personal number of check-outs as compared to the Ministry's goal where they act as a collective to maximize the total number of check-outs.

Let us illustrate resource sharing games with player specific value by considering the following example.

Example 1.2 (An illustrating example)**.** Let there be two players and two resources $J = \{1, 2\}$. Player 1 values resource 1 with value 2 and resource 2 with value 1, i.e $w_{1,1} = 2$ and $w_{1,2} = 1$. Player 2 values resource 1 with value 1 and resource 2 with value 2, i.e $w_{2,1} = 1$ and $w_{2,2} = 2$. Player 1 can choose at most both resources, i.e. the strategy set of player 1 is $\mathcal{S}_1 = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$, while player 2 can choose at most one resources, i.e. the strategy set of player 2 is $\mathcal{S}_2 = \{\emptyset, \{1\}, \{2\}\}$.

For the example with the no-sharing rule, the uniform sharing rule and the proportional sharing rule this results in the payoff matrices shown in Table 1, Table 2 and Table 3 respectively. Switching strategies can be viewed in the table as changing rows (but staying in the same column) for player 1 and changing columns (but staying in the same row) for

No-Sharing

		Player 2		
		$\emptyset$	$\{1\}$	$\{2\}$
Player 1	$\emptyset$	0, 0	0, 1	0, 2
	$\{1\}$	2, 0	$-\infty, -\infty$	$\underline{\mathbf{2, 2}}$
	$\{2\}$	1, 0	**1, 1**	$-\infty, -\infty$
	$\{1, 2\}$	**3, 0**	$-\infty, -\infty$	$-\infty, -\infty$
price of anarchy		2		

Table 1: Payoff matrixes of Example 1.2 with the no-sharing rule. The optimal allocation is underlined, pure Nash equilibria are made bold. The first value is the payoff for player 1 and the second value is the payoff of player 2. In the last row the price of anarchy of this particular instance is denoted.

player 2. In each table the optimum is underlined and the pure Nash equilibria are made bold. Furthermore in Figure 1 is shown how we will represent the resources in any figure and a representation of the optimum and a particular allocation with the different sharing rules.

Independent of the sharing rule the optimum of the resources is when player 1 chooses resource 1 and player 2 chooses resource 2, which has a total payoff of 4.

First, consider Example 1.2 with the no-sharing rule. See Table 1 for the payoff matrix. Whenever both players choose the same resource, the payoffs of both players is $-\infty$. Note in Table 1, when both players have a single resource or when player 1 has both resources and player 2 has none then both players can not improve their payoff by switching to a different set of resources. These are the pure Nash equilibria. In worst case, the players are in equilibrium when player 1 chooses resource 2 and player 2 chooses resource 1 with a total payoff of 2. Therefore the price of anarchy of this particular example equals $\frac{4}{2} = 2$.

Next, consider Example 1.2 with the uniform sharing rule. See Table 2 for the payoff matrix. Whenever both players choose the same resource they receive $\frac{1}{2}$ times their assigned value from that resource in their payoff. In case there were more than two players this value is dependent on the number of players choosing the same resource. Note in Table 2, that both players cannot do better by switching only when player 1 chooses both resources and player 2 chooses resource 2. Thus is is the only pure Nash equilibrium. Then player 1 receives a payoff of $2 + \frac{1}{2} \cdot 1$ and player 2 receives a payoff of $\frac{1}{2} \cdot 2$, the total payoff of this equilibrium is $3\frac{1}{2}$ and the price of anarchy for this particular example is $\frac{4}{3\frac{1}{2}} = 1\frac{1}{7}$.

Finally, consider Example 1.2 with the proportional sharing rule. See Table 3 for the payoff matrix. Since in Example 1.2 each resource is valued with 1 by some player and with 2 by the other player, whenever they both choose the same resource they receive respectively $\frac{1}{3}$rd of $1 (= \frac{1}{3})$ or $\frac{2}{3}$rds of 2 $(= 1\frac{1}{3})$. Note, that like with uniform sharing the only pure Nash equilibrium is when player 1 chooses both resources and player 2 chooses resource 2. Then player 1 receives a payoff of $2 + \frac{1}{3} \cdot 1$ and player 2 receives a payoff of $\frac{2}{3} \cdot 2$, the total payoff of this equilibrium is $3\frac{2}{3}$ and the price of anarchy for this particular example is $\frac{4}{3\frac{2}{3}} = 1\frac{1}{11}$.

Uniform Sharing

		Player 2		
		$\emptyset$	$\{1\}$	$\{2\}$
Player 1	$\emptyset$	0, 0	0, 1	0, 2
	$\{1\}$	2, 0	1, $\frac{1}{2}$	2, 2
	$\{2\}$	1, 0	1, 1	$\frac{1}{2}$, 1
	$\{1,2\}$	3, 0	2, $\frac{1}{2}$	$\mathbf{2\frac{1}{2}}$, 1
price of anarchy		$1\frac{1}{7}$		

Table 2: Payoff matrixes of Example 1.2 with the uniform sharing rule. The first value is the payoff for player 1 and the second value is the payoff of player 2. The optimal allocation is underlined, pure Nash equilibria are made bold. In the last row the price of anarchy of this particular instance is denoted.

Proportional Sharing

		Player 2		
		$\emptyset$	$\{1\}$	$\{2\}$
Player 1	$\emptyset$	0, 0	0, 1	0, 2
	$\{1\}$	2, 0	$1\frac{1}{3}$, $\frac{1}{3}$	2, 2
	$\{2\}$	1, 0	1, 1	$\frac{1}{3}$, $1\frac{1}{3}$
	$\{1,2\}$	3, 0	$2\frac{1}{3}$, $\frac{1}{3}$	$\mathbf{2\frac{1}{3}}$, $\mathbf{1\frac{1}{3}}$
price of anarchy		$1\frac{1}{11}$		

Table 3: Payoff matrixes of Example 1.2 with the proportional sharing rule. The first value is the payoff for player 1 and the second value is the payoff of player 2. The optimal allocation is underlined, pure Nash equilibria are made bold. In the last row the price of anarchy of this particular instance is denoted.

2 Related Work

The resource sharing games we consider have their roots in congestion and related games.

Congestion games are first introduced by Rosenthal[3]. In congestion games, players choose from a fixed number strategies. The strategies are subsets of a set of "primary factors", which can be interpreted as resources. In the version of Rosenthal, corresponding to their choice, the players have to pay the sum of costs of the resources in the strategy. The costs are are functions of the number of players choosing the resource and are not player-specific. Players try to minimize their cost. However this can also be changed to where the players maximize over the sum of payoffs, like in resource sharing games. For congestion games Rosenthal shows the existence of at least one pure Nash equilibrium. In his proof Rosenthal formulates the game as a set of equalities, and an objective function and then shows that any solution that satisfies the equalities and minimizes the objective function is a pure Nash equilibrium. The objective function in his integer program is also a potential function.

Potential functions and *potential games* are studied by Monderer and Shapley[4]. A potential function is a function over a strategy profile, such that when a player switches strategies the change in the payoff of that player is similar to the difference of the potential function over the old and new strategy profile. Every game where we can define a potential function is called a potential game. Monderer and Shapley show that potential games always have pure Nash equilibria, the proof follows from that it is shown that every potential game is a congestion game.

Congestion games with player-specific values also have been studied. When the payoff function are player-specific and decreasing with the number of players, Milchtaich[5] shows that there exist at least one Nash equilibrium when they are singleton, i.e strategies have only a single resource. In the non-singleton version when the strategy sets follow a certain combinatorial structure, namely when they are bases of a matroid. Matroids are not a topic in this thesis, so we will not discuss them in detail, but a well known example of a matroid are the forests in a finite undirected graph. Ackermann et al.[6] show that *player-specific matroid congestion games* have pure Nash equilibria. In general, however, Milchtaich[7] shows that it is not necessary true that pure Nash equilibria exist for any congestion game with player-specific values.

Dependent on the sharing rule resource sharing games with player-specific values extend (generalized) market sharing games and set packing games.

Market sharing games are introduced by Goemans et al.[8] and are a subclass of congestion games. Goemans et al. formulate it to investigate 3G wireless data networks. In market sharing games, each resource has a cost and a reward. Players have a budget and choose a subset of resources such that the total costs of resources fits within budget. When choosing a set of resources player gain a payoff, with an uniform sharing rule, this is the sum of the rewards of a resource divided by the number of players choosing it. The objective for the player is to maximize their payoff.

Brethouwer[9] generalises market sharing games with an uniform sharing rule by replacing budget and costs in favour of predefined downward-closed strategy sets. Brethouwer shows that *generalized market sharing games* are potential games, hence pure Nash equilibria exist and shows that the price of anarchy is $2 - \frac{1}{n}$ for a n player game.

Related to market sharing are *set packing games* introduced by De Jong and Uetz[10]. In set packing games, when resources are chosen by multiple players, they get a payoff of $-\infty$. This results in that the resources no longer are chosen by multiple players in Nash equilibria. Pure Nash equilibria always exist and it is shown that the class of set packing games has a price of anarchy of 2.

In comparison to both games, the extension by resource sharing games consist of considering them with player-specific values. In particular, resource sharing games with no-sharing rule extends set packing games and with uniform or proportional sharing extends generalized market sharing games.

A further generalization of market sharing games are *covering games* introduced by Gairing [11]. Strategy sets in covering games are no longer required to be downward closed. Gairing shows there exist some sharing rule such where the price of anarchy equals $\frac{e}{e-1} \approx 1.582$.

Market sharing games, set packing games and covering games are also *utility games*. Utility games are introduced by Vetta[12]. The payoff functions in utility games satisfy three conditions. The total payoff and the payoff of any player are in the same standard unit, the total payoff function is submodular and the payoff of the players are at least the change in total payoff. Utility games are shown to have a price of anarchy of at most 2. Resource sharing games without player-specific values are utility games, however this is no longer true for the games with player-specific values.

Another related problem are *budget games* introduced by Drees et al.[13]. In budget games each resource has a budget and players have a demand on that budget. When a player chooses a resource they gain some of the budget proportional to their demand and the total demand on the resource. A price of anarchy is shown of at most 2

When considering resource sharing games with player-specific values and proportional sharing, they are also an adaptation of budget games, where resources have a player-specific budget equal that is equal to the players' demand.

3 Preliminaries and Notation

We here fix some notation and the basic definitions.

Let $N := \{1, \ldots, n\}$ be the *set of n players*, let J be the *set of resources* and let $\mathbb{R}_{>0}$ be the positive real numbers. In the eyes of player $i \in N$ resource $j \in J$ has a *value* $w_{i,j} \in \mathbb{R}_{>0}$ that is *player-specific*, that means $w_{i,j}$ is not necessary equal to $w_{k,j}$ for all players $k \in N$ where $i \neq k$.

Simultaneously each player $i \in N$ chooses a subset of the resources in J. Each player i has a *strategy set* $\mathcal{S}_i \subseteq 2^J$ that determines between which subsets the player can choose. Each subset of resources in the strategy set is also referred to as a *strategy*. The strategy set is assumed to be *downward closed*, that is if $S_i \in \mathcal{S}_i$ then for all $T_i \subseteq S_i$ it implies that $T_i \in \mathcal{S}_i$. Or in other words if a player can choose some set of resources they can choose any subset of that set too.

A feasible allocation of the resource is a *strategy profile* $\mathbf{S} := (S_1, S_2, \ldots, S_n)$, such that $S_i \in \mathcal{S}_i$ for $i \in N$. Given a strategy profile $\mathbf{S}$ define $S_{-i} := (S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_n)$ as the *choices of all players except player i*. Sometimes we will write $\mathbf{S}$ as (S_i, S_{-i}).

In strategy profile $\mathbf{S}$, resources may appear in the chosen strategies of multiple players, we say these resources are *shared* between the relevant players. When resource j is shared, the payoff received of that resource is affected. We introduce a *sharing rule* $f_j(i, \mathbf{S})$. We consider three types of sharing rules, *no-sharing*, *uniform sharing* and *proportional sharing*. The exact definitions of the sharing rules are given respectively at the start of Section 5, Section 6 and Section 7. With notable exceptions like no-sharing, we generally define any sharing rule such that for all $\sum_{i \in N\colon j \in S_i} f_j(i, \mathbf{S}) = 1$.

