Ocean waves of maximal amplitude

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Contents

Introduction

A 'freak wave' is a wave that appears seemingly random and that is significantly larger than the surrounding waves. It is a phenomenon that occurs on many oceans around the world. These waves, that are known to have damaged and sunk some very large ships, will appear very sudden and disappear just a short time after.

Tales of freak waves were once dismissed as seafaring myths. But nowadays the height of the sea is continually monitored at many places such as oil rigs. One of these offshore platforms, the Draupner platform in the Norwegian section of the North Sea, has recorded a freak wave at 15:24 on 1st January 1995 [7]. (see figure 1)

We assume these freak waves originate from several smaller waves with different wavelengths that travel with different velocities. These smaller waves may at some point combine to one very big wave and almost immediately dissolve into smaller waves again.

In this report I'll try to find a description for the maximum wave height for all times, hoping to understand some of the phenomena of the ocean a bit better.

There are several partial differential equations that model wind generated water waves. In this report I will look at a few of them. For example in chapter 1 I'll use linear dispersive partial differential equations. The solution of such an equation, $\eta(x,t)$ with one spatial variable x and one time variable t, represents the wave height. A function called Maximal Spatial Amplitude, $MSA(t) = \max_x \eta(x,t)$, is used to describe the maximum of the wave height at a fixed time t. Though this formulation is simple, it is often not possible to get an explicit expression for the MSA. In chapter 1 I'll try to find some descriptions for the MSA using several mathematical analytical methods.

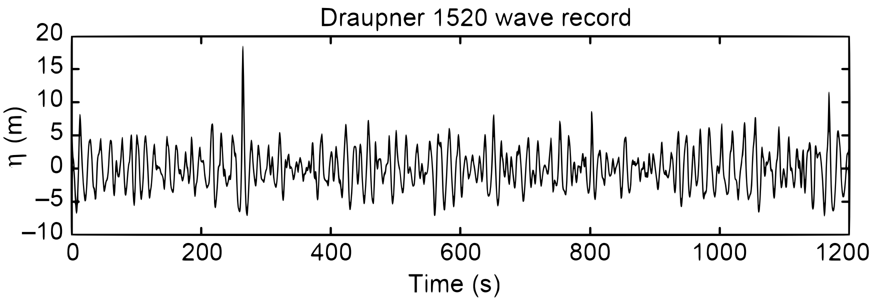

Figure 1: This plot shows a freak wave at approximately 270 seconds after 15:20 hours

The linear dispersive differential equations I use in the first chapter have several constants of motion including momentum and energy. In chapter 2 I will not use the differential equations to model water waves exactly, but instead I will only prescribe the amount of energy and momentum in the waves. Then I will look for the largest wave that satisfies these energy and momentum constraints. The mathematical solution is obtained by solving the ordinary differential equation that follows from Lagrange multiplier rule. We also look for periodic solutions of this differential equation.

In chapter 3 the largest wave from the previous chapter will be used as an initial condition in the linear dispersive differential equations I used in the first chapter. The solutions will be represented using the computer program Matlab. In this chapter I will also look at the linear evolution of the soliton solution of the Korteweg de Vries equation.

Chapter 1

From linear dispersive wave equation to Maximal Spatial Amplitude

1.1 Setting

First we investigate initial value problems for PDE's which are of first order in time, linear and dispersive:

$$\partial_t \eta (x, t) = p (\partial_x) \eta (x, t) \tag{1.1}$$

$$\eta (x, 0) = \eta_0 (x) \tag{1.2}$$

with $p (x)$ a polynomial. The dispersion relation for modes $e^{i(kx-\omega t)}$of the equation is then

$$\Omega (k) = ip (ik) \tag{1.3}$$

The PDE can then formally be given by

$$\partial_t \eta + i\Omega (-i\partial_x) \eta = 0 \tag{1.4}$$

With the initial condition $\eta (x, 0)$ given in a Fourier integral, the solution of the initial value problem is represented also by a Fourier integral

$$\eta (x, t) = \frac{1}{2\pi} \int g (k) e^{i(kx-\Omega(k)t)} dk \tag{1.5}$$

$$\eta (x, 0) = \frac{1}{2\pi} \int g (k) e^{ikx} dk \tag{1.6}$$

With an initial spectrum concentrated around k_0 the expression for $\eta (x, t)$ can be simplified using a Taylor series for $\Omega (k)$ around k_0. We need at least second order to get a variation in the maximum water height. Take $\Omega (k) = \Omega (k_0) + \Omega' (k_0) (k - k_0) + \frac{1}{2}\Omega'' (k_0) (k - k_0)^2$ then $\eta (x, t)$ becomes

$$\begin{aligned}\eta (x, t) &= \frac{1}{2\pi} \int g (k) e^{i\left(kx-\Omega(k_0)t-\Omega'(k_0)(k-k_0)t-\frac{1}{2}\Omega''(k_0)(k-k_0)^2 t\right)} dk \\ &= e^{ik_0\left(x-\frac{\Omega(k_0)}{k_0}t\right)} \frac{1}{2\pi} \int g (\kappa + k_0) e^{i\left(\kappa x-\Omega'(k_0)\kappa t-\frac{1}{2}\Omega''(k_0)\kappa^2 t\right)} d\kappa \end{aligned} \tag{1.7}$$

which is a combination of a carrier wave $e^{ik_0\left(x-\frac{\Omega(k_0)}{k_0}t\right)}$ and an amplitude

$$A\left(x,t\right)=\frac{1}{2\pi}\int g\left(\kappa+k_0\right)e^{i\left(\kappa x-\Omega'(k_0)\kappa t-\frac{1}{2}\Omega''(k_0)\kappa^2 t\right)}d\kappa \tag{1.8}$$

With the transformation to a moving frame:

$$\begin{cases}\xi=x-\Omega'\left(k_0\right)t\\ \tau=t\end{cases} \tag{1.9}$$

the amplitude becomes

$$A\left(\xi,\tau\right)=\frac{1}{2\pi}\int g\left(\kappa+k_0\right)e^{i\left(\kappa\xi-\frac{1}{2}\Omega''(k_0)\kappa^2\tau\right)}d\kappa \tag{1.10}$$

Remember $\eta\left(x,t\right)$ was given in the form of a Fourier integral (1.5) and as the solution of a PDE (1.4). Since the amplitude is given in a similar integral, it can also be seen as the solution of a PDE:

$$\begin{aligned}\partial_\tau A+i\frac{1}{2}\Omega''\left(k_0\right)\left(-i\partial_\xi\right)^2A=0\\ \partial_\tau A-i\frac{1}{2}\Omega''\left(k_0\right)\partial_{\xi\xi}A=0\end{aligned} \tag{1.11}$$

The envelope of $\eta\left(x,t\right)$ is defined by:

$$envelope\left(\xi,\tau\right)=\left|A\left(\xi,\tau\right)\right| \tag{1.12}$$

Since the solution is smaller than the envelope for every ξ and τ, the Maximal Spatial Amplitude (MSA) is smaller than the maximum of the envelope for every τ:

$$MSA\left(\tau\right)=\max_x\left\{\eta\left(x,t\right)\right\}\leq\max_\xi\left\{envelope\left(\xi,\tau\right)\right\}=MSE\left(\tau\right) \tag{1.13}$$

where MSE stands for Maximal Spatial Envelope.

1.2 Simple example: Gaussian spectrum

In this section I will illustrate the material of the previous section for certain PDE and initial condition.

Use a Gaussian spectrum

$$g\left(k\right)=\frac{1}{\sigma\sqrt{\pi}}e^{-\frac{1}{2}\frac{(k-k_0)^2}{\sigma^2}} \tag{1.14}$$

for the initial condition (1.6)

$$\begin{aligned}\eta(x,0) &= \frac{1}{2\pi}\int g(k)\,e^{ikx}dk \\ &= \frac{1}{2\pi}\int \frac{1}{\sigma\sqrt{\pi}}e^{-\frac{1}{2}\frac{(k-k_0)^2}{\sigma^2}}e^{ikx}dk \end{aligned} \tag{1.15}$$

The solution $\eta(x,t)$ then becomes

$$\eta(x,t) = e^{ik_0\left(x-\frac{\Omega(k_0)}{k_0}t\right)}A(x,t) \tag{1.16}$$

with amplitude (using the coordinate transformation (1.9))

$$\begin{aligned}A(\xi,\tau) &= \frac{1}{2\pi}\int \frac{1}{\sigma\sqrt{\pi}}e^{i\kappa\xi-\frac{1}{2}\kappa^2\left(\frac{1}{\sigma^2}+i\Omega''(k_0)\tau\right)}d\kappa \\ &= \frac{1}{\sqrt{2}\pi}\frac{e^{-\frac{1}{2}\xi^2\sigma^2/\left(1+i\sigma^2\Omega''(k_0)\tau\right)}}{\sqrt{1+i\sigma^2\Omega''(k_0)\tau}}\end{aligned} \tag{1.17}$$

This is a complex solution. Therefore I have to take the real part of $\eta(x,t)$ before plotting and before calculating the Maximal Spatial Amplitude. The envelope of the solution $\eta(x,t)$ is:

$$envelope(\xi,\tau) = |A(\xi,\tau)| \tag{1.18}$$

$$= \frac{1}{\sqrt{2}\pi}\frac{e^{-\frac{1}{2}\xi^2\sigma^2/\left(1+\left(\sigma^2\Omega''(k_0)\tau\right)^2\right)}}{\left(1+\left(\sigma^2\Omega''(k_0)\tau\right)^2\right)^{\frac{1}{4}}} \tag{1.19}$$

This envelope is maximal at $\xi = 0$ for all τ (first derivative $= 0$ and second derivative ≤ 0).

If at time t the carrier wave has a top in $\xi = 0$ (which corresponds to $x = \Omega'(k_0)\,t$), the maximum of $\eta(x,t)$ is the same as the maximum of the envelope. If for a given t the carrier wave has no top in $\xi = 0$, the solution $\eta(x,t)$ has it's maximum a short distance away from $x = \Omega'(k_0)\,t$, and therefore the maximum of $\eta(x,t)$ is a bit smaller than the maximum of the envelope (see figure 1.1). We can use the maximum of the envelope, which I denote $MSE(\tau)$, as the general form of the MSA (see figure 1.2):

$$MSA(\tau) \leq \frac{1}{\sqrt{2}\pi}\frac{1}{\left(1+\left(\sigma^2\Omega''(k_0)\tau\right)^2\right)^{\frac{1}{4}}} = MSE(\tau) \tag{1.20}$$

In order to make some plots we need to specify the PDE. For the plots below (figure 1.1 - 1.3) I used the linear version of the Korteweg-de Vries equation (see [3])

$$\partial_t\eta + \partial_x\eta + \frac{1}{6}\partial_{xxx}\eta = 0 \tag{1.21}$$

This PDE has the dispersion relation

$$\Omega(k) = k - \frac{1}{6}k^3 \tag{1.22}$$

For the initial conditions I used the Gaussian spectrum (1.14) with $k_0 = 1$ and $\sigma = 0.2$

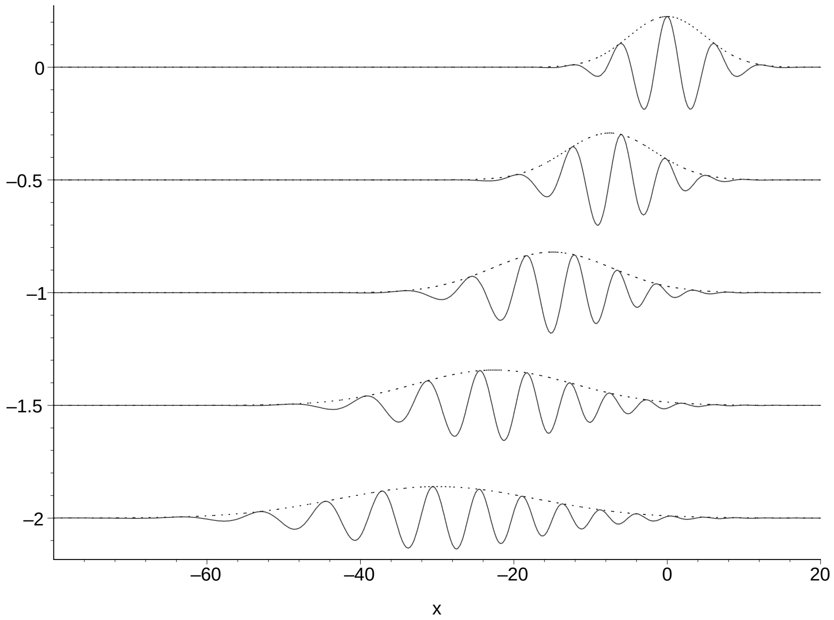

Figure 1.1: $\Re\left(\eta\left(x,t\right)\right)$ (solid) and the envelope (dotted) at times between $t=-60$ (bottom) and $t=0$ (top).