Each player aims to maximize their payoff. The *payoff of player i* is defined as

$$w_i(S_i, S_{-i}) = \sum_{j \in S_i} f_j(i, \mathbf{S}) \cdot w_{i,j}. \tag{1}$$

We are interested in pure Nash equilibria. A strategy profile $\mathbf{S}$ is a *pure Nash equilibrium* if for all players $i \in N$ it is true that

$$w_i(S_i, S_{-i}) \geq w_i(T_i, S_{-i}) \qquad \text{for all} \qquad T_i \in \mathcal{S}_i. \tag{2}$$

Additionally, let for a strategy profile $\mathbf{S}$ the *total payoff* be defined as

$$w(\mathbf{S}) = \sum_{i=1}^{n} w_i(S_i, S_{-i}). \tag{3}$$

Furthermore, for a strategy profile $\mathbf{S}$ and $J' \subseteq J$ let the *total payoff of all resources in subset J'* be

$$w(\mathbf{S} \mid J') = \sum_{i=1}^{n} w_i(S_i \cap J', S_{-i}). \tag{4}$$

Let $\mathcal{I}$ be a class games, in our problem $\mathcal{I}$ is the class of resource sharing games with player specific values with the respective sharing rule. Let $\mathrm{NE}(I)$ denotes the set of all pure Nash equilibria of instance $I \in \mathcal{I}$. When a strategy profile maximizes the total payoff, it is an optimum and we write $\mathbf{OPT}_I$ for an optimal allocation of the resources for instance I. The *price of anarchy*[1] PoA for a class of games $\mathcal{I}$ is the ratio

$$\mathrm{PoA} = \sup_{I \in \mathcal{I}} \sup_{\mathbf{S} \in \mathrm{NE}(I)} \frac{w(\mathbf{OPT}_I)}{w(\mathbf{S})}. \tag{5}$$

Which measures how much worse at most in the class of games the total payoff gets of in case of the worst Nash equilibrium as compared to the optimal allocation.

3.1 Generalized Mediant Inequality

The mediant and mediant inequality has been studied in classical Greece and has reappeared in works of Chuquet, Wallis, Taylor and Chauchy [14, 15]. In Section 5, we will make use of the unweighted generalised version.

Let $F = \{\frac{a_i}{b_i} : i \in \{1, 2, \ldots, n\}\}$ be a (finite) set of ratios, where $a_i, b_i \in \mathbb{R}_{>0}$.

Definition 3.1 (Generalized mediant)**.** The *generalized mediant* of F is defined as

$$\sigma(F) = \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i}$$

In Appendix B, we proof the generalized mediant inequality.

Lemma 3.1 (Generalized mediant inequality)**.** For a (finite) set F or ratios it holds that $\sup_{i \in \{1,\ldots,n\}} \frac{a_i}{b_i} \geq \sigma(F) \geq \inf_{i \in \{1,\ldots,n\}} \frac{a_i}{b_i}$.

3.2 Potential Function

A game where we can define a potential function is a *potential game*. They are a useful tool to say something about the existence of pure Nash equilibria. There are various types of potential functions. We will use the exact potential function in Section 7.

Definition 3.2 (Exact potential function)**.** A function $\pi \colon \mathcal{S}_1 \times \ldots \times \mathcal{S}_n \to \mathbb{R}$ is an *exact potential function* if for all players $i \in N$ and $S_i, T_i \in \mathcal{S}_i$ it holds that

$$\pi(S_i, S_{-i}) - \pi(T_i, S_{-i}) = w_i(S_i, S_{-i}) - w_i(T_i, S_{-i}).$$

It is shown by Monderer and Shapley[4] is that every potential game has at least one pure Nash equilibrium.

Lemma 3.2. Every potential game has at least one pure Nash equilibrium.

3.3 Sedrakyan's Inequality

In 1997 Nairi Sedrakyan published an article called "On the applications of a useful inequality,"[16, 17]. Sedrakyan proves an inequality what we will refer to as Sedrakyan's inequality. We will make use of this inequality in Section 7.

Lemma 3.3 (Sedrakyan's inequality)**.** For any real numbers $a_1, a_2, \ldots, a_n$ and any positive real numbers $b_1, b_2, \ldots, b_n$, it holds that

$$\sum_{i=1}^{n} \frac{a_i^2}{b_i} \geq \frac{(\sum_{i=1}^{n} a_i)^2}{\sum_{i=1}^{n} b_i}.$$

4 Contribution

Let θ be the maximum ratio of values of assigned to any resource. We show the price of anarchy for (n player) set packing games with player specific values is $\theta + 1$.

While pure Nash equilibria are not guaranteed to exist in general for games with uniform sharing, for the subclass of (n player) games that have pure Nash equilibria we show they have a price of anarchy of n.

We show that pure Nash equilibria exist for 2 player games with proportional sharing, but that this is not guaranteed for more than 2 players. We show that 2 player games have a price of anarchy of φ, the golden ratio. For the class of n player games with pure Nash equilibria, we conjecture that the price of anarchy is $\frac{1+\sqrt{4(n-1)+1}}{2}$ and we show a matching lower bound.

5 No-Sharing: Set Packing Games with Player-Specific Values

We start with extending set packing games discussed by De Jong and Uetz[10] to set packing games with player-specific values. It relates to resource allocation problems where for instance the resources are indivisible and consumable (that is, the resource disappears after a player receives it). First follows a definition.

Definition 5.1. *Set packing games with player-specific values* (hereafter also referred to as set packing games) is a class of resource sharing games with player-specific values, where the *no-sharing rule* is defined as

$$f_j(i, \mathbf{S}) = \begin{cases} 1 & \text{if there is no } k \in N \text{ such that } i \neq k \text{ and } j \in S_k \\ -\infty & \text{otherwise} \end{cases}$$

Effectively this means that any player that shares at least one resource, their payoff will be $-\infty$. In the following section we show that pure Nash equilibria always exist. By setting the payoffs received by the players to $-\infty$ when at least one resource is shared, we rule out sharing in a Nash equilibrium as having no resources (which has a payoff of 0) is always preferred by any player over sharing. Of course, while this perfectly models what we want, it is a very artificial result, since by assumption, the players choose their strategies simultaneously. Therefore they would have no way of knowing what resources the other players choose.

5.1 Existence of pure Nash equilibria

Hereafter we show that the existence of Nash equilibria is guaranteed by showing that the optimum is a Nash equilibrium.

Theorem 5.1. In set packing games with player-specific values the optimum is a Nash equilibrium

Proof. Proof by Contradiction
Take any instance I. Let $\mathbf{OPT}$ be the optimum and let OPT_i be the strategy chosen by player $i \in N$. Assume for some player i, there exists some strategy $S_i \in \mathcal{S}_i$, such that

$$w_i(S_i, \text{OPT}_{-i}) > w_i(\text{OPT}_i, \text{OPT}_{-i})$$

Note that, since $w_i(\text{OPT}_i, \text{OPT}_{-i}) \geq -\infty$, it must holds that $w_i(S_i, \text{OPT}_{-i}) > -\infty$. Hence S_i cannot contain any resources also in some strategy in OPT_{-i}. Therefore, the payoffs of the other players can only be made better by player i switching to S_i. Hence, if the inequality holds, the new total payoff equals

$$\begin{aligned} w(S_i, \text{OPT}_{-i}) &\geq w(\mathbf{OPT}) - w_i(\text{OPT}_i, \text{OPT}_{-i}) + w_i(S_i, \text{OPT}_{-i}) \\ &> w(\mathbf{OPT}) \end{aligned}$$

Which implies that $\mathbf{OPT}$ is not the optimum, which yields a contradiction and $\mathbf{OPT}$ has to be a Nash equilibrium. □

5.2 Price of Anarchy

Let $\delta_i(J) := \bigcup_{S_i \in \mathcal{S}_i} S_i$ be all resources that can be chosen by player i. We define θ as the largest ratio of values.

Definition 5.2 (Largest ratio of values)**.** The *largest ratio of values* of any resource $j \in J$ is defined as

$$\theta = \sup_{i \in N, k \in N, j \in \delta_i(J) \cap \delta_k(J)} \frac{w_{i,j}}{w_{k,j}}.$$

We will show that the price of anarchy PoA $= \theta + 1$ for set packing games with player-specific values. Note that $\theta = 1$, if the values are not player-specific. Then the price of anarchy PoA $= 2$. This is the same result as by De Jong and Uetz [10]. Furthermore if there exist a $j \in J$ such that $w_{i,j} > w_{k,j}$ for some $i, k \in N$, then $\theta > 1$ and we show that PoA > 2.

First consider the following lemma, which shows a relation between θ and and any two pure Nash equilibria. Let $S := \bigcup_{j \in N} S_i$ be the set of all resources chosen in any given strategy profile $\mathbf{S}$. Recall that with strategy profiles $\mathbf{S}$ and $\mathbf{T}$, $w(\mathbf{T} \mid S)$ is the total payoff of the resources chosen by at least one player in $\mathbf{S}$ when allocated as in $\mathbf{T}$.

Lemma 5.2. It holds for set packing games with player-specific values with pure Nash equilibria $\mathbf{T}$ and $\mathbf{S}$, that $\theta \geq \frac{w(\mathbf{T}|S)}{w(\mathbf{S})}$.

Proof. Let T_i and S_i be the resources chosen by player i in $\mathbf{T}$ and $\mathbf{S}$ respectively and let $T = \bigcup_{i \in N} T_i$ and $S = \bigcup_{i \in N} S_i$ be the set of all resources chosen in strategy set $\mathbf{T}$ and $\mathbf{S}$ respectively. Furthermore let $F = \{\frac{w(\mathbf{T}|\{j\})}{w(\mathbf{S}|\{j\})} : j \in S\}$.

- Since for all $j \in S \setminus T$, i.e, resources only chosen by some player in strategy profile $\mathbf{S}$, no resource are chosen by more than one player in a pure Nash equilibrium, then $\frac{w(\mathbf{T}|\{j\})}{w(\mathbf{S}|\{j\})} = 0$
- Since for all $j \in T \cap S$, where $j \in T_i$ and $j \in S_i$, i.e. resources j chosen by player i in both strategy profiles, no resource are chosen by more than one player in a pure Nash equilibrium, then $\frac{w(\mathbf{T}|\{j\})}{w(\mathbf{S}|\{j\})} = 1$.
- Since for all $j \in T \cap S$, where $j \in T_i$ and $j \in S_k$ where $i \neq k$, i.e. resources j chosen different players in $\mathbf{S}$ and $\mathbf{T}$, no resources are chosen by more than one player in a pure Nash equilibrium, then $\frac{w(\mathbf{T}|\{j\})}{w(\mathbf{S}|\{j\})} = \frac{w_{ij}}{w_{kj}}$.

Hence $\theta \geq \frac{w(\mathbf{T}|\{j\})}{w(\mathbf{S}|\{j\})}$ for all $j \in S$. We have that $\frac{\sum_{j \in S} w(\mathbf{T}|\{j\})}{\sum_{j \in S} w(\mathbf{S}|\{j\})} = \frac{w(\mathbf{T}|S)}{w(\mathbf{S})}$ is the generalized mediant of F, hence by the generalized mediant inequality (Lemma 3.1) we can conclude that

$$\theta \geq \frac{w(\mathbf{T} \mid S)}{w(\mathbf{S})}.$$

□

Now we show the upper bound for the price of Anarchy, which is proved in a similar fashion as in [10].

Theorem 5.3. The price of anarchy PoA $\leq \theta+1$ for set packing games with player-specific values.

Proof. Take any instance I with optimal solution $\mathbf{OPT}$ and Nash equilibrium $\mathbf{S}$, and let S_i and OPT_i, for $i \in N$, be the resources selected by player i in $\mathbf{S}$ and $\mathbf{OPT}$, respectively.

Let $S = \bigcup_{i \in N} S_i$ be the set of all resources chosen in the Nash equilibrium. Since all resources in $\overline{S} = J \setminus S$ are available, and all resources in OPT_i are feasible for player i and OPT_i is downward closed, by definition of Nash equilibrium (Equation 2) we have for all players i that

$$w_i(S_i, S_{-i}) \geq w_i(\mathrm{OPT}_i \cap \overline{S}, S_{-i}) \tag{6}$$

Since $\overline{S} \cap S_i = \emptyset$ and as a result of the no-sharing rule it follows that $\mathrm{OPT}_i \cap \mathrm{OPT}_k = \emptyset$ for all $k \neq i$, and therefore

$$w_i(\mathrm{OPT}_i \cap \overline{S}, S_{-i}) = w_i(\mathrm{OPT}_i \cap \overline{S}, \mathrm{OPT}_{-i}). \tag{7}$$

Furthermore because of Lemma 5.2, we have that $\theta \geq \frac{w(\mathbf{OPT}|S)}{w(\mathbf{S})}$ and therefore

$$\theta w(\mathbf{S}) \geq w(\mathbf{OPT} \mid S). \tag{8}$$

Now we get by the fact that for any $i \neq k$ also $S_i \cap S_k = \emptyset$ and from (8) it follows that

$$\begin{aligned}(\theta+1)w(\mathbf{S}) &\geq w(\mathbf{OPT} \mid S) + w(\mathbf{S}) \\ &= w(\mathbf{OPT} \mid S) + \sum_{i=1}^{n} w_i(S_i, S_{-i})\end{aligned} \tag{9}$$

By (6) and (7),

$$\begin{aligned}w(\mathbf{OPT} \mid S) + \sum_{i=1}^{n} w_i(S_i, S_{-i}) &\geq w(\mathbf{OPT} \mid S) + \sum_{i=1}^{n} w_i(\mathrm{OPT}_i \cap \overline{S}, \mathrm{OPT}_{-i}) \\ &= w(\mathbf{OPT} \mid S) + w(\mathbf{OPT} \mid \overline{S}) \\ &= w(\mathbf{OPT})\end{aligned} \tag{10}$$

Hence by (9) and (10) can we conclude that $(\theta+1)w(\mathbf{S}) \geq w(\mathbf{OPT})$, therefore PoA $\leq \theta + 1$. □

For the lower bound consider the following example.