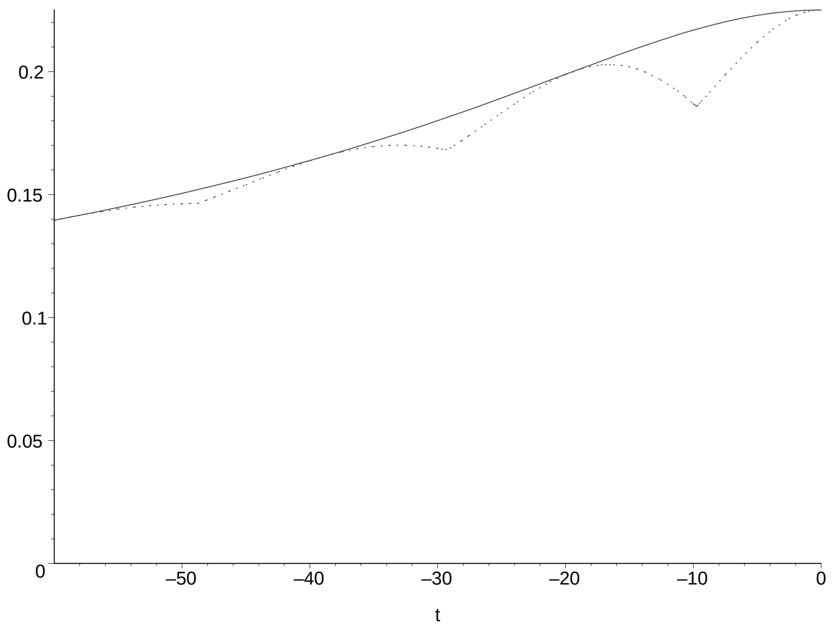

Figure 1.2: $MSE\left(t\right)$ (solid) and a sketch of $MSA\left(t\right)$ (dotted).

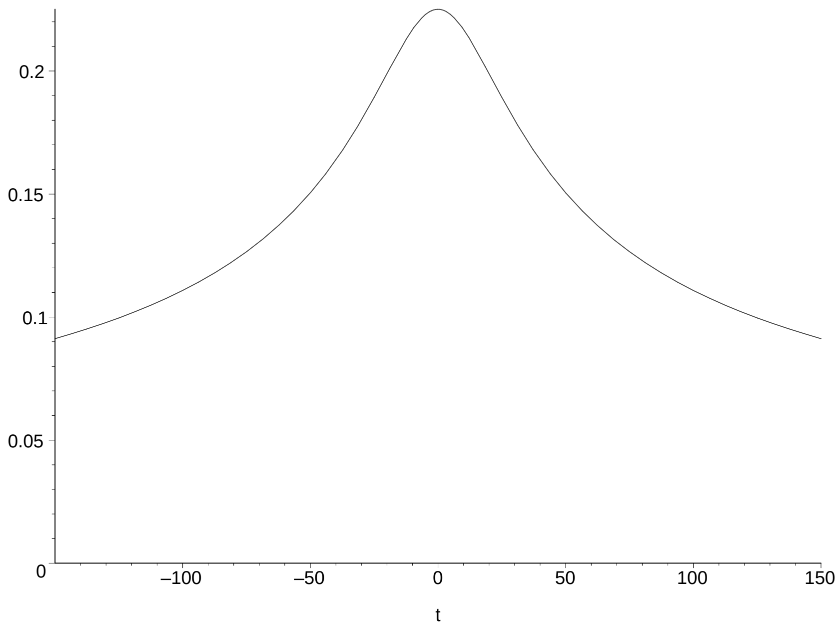

Figure 1.3: $MSE\left(t\right)$ over a longer period.

As figure 1.1 and 1.3 show, $\eta\left(x,t\right)$ attains the highest maximum if $t=0$ and $x=0$. This will be the case in general, because the term $e^{i(kx-\Omega(k)t)}$ in 1.5 is always smaller or equal to 1, and it is exactly 1 for all values of k only if x and t are both zero, resulting in the highest value of the wave height: $\eta\left(0,0\right)=\frac{1}{2\pi}\int g\left(k\right)dk$.

In the remainder of this report I will not look at the actual $MSA\left(t\right)$ anymore, because the function $MSE\left(t\right)$ gives a good description of the general form of $MSA\left(t\right)$ and is easier to calculate.

1.3 The influence of a phase difference

In this section I'll start with a different formula for $\eta\left(x,t\right)$, but I'll follow the same steps as in section 1.1, so the results from both sections can be compared.

As seen in the example from the previous section all the different waves had a maximum in $x=0$ at $t=0$. All these maximums in exactly the same place at the same time give the MSA it's maximum at $t=0$. In general the waves will have a phase difference though. This is modeled by adding the term $e^{i\theta(k)}$ to formula (1.5):

$$\eta\left(x,t\right)=\frac{1}{2\pi}\int g\left(k\right)e^{i\theta(k)}e^{i(kx-\Omega(k)t)}dk \tag{1.23}$$

You can see that choosing $\theta\left(k\right)=0$ gives the same formula as formula (1.5). Therefore choosing $\theta\left(k\right)=0$ in the results of this section, should give the same results as in the previous section.

In section 1.1 the dispersion relation $\Omega(k)$ was approximated by the second-order Taylor expansion. We can do the same for the phase difference $\theta(k) \approx \theta(k_0) + \theta'(k_0)(k-k_0) + \frac{1}{2}\theta''(k_0)(k-k_0)^2$, which will result in an approximation to the real solutions only if the spectrum $g(k)$ is concentrated around k_0. The solution $\eta(x,t)$ now becomes

$$\eta(x,t) = e^{ik_0\left(x-\frac{\Omega(k_0)}{k_0}t\right)} e^{i\theta(k_0)} \frac{1}{2\pi} \int g(\kappa+k_0)\, e^{i\left(\kappa x - \Omega'(k_0)\kappa t - \frac{1}{2}\Omega''(k_0)\kappa^2 t + \theta'(k_0)\kappa + \frac{1}{2}\theta''(k_0)\kappa^2\right)} d\kappa \tag{1.24}$$

This is again a combination of a carrier wave and an amplitude. By using the transformation (1.9) the amplitude simplifies to

$$A(\xi,\tau) = \frac{1}{2\pi}\int g(\kappa+k_0)\, e^{i\left(\kappa(\xi+\theta'(k_0)) - \frac{1}{2}\Omega''(k_0)\kappa^2\left(\tau - \frac{\theta''(k_0)}{\Omega''(k_0)}\right)\right)} d\kappa \tag{1.25}$$

Comparing this with equation (1.10) shows that the amplitude is now shifted in space and time, as is the envelope, which is the absolute value of this amplitude. Taking the maximum over all ξ for each τ results in a function $MSE(\tau)$ which is just shifted in time compared to the same function in the previous section:

$$MSE(\tau) = MSE_{\theta=0}\left(\tau - \frac{\theta''(k_0)}{\Omega''(k_0)}\right) \tag{1.26}$$

But remember we assumed the spectrum to be concentrated around $k = k_0$ in order to use the Taylor expansions, so this result might not be true for other spectra.

For some specific phases, however, this result of a MSE shifted in time, is valid for any spectrum. For example if the phase is a constant $\theta(k) = \alpha$: The term $e^{i\alpha}$ can be put in front of the integral. And since the absolute value $|e^{i\alpha}| = 1$, the envelope of the wave height with a constant phase is the same as the envelope of the wave height with no phase.

Another possibility is a linear phase $\theta(k) = \beta k$. In this case the solution 1.23 will be

$$\eta(x,t) = \frac{1}{2\pi}\int g(k)\, e^{i\beta k} e^{i(kx-\Omega(k)t)} dk \tag{1.27}$$

$$= \frac{1}{2\pi}\int g(k)\, e^{i(k(x+\beta)-\Omega(k)t)} dk \tag{1.28}$$

$$\tag{1.29}$$

Therefore a linear phase is the same as a translation of the spatial variable. So the $MSE(t)$ stays the same.

Finally if the phase is a multiple of the dispersion relation $\theta(k) = \gamma\Omega(k)$. Then the solution can be written as

$$\eta(x,t) = \frac{1}{2\pi}\int g(k)\, e^{i\gamma\Omega(k)} e^{i(kx-\Omega(k)t)} dk \tag{1.30}$$

$$= \frac{1}{2\pi}\int g(k)\, e^{i(kx-\Omega(k)(t-\gamma))} dk \tag{1.31}$$

$$\tag{1.32}$$

This formula shows that a phase which is a multiple of the dispersion relation, gives the same results as a linear translation of the time variable. Therefore the MSE will be shifted in time with respect to the case with zero phase.

Of course any linear combinations of these three types of phases also result in a shifted Maximal Spatial Envelope:

$$MSE_{\theta=\alpha+\beta k+\gamma\Omega(k)}\left(t\right) = MSE_{\theta=0}\left(t-\gamma\right) \tag{1.33}$$

1.4 Various descriptions for the maximum of the envelope

If the envelope has it's maximum at $\xi = 0$ for all τ, various descriptions of this maximum can be found by expressing $A\left(0,\tau\right)$ in different ways. With a small change in notation the amplitude (1.10) becomes

$$A\left(\xi,\tau\right) = \frac{1}{2\pi}\int \widehat{A_0}\left(\kappa\right) e^{i\left(\kappa\xi-\nu(\kappa)\tau\right)} d\kappa \tag{1.34}$$

Now rewrite $A\left(0,\tau\right)$ using the fact that $\widehat{A_0}\left(\kappa\right)$ is the Fourier transform of $A\left(\xi,0\right)$

$$\begin{aligned}
A\left(0,\tau\right) &= \frac{1}{2\pi}\int \widehat{A_0}\left(\kappa\right) e^{-i\nu(\kappa)\tau} d\kappa && (1.35)\\
&= \frac{1}{2\pi}\int \left(\int A_0\left(\xi\right) e^{-i\kappa\xi} d\xi\right) e^{-i\nu(\kappa)\tau} d\kappa && (1.36)\\
&= \int A_0\left(\xi\right)\left(\frac{1}{2\pi}\int e^{-i\left(\kappa\xi+\nu(\kappa)\tau\right)} d\kappa\right) d\xi && (1.37)\\
&= \int A_0\left(\xi\right) S\left(\xi,\tau\right) d\xi && (1.38)
\end{aligned}$$

Similar to $\eta\left(x,t\right)$ and $A\left(\xi,\tau\right)$ in section 1.1, $S\left(\xi,\tau\right) = \frac{1}{2\pi}\int e^{-i\left(\kappa\xi+\nu(\kappa)\tau\right)} d\kappa$ is also the solution of a PDE

$$\partial_\tau S + i\nu\left(i\partial_\xi\right) S = 0 \tag{1.39}$$

with initial condition

$$S\left(\xi,0\right) = \frac{1}{2\pi}\int e^{-i\kappa\xi} d\kappa = \delta\left(\xi\right) \tag{1.40}$$

If from $\nu\left(\kappa\right)$, κ can be written as a function of ν, the expression (1.35) can also be rewritten to

$$A(0,\tau) = \frac{1}{2\pi}\int \widehat{A_0}(\kappa)\, e^{-i\nu(\kappa)\tau} d\kappa \tag{1.41}$$

$$= \frac{1}{2\pi}\int \widehat{A_0}(\kappa(\nu))\, e^{-i\nu\tau}\frac{d\kappa(\nu)}{d\nu} d\nu \tag{1.42}$$

$$= \frac{1}{2\pi}\int \left(\int A_0(\xi)\, e^{-i\kappa(\nu)\xi} d\xi\right) e^{-i\nu\tau}\frac{d\kappa(\nu)}{d\nu} d\nu \tag{1.43}$$

$$= \int A_0(\xi)\left(\frac{1}{2\pi}\int \frac{d\kappa(\nu)}{d\nu} e^{-i(\kappa(\nu)\xi+\nu\tau)} d\nu\right) d\xi \tag{1.44}$$

$$= \int A_0(\xi)\, G(\xi,\tau)\, d\xi \tag{1.45}$$

Just like $S(\xi,\tau)$ also $G(\xi,\tau) = \frac{1}{2\pi}\int \frac{d\kappa(\nu)}{d\nu} e^{-i(\kappa(\nu)\xi+\nu\tau)} d\nu$ is the solution of a PDE

$$\partial_\xi G + i\kappa(i\partial_\tau)\, G = 0 \tag{1.46}$$

$$G(0,\tau) = \int \frac{1}{2\pi}\frac{d\kappa(\nu)}{d\nu} e^{-i\nu\tau} d\nu \tag{1.47}$$

So if the envelope has it's maximum at $\xi = 0$ for all τ, it can be expressed as the absolute value of an integral $\int A_0(\xi)\, f(\xi,\tau)\, d\xi$. Where $f(\xi,\tau)$ is the solution of the initial value problem (1.39) and (1.40) or boundary value problem (1.46) and (1.47).