Example 5.1. We have n players and two disjoint sets of resources P and Q with $|P| = |Q| = n-1$. Player 1 can choose at most $n-1$ resources from $P \cup Q$, for player 1 the resources have value 1, i.e. $w_{1j} = 1$ for all $j \in P \cup Q$. The other $n-1$ players are restricted to choosing at most a single resource from Q, for the other players the resources have value $c \geq 1$, an arbitrary constant, i.e. $w_{ij} = c$ for $i \in \{2, \ldots, n\}$ and $j \in Q$.

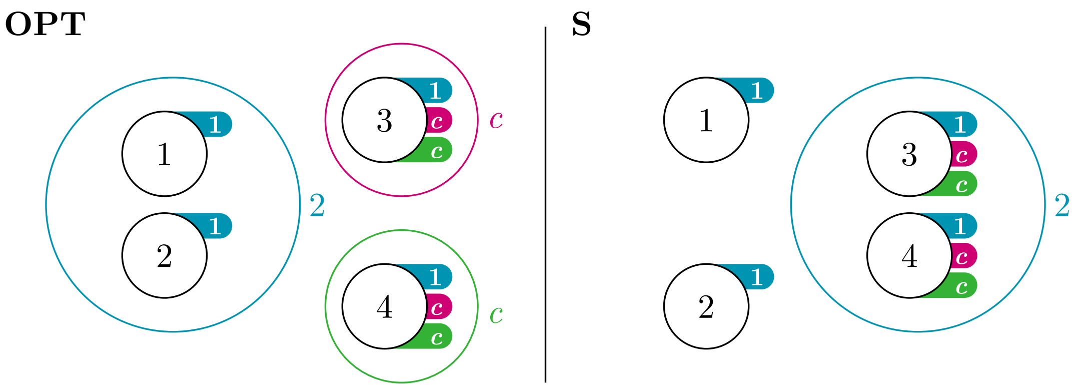


FIGURE 2: Illustration corresponding to Example 5.1 with $n = 3$ players, $P = \{1, 2\}$, $Q = \{3, 4\}$ and $J = P \cup Q$. Depicted are the optimum **OPT** (left) and the worst Nash equilibrium **S** (right). Strategies, values and payoffs of player 1 is associated with the colour blue, of player 2 with the colour magenta and of player 3 with the colour green.

Theorem 5.4. The price of anarchy PoA $\geq \theta+1$ for set packing games with player-specific values.

Proof. Consider Example 5.1. In the optimum all resources of P are chosen by player 1 and each other player choses a single different resource from Q. In this instance $w(\mathbf{OPT}) = n-1+c(n-1)$. Consider the solution where all resources of Q are chosen by player 1 and no resources are chosen by the other players. Then player 1 receives a payoff of $n-1$, the maximum that player can obtain. Given that all resources from Q are chosen by player 1, players $2, \ldots, n$ cannot do better than a value 0, as none of the resources from P are feasible for these players. Hence for this solution, the Nash condition (Equation 2) holds for all players. Since $\theta = c$, we conclude that PoA $\geq w(\mathbf{OPT})/w(\mathbf{S}) = (n-1+c(n-1))/(n-1) = c+1 = \theta+1$. □

Using previous two theorems we can conclude that that PoA $= \theta + 1$ for the class of set packing games with player-specific values.

Corollary 5.5. The price of anarchy PoA $= \theta + 1$ for set packing games with player-specific values.

6 Uniform Sharing

In case the resources are divisible or are not consumable, we can wonder when multiple players want a resource, giving them a fraction of the resource or giving them access to the resource for a fraction of the time improves the total payoff. First, we consider resource sharing games, whenever a resource is used by multiple players, each player receives an equal amount or has the resource for a equal amount of time.

This extends generalised market sharing games a studied by Brethouwer[9] to uniform resource sharing games with player specific-values.

Definition 6.1. *Uniform resource sharing games with player specific-values* (hereafter also referred to as uniform resource sharing games) is a class of resource sharing game with player specific-values, where the *uniform sharing rule* is the defined as

$$f_j(i, \mathbf{S}) = \frac{1}{1 + \sum_{k \in N \setminus \{i\} \colon j \in S_k} 1}.$$

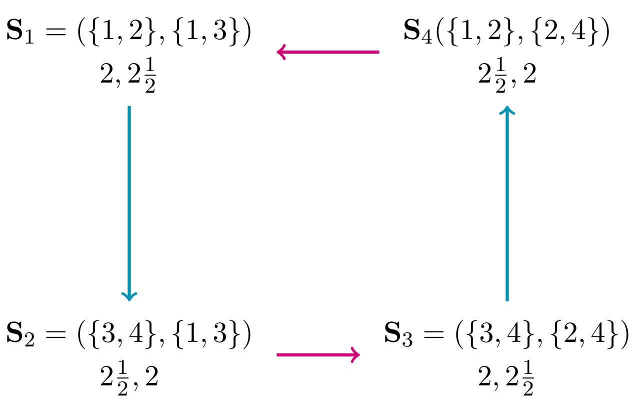
6.1 Non-Existence of pure Nash equilibria

Unlike set packing games, Nash equilibria are not guaranteed to exist for uniform resource sharing games. The result follows from [7]. Alternatively consider the following example.

Example 6.1. We have two players and four resources $J = \{1, 2, 3, 4\}$. The strategy set of player 1 is $\mathcal{S}_1 = \{\{1, 2\}, \{3, 4\}\}$ and the values of the resources are respectively $w_{1,1} = 2$, $w_{1,2} = 1$, $w_{1,3} = 1$ and $w_{1,4} = 2$. The strategy set of player 2 is $\mathcal{S}_2 = \{\{1, 3\}, \{2, 4\}\}$ and the values of the resources are respectively $w_{2,1} = 1$, $w_{2,2} = 2$, $w_{2,3} = 2$ and $w_{2,4} = 1$.

For our convenience we ignored the downward closed requirement for the strategy sets in the above example. However note that when all player specific-values are strictly positive then for all $S_i \in \mathcal{S}_i$, if we introduce all strict subsets $T_i \subset S_i$ to $\mathcal{S}_i$, it holds that $w_i(S_i, S_{-i}) = w_i(T_i, S_{-i}) + w_i(S_i \setminus T_i, S_{-i})$, which implies that $w_i(S_i, S_{-i}) > w_i(T_i, S_{-i})$. Therefore strict subsets T_i cannot be chosen in Nash equilibria. While the same is not true in general for the optimum (see for instance Example 1.2), in this example we are only interested in the existence of pure Nash equilibria.

Consider the following graph. Each vertex represents a strategy profile with corresponding payoffs. The edges point to strategy profiles one player improves to by switching strategies; the blue corresponding to player 1 and the magenta to player 2.



We can clearly see that there is a cycle between all states. Therefore, no Nash equilibria exist.

6.2 Price of Anarchy

From now on we restrict ourselves to the subclass of uniform resource sharing games that have Nash equilibria.

First, we show a upper bound.

Theorem 6.1. The price of anarchy $\text{PoA} \leq n$ for resource uniform sharing game with player specific-values.

Proof. Take any instance with optimal solution $\mathbf{OPT}$ and Nash equilibrium $\mathbf{S}$, and let S_i and OPT_i, $i \in N$, be the resources selected by player i in $\mathbf{S}$ and $\mathbf{OPT}$, respectively.

Since in a Nash equilibrium, no player can improve, but OPT_i is still available to player i. In worst case player i gets at least $\frac{1}{n} w_{i,j}$ for all resources $j \in \text{OPT}_i$ (and at most full value). Therefore,

$$\begin{aligned} w(\mathbf{S}) &= \sum_{i=1}^{n} w_i(S_i, S_{-i}) \\ &\geq \sum_{i=1}^{n} w_i(\text{OPT}_i, S_{-i}) \\ &\geq \frac{1}{n} \sum_{i \in N} \sum_{j \in \text{OPT}_i} w_{i,j} \\ &= \frac{1}{n} w(\mathbf{OPT}) \end{aligned}$$

Here the first inequality follows from the Nash condition. Thus we can conclude that $\text{PoA} \leq n$. □

Next we show, maybe surprisingly, a matching lower bound exists. First, consider the following example.

Example 6.2. Let $J = \{1, 2, \ldots, n$. Player $i \in N$ can either choose resource i or (a subset of) the set of all resources except resource i. Furthermore player i values the ith resource with a value of n and the other resources with a value of 1, i.e. $w_{i,i} = n$ and $w_{i,j} = 1$ for $j \in \{1, \ldots, i-1, i+1, \ldots, n\}$.

Theorem 6.2. The price of anarchy $\text{PoA} \geq n$ for resource uniform sharing game with player specific-values.

Proof. Consider Example 6.2. In the optimum each player $i \in N$ chooses resource i for all $i \in N$ and therefore $w(\mathbf{OPT}) = n^2$. Consider the Nash equilibrium where player i chooses all resources except resource i for all $i \in \{1, \ldots, n\}$. Since each player now chooses $n-1$ resources and each resource is chosen by $n-1$ players, player i receives payoff $(n-1)\frac{1}{n-1} = 1$. To see that this a Nash equilibrium, observe that, when switching to only resource i, the resource now is chosen by n players and player i still receive a payoff of 1. Furthermore switching to any subset of $\{1, \ldots, i-1, i+1, \ldots, n\}$ leads to a payoff of less then 1. Hence the Nash condition (Equation 2) holds for all players. We can conclude that $\text{PoA} \geq w(\mathbf{OPT})/w(\mathbf{S}) = n^2/n = n$. □

From Theorem 6.1 and Theorem 6.2 we can conclude that $\text{PoA} = n$ for uniform resource sharing games.

Corollary 6.3. The price of anarchy $\text{PoA} = n$ for resource uniform sharing game with player specific-values.

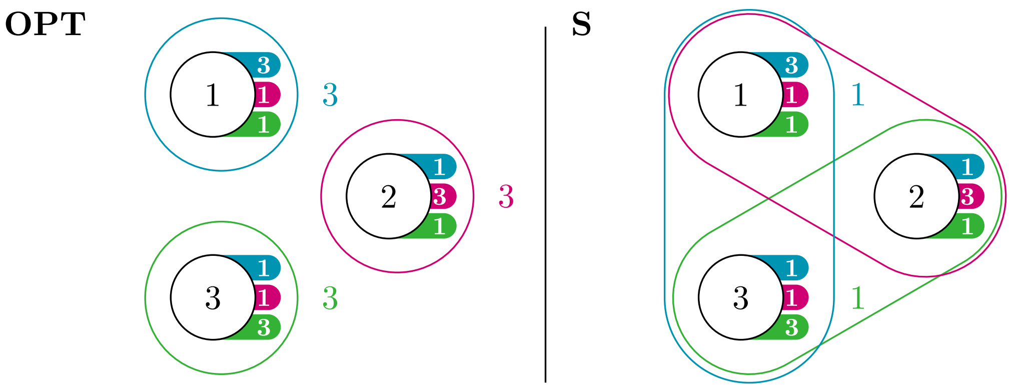


FIGURE 3: Illustration corresponding to Example 6.2. Depicted are the optimum (left) and the Nash equilibrium (right). Strategies, values and payoffs of player 1 is associated with the colour blue, of player 2 with the colour magenta and of player 3 with the colour green.

7 Proportional Sharing

While uniform sharing seems like a fair rule, it might not align with a secondary goal. For instance consider Example 1.1, where players are libraries choosing books, and the values are expected number of times the book is borrowed per year. Next to maximizing their own number of borrowed books in a year, maybe informally they discussed to also still try to make in general the total number of borrowed books as high as possible. Perhaps then, it is more logical allowing libraries where the book is going to be borrowed more to hold on it for a longer period of time.

We therefore introduce proportional resource sharing games with player specific values.

Definition 7.1. *Proportional resource sharing games with player specific values* (hereafter also referred to a proportional resource sharing games) is a class of resource sharing games with player specific value, where the *proportional sharing rule* is defined as

$$f_j(i, \mathbf{S}) = \frac{w_{i,j}}{w_{i,j} + \sum_{k \in N \setminus \{i\} \colon j \in S_k} w_{k,j}}$$

Note that when for some $w_j \in \mathbb{R}_{\geq 0}$, when $w_{i,j} = w_j$ for all $i \in N$, then

$$\frac{w_{i,j}}{w_{i,j} + \sum_{k \in N \setminus \{i\} \colon j \in S_k} w_{k,j}} = \frac{1}{1 + \sum_{k \in N \setminus \{i\} \colon j \in S_k} 1}.$$

Hence this also extends the generalized market sharing games as discussed by Brethouwer[9].

Surprisingly 2-player proportional resource sharing games are potential games as we will show below. However this is no longer true that for more than two players pure Nash equilibria are guaranteed. Hence we will cover them separately.

7.1 Two-Player Proportional Resource Sharing Games

7.1.1 Existence of pure Nash equilibria

Here we show that 2-player proportional resource sharing games with player specific values are potential games and therefore pure Nash equilibria always exist.

Theorem 7.1. 2-player proportional resource sharing games with player specific values are potential games.

Proof. The idea of a potential game is the existence of a potential function. Recall that a function $\pi(\mathbf{S})$ is an exact potential if it holds that

$$\pi(S_i, S_{-i}) - \pi(T_i, S_{-i}) = w_i(S_i, S_{-i}) - w_i(T_i, S_{-i})$$

We now define such a function and show it is an exact potential function. A potential function for any 2-player proportional resource sharing games can be defined as

$$\pi_j(\mathbf{S}) = \begin{cases} w_{i,j} & \text{if } j \in S_i \text{ and } j \notin S_{-i} \text{ for some } i \in \{1,2\} \\ \frac{w_{i,j}^2 + w_{i,j}w_{-i,j} + w_{-i,j}^2}{w_{i,j} + w_{-i,j}} & \text{if } j \in S_1 \text{ and } j \in S_2 \\ 0 & \text{otherwise} \end{cases}$$

$$\pi(\mathbf{S}) = \sum_{j \in J} \pi_j(\mathbf{S}).$$

Assume player i switches from strategy S_i to strategy T_i.

Since $\pi_j(T_i, S_{-i}) - \pi_j(S_i, S_{-i})$ is only non-zero when after player i switches strategies, the number of players choosing resource j changes. That is, when $j \in T_i \setminus S_i$ or when $j \in S_i \setminus T_i$. It follows that,

$$\begin{aligned} \pi(T_i, S_{-i}) - \pi(S_i, S_{-i}) = & \sum_{j \in T_i \setminus S_i} \left(\pi_j(T_i, S_{-i}) - \pi_j(S_i, S_{-i})\right) \\ & + \sum_{j \in S_i \setminus T_i} \left(\pi_j(T_i, S_{-i}) - \pi_j(S_i, S_{-i})\right) \end{aligned} \tag{11}$$

We can split $T_i \setminus S_i$ by whether or not $j \in S_{-i}$. Hence,

$$\begin{aligned} \sum_{j \in T_i \setminus S_i} \left(\pi_j(T_i, S_{-i}) - \pi_j(S_i, S_{-i})\right) = & \sum_{j \in (T_i \setminus S_i) \cap S_{-i}} \left(\pi_j(T_i, S_{-i}) - \pi_j(S_i, S_{-i})\right) \\ & + \sum_{j \in (T_i \setminus S_i) \cap \overline{S}_{-i}} \left(\pi_j(T_i, S_{-i}) - \pi_j(S_i, S_{-i})\right) \end{aligned} \tag{12}$$

When $j \in (T_i \setminus S_i) \cap S_{-i}$, then $\pi_j(T_i, S_{-i}) = \frac{w_{i,j}^2 + w_{i,j}w_{-i,j} + w_{-i,j}^2}{w_{i,j} + w_{-i,j}}$ and $\pi_j(S_i, S_{-i}) = w_{-i,j}$. Therefore,

$$\pi_j(T_i, S_{-i}) - \pi_j(S_i, S_{-i}) = \frac{w_{i,j}^2 + w_{i,j}w_{-i,j} + w_{-i,j}^2}{w_{i,j} + w_{-i,j}} - w_{-i,j} = \frac{w_{i,j}^2}{w_{i,j} + w_{-i,j}}. \tag{13}$$

When $j \in (T_i \setminus S_i) \cap \overline{S}_{-i}$, then $\pi_j(T_i, S_{-i}) = w_{i,j}$ and $\pi_j(S_i, S_{-i}) = 0$. Hence

$$\pi_j(T_i, S_{-i}) - \pi_j(S_i, S_{-i}) = w_{i,j} \tag{14}$$

Thus by (12), (13) and (14),

$$\sum_{j\in T_i\setminus S_i}\left(\pi_j(T_i,S_{-i})-\pi_j(S_i,S_{-i})\right)=\sum_{j\in(T_i\setminus S_i)\cap S_{-i}}\frac{w_{i,j}^2}{w_{i,j}+w_{-i,j}}+\sum_{j\in(T_i\setminus S_i)\cap\overline{S}_{-i}}w_{i,j}. \tag{15}$$

Furthermore since $w_i(T_i,S_{-i})=w_i(T_i\setminus S_i,S_{-i})+w_i(T_i\cap S_{-i}\setminus S_i,S_{-i})$ and

$$\begin{aligned}w_i(T_i\setminus S_i,S_{-i})&=\sum_{j\in(T_i\setminus S_i)\cap S_{-i}}\frac{w_{i,j}^2}{w_{i,j}+w_{-i,j}}+\sum_{j\in(T_i\setminus S_i)\cap\overline{S}_{-i}}w_{i,j}\\&=\sum_{j\in T_i\setminus S_i}\left(\pi_j(T_i,S_{-i})-\pi_j(S_i,S_{-i})\right),\end{aligned}$$

where the second equality follows from (15), it follows that,

$$\sum_{j\in T_i\setminus S_i}\left(\pi_j(T_i,S_{-i})-\pi_j(S_i,S_{-i})\right)=w_1(T_i,S_{-i})-w_1(S_i\cap T_i,S_{-i}). \tag{16}$$

Likewise, we can split $S_i\setminus T_i$ by whether or not $j\in S_{-i}$. Hence,

$$\begin{aligned}\sum_{j\in S_i\setminus T_i}\left(\pi_j(T_i,S_{-i})-\pi_j(S_i,S_{-i})\right)=&\sum_{j\in(S_i\setminus T_i)\cap S_{-i}}\left(\pi_j(T_i,S_{-i})-\pi_j(S_i,S_{-i})\right)\\&+\sum_{j\in(S_i\setminus T_i)\cap\overline{S}_{-i}}\left(\pi_j(T_i,S_{-i})-\pi_j(S_i,S_{-i})\right)\end{aligned} \tag{17}$$

When $j\in(S_i\setminus T_i)\cap S_{-i}$, then $\pi_j(T_i,S_{-i})=w_{-i,j}$ and $\pi_j(S_i,S_{-i})=\frac{w_{i,j}^2+w_{i,j}w_{-i,j}+w_{-i,j}^2}{w_{i,j}+w_{-i,j}}$. Therfore,

$$\pi_j(T_i,S_{-i})-\pi_j(S_i,S_{-i})=w_{-i,j}-\frac{w_{i,j}^2+w_{i,j}w_{-i,j}+w_{-i,j}^2}{w_{i,j}+w_{-i,j}}=-\frac{w_{i,j}^2}{w_{i,j}+w_{-i,j}} \tag{18}$$

When $j\in(S_i\setminus T_i)\cap\overline{S}_{-i}$, then $\pi_j(T_i,S_{-i})=0$ and $\pi_j(S_i,S_{-i})=w_{i,j}$. Hence

$$\pi_j(T_i,S_{-i})-\pi_j(S_i,S_{-i})=-w_{i,j} \tag{19}$$

Thus by (17), (18) and (19), it follows that

$$\sum_{j\in T_i\setminus S_i}\left(\pi_j(T_i,S_{-i})-\pi_j(S_i,S_{-i})\right)=-\sum_{j\in(T_i\setminus S_i)\cap S_{-i}}\frac{w_{i,j}^2}{w_{i,j}+w_{-i,j}}-\sum_{j\in(T_i\setminus S_i)\cap\overline{S}_{-i}}w_{i,j}. \tag{20}$$

Furthermore since $w_i(S_i,S_{-i})=w_i(S_i\setminus T_i,S_{-i})+w_i(T_i\cap S_{-i}\setminus S_i,S_{-i})$ and

$$\begin{aligned}-w_i(S_i\setminus T_i,S_{-i})&=-\sum_{j\in(S_i\setminus T_i)\cap S_{-i}}\frac{w_{i,j}^2}{w_{i,j}+w_{-i,j}}-\sum_{j\in(S_i\setminus T_i)\cap\overline{S}_{-i}}w_{i,j}\\&=\sum_{j\in S_i\setminus T_i}\left(\pi_j(T_i,S_{-i})-\pi_j(S_i,S_{-i})\right),\end{aligned}$$

where the second equality follows from (20), it follows that

$$\sum_{j\in S_i\setminus T_i}\left(\pi_j(T_i,S_{-i})-\pi_j(S_i,S_{-i})\right)=-w_1(S_i,S_{-i})+w_1(S_i\cap T_i,S_{-i}). \tag{21}$$

Hence from (11), (16) and (21), we can conclude that

$$\begin{aligned}\pi(T_i, S_{-i}) - \pi(S_i, S_{-i}) &= w_1(T_i, S_{-i}) - w_1(S_i \cap T_i, S_{-i}) - w_1(S_i, S_{-i}) + w_1(S_i \cap T_i, S_{-i}) \\ &= w_1(T_i, S_{-i}) - w_1(S_i, S_{-i}).\end{aligned}$$

Therefore we have shown that $\pi(\mathbf{S})$ is an exact potential function. Thus any 2-player proportional resource sharing game is an exact potential game. □

Then from Theorem 7.1 and Lemma 3.2, it follows that there is at least one Nash equilibrium in any proportional resource sharing game.

Corollary 7.2. Two-player proportional resource sharing games with player specific values always have a pure Nash equilibrium.

7.1.2 Price of Anarchy

Let $\varphi = \frac{1+\sqrt{5}}{2}$ be the golden ratio. A property of the golden ratio is that $\varphi^2 = \varphi + 1$. We will show that that for 2-player proportional resource sharing games the Price of anarchy $\text{PoA} = \varphi$.

First, we define $v_j(\lambda, \mu) := -\lambda w(\mathbf{OPT} \mid \{j\}) - \mu w(\mathbf{S} \mid \{j\}) + \sum_{i=1}^{2} w_i(\text{OPT}_i \cap \{j\}, S_{-i})$. In the following lengthy lemma we show that $v_j(\lambda, \mu) \geq 0$ for any $j \in J$ when $\lambda = \frac{2\varphi-1}{\varphi^2}$ and $\mu = -\frac{1}{\varphi^2}$. We will do this, to later be able to show that $w(\mathbf{S}) \geq \lambda w(\mathbf{OPT}) + \mu w(\mathbf{S})$, and thus conclude that $\frac{1-\mu}{\lambda} \geq \text{PoA}$.

Lemma 7.3. When $\lambda = \frac{2\varphi-1}{\varphi^2}$ and $\mu = -\frac{1}{\varphi^2}$, for any $j \in J$ it holds that

$$v_j(\lambda, \mu) = -\lambda w(\mathbf{OPT} \mid \{j\}) - \mu w(\mathbf{S} \mid \{j\}) + \sum_{i \in \{1,2\}} w_i(\text{OPT}_i \cap \{j\}, S_{-i}) \geq 0.$$

Proof. Take any instance with optimal solution **OPT** and Nash equilibrium **S**, and let S_i and OPT_i, $i = 1, 2$, be the resources selected by player i in **S** and **OPT**, respectively. Since we have 2 players $\text{OPT}_{-1} = \text{OPT}_2$, $\text{OPT}_{-2} = \text{OPT}_1$, $S_{-1} = S_2$ and $S_{-2} = S_1$. Similarly let $w_{-1,j} = w_{2,j}$ and let $w_{-2,j} = w_{1,j}$.

Proof by Case Distinction
Case I: $j \in \overline{\text{OPT}}_1 \cap \overline{\text{OPT}}_2 \cap \overline{S}_1 \cap \overline{S}_2$.

Since $w(\mathbf{OPT} \mid \{j\}) = w(\mathbf{S} \mid \{j\}) = \sum_{i \in \{1,2\}} w_i(\text{OPT}_i \cap \{j\}, S_{-i}) = 0$. It follows that

$$v_j(\lambda, \mu) = 0.$$

Hence the statement is true when $j \in \overline{\text{OPT}}_1 \cap \overline{\text{OPT}}_2 \cap \overline{S}_1 \cap \overline{S}_2$.

Case II: $j \in \mathrm{OPT}_1 \cap \mathrm{OPT}_2 \cap S_1 \cap S_2$.