1.5 Spectra based on field data

In this section I will look at the behavior of the maximum amplitude when using spectra other than the Gaussian spectrum I used in section 1.2. In particular the Pierson-Moskowitz and the JONSWAP spectrum which can be found for instance in the book of S.R.Massel [5]. These spectra are based on theoretical discoveries combined with field data of wind generated surface waves on different seas., hence more realistic than a Gaussian spectrum. Because these are frequency spectra the Fourier integral should be taken over ω instead of k. Another important difference is that these spectra are power spectra, but the Fourier integral needs an amplitude spectrum. Therefore I should use the square root of these power spectra.

$$\eta(x,t) = \frac{1}{2\pi}\int \sqrt{S(\omega)} e^{i(K(\omega)x-\omega t)} d\omega \tag{1.48}$$

where $K(\omega)$ is the inverse of the dispersion relation $\Omega(k) = \sqrt{k\tanh(k)}$. This is the 'exact' dispersion relation; the one I used in section 1.2 is an approximation of this one. I don't consider any phase in this equation. Therefore the highest maximum of $\eta(x,t)$ will be obtained when x and t are both zero. The value of this maximum is $\eta(0,0) = \int \sqrt{S(\omega)} d\omega$. The absolute value $|\eta(x,t)|$ is the envelope of $\eta(x,t)$. Taking the maximum over t of this envelope results in the Maximal Temporal Envelope $MTE(x)$.

The spectrum proposed by Pierson and Moskowitz in 1964 is [6]

$$S(\omega) = \alpha g^2 \omega^{-5} e^{-B\left(\frac{g}{\omega U}\right)^4} \tag{1.49}$$

where $\alpha = 8.1\ 10^{-3}$, $B = 0.74$ and U is the wind speed at a height of 19.5 m above the sea surface. This spectrum was proposed for fully-developed sea.

I calculated the normalized spectra with different values of the wind speed U and I plotted these normalized spectra and the corresponding $MTE(x)$ (see figure 1.4 and 1.5).

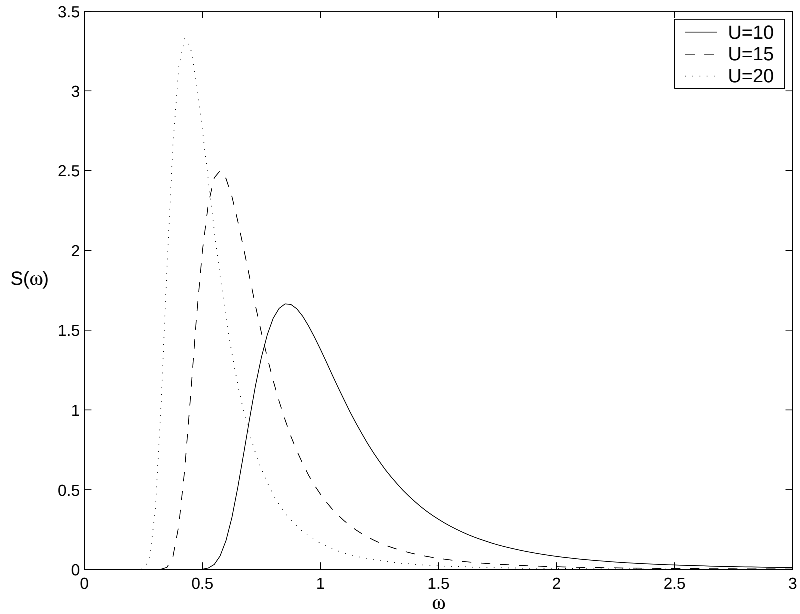

Figure 1.4: Pierson-Moskowitz spectrum

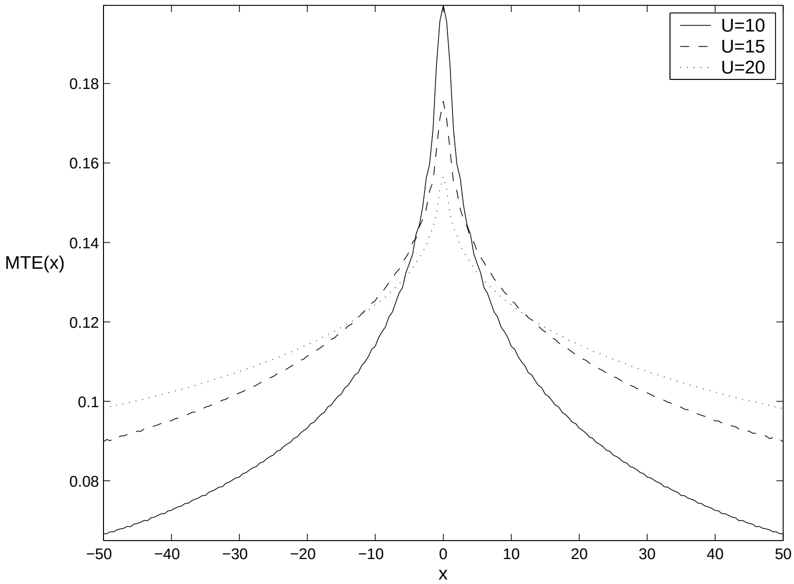

Figure 1.5: $MTE\,(x)$ corresponding to Pierson-Moskowitz spectrum

The experimental spectra given by Pierson and Moskowitz yield $\frac{U\omega_p}{g} = 0.879$. Substitution into (1.49) leads to

$$S\,(\omega) = \alpha g^2 \omega^{-5} e^{-\frac{5}{4}\left(\frac{\omega}{\omega_p}\right)^{-4}} \tag{1.50}$$

The JONSWAP spectrum extends this form of the Pierson-Moskowitz spectrum to include fetch-limited seas. JONSWAP stands for Joint North Sea Wave Project. It is a wave measurement program carried out in 1968 and 1969 in the North Sea [4]. The JONSWAP spectrum model that is based on this program takes the form

$$S\,(\omega) = \alpha g^2 \omega^{-5} e^{-\frac{5}{4}\left(\frac{\omega}{\omega_p}\right)^{-4}} \gamma^{e^{-\frac{(\omega-\omega_p)^2}{2\sigma_0^2\omega_p^2}}} \tag{1.51}$$

The mean JONSWAP spectrum yields $\gamma = 3.3$, $\sigma_0' = 0.07$, $\sigma_0'' = 0.09$ and

$$\alpha = 0.076 \left(\frac{gX}{U^2}\right)^{-0.22} \tag{1.52}$$

$$\omega_p = 7\pi \frac{g}{U} \left(\frac{gX}{U^2}\right)^{-0.33} \tag{1.53}$$

For comparison with the Pierson-Moskowitz spectrum I plotted the normalized JONSWAP spectra and the corresponding $MTE\,(x)$ for a fully developed sea ($X = 200$ km) with different wind velocities U (figure 1.6 and 1.7).

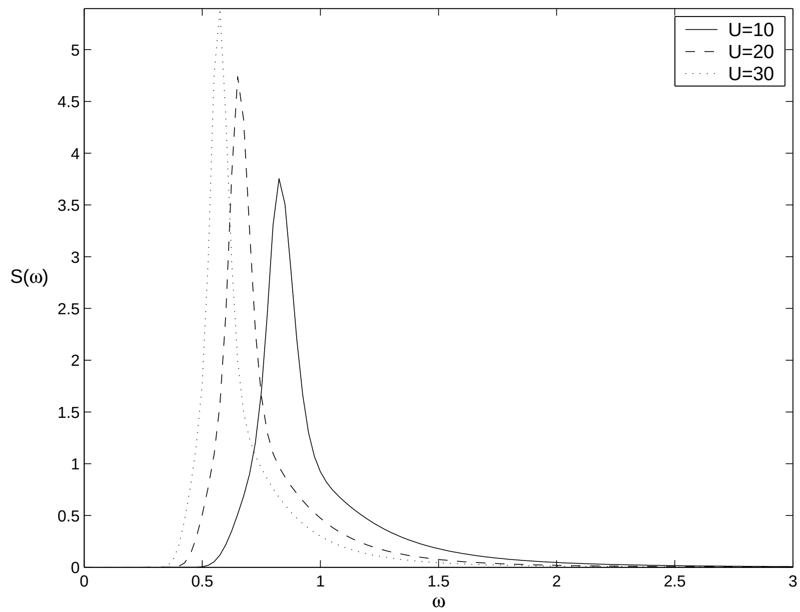

Figure 1.6: JONSWAP spectrum for fully developed sea

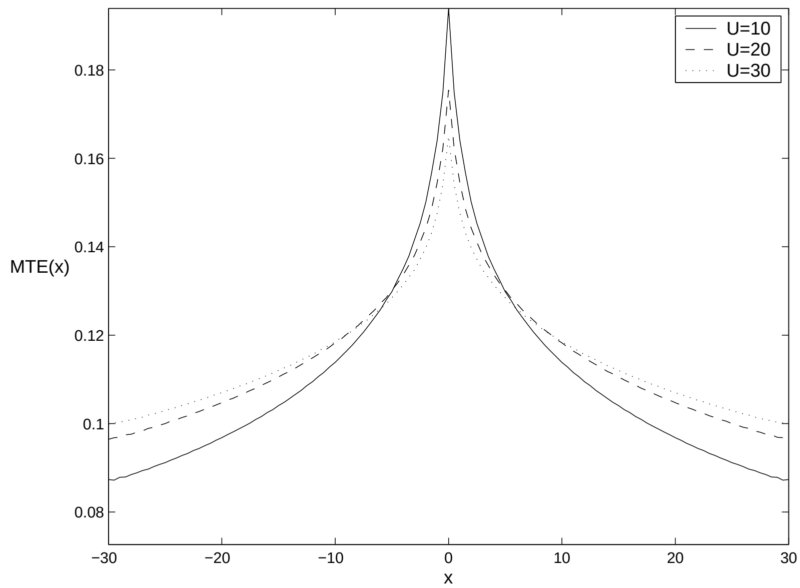

Figure 1.7: Corresponding $MTE\,(x)$

As you can see in figure 1.7, a lower wind speed results in a higher and sharper peak in the $MTE\,(x)$. Just like it did with the Pierson-Moskowitz spectrum (figure 1.5)

I also plotted the JONSWAP spectrum and the MTE for fetch X ranging from $X = 25$ km (fetch-limited seas) to $X = 200$ km (fully developed sea) with $U = 10$ m/s (figure 1.8 and 1.9).