Since $w(\mathbf{OPT} \mid \{j\}) = w(\mathbf{S} \mid \{j\}) = \sum_{i\in\{1,2\}} w_i(\mathrm{OPT}_i \cap \{j\}, S_{-i}) = \frac{w_{1,j}^2}{w_{1,j}+w_{2,j}} + \frac{w_{2,j}^2}{w_{1,j}+w_{2,j}}$. It follows that

$$v_j(\lambda,\mu) = (1-\lambda-\mu)\Big(\frac{w_{1,j}^2}{w_{1,j}+w_{2,j}} + \frac{w_{2,j}^2}{w_{1,j}+w_{2,j}}\Big).$$

By definition $1-\lambda-\mu \approx 0.527864 > 0$ and $\frac{w_{1,j}^2}{w_{1,j}+w_{2,j}} + \frac{w_{2,j}^2}{w_{1,j}+w_{2,j}} > 0$. Therefore $v_j(\lambda,\mu) \geq 0$. Hence the statement holds if $j \in \mathrm{OPT}_1 \cap \mathrm{OPT}_2 \cap S_1 \cap S_2$.

Case III: $j \in \mathrm{OPT}_i \cap \overline{\mathrm{OPT}}_{-i} \cap S_i \cap \overline{S}_{-i}$ for some $i \in \{1,2\}$.

Since $w(\mathbf{OPT} \mid \{j\}) = w(\mathbf{S} \mid \{j\}) = \sum_{i\in\{1,2\}} w_i(\mathrm{OPT}_i \cap \{j\}, S_{-i}) = w_{i,j}$. It follows that

$$v_j(\lambda,\mu) = (1-\lambda-\mu)w_{i,j}$$

Since by definition $1-\lambda-\mu \approx 0.527864 > 0$ and $w_{i,j} > 0$, it follows that $v_j(\lambda,\mu) \geq 0$. Hence the statement holds if $j \in \mathrm{OPT}_i \cap \overline{\mathrm{OPT}}_{-i} \cap S_i \cap \overline{S}_{-i}$ for some $i \in \{1,2\}$.

Case IV: $j \in \mathrm{OPT}_i \cap \overline{\mathrm{OPT}}_{-i} \cap \overline{S}_1 \cap \overline{S}_2$ for some $i \in \{1,2\}$.

Since $w(\mathbf{OPT} \mid \{j\}) = \sum_{i\in\{1,2\}} w_i(\mathrm{OPT}_i \cap \{j\}, S_{-i}) = w_{i,j}$ and $w(\mathbf{S} \mid \{j\}) = 0$, it follows that

$$v_j(\lambda,\mu) = (1-\lambda)w_{i,j}.$$

By definition $1-\lambda \approx 0.145898 > 0$ and $w_{i,j} > 0$. Therefore $v_j(\lambda,\mu) \geq 0$. Hence the statement holds if $j \in \mathrm{OPT}_i \cap \overline{\mathrm{OPT}}_{-i} \cap \overline{S}_1 \cap \overline{S}_2$ for some $i \in \{1,2\}$.

Case V: $j \in \mathrm{OPT}_1 \cap \mathrm{OPT}_1 \cap \overline{S}_1 \cap \overline{S}_2$.

Since $w(\mathbf{OPT} \mid \{j\}) = \frac{w_{1,j}^2}{w_{1,j}+w_{2,j}} + \frac{w_{2,j}^2}{w_{1,j}+w_{2,j}}$, $\sum_{i\in\{1,2\}} w_i(\mathrm{OPT}_i \cap \{j\}, S_{-i}) = w_{1,j} + w_{2,j}$ and $w(\mathbf{S} \mid \{j\}) = 0$. It follows that

$$v_j(\lambda,\mu) = w_{1,j} + w_{2,j} - \lambda\Big(\frac{w_{1,j}^2}{w_{1,j}+w_{2,j}} + \frac{w_{2,j}^2}{w_{1,j}+w_{2,j}}\Big).$$

By definition $\lambda \approx 0.854102 < 1$ and $w_{i,j} > \frac{w_{i,j}^2}{w_{1,j}+w_{2,j}}$. Therefore

$$\begin{aligned} w_{1,j} + w_{2,j} &\geq \frac{w_{1,j}^2}{w_{1,j}+w_{2,j}} + \frac{w_{2,j}^2}{w_{1,j}+w_{2,j}} \\ &\geq \lambda(\frac{w_{1,j}^2}{w_{1,j}+w_{2,j}} + \frac{w_{2,j}^2}{w_{1,j}+w_{2,j}}) \end{aligned}$$

$$\implies \quad w_{1,j} + w_{2,j} - \lambda\left(\frac{w_{1,j}^2}{w_{1,j}+w_{2,j}} + \frac{w_{2,j}^2}{w_{1,j}+w_{2,j}}\right) \geq 0.$$

Hence the statement holds if $j \in \mathrm{OPT}_1 \cap \mathrm{OPT}_1 \cap \overline{S}_1 \cap \overline{S}_2$.

Case VI: $j \in \overline{\mathrm{OPT}}_1 \cap \overline{\mathrm{OPT}}_2 \cap S_i \cap \overline{S}_{-i}$ for some $i \in \{1,2\}$.

Since $w(\mathbf{OPT} \mid \{j\}) = \sum_{i\in\{1,2\}} w_i(\mathrm{OPT}_i \cap \{j\}, S_{-i}) = 0$ and $w(\mathbf{S} \mid \{j\}) = w_{i,j}$, it follows that

$$v(\lambda, \mu) = -\mu w_{i,j}.$$

By definition $-\mu \approx 0.381966$ and $w_{i,j} > 0$. Therefore $v_j(\lambda, \mu) \geq 0$. Hence the statement holds if $j \in \overline{\mathrm{OPT}}_1 \cap \overline{\mathrm{OPT}}_2 \cap S_i \cap \overline{S}_{-i}$ for some $i \in \{1,2\}$.

Let $\mathbb{R}_{>0}$ be the positive real numbers. Before we continue to Case VII, we claim that for all $(x,y) \in \mathbb{R}_{>0}^2$,

$$-\mu x - \lambda y + \frac{y^2}{x+y} \geq 0. \tag{22}$$

Let $g(x,y) = -\mu x - \lambda y + \frac{y^2}{x+y}$. The proof of our claim follows by showing convexity of $g(x,y)$. First we show that $g(x,y)$ is convex on $\mathbb{R}_{>0}^2$. Let $\tau \in [0,1]$ be given

$$\begin{aligned} \tau g(x_1,y_1) + (1-\tau)g(x_2,y_2) &= \tau(-\mu x_1 - \lambda y_1 + \frac{y_1^2}{x_1+y_1}) + (1-\tau)(-\mu x_2 - \lambda y_2 + \frac{y_2^2}{x_2+y_2}) \\ &= -\mu(_1{+}(1-\tau)x_2) - \lambda(\tau y_1 + (1-t)y_2) \\ &+ \tau\frac{y_1^2}{x_1+y_1} + (1-\tau)\frac{y_2^2}{x_2+y_2} \\ &= -\mu(\tau x_1 + (1-\tau)x_2) - \lambda(\tau y_1 + (1-\tau)y_2) \\ &+ \frac{(\tau y_1)^2}{\tau x_1 + \tau y_1} + \frac{\big((1-\tau)y_2\big)^2}{(1-\tau)x_2 + (1-\tau)y_2} \end{aligned}$$

From Sedrakyan's inequality (Lemma 3.3), it follows that

$$\frac{(\tau y_1)^2}{\tau x_1 + \tau y_1} + \frac{\big((1-\tau)y_2\big)^2}{(1-\tau)x_2 + (1-\tau)y_2} \geq \frac{(\tau y_1 + (1-\tau)y_2)^2}{\tau x_1 + (1-\tau)x_2 + \tau y_1 + (1-\tau)y_2}.$$

Hence,

$$\begin{aligned}
\tau g(x_1,y_1)+(1-\tau)g(x_2,y_2) &= -\mu(\tau x_1+(1-\tau)x_2)-\lambda(\tau y_1+(1-\tau)y_2)\\
&\quad+\frac{(\tau y_1)^2}{\tau x_1+\tau y_1}+\frac{((1-\tau)y_2)^2}{(1-\tau)x_2+(1-\tau)y_2}\\
&\geq -\mu(\tau x_1+(1-\tau)x_2)-\lambda(\tau y_1+(1-\tau)y_2)\\
&\quad+\frac{(\tau y_1+(1-\tau)y_2)^2}{\tau x_1+(1-\tau)x_2+\tau y_1+(1-\tau)y_2}\\
&= g(\tau x_1+(1-\tau)x_2,\tau y_1+(1-\tau)y_2)
\end{aligned}$$

Therefore we have shown that $g(x,y)$ is convex. The gradient and the Hessian of g are respectively

$$\nabla_g(x,y)=\begin{pmatrix}-\mu-\frac{y^2}{(x+y)^2}\\ -\lambda-\frac{y^2}{(x+y)^2}+\frac{2y}{x+y}\end{pmatrix}\qquad\text{and}$$

$$H_g(x,y)=\begin{pmatrix}\frac{2y^2}{(x+y)^3} & \frac{2y^2}{(x+y)^3}-\frac{2y}{(x+y)^2}\\ \frac{2y^2}{(x+y)^3}-\frac{2y}{(x+y)^2} & \frac{2y^2}{(x+y)^3}-\frac{4y}{(x+y)^2}+\frac{2}{x+y}\end{pmatrix}$$

When $y=\varphi x$, the gradient

$$\nabla_g(x,\varphi x)=\begin{pmatrix}-(-\frac{1}{\varphi^2})-\frac{\varphi^2x^2}{(\varphi+1)^2x^2}\\ -\frac{2\varphi-1}{\varphi^2}-\frac{1}{\varphi^2}+\frac{2\varphi x}{(\varphi+1)x}\end{pmatrix}=\begin{pmatrix}\frac{1}{\varphi^2}-\frac{1}{\varphi^2}\\ \frac{1-2\varphi}{\varphi^2}-\frac{1}{\varphi^2}+\frac{2\varphi}{\varphi^2}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$

Furthermore the Hessian, when $y=\varphi x$,

$$H_g(x,\varphi x)=\begin{pmatrix}\frac{2\varphi^2x^2}{(\varphi+1)^3x^3} & \frac{2}{\varphi^4x}-\frac{2\varphi x}{(\varphi+1)^2x^2}\\ \frac{2}{\varphi^4x}-\frac{2\varphi x}{(\varphi+1)^2x^2} & \frac{2-4\varphi+2\varphi^2}{\varphi^4x}\end{pmatrix}=\begin{pmatrix}\frac{2}{\varphi^4x} & \frac{4-4\varphi}{\varphi^4x}\\ \frac{4-4\varphi}{\varphi^4x} & \frac{4-2\varphi}{\varphi^4x}\end{pmatrix}$$

Let $\mathbf{z}=\begin{pmatrix}z_1\\z_2\end{pmatrix}\in\mathbb{R}^2_+$, then $\mathbf{z}^TH_g(x,\varphi x)\mathbf{z}$ equals

$$\begin{aligned}
\begin{pmatrix}z_1 & z_2\end{pmatrix}H_g(x,\varphi x)\begin{pmatrix}z_1\\z_2\end{pmatrix} &= \frac{2}{\varphi^4x}z_1^2+\frac{4-4\varphi}{\varphi^4x}z_1z_2+\frac{4-2\varphi}{\varphi^4x}z_2^2\\
&= \frac{2}{\varphi^4x}(z_1+(1-\varphi)z_2)^2 \qquad \geq 0
\end{aligned}$$

Hence $H_g(x,y)$ is positive semi definite at $y=\varphi x$. Since $g(x,y)$ is convex, this therefore implies $g(x,y)$ has a global minimum at $y=\varphi x$. Since

$$g(x,\varphi x)=\frac{1}{\varphi^2}x-\frac{2\varphi^2-\varphi}{\varphi^2}x+\frac{\varphi^2}{\varphi^2}x\qquad=\frac{1-(\varphi+2)+\varphi+1}{\varphi^2}x\qquad=0$$

It follows that $g(x,y)\geq 0$, for all $(x,y)\in\mathbb{R}^2_+$.

Now we continue with Case VII.

Case VII: $j \in \mathrm{OPT}_i \cap \overline{\mathrm{OPT}}_{-i} \cap \overline{S}_i \cap S_{-i}$ for some $i \in \{1, 2\}$.

Since $w(\mathbf{OPT} \mid \{j\}) = w_{i,j}$, $\sum_{i \in \{1,2\}} w_i(\mathrm{OPT}_i \cap \{j\}, S_{-i}) = \frac{w_{i,j}^2}{w_{1,j}+w_{2,j}}$ and $w(\mathbf{S} \mid \{j\}) = w_{-i,j}$, it follows that

$$v_j(\lambda, \mu) = \frac{w_{i,j}^2}{w_{1,j} + w_{2,j}} - \mu w_{-i,j} - \lambda w_{i,j}.$$

From our previous claim (22) it follows that

$$\frac{w_{i,j}^2}{w_1 + w_2} - \mu w_{-i,j} - \lambda w_{i,j} > 0.$$

Therefore $v_j(\lambda, \mu) \geq 0$. Hence the statement is true for $j \in \mathrm{OPT}_i \cap \overline{\mathrm{OPT}}_{-i} \cap \overline{S}_i \cap S_{-i}$.

Case VIII: $j \in \mathrm{OPT}_1 \cap \mathrm{OPT}_2 \cap S_i \cap \overline{S}_{-i}$ for some $i \in \{1, 2\}$.

Since $w(\mathbf{OPT} \mid \{j\}) = \frac{w_{i,j}^2}{w_{1,j}+w_{2,j}} + \frac{w_{-i,j}^2}{w_{1,j}+w_{2,j}}$, $\sum_{i \in \{1,2\}} w_i(\mathrm{OPT}_i, S_{-i}) = w_{i,j} + \frac{w_{-i,j}^2}{w_{1,j}+w_{2,j}}$ and $w(\mathbf{S} \mid \{j\}) = w_{i,j}$. It follows that

$$v_j(\lambda, \mu) = (1 - \mu) w_{i,j} + (1 - \lambda) \frac{w_{-i,j}^2}{w_{1,j} + w_{2,j}} - \lambda \frac{w_{i,j}^2}{w_{1,j} + w_{2,j}}$$

By definition $1 - \mu = 1.38197 > 1$, $\lambda \approx 0.854102 < 1$ and $w_{i,j} \geq \frac{w_{i,j}^2}{w_{1,j}+w_{2,j}}$, it follows that

$$(1 - \mu) w_{i,j} \geq w_{i,j} \geq \frac{w_{i,j}^2}{w_{1,j} + w_{2,j}} \geq \lambda \frac{w_{i,j}^2}{w_{1,j} + w_{2,j}}$$

$$(1 - \mu) w_{i,j} - \lambda \frac{w_{i,j}^2}{w_{1,j} + w_{2,j}} \geq 0$$

Furthermore since $1 - \lambda \approx 0.145898 > 0$ and $\frac{w_{-i,j}^2}{w_{1,j}+w_{2,j}} > 0$, it follows that

$$(1 - \lambda) \frac{w_{-i,j}^2}{w_{1,j} + w_{2,j}} \geq 0$$

Hence we can conclude that

$$(1 - \mu) w_{i,j} + (1 - \lambda) \frac{w_{-i,j}^2}{w_{1,j} + w_{2,j}} - \lambda \frac{w_{i,j}^2}{w_{1,j} + w_{2,j}} > 0.$$

Hence the statement is true for $j \in \mathrm{OPT}_1 \cap \mathrm{OPT}_2 \cap S_i \cap \overline{S}_{-i}$ for some $i \in \{1, 2\}$.

Case IX: $j \in \overline{\mathrm{OPT}}_1 \cap \overline{\mathrm{OPT}}_2 \cap S_1 \cap S_2$.

Since $w(\mathbf{OPT} \mid \{j\}) = \sum_{i\in\{1,2\}} w_i(\mathrm{OPT}_i \cap \{j\}, S_{-i}) = 0$ and $w(\mathbf{S} \mid \{j\}) = \frac{w_{1,j}^2}{w_{1,j}+w_{2,j}} + \frac{w_{2,j}^2}{w_{1,j}+w_{2,j}}$. It follows that

$$v_j(\lambda, \mu) = -\mu(\frac{w_{1,j}^2}{w_{1,j} + w_{2,j}} + \frac{w_{2,j}^2}{w_{1,j} + w_{2,j}})$$

By definition $-\mu \approx 0.381966$ and $\frac{w_{i,j}^2}{w_{1,j}+w_{2,j}} > 0$ for $i \in \{1, 2\}$. Hence the statement holds if $j \in \overline{\mathrm{OPT}}_1 \cap \overline{\mathrm{OPT}}_2 \cap S_1 \cap S_2$.

Case X: $j \in \mathrm{OPT}_i \cap \overline{\mathrm{OPT}}_{-i} \cap S_1 \cap S_2$ for some $i \in \{1, 2\}$.

Since $w(\mathbf{OPT} \mid \{j\}) = w_{i,j}$, $\sum_{i\in\{1,2\}} w_i(\mathrm{OPT}_i \cap \{j\} = \frac{w_{i,j}^2}{w_{1,j}+w_{2,j}}$ and $w(\mathbf{S} \mid \{j\}) = \frac{w_{i,j}^2}{w_{1,j}+w_{2,j}} + \frac{w_{-i,j}^2}{w_{1,j}+w_{2,j}}$. It follows that

$$v_j(\lambda, \mu) = (1-\mu)\frac{w_{i,j}^2}{w_{1,j} + w_{2,j}} - \mu\frac{w_{-i,j}^2}{w_{1,j} + w_{2,j}} - \lambda w_{i,j}$$

Then,

$$\begin{aligned}(1-\mu)\frac{w_{i,j}^2}{w_{1,j} + w_{2,j}} - \mu\frac{w_{-i,j}^2}{w_{1,j} + w_{2,j}} - \lambda w_{i,j} &= \frac{(1-\lambda-\mu)w_{i,j}^2 - \lambda w_{i,j}w_{-i,j} - \mu w_{-i,j}^2}{w_{1,j} + w_{2,j}} \\ &= \frac{(1-\lambda-\mu)(w_{i,j} - \frac{\lambda}{2(1-\lambda-\mu)}w_{-i,j})^2}{w_{1,j} + w_{2,j}} \\ &- \frac{(\mu + \frac{\lambda^2}{4(1-\lambda-\mu)})w_{-i,j}^2}{w_{1,j} + w_{2,j}}\end{aligned}$$

Since $1 - \lambda - \mu = 0.527864 > 0$ and $(w_{i,j} - \frac{\lambda}{2(1-\lambda-\mu)}w_{-i,j})^2 \geq 0$, therefore $\frac{(1-\lambda-\mu)(w_{i,j} - \frac{\lambda}{2(1-\lambda-\mu)}w_{-i,j})^2}{w_{1,j}+w_{2,j}} \geq 0$. Furthermore since $-(\mu + \frac{\lambda^2}{4(1-\lambda-\mu)}) \approx 0.0364745 > 0$, therefore $-\frac{(\mu+\frac{\lambda^2}{4(1-\lambda-\mu)})w_{-i,j}^2}{w_{1,j}+w_{2,j}} > 0$. Hence,

$$(1-\mu)\frac{w_{i,j}^2}{w_{1,j} + w_{2,j}} - \mu\frac{w_{-i,j}^2}{w_{1,j} + w_{2,j}} - \lambda w_{i,j} > 0.$$

Therefore $v_j(\lambda, \mu) \geq 0$. And thus the statement is true for $j \in \mathrm{OPT}_i \cap \overline{\mathrm{OPT}}_{-i} \cap S_1 \cap S_2$ for some $i \in \{1, 2\}$.

Now we can conclude that when $\lambda = \frac{2\varphi-1}{\varphi^2}$ and $\mu = -\frac{1}{\varphi^2}$, for any $j \in J$ it holds that

$$v_j(\lambda, \mu) \geq 0.$$

□

Now we can show the upper bound for the price of anarchy.

Theorem 7.4. The price of anarchy PoA $\leq \varphi$ for for 2-player proportional resource sharing games with player specific values.

Proof. Take any instance I with optimal solution $\mathbf{OPT}$ and Nash equilibrium $\mathbf{S}$, and let S_i and OPT_i, $i = 1, 2$, be the resources selected by player i in $\mathbf{S}$ and $\mathbf{OPT}$, respectively. Since we have 2 players $\mathrm{OPT}_{-1} = \mathrm{OPT}_2$, $\mathrm{OPT}_{-2} = \mathrm{OPT}_1$, $S_{-1} = S_2$ and $S_{-2} = S_1$.

Let $v_j(\lambda, \mu) = -\lambda w(\mathbf{OPT} \mid \{j\}) - \mu w(\mathbf{S} \mid \{j\}) + \sum_{i \in \{1,2\}} w_i(\mathrm{OPT}_i \cap \{j\}, S_{-i})$. Using the Nash condition and then for all λ and μ, it holds that

$$\begin{aligned} w(\mathbf{S}) &\geq \sum_{i \in \{1,2\}} w_i(\mathrm{OPT}_i, S_{-i}) \\ &= \lambda w(\mathbf{OPT}) + \mu w(\mathbf{S}) + \sum_{j \in J} v_j(\lambda, \mu) \end{aligned}$$

Let $\lambda = \frac{2\varphi - 1}{\varphi^2}$ and $\mu = -\frac{1}{\varphi^2}$. By Lemma 7.3 it holds $v_j(\lambda, \mu) \geq 0$ for all $j \in J$. Then it follows that

$$\begin{aligned} && w(\mathbf{S}) &\geq \lambda w(\mathbf{OPT}) + \mu w(\mathbf{S}) + \sum_{j \in J} v_j(\lambda, \mu) \\ && &\geq \lambda w(\mathbf{OPT}) + \mu w(\mathbf{S}) \\ \implies && (1 - \mu) w(\mathbf{S}) &\geq \lambda w(\mathbf{S}) \\ \implies && \frac{1 - \mu}{\lambda} &\geq \frac{w(\mathbf{OPT})}{w(\mathbf{S})} = \mathrm{PoA} \end{aligned}$$

Filling in $\lambda = \frac{2\varphi - 1}{\varphi^2}$ and $\mu = -\frac{1}{\varphi^2}$, and since $\varphi^2 = \varphi + 1$, the left-hand side becomes

$$\begin{aligned} \frac{1 - \mu}{\lambda} &= \frac{1 + \frac{1}{\varphi^2}}{\frac{2\varphi - 1}{\varphi^2}} = \frac{\varphi^2 + 1}{2\varphi - 1} = \frac{\varphi + 2}{2\varphi - 1} \\ &= \frac{\frac{1+\sqrt{5}}{2} + 2}{2\frac{1+\sqrt{5}}{2} - 1} = \frac{1}{2} \frac{5 + \sqrt{5}}{\sqrt{5}} = \frac{1}{2} \frac{5 + 5\sqrt{5}}{5} = \frac{1 + \sqrt{5}}{2} = \varphi \end{aligned}$$

Thus we can conclude that $\varphi \geq$ PoA. □

To show the lower bound for the price of anarchy, consider the the following example and theorem.

Example 7.1. Let there be two resources. Both players can only choose one of the resources. Let $w_{1,1} = \varphi$, $w_{1,2} = 1$, $w_{2,1} = 1$ and let $w_{2,2} = \varphi$.

Theorem 7.5. PoA $\geq \varphi$ for 2-player proportional resource sharing games with player specific values.

Proof. Consider Example 7.1. In the optimum the first resource is chosen by player 1 and the second resource is chosen by player 2. In this instance $w(\mathbf{OPT}) = 2\varphi$. Consider the Nash equilibrium where player 1 chooses the second resource and player 2 chooses the first resource. To see that is is a Nash equilibrium. If player 2 chooses the first resource, player 1 receives value 1 regardless of what resource they choose. Since resource 2 has full value 1 for player 1 and resource 1 has value $\frac{\varphi^2}{\varphi+1} = \frac{\varphi+1}{\varphi+1} = 1$ when shared. Likewise if player 2 chooses the second resource, player 2 receives value 1 regardless if what resource they choose. Hence the Nash condition (Equation 2) holds for all players. We can conclude that PoA $\geq w(\mathbf{OPT})/w(\mathbf{S}) = 2\varphi/2 = \varphi$. □

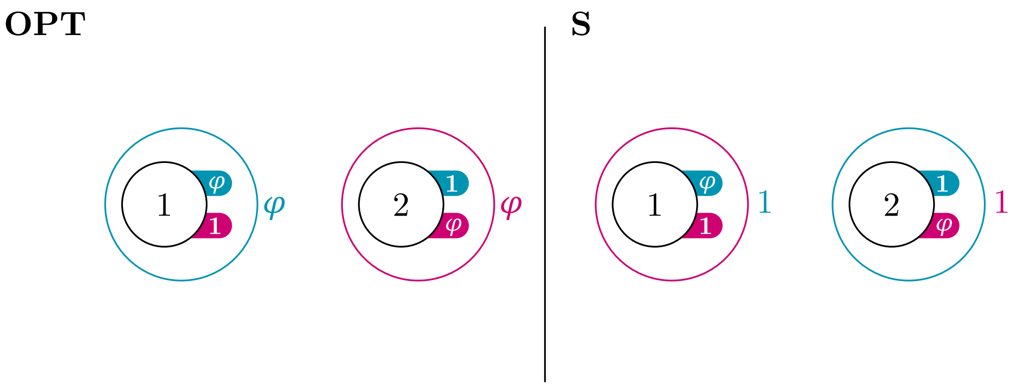


FIGURE 4: Illustration corresponding to Example 7.1Depicted are the optimum (left) and the Nash equilibrium (right). Strategies, values and payoffs of player 1 is associated with the colour blue and of player 2 with the colour magenta.