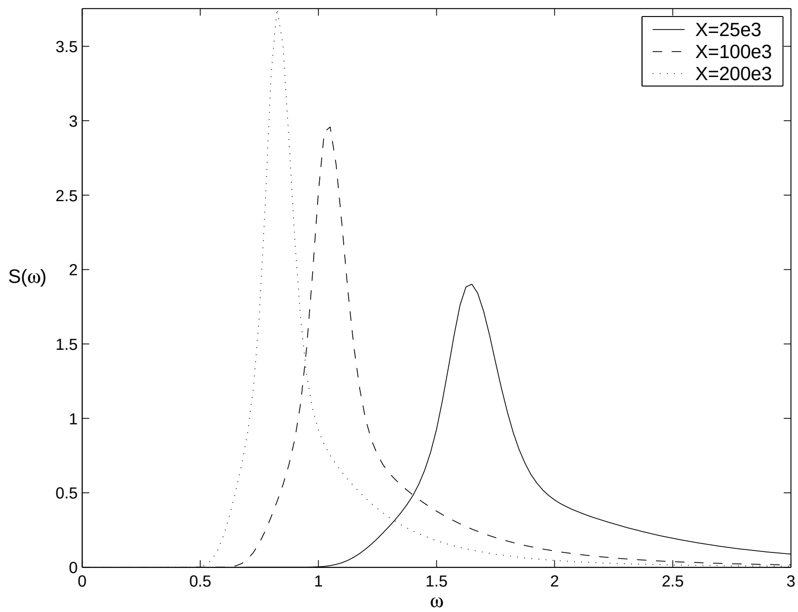

Figure 1.8: JONSWAP spectrum for different fetch with wind velocity $U = 10$ m/s

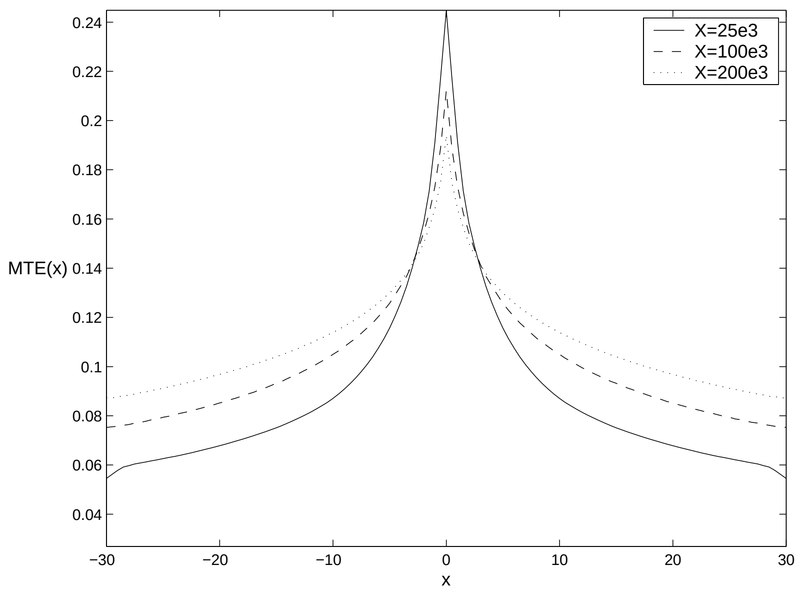

Figure 1.9: Corresponding $MTE\,(x)$

This figure shows that on fetch limited seas the MTE has a higher and sharper peak than on fully developed seas.

Now I will observe how a change in some of the other parameters effects the spectrum and the $MTE\,(x)$. First I will look at γ which describes the degree of peakedness of the JONSWAP spectrum. A higher value of γ results in a spectrum that is more peaked, and the corresponding $MTE\,(x)$ will be wider (see figure 1.10 and 1.11).

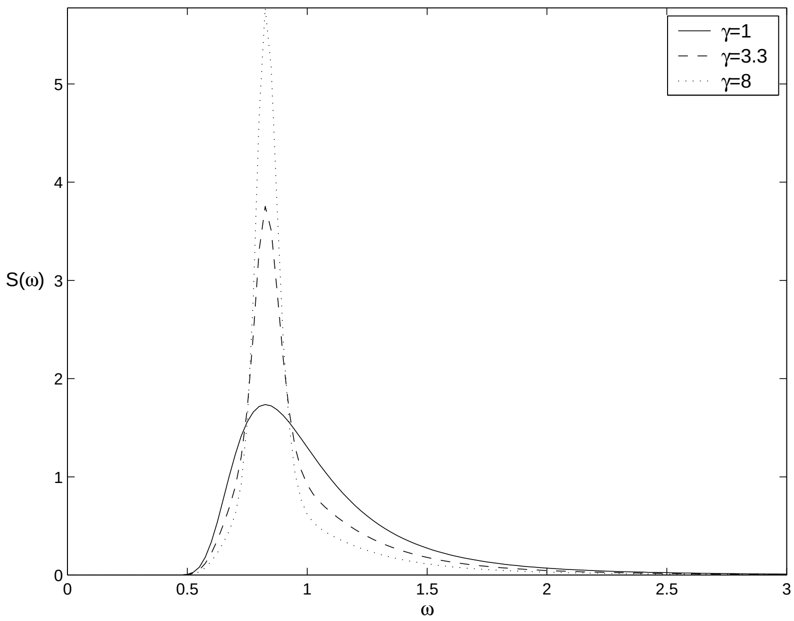

Figure 1.10: JONSWAP spectrum for different values of γ

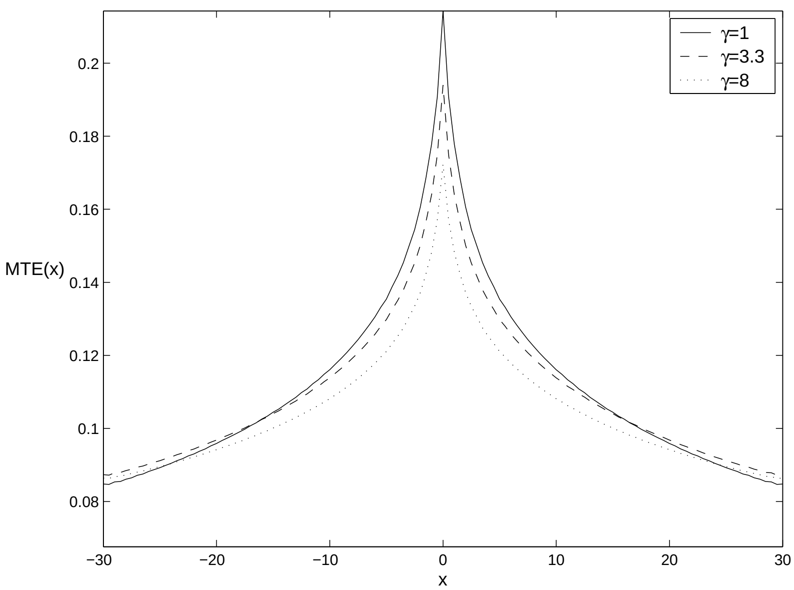

Figure 1.11: Corresponding $MTE\,(x)$

You can see in figures 1.10 and 1.11 that a spectrum which is more peaked, due to a higher value of γ, results in a MTE which is less peaked (the dotted line).

The width of the peak region will change when changing σ_0' and σ_0'': Smaller values of σ_0' and σ_0'' result in a smaller peak of the spectrum (and a wider base after normalizing the spectrum). The corresponding $MTE\,(x)$ will be more peaked (figure 1.12 and 1.13).

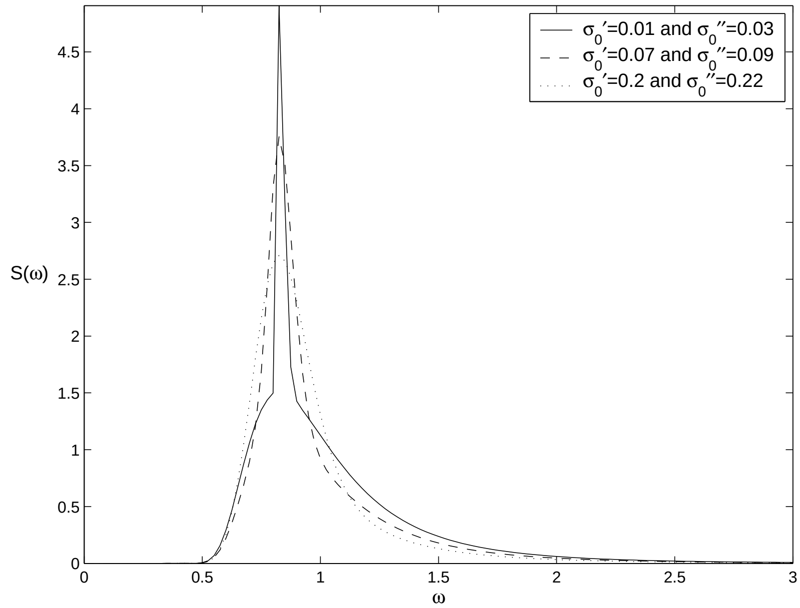

Figure 1.12: JONSWAP spectrum for different values of σ_0' and σ_0''

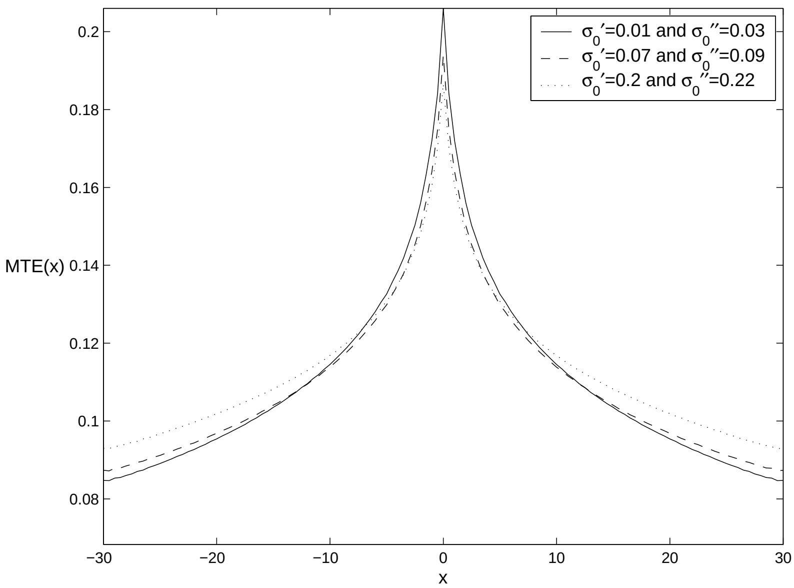

Figure 1.13: Corresponding $MTE\,(x)$

For a comparison with the Gaussian spectrum I used in a previous section, I normalized a Gaussian spectrum, the Pierson-Moskowitz spectrum and the JONSWAP spectrum on a fully developed sea with a wind speed $U = 10$ m/s. Next I calculated the $MTE\,(x)$ for each of those three spectra, and plotted them in one figure (see figure 1.14 and 1.15).

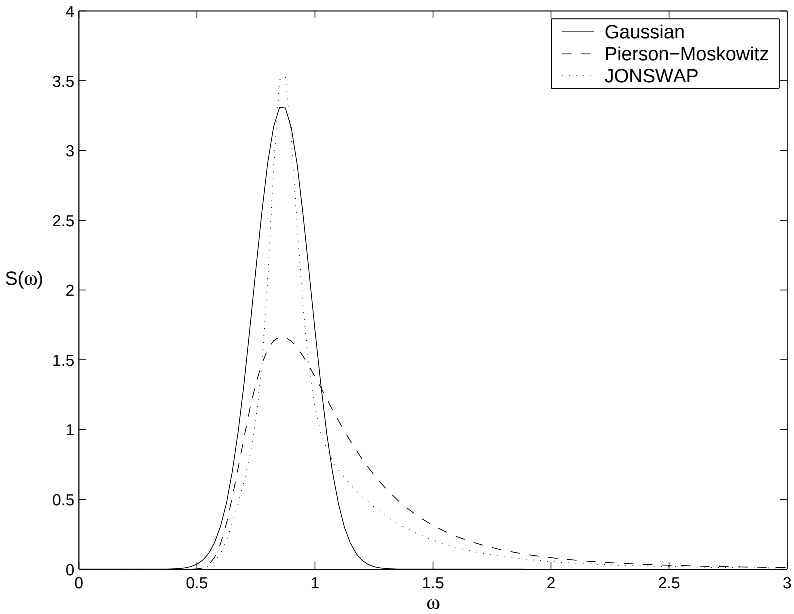

Figure 1.14: various spectra

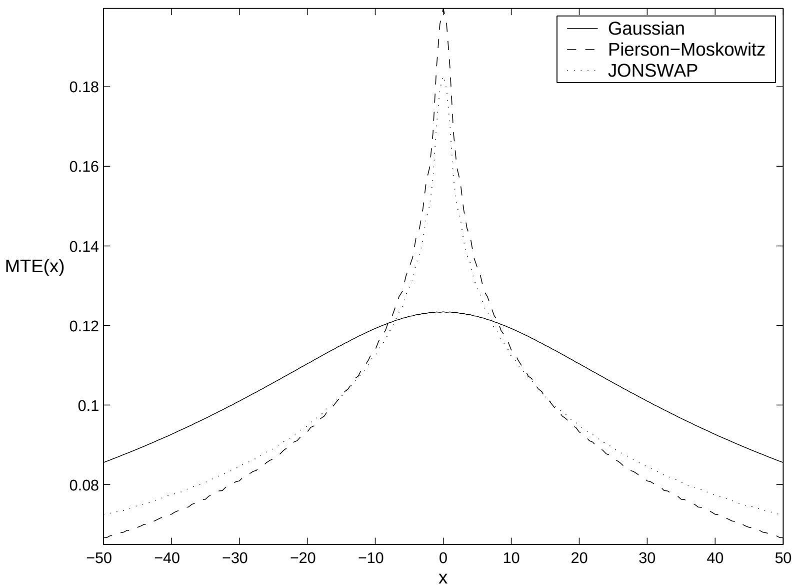

Figure 1.15: $MTE\,(x)$ corresponding to various spectra

As you can see in this last picture, the tail of the JONSWAP and the Pierson-Moskowits spectrum results in a sharper peak for the $MTE\,(x)$.