From Theorem 7.4 and Theorem 7.5 it follows that PoA $= \varphi$ for 2-player proportional resource sharing games.

Corollary 7.6. The price of anarchy PoA $= \varphi$ for 2-player proportional resource sharing games with player specific values.

7.2 Proportional resource sharing games with more than two players

7.2.1 Non-existence of pure Nash equilibria

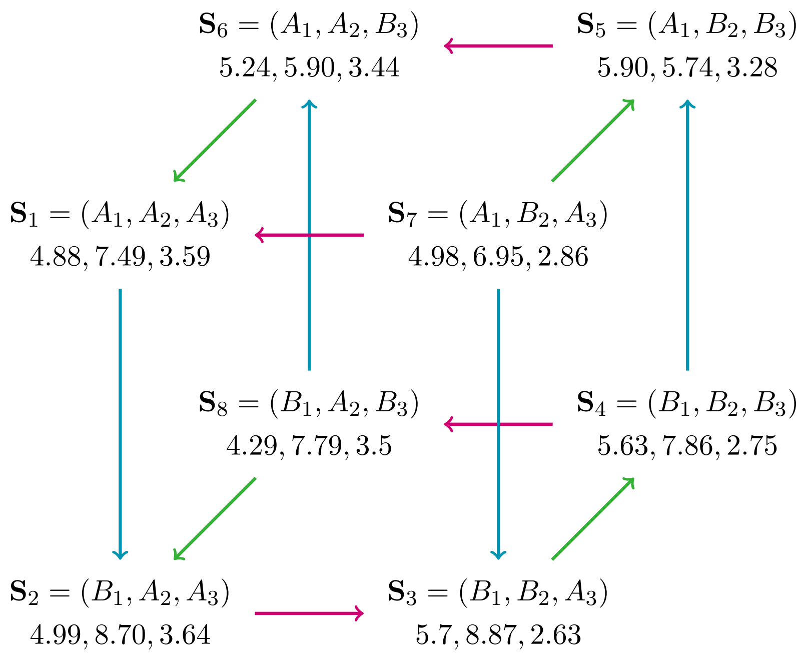
Unlike with 2-players in general for more than two players, proportional resource sharing games are not guaranteed to have pure Nash equilibria. Consider the following example

Example 7.2. Let there be three players and six resources $J = \{1, 2, 3, 4, 5, 6\}$. The values of the players are summarised in the following table.

$i \setminus j$	1	2	3	4	5	6
1	2	5	1	1	2	5
2	4	3	1	4	3	1
3	1	1	3	4	1	2

Let $A_1 = \{1, 2, 4, 5\}$, $B_1 = \{1, 3, 6\}$, $A_2 = \{1, 2, 4, 6\}$, $B_2 = \{2, 3, 4, 5, 6\}$, $A_3 = \{1, 2, 3, 5\}$ and $B_3 = \{4, 5, 6\}$. Player i has strategy set $\mathcal{S}_i = \{A_i, B_i\}$.

Consider the following graph. Each vertex represents a strategy profile with corresponding payoffs (rounded to two decimals, for the full calculation of the payoffs see Appendix C). The edges point to strategy profiles one player improves to by switching strategies; the blue corresponding to player 1, magenta to player 2 and green to player 3.



Note that no vertex has three incoming edges, hence it follows that this example has no Nash equilibrium.

7.2.2 Price of Anarchy

We now only consider the instances of n-player games that do have pure Nash equilibria. First, we define an extension to the golden ratio. Define $\phi_n := \frac{1+\sqrt{4(n-1)+1}}{2}$. Note that $\phi_n^2 = \phi_n + n - 1$ and for $n = 2$ this is exactly the golden ratio.

We show a lower bound of ϕ_n for the price of anarchy for n-player proportional resource sharing games with player specific values. Consider the following example.

Example 7.3. Let there be n resources $J = \{1, 2, \ldots, n\}$. For $i = 1, \ldots, n$. Player i can either resource i or (a subset of) the set of all resources except resource i. Player i prefers the ith resource with value ϕ_n and the other resource with value 1, i.e. for all $i \in N$, $w_{i,i} = \phi_n$ and $w_{i,j} = 1$ for all $j \in \{1, \ldots, i-1, i+1, \ldots, n\}$.

Theorem 7.7. The price of anarchy PoA $\geq \phi_n$ for proportional resource sharing games with player specific values.

Proof. Consider Example 6.2. In the optimum is, player i chooses resource i for all $i \in N$ and therefore $w(\mathbf{OPT}) = n\phi_n$. Consider the Nash equilibrium where player i chooses all resources except resource i for all $i \in \{1, \ldots, n\}$. Since each player now chooses $n-1$ resources and each resource is chosen by $n-1$ players, player i receives payoff $(n-1)\frac{1^2}{n-1} = 1$. We show that this is a Nash equilibrium. When switching to only resource i, the resource now is chosen by n players and player i receives a payoff of $\frac{\phi_n^2}{\phi_n+n-1} = 1$. Furthermore switching to any subset of $\{1, \ldots, i-1, i+1, \ldots, n\}$ leads to a payoff of less then 1. Hence the Nash condition (Equation 2) holds for all players. We can conclude that PoA $\geq w(\mathbf{OPT})/w(\mathbf{S}) = n\phi_n/n = \phi_n$. □

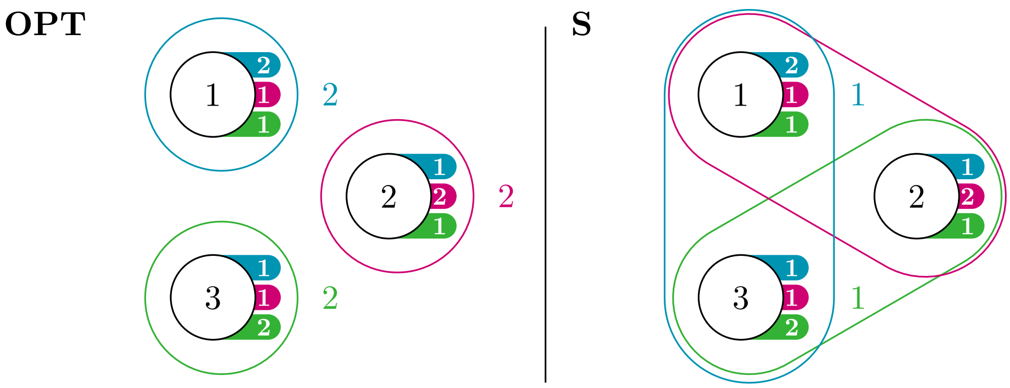


FIGURE 5: Illustration corresponding to Example 7.3. Since $\phi_3 = \frac{1+\sqrt{4 \cdot 2+1}}{2} = 2$. Depicted are the optimum (left) and the Nash equilibrium (right). Strategies and values of player 1 is associated with the colour blue, strategies and values of player 2 with the colour magenta and strategies and values of player 3 with the colour green.

Furthermore, we conjecture that the price of anarchy PoA equals ϕ_n for n-player proportional resource sharing games. The conjecture follows from our observation that the lower bound example for 2-player uniform resource sharing games and proportional resource sharing games follow the same structure. Where with $w_{i,i} = x$ and $w_{k,i} = 1$ for $i, k \in N$ and $i \neq k$, $x =$ PoA solves the equation

$$f_i \cdot x = 1. \tag{23}$$

Where f_i is the value of the sharing rule $f_i(i, \mathbf{S})$ when all players choose resource i, therefore $f_i = \frac{1}{2}$ for 2-player uniform resource sharing games and $f_i = \frac{x}{x+1}$ for 2-player proportional sharing games. Likewise for n player uniform sharing games $x = n$ solves the equation. We suspect the same to be true for n-player resource sharing games, where $x = \phi_n$ solves Equation 23 and the lower bound is shown using an example similar in structure as for n player uniform resource sharing games.

Conjecture 7.1. The price of anarchy PoA $= \phi_n$ for n player proportional resource sharing games with player specific values.

8 Conclusion and Discussion

We showed that the price of anarchy for (n player) set packing games with player specific values is $\theta + 1$, where θ is the maximum ratio of values of assigned to a resource.

Furthermore, we showed that while pure Nash equilibria are not guaranteed to exist in general for games with uniform sharing, the subclass of (n player) games that do have pure Nash equilibria has a price of anarchy of n.

Lastly, we showed that pure Nash equilibria exist for 2-player games with proportional sharing games, but that this is not guaranteed for three or more players. We show that 2 player games have a price of anarchy of φ, the golden ratio. For the class of n player games with pure Nash equilibria, we conjecture that the price of anarchy is $\frac{1+\sqrt{4(n-1)+1}}{2}$ and we showed a matching lower bound.

Depending on n and if θ is smaller, equal or bigger than $n-1$, the price of anarchy of either set packing games and uniform resource sharing games can be equal or worse than the other one. But, since $\theta \geq 1$, we see that the class of 2-player proportional sharing always has the lowest price of anarchy for all 2-player variants.

Perhaps, this implies that when giving the player with the highest value more of the resource improves the price of anarchy. Further research could go into a "winner takes all" scenario, where the player with the highest value choosing the resource gains full value and the others gain no value.

Moreover, between the two sharing rules where all players get a non-zero fraction of the value of all resources they choose, i.e. uniform and proportional sharing. Proportional sharing has the benefit of that pure Nash equilibria exist for two player games. Additional further research could go if there exist a sharing rule such that all players get a non-zero fraction of the value of all resources they choose and pure Nash equilibria always exist.

Also, because pure Nash equilibria are never guaranteed to exist, for uniform resource sharing games and proportional resource sharing games (for more than two players) we then restrict ourselves to only the instances where they do exist. We do not discuss anything about the time complexity of deciding whether or not they exist and leave this open for further research.

Examples 6.2 and 7.3, do not make a lot of sense in context of the library book allocation in Example 1.1 and would imply that books have variable sizes depending on which library would choose it. Perhaps, an additional property can be assigned to the strategy sets to better reflect the books taking up the same amount of space in each library. Then, it could also be investigated if this also improves the existence of Nash equilibria and/or the price of anarchy.

Additionally, further research could go into proving the price of anarchy of n player proportional resource sharing games. If our conjecture is true, perhaps this also implies for certain sharing rules the price of anarchy can be found by solving for x equation 23. Regardless if the conjecture is true, for which sharing rules this does hold is also possible further research.

For no-sharing, we choose to assign $-\infty$ to all shared resources, mainly because that is what is used in the literature, and because conceptually it makes sense as it prevents sharing in pure Nash equilibria.

Alternatively, future research could look into a sharing rule where instead of both players receiving $-\infty$ they receive 0 for shared resources. One can easily alter our proof to show the existence of pure Nash equilibria and our example still works for the lower bound, but it is unclear is it has the same or a worse price of anarchy.

Lastly, we can define many more sharing rules, see examples in Appendix D, of which the existence of pure Nash equilibria and price of anarchy also could still be investigated.

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Appendix

A Table of Variable Names and Symbols

In alphabetical order

Latin alphabet

A_i	one of two strategies of player i
B_i	one of two strategies of player i
a_i	numerator of the ith ratio
b_i	denominator of the ith ratio
c	some constant
F	set of ratios
$f_j(i, \mathbf{S})$	Sharing rule
f_i	Value of sharing rule for resource i and player i, when all players choose i and $w_{i,i} = x$ and $w_{k,i}$
$g(x, y)$	a function over variables x and y
$\mathcal{I}$	class of games
I	instance
i	player
J	set of resources
J'	subset of resources
j	resource
H_g	Hessian of g
k	(other) player
N	set of n players
$\mathbf{S}$	allocation, strategy profile, (pure) Nash equilibrium
$\mathrm{NE}(I)$	set of pure Nash equilibria in instance I
$\mathbf{OPT}$	optimum
OPT_i	Strategy of player i in optimum
PoA	price of anarchy
P	set of resources
Q	set of resources
$\mathbb{R}_{>0}$	strictly positive real numbers
$\mathcal{S}_i$	Strategy set of player i
S	all resources chosen by at least one player in $\mathbf{S}$
$\overline{S}$	all resources (in J) not chosen by any players in $\mathbf{S}$
S_i	(Nash) strategy of player i
S_{-i}	all strategies chosen by all players except player i in $\mathbf{S}$

$\mathbf{T}$	(other) strategy profile
T_i	(other) strategy of player i
$v_j(\lambda, \mu)$	difference per resource j of $\lambda w(\mathbf{OPT}) + \mu w(\mathbf{S})$ and $\sum_{i=1}^{n} w_i(\mathrm{OPT}_i, S_{-i})$
$w_{i,j}$	value of resource j for player i
$w(\mathbf{S})$	total payoff
$w(\mathbf{S} \mid J')$	total payoff of all resources in $J' \subseteq J$
$w_i(S_i, S_{-i})$	payoff of player i
x	variable
y	variable
z	variable

Greek alphabet

$\delta_i(J)$	all resources in J that can be chosen by player i
θ	largest ratio of values
λ	variable, such that $\frac{1-\mu}{\lambda} = \frac{w(\mathbf{OPT})}{w(\mathbf{S})=\mathrm{PoA}}$
μ	variable, such that $\frac{1-\mu}{\lambda} = \frac{w(\mathbf{OPT})}{w(\mathbf{S})=\mathrm{PoA}}$
$\pi(\mathbf{S})$	potential function
$\sigma(F)$	generalized mediant of F
τ	variable
φ	golden ration
ϕ_n	extension of the golden ratio, such that $\phi_n^2 = \phi_n + n - 1$

Other

∇_g	gradient of g

B Generalised Mediant Inequality

Let $F = \{\frac{a_i}{b_i} : i \in \{1, \dots n\}\}$ be a (finite) set of ratios, where $a_i, b_i \in \mathbb{R}_{>0}$

Lemma B.1 (Generalized mediant inequality)**.** For a (finite) set F or ratios it holds that $\sup_{i\in\{1,\dots,n\}} \frac{a_i}{b_i} \geq \sigma(F) \geq \inf_{i\in\{1,\dots,n\}} \frac{a_i}{b_i}$.