Chapter 2

Largest possible wave given certain momentum and energy

The differential equations I used in the previous chapter have several constants of motion including momentum and energy:

$$I(\eta) = \int \eta^2 dx \tag{2.1}$$

$$H(\eta) = \int (\partial_x \eta)^2 \, dx \tag{2.2}$$

So the momentum and energy of the initial condition are the same as the momentum and energy of the wave height at any other time. In this chapter I want to investigate which wave has the largest wave hight given only the momentum and energy.

I have to specify both the momentum and energy, because if I only specify one of them, the solution with the largest wave height has a maximum which goes to infinity.

2.1 Formulation of the optimization problem

Consider the following problem with two constraints: momentum and energy.

$$\max_{\eta} \left\{ \max_{x} \eta(x) \,\middle|\, \int \eta^2(x)\, dx = \gamma_1, \int (\partial_x \eta(x))^2 \, dx = \gamma_2 \right\} \tag{2.3}$$

In this problem there are three functionals

$$F(\eta) = \max_{x} \{\eta(x)\} \tag{2.4}$$

$$G_1(\eta) = \int \eta^2(x)\, dx \tag{2.5}$$

$$G_2(\eta) = \int (\partial_x \eta(x))^2 \, dx \tag{2.6}$$

We can translate this constrained variational problem into a differential equation by calculating the variational derivative for each functional and applying Lagrange multiplier rule.

The first variation of $G_1(\eta)$ is

$$\begin{aligned}
\delta G_1(\eta; v) &= \frac{d}{d\epsilon} G_1(\eta + \epsilon v)\Big|_{\epsilon=0} \\
&= \frac{d}{d\epsilon} \int (\eta(t) + \epsilon v(x))^2 \, dx\Big|_{\epsilon=0} \\
&= \int 2\eta(x) v(x) \, dx
\end{aligned} \tag{2.7}$$

Which is the L_2 inner product of $v(t)$ and the variational derivative

$$\delta G_1(\eta) = 2\eta \tag{2.8}$$

The first variation of $G_2(\eta)$ is

$$\begin{aligned}
\delta G_2(\eta; v) &= \frac{d}{d\epsilon} \int (\partial_x \eta(x) + \epsilon \partial_x v(x))^2 \, dx\Big|_{\epsilon=0} \\
&= \int 2\partial_x \eta(x) \partial_x v(x) \, dx \\
&= -\int 2\partial_{xx} \eta(x) v(x) \, dx + [2\partial_x \eta(x) v(x)]_{-\infty}^{\infty}
\end{aligned} \tag{2.9}$$

if $\eta(x)$ goes to 0 for $x \to \pm\infty$, so does $v(x)$ and so the boundary term vanishes. The related variational derivative is

$$\delta G_2(\eta) = 2\partial_{xx}\eta \tag{2.10}$$

In order to calculate the first variational of the functional $F(\eta)$, I have to rewrite it as follows:

$$\begin{aligned}
F(\eta + \epsilon v) &= \max_x \{(\eta + \epsilon v)(x)\} \\
&= (\eta + \epsilon v)(x^*(\epsilon))
\end{aligned} \tag{2.11}$$

where $x^*(\epsilon)$ is the value of x for which $\eta(x) + \epsilon v(x)$ has its maximum. The first variation of $F(\eta)$ is the derivative of $F(\eta + \epsilon v)$ with respect to ϵ at $\epsilon = 0$. For continuous functions η and v this is

$$\begin{aligned}
\delta F(\eta; v) &= \frac{d}{d\epsilon} (\eta + \epsilon v)(x^*(\epsilon))\Big|_{\epsilon=0} \\
&= v(x^*(0))
\end{aligned} \tag{2.12}$$

In order to calculate the variational derivative, the first variation must be in the form of an integral

$$\begin{aligned} \delta F(\eta; v) &= v(x^*(0)) \\ &= \int v(x)\,\delta(x - x^*(0))\,dx \end{aligned} \tag{2.13}$$

This first variation is again the inner product of $v(x)$ and the variational derivative

$$\delta F(\eta) = \delta(x - x^*) \tag{2.14}$$

where $x^* = x^*(0)$ is the value of x for which $\eta(x)$ has its maximum.

Applying Lagrange multiplier rule to the three variational derivatives calculated above

$$\sigma \delta F(\eta) = \lambda_1 \delta G_1(\eta) + \lambda_2 \delta G_2(\eta) \tag{2.15}$$

gives a differential equation

$$\boxed{\sigma \delta(x - x^*) = 2\lambda_1 \eta(x) - 2\lambda_2 \partial_{xx} \eta(x)} \tag{2.16}$$

2.2 Single peak solutions

In this section I want to find single solutions of the differential equation (2.16). In section 2.4 I'll look for periodic solutions. The differential equation derived in the previous section is

$$\sigma \delta(x - x^*) = 2\lambda_1 \eta(x) - 2\lambda_2 \partial_{xx} \eta(x) \tag{2.17}$$

And the solution to this differential equation should of course satisfy the constraints in the constrained variational problem (2.3)

$$\int_{-\infty}^{\infty} \eta^2(x)\,dx = \gamma_1 \tag{2.18}$$

$$\int_{-\infty}^{\infty} (\partial_x \eta(x))^2\,dx = \gamma_2 \tag{2.19}$$

Because the first constraint $\int \eta^2(x)\,dx = \gamma_1 < \infty$ we must assume the boundary conditions $\lim_{x \to -\infty} \eta(x) = 0$ and $\lim_{x \to \infty} \eta(x) = 0$.

If $\sigma = 0$ the differential equation reduces to $\lambda_1 \eta(x) - \lambda_2 \partial_{xx} \eta(x) = 0$. The only solution to this differential equation that satisfies the boundary conditions above is the trivial solution $\eta(x) = 0$. Therefore I take $\sigma = 1$ (for any other value I can divide the entire differential equation by σ.).

For $x < x^*$ we have the differential equation $\lambda_1 \eta(x) - \lambda_2 \partial_{xx} \eta(x) = 0$. Because of the first boundary condition, the solution to this differential equation is

$$\eta(x) = c_1 e^{\sqrt{\frac{\lambda_1}{\lambda_2}}(x - x^*)} \text{ for } x < x^* \tag{2.20}$$

where λ_1 and λ_2 are both positive or both negative. In a similar way for $x > x^*$:

$$\eta(x) = c_2 e^{-\sqrt{\frac{\lambda_1}{\lambda_2}}(x-x^*)} \text{ for } x > x^* \tag{2.21}$$

$\eta(x)$ must be continuous in $x = x^*$, so $c_2 = c_1$. Now for the δ-function it holds that $\int_a^b \delta(x - x^*)\, dx = 1$ for any $a < x^*$ and $b > x^*$. Therefore

$$\begin{aligned}
1 &= \int_a^b 2\lambda_1 \eta(x) - 2\lambda_2 \partial_{xx} \eta(x)\, dx \\
&= 2\lambda_1 \int_a^b \eta(x)\, dx + 2\lambda_2 \left(\partial_x \eta(a) - \partial_x \eta(b)\right) \\
&= 2\lambda_1 \int_a^b \eta(x)\, dx + 2\lambda_2 \sqrt{\frac{\lambda_1}{\lambda_2}} c_1 \left(e^{\sqrt{\frac{\lambda_1}{\lambda_2}}(a-x^*)} + e^{-\sqrt{\frac{\lambda_1}{\lambda_2}}(b-x^*)}\right)
\end{aligned} \tag{2.22}$$

for any $a < x^*$ and $b > x^*$. In the limit where a and b go to x^*, the integral term tends to zero, and this equation reduces to $1 = 4\lambda_2 \sqrt{\frac{\lambda_1}{\lambda_2}} c_1$, so c_1 must be $c_1 = \pm \frac{1}{4\sqrt{\lambda_1 \lambda_2}}$, where the negative sign is used when λ_1 and λ_2 are negative.

It turns out that a negative c_1 produces a negative solution $\eta(x)$ with a minimum in $x = x^*$. Therefore the solution we are looking for is the one with the positive value of c_1. Now substitute this value for c_1 and the same value for c_2 into the solution (2.20) and (2.21):

$$\eta(t) = \frac{e^{-\sqrt{\frac{\lambda_1}{\lambda_2}}|x-x^*|}}{4\sqrt{\lambda_1 \lambda_2}} \tag{2.23}$$

Substituting this solution in the constraints gives two equations from which λ_1 and λ_2 can be expressed in γ_1 and γ_2. The first constraint (2.18) gives:

$$\begin{aligned}
\gamma_1 &= \int \eta^2(x)\, dx \\
&= \int_{-\infty}^{x^*} \left(\frac{e^{\sqrt{\frac{\lambda_1}{\lambda_2}}(x-x^*)}}{4\sqrt{\lambda_1\lambda_2}}\right)^2 dx + \int_{x^*}^{\infty} \left(\frac{e^{-\sqrt{\frac{\lambda_1}{\lambda_2}}(x-x^*)}}{4\sqrt{\lambda_1\lambda_2}}\right)^2 dx \\
&= \int_{-\infty}^{x^*} \frac{e^{2\sqrt{\frac{\lambda_1}{\lambda_2}}(x-x^*)}}{16\lambda_1\lambda_2} dx + \int_{x^*}^{\infty} \frac{e^{-2\sqrt{\frac{\lambda_1}{\lambda_2}}(x-x^*)}}{16\lambda_1\lambda_2} dx \\
&= \left[\frac{e^{2\sqrt{\frac{\lambda_1}{\lambda_2}}(x-x^*)}}{32\lambda_1\lambda_2\sqrt{\frac{\lambda_1}{\lambda_2}}}\right]_{-\infty}^{x^*} - \left[\frac{e^{-2\sqrt{\frac{\lambda_1}{\lambda_2}}(x-x^*)}}{32\lambda_1\lambda_2\sqrt{\frac{\lambda_1}{\lambda_2}}}\right]_{x^*}^{\infty} \\
&= \pm \frac{1}{16\lambda_1\sqrt{\lambda_1\lambda_2}}
\end{aligned} \tag{2.24}$$

Similar for the second constraint (2.19):

$$\gamma_2 = \pm \frac{1}{16\lambda_1\sqrt{\lambda_1\lambda_2}} \tag{2.25}$$

where the negative sign is used when λ_1 and λ_2 are negative. Combining equations (2.24) and (2.25) gives the following expressions for λ_1 and λ_2:

$$\lambda_1 = \pm \frac{\gamma_2^{1/4}}{4\gamma_1^{3/4}} \tag{2.26}$$

$$\lambda_2 = \pm \frac{\gamma_1^{1/4}}{4\gamma_2^{3/4}} \tag{2.27}$$

with λ_1 and λ_2 both positive or both negative. The solution (2.23) can now be written in terms of γ_1 and γ_2.

$$\eta(x) = (\gamma_1\gamma_2)^{1/4} e^{-\left(\frac{\gamma_2}{\gamma_1}\right)^{1/2}|x-x^*|} \tag{2.28}$$

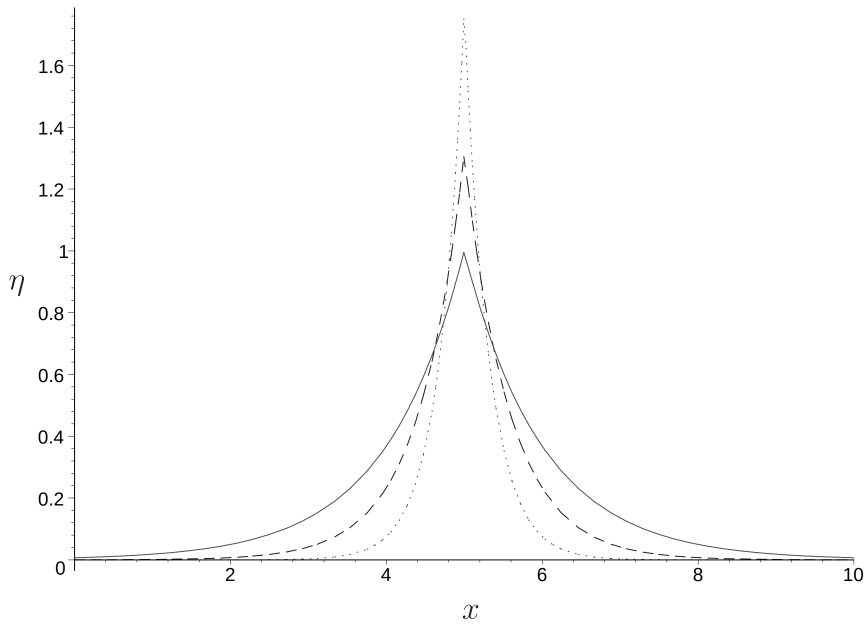

Figure 2.1: $\eta(x)$ for $\gamma_2 = 1$ (solid), $\gamma_2 = 3$ (dashed) and $\gamma_2 = 10$ (dotted).

In this plot I used $x^* = 5$, but any other value of x^* results only in a linear translation along the x-axis. As you can see in this plot, a higher value of γ_2 gives a solution $\eta(x)$ with a smaller, but higher peak. This can be derived from the formulas as well. First the height of the peak is the maximum of 2.28:

$$\max_x (\eta(x)) = \eta(x^*) = (\gamma_1\gamma_2)^{1/4} \tag{2.29}$$

and second the vertex angle, which is a measure for the width of the peak, is

$$\text{vertex angle} = 2\arctan\left(\gamma_1^{1/4}\gamma_2^{-3/4}\right) \tag{2.30}$$

2.3 Verification by Fourier transformation

We can check whether the calculations in the previous sections are correct using Fourier transform on the constrained variational problem we started with (2.3), on the differential equation we found halfway through (2.16) and on the solution we found in the end (2.28). To simplify the calculations I'll take $x^* = 0$ in this section. In that case the constrained variational problem (2.3) is

$$\max_{\eta}\left\{\max_{x}\eta\left(x\right)\middle|\int\eta^{2}\left(x\right)dx=\gamma_{1},\int\left(\partial_{x}\eta\left(x\right)\right)^{2}dx=\gamma_{2}\right\} \tag{2.31}$$

The differential equation (2.16) reduces to

$$\delta\left(x\right)=2\lambda_{1}\eta\left(x\right)-2\lambda_{2}\partial_{xx}\eta\left(x\right) \tag{2.32}$$

And the solution (2.28) reduces to

$$\eta\left(x\right)=\left(\gamma_{1}\gamma_{2}\right)^{1/4}e^{-\left(\frac{\gamma_{2}}{\gamma_{1}}\right)^{1/2}\left|x\right|} \tag{2.33}$$

If we express the constrained variational problem in the Fourier transform $\hat{\eta}$ of η, and solve this problem, we should get a solution which is the Fourier transform of the solution in the previous section.

Let's take a look at the three functionals of the constrained variational problem. The first one is $\max\limits_{x}\eta\left(x\right)=\eta\left(0\right)$. Expressed in $\hat{\eta}$ this is

$$\eta\left(0\right)=\frac{1}{2\pi}\int\hat{\eta}\left(k\right)e^{ik0}dk=\frac{1}{2\pi}\int\hat{\eta}\left(k\right)dk \tag{2.34}$$

The second functional can be expressed in $\hat{\eta}$ using Parseval's theorem.

$$\int\eta^{2}\left(x\right)dx=\frac{1}{2\pi}\int\left|\hat{\eta}\left(k\right)\right|^{2}dk \tag{2.35}$$

For the last functional the calculations can be found in appendix A.

$$\int\left(\partial_{x}\eta\left(x\right)\right)^{2}dx=\frac{1}{2\pi}\int k^{2}\left|\hat{\eta}\left(k\right)\right|^{2}dk \tag{2.36}$$

With these three functionals the constrained variational problem (2.31) can be expressed in $\hat{\eta}$

$$\max_{\hat{\eta}}\left\{\frac{1}{2\pi}\int\hat{\eta}\left(k\right)dk\middle|\frac{1}{2\pi}\int\left|\hat{\eta}\left(k\right)\right|^{2}dk=\gamma_{1},\frac{1}{2\pi}\int k^{2}\left|\hat{\eta}\left(k\right)\right|^{2}dk=\gamma_{2}\right\} \tag{2.37}$$

Now we need to find the variational derivatives of the three functionals and apply Lagrange multiplier rule. The variational derivative of $\frac{1}{2\pi}\int\hat{\eta}\left(k\right)dk$ is $\frac{1}{2\pi}$. Assuming $\hat{\eta}$ is real, the variational derivative of $\frac{1}{2\pi}\int |\hat{\eta}(k)|^2 dk$ is $\frac{1}{2\pi}2\hat{\eta}(k)$ and the variational derivative of $\frac{1}{2\pi}\int k^2 |\hat{\eta}(k)|^2 dk$ is $\frac{1}{2\pi}2k^2\hat{\eta}(k)$. Applying Lagrange multiplier rule gives the following equation

$$1 = 2\lambda_1\hat{\eta}(k) + 2\lambda_2 k^2\hat{\eta}(k) \tag{2.38}$$

This is indeed the Fourier transform of the differential equation (2.32). The solution of this equation is

$$\hat{\eta}(k) = \frac{1}{2(\lambda_1 + \lambda_2 k^2)} \tag{2.39}$$

By substituting this solution in the constraints we get:

$$\gamma_1 = \pm\frac{1}{16\lambda_1\sqrt{\lambda_1\lambda_2}} \tag{2.40}$$

$$\gamma_2 = \pm\frac{1}{16\lambda_1\sqrt{\lambda_1\lambda_2}} \tag{2.41}$$

where the negative sign is used when λ_1 and λ_2 are negative. From these two equations λ_1 and λ_2 can be expressed in γ_1 and γ_2:

$$\lambda_1 = \pm\frac{\gamma_2^{1/4}}{4\gamma_1^{3/4}} \tag{2.42}$$

$$\lambda_2 = \pm\frac{\gamma_1^{1/4}}{4\gamma_2^{3/4}} \tag{2.43}$$

with λ_1 and λ_2 both positive or both negative. Since we want to find the maximal solution, not the minimal one, we have to use the positive λ_1 and λ_2. Then the solution $\hat{\eta}$ expressed in γ_1 and γ_2 is

$$\hat{\eta}(k) = \frac{2(\gamma_1\gamma_2)^{3/4}}{\gamma_2 + \gamma_1 k^2} \tag{2.44}$$

The inverse Fourier transform of this solution is

$$\eta(x) = (\gamma_1\gamma_2)^{1/4} e^{-\left(\frac{\gamma_2}{\gamma_1}\right)^{1/2}|x|} \tag{2.45}$$

which is indeed the solution (2.33) we found in solving the constrained variational problem in η.

2.4 Periodic solutions

In this section I'll try to find periodic solutions of the differential equation (2.16)

$$\sigma\delta\left(x-x^{*}\right)=2\lambda_{1}\eta\left(x\right)-2\lambda_{2}\partial_{xx}\eta\left(x\right) \tag{2.46}$$

For periodic solutions there is of course more than one x^* where η attains it's maximum. I can choose $x^* = 0$ to simplify the calculations. I'll try to find a solution in the interval $[0, X]$ where X is the period of the solution $\eta(x)$. Because I choose $x^* = 0$, the differential equation reduces to

$$2\lambda_{1}\eta\left(x\right)-2\lambda_{2}\partial_{xx}\eta\left(x\right)=0 \tag{2.47}$$

in the interior of the interval $[0, X]$.

If λ_1 and λ_2 have the same sign, the solution of this differential equation is

$$\eta\left(x\right)=c_{1}e^{sx}+c_{2}e^{-sx} \tag{2.48}$$

where $s=\sqrt{\lambda_1/\lambda_2}$. This is a solution on one period, which I can extend to the entire real line, but for the extended solution to be continuous I must hold that

$$\eta\left(0\right)=\eta\left(X\right) \tag{2.49}$$

If $\sigma = 0$, and λ_1 and λ_2 have the same sign, the only possible periodic solution of the differential equation 2.46 is $\eta(x) = 0$. Therefore I can take $\sigma = 1$. We assumed the maximum to be at $x = 0$. At this maximum the delta-function in the differential equation (2.46) is not equal to zero, so for the solution (2.48) to satisfy the differential equation at $x = 0$ it must satisfy the following equation

$$\begin{aligned}
1&=\int_{-0}^{+0}\delta\left(x\right)dx\\
1&=\int_{-0}^{+0}2\lambda_{1}\eta\left(x\right)dx-\int_{-0}^{+0}2\lambda_{2}\partial_{xx}\eta\left(x\right)dx\\
1&=0-2\lambda_{2}\left(\partial_{x}\eta\left(+0\right)-\partial_{x}\eta\left(-0\right)\right)\\
1&=-2\lambda_{2}\left(\partial_{x}\eta\left(0\right)-\partial_{x}\eta\left(X\right)\right)
\end{aligned} \tag{2.50}$$

With these two equations I can find expressions for c_1 and c_2, and the solution (2.48) becomes

$$\eta\left(x\right)=\frac{-e^{-s\frac{X}{2}}}{2\lambda_{2}s\left(e^{-sX}-1\right)}\cosh\left(s\left(x-\frac{X}{2}\right)\right) \tag{2.51}$$

with $s=\sqrt{\frac{\lambda_1}{\lambda_2}}$.

If I substitute this solution in the two constraints

$$\int_0^X \eta^2(x)\, dx = \gamma_1 \tag{2.52}$$

$$\int_0^X (\partial_x \eta(x))^2\, dx = \gamma_2 \tag{2.53}$$

I get two equation which implicitly express λ_1 and λ_2 in γ_1 and γ_2.

$$\gamma_1 = \frac{e^{2sX} + 2e^{sX} sX - 1}{16\lambda_1\lambda_2 s\left(e^{sX} - 1\right)^2} \tag{2.54}$$

$$\gamma_2 = \frac{-e^{2sX} + 2e^{sX} sX + 1}{16\lambda_2^2 s\left(e^{sX} - 1\right)^2} \tag{2.55}$$

with $s = \sqrt{\frac{\lambda_1}{\lambda_2}}$.

Now for some plots I choose $X = \pi$ and $\gamma_1 = 1$.

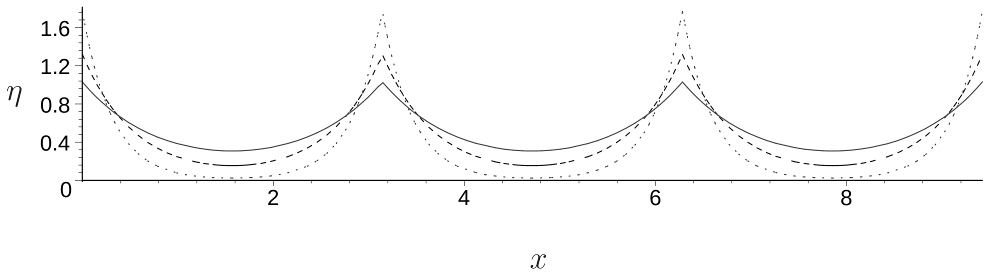

Figure 2.2: $\eta(x)$ for $\gamma_2 = 1$ (solid), $\gamma_2 = 3$ (dashed) and $\gamma_2 = 10$ (dotted).

As you can see in this plot, for a fixed period $X = \pi$ and a fixed value of $\gamma_1 = 1$, a higher value of γ_2 results in a higher maximum of the solution. The next plot shows the relation between this maximum and γ_2.

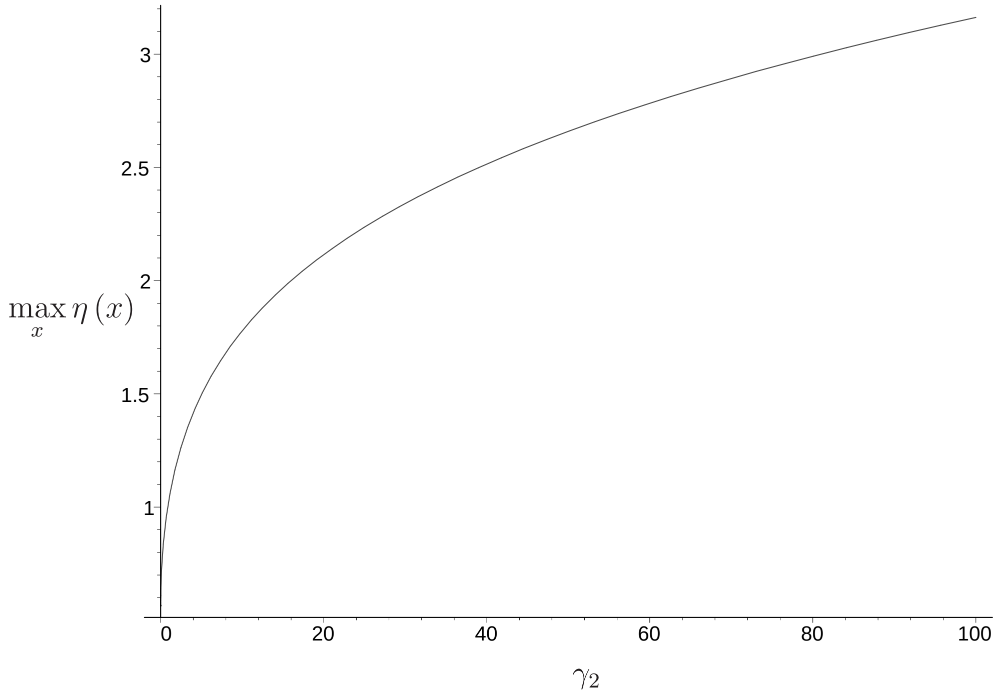

Figure 2.3: The maximum of $\eta(x)$ as a function of γ_2.