Proof. w.l.o.g. assume that F is ordered such that $\frac{a_i}{b_i} \geq \frac{a_{i-1}}{b_{i-1}}$ for all $i \in \{2, \dots, n\}$. Then $\inf_{i\in\{1,\dots,n\}} \frac{a_i}{b_i} = \frac{a_1}{b_1}$ and $\sup_{i\in\{1,\dots,n\}} \frac{a_i}{b_i} = \frac{a_n}{b_n}$

By induction we will show for $k \in \{1, \dots, n\}$, that.

$$\frac{a_k}{b_k} \geq \frac{\sum_{i=1}^{k} a_i}{\sum_{i=1}^{k} b_i} \geq \frac{a_1}{b_1}.$$

Proof by Induction

Base Case: Obviously, $\frac{a_1}{b_1} \geq \frac{a_1}{b_1} \geq \frac{a_1}{b_1}$. Hence the statement holds for the base case.

Induction Hypothesis: Now assume that $\frac{a_{k-1}}{b_{k-1}} \geq \frac{\sum_{i=1}^{k-1} a_i}{\sum_{i=1}^{k-1} b_i} \geq \frac{a_1}{b_1}$ is true.

INDUCTIVE STEP: Since $\frac{a_k}{b_k} \geq \frac{a_{k-1}}{b_{k-1}}$ and by induction hypothesis, it follows that

$$\begin{aligned}
& \frac{a_k}{b_k} \geq \frac{\sum_{i=1}^{k-1} a_i}{\sum_{i=1}^{k-1} b_i} && \text{and} && \frac{a_k}{b_k} \geq \frac{\sum_{i=1}^{k-1} a_i}{\sum_{i=1}^{k-1} b_i} \\
\implies & \frac{a_k \sum_{i=1}^{k-1} b_i}{b_k} \geq \sum_{i=1}^{k-1} a_i && \text{and} && a_k \geq \frac{\sum_{i=1}^{k-1} a_i b_k}{\sum_{i=1}^{k-1} b_i} \\
\implies & \frac{a_k(b_k + \sum_{i=1}^{k-1} b_i)}{\sum_{i=1}^{k-1} b_i} \geq a_k + \sum_{i=1}^{k-1} a_i && \text{and} && a_k + \sum_{i=1}^{k-1} a_i \geq \frac{\sum_{i=1}^{k-1} a_i(b_k + \sum_{i=1}^{k-1} b_i)}{\sum_{i=1}^{k-1} b_i} \\
\implies & \frac{a_k(b_k + \sum_{i=1}^{k-1} b_i)}{b_k(b_k + \sum_{i=1}^{k-1} b_i)} \geq \frac{a_k + \sum_{i=1}^{k-1} a_i}{b_k + \sum_{i=1}^{k-1} b_i} && \text{and} && \frac{a_k + \sum_{i=1}^{k-1} a_i}{b_k + \sum_{i=1}^{k-1} b_i} \geq \frac{\sum_{i=1}^{k-1} a_i(b_k + \sum_{i=1}^{k-1} b_i)}{\sum_{i=1}^{k-1} b_i(b_k + \sum_{i=1}^{k-1} b_i)} \\
\implies & \frac{a_k}{b_k} \geq \frac{a_k + \sum_{i=1}^{k-1} a_i}{b_k + \sum_{i=1}^{k-1} b_i} && \text{and} && \frac{a_k + \sum_{i=1}^{k-1} a_i}{b_k + \sum_{i=1}^{k-1} b_i} \geq \frac{\sum_{i=1}^{k-1} a_i}{\sum_{i=1}^{k-1} b_i}
\end{aligned}$$

Hence it follows that $\frac{a_k}{b_k} \geq \frac{\sum_{i=1}^{k} a_i}{\sum_{i=1}^{k} b_i} \geq \frac{\sum_{i=1}^{k-1} a_i}{\sum_{i=1}^{k-1} b_i} \geq \frac{a_1}{b_1}$.

Therefore, $\frac{a_n}{b_n} \geq \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i} \geq \frac{a_1}{b_1}$. Thus we can conclude that

$$\sup_{i \in \{1,\dots,n\}} \frac{a_i}{b_i} \geq \sigma(F) \geq \inf_{i \in \{1,\dots,n\}} \frac{a_i}{b_i}.$$

□

C Full calculation of payoffs of players in example 7.2

For each strategy profile $\mathbf{S}$, in the following calculations, the values gained from the same resource are visually aligned between the players.

Let $A_1 = \{1,2,4,5\}$, $B_1 = \{1,3,6\}$, $A_2 = \{1,2,4,6\}$, $B_2 = \{2,3,4,5,6\}$, $A_3 = \{1,2,3,5\}$ and $B_3 = \{4,5,6\}$. Player i has strategy set $\mathcal{S}_i = \{A_i, B_i\}$.

$\mathbf{S}_1 = (A_1, A_2, A_3)$

$$\begin{aligned}
w_1(\mathbf{S}_1) &= \frac{2^2}{2+4+1} + \frac{5^2}{5+3+1} + \frac{1^2}{1+4} + \frac{2^2}{2+1} = \frac{1538}{315} \approx 4.8825 \\
w_2(\mathbf{S}_1) &= \frac{4^2}{2+4+1} + \frac{3^2}{5+3+1} + \frac{4^2}{1+4} + 1 = \frac{262}{35} \approx 7.4857 \\
w_3(\mathbf{S}_1) &= \frac{1^2}{2+4+1} + \frac{1^2}{5+3+1} + 3 + \frac{1^2}{2+1} = \frac{226}{63} \approx 3.5873
\end{aligned}$$

$\mathbf{S}_2 = (B_1, A_2, A_3)$

$$\begin{aligned}
w_1(\mathbf{S}_2) &= \frac{2^2}{2+4+1} + \frac{1^2}{1+3} + \frac{5^2}{5+1} = \frac{419}{84} \approx 4.9881 \\
w_2(\mathbf{S}_2) &= \frac{4^2}{2+4+1} + \frac{3^2}{3+1} + 4 + \frac{1^2}{5+1} = \frac{173}{84} \approx 8.7024 \\
w_3(\mathbf{S}_2) &= \frac{1^2}{2+4+1} + \frac{1^2}{3+1} + \frac{3^2}{1+3} + 1 = \frac{51}{14} \approx 3.6429
\end{aligned}$$

$\mathbf{S}_3 = (B_1, B_2, A_3)$

$$w_1(\mathbf{S}_3) = \frac{2^2}{2+1} + \frac{1^2}{1+1+3} + \frac{5^2}{5+1} = \frac{57}{10} = 5.7$$
$$w_2(\mathbf{S}_3) = \frac{3^2}{3+1} + \frac{1^2}{1+1+3} + 4 + \frac{3^2}{3+1} + \frac{1^2}{5+1} = \frac{133}{15} \approx 8.8667$$
$$w_3(\mathbf{S}_3) = \frac{1^2}{2+1} + \frac{1^2}{3+1} + \frac{3^2}{1+1+3} + \frac{1^2}{3+1} = \frac{79}{30} \approx 2.6333$$

$\mathbf{S}_4 = (B_1, B_2, B_3)$

$$w_1(\mathbf{S}_4) = 2 + \frac{1^2}{1+1} + \frac{5^2}{5+1+2} = \frac{45}{8} = 5.625$$
$$w_2(\mathbf{S}_4) = 3 + \frac{1^2}{1+1} + \frac{4^2}{4+4} + \frac{3^2}{3+1} + \frac{1^2}{5+1+2} = \frac{63}{8} = 7.875$$
$$w_3(\mathbf{S}_4) = \frac{4^2}{4+4} + \frac{1^2}{3+1} + \frac{2^2}{5+1+2} = \frac{11}{4} = 2.75$$

$\mathbf{S}_5 = (A_1, B_2, B_3)$

$$w_1(\mathbf{S}_5) = 2 + \frac{5^2}{5+3} + \frac{1^2}{1+4+4} + \frac{2^2}{2+3+1} = \frac{425}{72} \approx 5.9028$$
$$w_2(\mathbf{S}_5) = \frac{3^2}{5+3} + 1 + \frac{4^2}{1+4+4} + \frac{3^2}{2+3+1} + \frac{1^2}{1+2} = \frac{413}{72} \approx 5.7361$$
$$w_3(\mathbf{S}_5) = + \frac{4^2}{1+4+4} + \frac{1^2}{2+3+1} + \frac{2^2}{1+2} = \frac{59}{18} \approx 3.2778$$

$\mathbf{S}_6 = (A_1, A_2, B_3)$

$$w_1(\mathbf{S}_6) = \frac{2^2}{2+4} + \frac{5^2}{5+3} + \frac{1^2}{1+4+4} + \frac{2^2}{2+1} = \frac{377}{72} \approx 5.2361$$
$$w_2(\mathbf{S}_6) = \frac{4^2}{2+4} + \frac{3^2}{5+3} + \frac{4^2}{1+4+4} + \frac{1^2}{1+2} = \frac{425}{72} \approx 5.9028$$
$$w_3(\mathbf{S}_6) = \frac{4^2}{1+4+4} + \frac{1^2}{2+1} + \frac{2^2}{1+2} = \frac{31}{9} \approx 3.4444$$

$\mathbf{S}_7 = (A_1, B_2, A_3)$

$$w_1(\mathbf{S}_7) = \frac{2^2}{2+1} + \frac{5^2}{5+3+1} + \frac{1^2}{1+4} + \frac{2^2}{2+3+1} = \frac{224}{45} \approx 4.9778$$
$$w_2(\mathbf{S}_7) = \frac{3^2}{5+3+1} + \frac{1^2}{1+3} + \frac{4^2}{1+4} + \frac{3^2}{2+3+1} + 1 = \frac{139}{20} = 6.95$$
$$w_3(\mathbf{S}_7) = \frac{1^2}{2+1} + \frac{1^2}{5+3+1} + \frac{3^2}{1+3} + \frac{1^2}{2+3+1} = \frac{103}{36} \approx 2.8611$$

$\mathbf{S}_8 = (B_1, A_2, B_3)$

$$
\begin{aligned}
w_1(\mathbf{S}_8) &= \frac{2^2}{2+4} + \frac{1^2}{1+1} + \frac{5^2}{5+1+2} = \frac{103}{24} \approx 4.2917 \\
w_2(\mathbf{S}_8) &= \frac{4^2}{2+4} + 3 + \frac{4^2}{4+4} + \frac{1^2}{5+1+2} = \frac{187}{24} \approx 7.7917 \\
w_3(\mathbf{S}_8) &= \frac{4^2}{4+4} + 1 + \frac{2^2}{5+1+2} = \frac{7}{2} = 3.5
\end{aligned}
$$

D Other Sharing Rules

PROPOSED NAME	$f_j(i, \mathbf{S}) =$
full sharing	1
winner takes all	$\begin{cases} 1 & \text{if there exist no } k \in N \text{ where } i \neq k, j \in S_k \text{ and } w_{k,j} \geq w_{i,j} \\ 0 & \text{otherwise} \end{cases}$
refused sharing	$\begin{cases} 1 & \text{if there exist no } k \in N \text{ where } i \neq k \text{ and } j \in S_k \\ 0 & \text{otherwise} \end{cases}$
anti-proportional sharing I	$\frac{\sum_{k \in N \setminus \{i\}} w_{k,j}}{(n-1)(w_{i,j} + \sum_{k \in N \setminus \{i\}} w_{k,j})}$
anti-proportional sharing II	$\frac{\prod_{k \in N \setminus \{i\}} w_{k,j}}{\prod_{k \in N \setminus \{i\}} w_{k,j} + \sum_{k \in N \setminus \{i\}} \prod_{k' \in N \setminus \{k\}} w_{k',j}}$