If I make a plot of the maximum of the solution on the entire real line, as I calculated in section 2.2, it will be the same as figure 2.3 for large values of γ_2 only. For small values of γ_2 the maximum of the solution on the entire real line is a little bit smaller than the maximum for the periodic case.

In the beginning of this section I assumed that λ_1 and λ_2 have the same sign. If I assume they have a different sign, I can find a solution $\eta(x)$ which has a lover maximum than the solution in this section. Therefore it is not the solution of the constrained variational problem (2.3)

$$\max_\eta \left\{ \max_x \eta(x) \,\middle|\, \int \eta^2(x)\, dx = \gamma_1, \int (\partial_x \eta(x))^2\, dx = \gamma_2 \right\}$$

but it is a solution of

$$\operatorname*{crit}_\eta \left\{ \max_x \eta(x) \,\middle|\, \int \eta^2(x)\, dx = \gamma_1, \int (\partial_x \eta(x))^2\, dx = \gamma_2 \right\}$$

Calculations and plots for this case can be found in appendix B.

Chapter 3

Physical constraints related to differential equations

In chapter 1 we used a linear dispersive differential equation to model ocean waves (1.4)

$$\partial_t \eta + i\Omega\left(-i\partial_x\right)\eta = 0 \tag{3.1}$$

This differential equations has several constants of motion including momentum and energy:

$$I\left(\eta\right) = \int \eta^2 dx \tag{3.2}$$

$$H\left(\eta\right) = \int \left(\partial_x \eta\right)^2 dx \tag{3.3}$$

So the momentum and energy of the initial condition are the same as the momentum and energy of the wave height at any other time. In chapter 2 I found the wave with the highest possible maximum which satisfied certain momentum and energy by solving the following constrained problem

$$\max_{\eta}\left\{\max_{x} \eta\left(x\right) | I\left(\eta\right) = \gamma_1, H\left(\eta\right) = \gamma_2\right\} \tag{3.4}$$

In this chapter I will use the solution to this problem as an initial condition for the differential equation.

3.1 Linear evolution of the largest possible wave

The two constraints in problem (3.4) represent momentum and energy.

$$I\left(\eta\right) = \int \eta^2 dx \tag{3.5}$$

$$H\left(\eta\right) = \int \left(\partial_x \eta\right)^2 dx \tag{3.6}$$

I will show that these constraints are constants of motion for the differential equation (3.1) with dispersion relation $\Omega(k) = k - \frac{1}{6}k^3$. Taking the partial derivative of $I(\eta)$:

$$\partial_t (I(\eta)) = \partial_t \left(\int_{-\infty}^{\infty} \eta^2 dx \right) \tag{3.7}$$

$$= \int_{-\infty}^{\infty} 2\eta\eta_t \; dx \tag{3.8}$$

The term η_t can be substituted using the differential equation (3.1)

$$\partial_t (I(\eta)) = \int_{-\infty}^{\infty} -2\eta \left(\eta_x + \frac{1}{6}\eta_{xxx} \right) dx \tag{3.9}$$

$$= \int_{-\infty}^{\infty} -2\eta\eta_x - \frac{1}{3}\eta\eta_{xxx} \; dx \tag{3.10}$$

Using integration by parts on the second term we get

$$\partial_t (I(\eta)) = \int_{-\infty}^{\infty} -2\eta\eta_x dx + \int_{-\infty}^{\infty} \frac{1}{3}\eta_x\eta_{xx} \; dx + \left[-\frac{1}{3}\eta\eta_{xx} \right]_{-\infty}^{\infty} \tag{3.11}$$

$$= \left[-\eta^2 + \frac{1}{6}(\eta_x)^2 - \frac{1}{3}\eta\eta_{xx} \right]_{-\infty}^{\infty} \tag{3.12}$$

This is equal to zero for the solutions of the optimization problem in section 2.2, because η and all it's derivatives go to zero for $x \to \pm\infty$. Therefore $I(\eta)$ is a constant of motion.

Similarly for the second constraint we have

$$\partial_t (H(\eta)) = \partial_t \left(\int_{-\infty}^{\infty} \eta_x^2 dx \right) \tag{3.13}$$

$$= \int_{-\infty}^{\infty} 2\eta_x\eta_{xt} \; dx \tag{3.14}$$

$$= \int_{-\infty}^{\infty} -2\eta_x \left(\eta_{xx} + \frac{1}{6}\eta_{xxxx} \right) dx \tag{3.15}$$

$$= \left[-\eta_x^2 \right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{3}\eta_x x\eta_{xxx} \; dx + \left[-\frac{1}{3}\eta_x\eta_{xxx} \right]_{-\infty}^{\infty} \tag{3.16}$$

$$= \left[-\eta_x^2 + \frac{1}{6}(\eta_{xx})^2 - \frac{1}{3}\eta_x\eta_{xxx} \right]_{-\infty}^{\infty} \tag{3.17}$$

$$= 0 \tag{3.18}$$

So the second constraint is also a constant of motion.

Since the two constraints in the optimization problem (3.4) are constants of motion for the differential equation (3.1), we can use the optimal solution (2.28) of the constrained problem as an initial condition.

$$\eta(x, 0) = (\gamma_1\gamma_2)^{\frac{1}{4}} e^{-\left(\frac{\gamma_2}{\gamma_1}\right)^{\frac{1}{2}}|x|} \tag{3.19}$$

Next is a plot of this initial condition for $I(\eta) = \gamma_1 = 1$ and $H(\eta) = \gamma_2$.

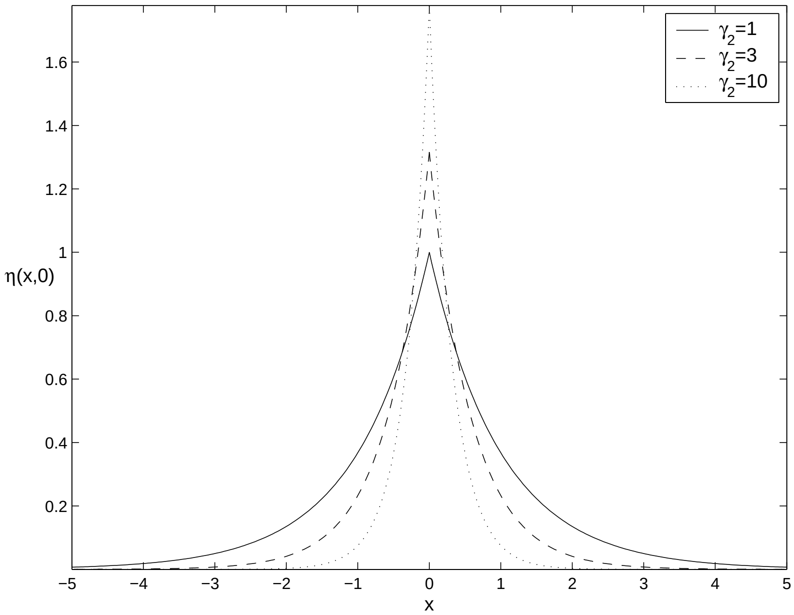

Figure 3.1: Initial condition

In this plot, and all the following plots in this section, the solid line corresponds to the initial condition with $\gamma_2 = 1$, the dashed line corresponds to $\gamma_2 = 3$ and the dotted line to $\gamma_2 = 10$.

In chapter 1 the solution of the differential equation is

$$\eta\left(x,t\right) = \frac{1}{2\pi}\int \hat{\eta}\left(k\right) e^{i(kx-\Omega(k)t)} dk \tag{3.20}$$

where $\hat{\eta}\left(k\right)$ is the Fourier transform of the initial condition $\eta\left(x,0\right)$. For the optimal solution of the constrained problem the Fourier transform was calculated in section 2.3:

$$\hat{\eta}\left(k\right) = \frac{2\left(\gamma_1\gamma_2\right)^{3/4}}{\gamma_2 + \gamma_1 k^2} \tag{3.21}$$

In figure 3.2 is a plot of this Fourier transform with $\gamma_1 = 1$.

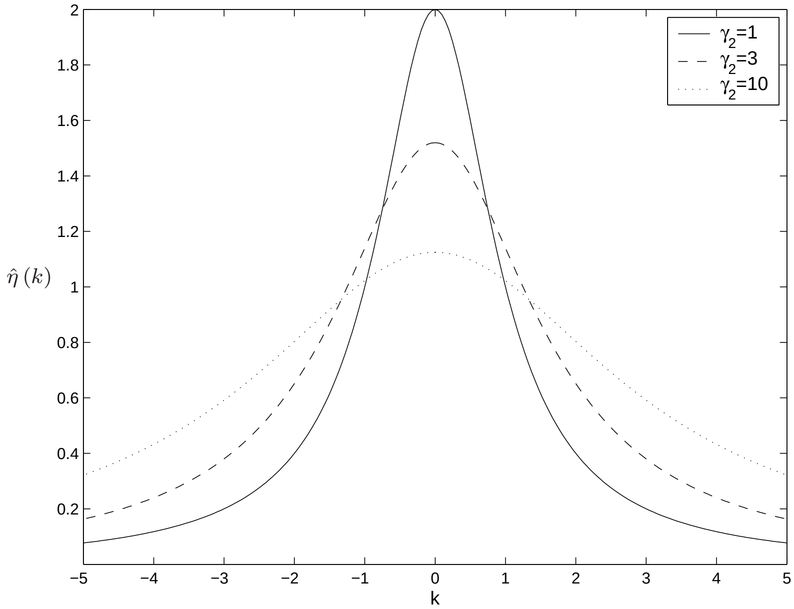

Figure 3.2: Spectrum

The term $\Omega(k)$ in equation (3.20) is the dispersion relation. In the following plots I'll use the 'exact' dispersion relation $\Omega(k) = k\sqrt{\frac{\tanh k}{k}}$.

We can't work out the integral in equation (3.20) to get an explicit formula for the wave height. Still we are able to make some plots (with the use of Matlab) of the solution $\eta(x,t)$ at different times. I added a constant for the solution at $t - 10$ and at $t = 10$ so that the plots at different times do not overlap each other.

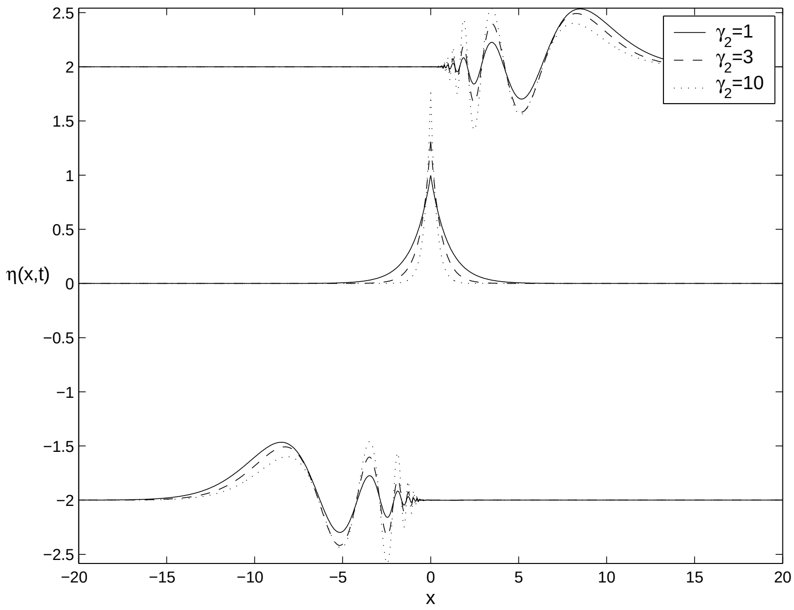

Figure 3.3: $\eta(x,t)$ for $t=-10$ (bottom), $t=0$ (middle) and $t=10$ (top)

The phase velocity for these waves is $V_{ph}(k)=\frac{\Omega(k)}{k}=\sqrt{\frac{\tanh k}{k}}$. This means that short waves, which have a large wavenumber k, travel slow and large waves travel fast. You can also see this in the figure 3.3. For $t<0$ the short waves are close to zero and the long waves are further away to the left. But since these long waves travel faster, they will catch up with the short waves, so that they all add up at $x=0$, generating a 'freak wave'. This is called 'phase focussing'.

We are really interested in the Maximal Spatial Envelope, $MSE(t)$, which is for every t the maximum over all x of the envelope of the solution:

$$MSE(t)=\max_x |\eta(x,t)| \tag{3.22}$$

For the $\eta(x,t)$ I plotted in figure 3.3 the MSE is plotted in figure 3.4.

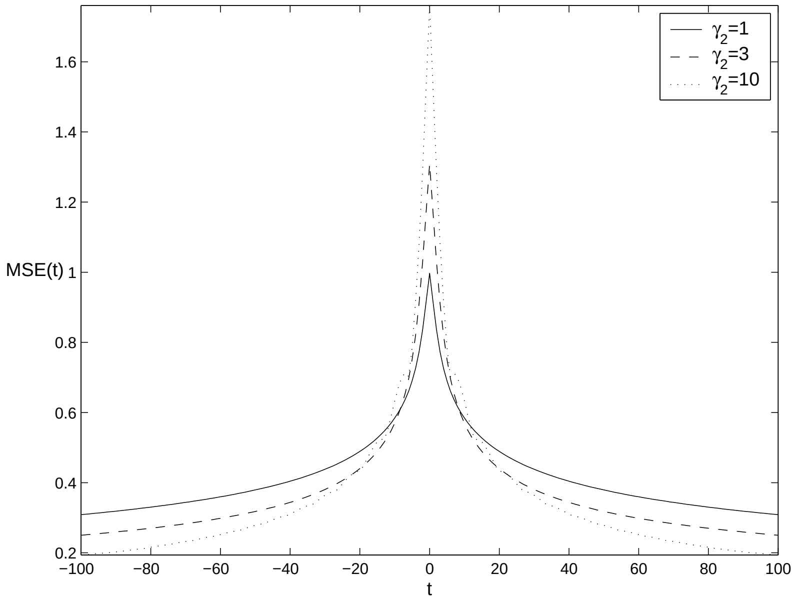

Figure 3.4: Maximal Spatial Envelope

3.2 Linear evolution of a soliton

The differential equation (3.1) I used in the previous section and in chapter 1 is the linearized KdV-equation. The KdV-equation in normalized form (see [3]):

$$\partial_t \eta = -\partial_x \left(\eta_{xx} + \eta^2\right) \tag{3.23}$$

has several constants of motion, including:

$$I(\eta) = \int \eta^2(x)\, dx \tag{3.24}$$

$$H(\eta) = \int \frac{\eta_x^2}{2} - \frac{\eta^3}{3} dx \tag{3.25}$$

This KdV-equation has many soliton solutions:

$$\eta(x, 0) = \frac{3V}{2\cosh^2\left(\frac{1}{2}\sqrt{V}\,(x - Vt)\right)} \tag{3.26}$$

where V denotes the velocity of the soliton.

Now look at what happens if I use the constants of motion as constraints in the optimization problem (3.4)

$$\max_{\eta} \left\{ \max_{x} \eta(x) \,|\, I(\eta) = \gamma_1, H(\eta) = \gamma_2 \right\} \tag{3.27}$$

This constraint problem can be solved in a similar way as the one in chapter 2, resulting in a differential equation:

$$\sigma \delta F(\eta) = \lambda_1 \delta G_1(\eta) + \lambda_2 \delta G_2(\eta) \tag{3.28}$$

For very special values of γ_1 and γ_2 the σ will be zero. The solution of this problem is then a single soliton:

$$\eta(x) = \frac{3V}{2\cosh^2\left(\frac{1}{2}\sqrt{V}x\right)} \tag{3.29}$$

In general $\sigma \neq 0$, but the left hand side of the differential equation is equal to zero for $x \neq 0$. So the solution of this problem is part of a soliton on $x > 0$ and also on $x < 0$, combined at $x = 0$ to a continuous but non-smooth function. I will look at the smooth soliton first.

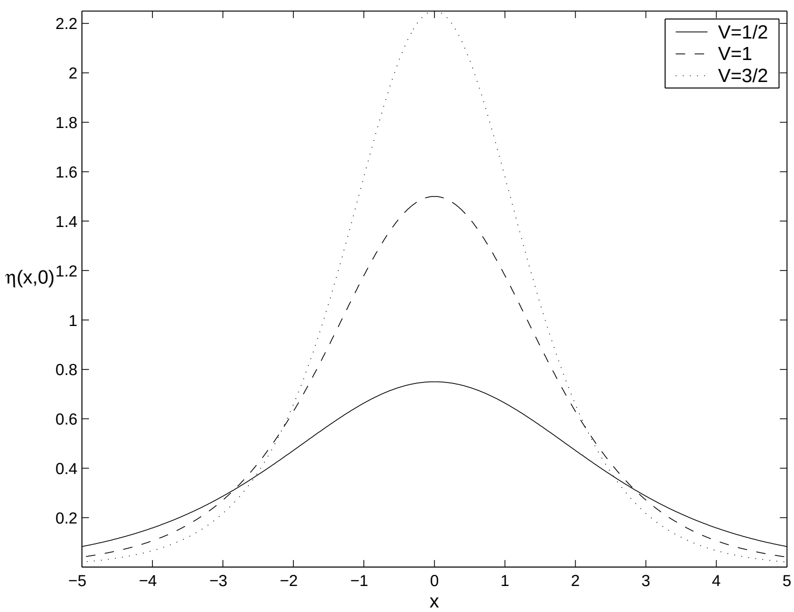

Figure 3.5: Initial condition: soliton profile

This smooth soliton will travel undisturbed with velocity V. Therefore the maximum will be the same for all time, so the Maximal Spatial Amplitude will be a constant.

It is also interested to see what will happen if we use this soliton as an initial condition for the linear differential equation we used in the previous section. I made some plots again with Matlab. In all of the plots in this section the solid line corresponds to the soliton with $V = \frac{1}{2}$, the dashed line to $V = 1$ and the dotted one to $V = \frac{3}{2}$.

First I plot the spectrum.

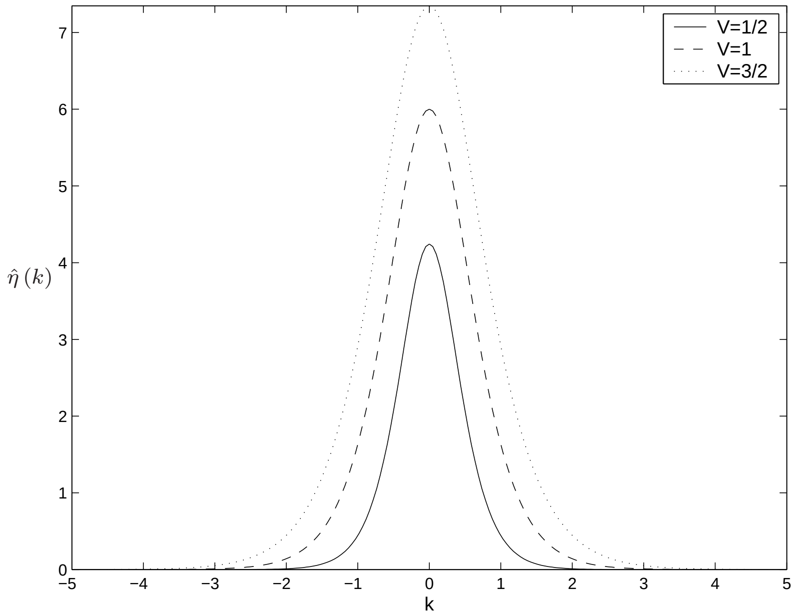

Figure 3.6: Spectrum of the soliton profile

Next is the solution $\eta(x,t)$ at different times for the linear evolution according to the dispersion relation $\Omega(k) = k\sqrt{\frac{\tanh k}{k}}$:

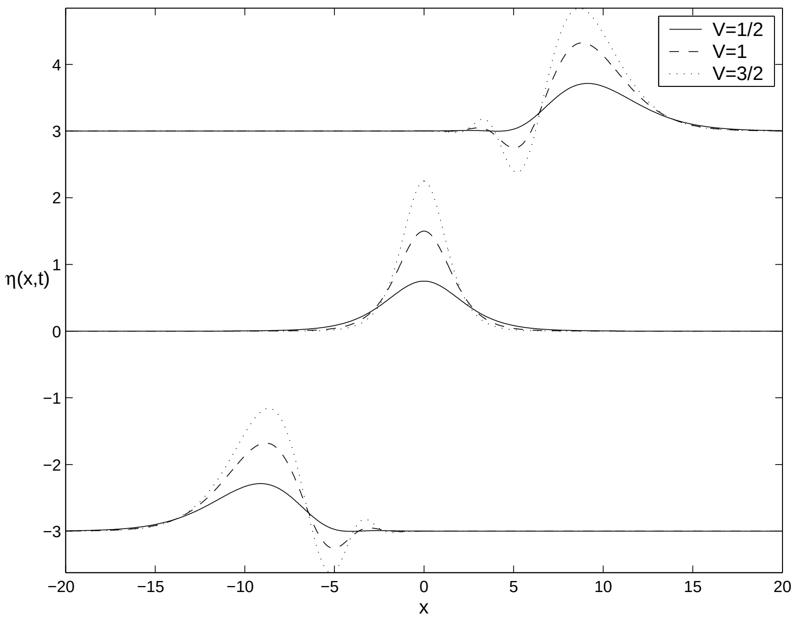

Figure 3.7: $\eta(x,t)$ for $t = -10$ (bottom), $t = 0$ (middle) and $t = 10$ (top)

And finally a plot of the Maximal Spatial Envelope:

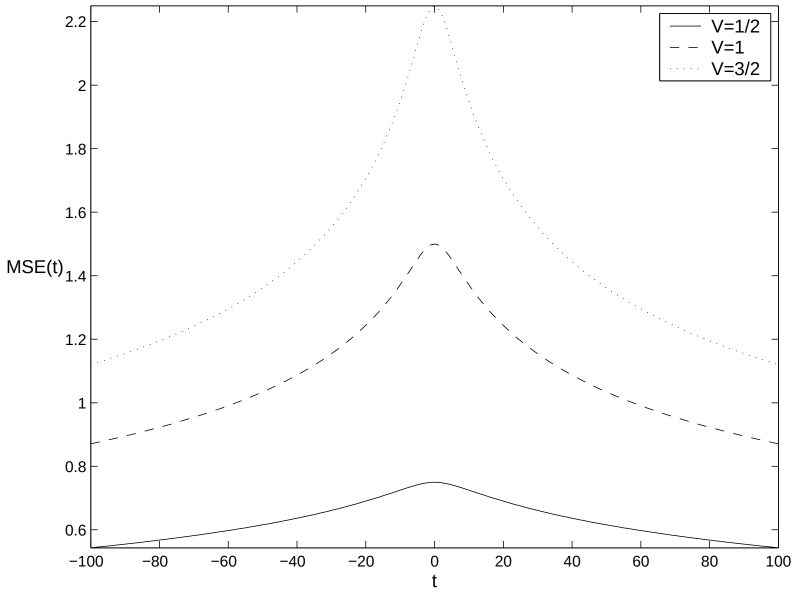

Figure 3.8: Maximal Spatial Envelope

In figure 3.7 you can see the 'phase focussing' again, which I spoke about in the previous section (figure 3.3)

In figure 3.8 you can see, the MSE is now not a constant. But it also does not descent as fast as the MSE in the previous section. The next section gives a better comparison of the two.

3.3 Comparison of cornered and smooth initial profiles

For a better comparison of the solutions in the last two sections I will choose certain values for the different constants (γ_1, γ_2 and V) in such way that the asymptotic behavior of the initial condition and the momentum are the same in both cases.

The solution in section 3.1 has momentum $\int \eta^2 dx = \gamma_1$. I choose the same value for the momentum of the soliton:

$$\gamma_1 = \int \eta^2 dx \tag{3.30}$$

$$= \int \frac{9V^2}{4\cosh^4\left(\frac{1}{2}\sqrt{V}x\right)} dx \tag{3.31}$$

$$= 6V^{\frac{3}{2}} \tag{3.32}$$

The initial condition (3.19) in section 3.1 has asymptotic behavior $e^{-(\gamma_2/\gamma_1)^{1/2}x}$. The soliton, which is used as initial condition in section 3.2, is for large values of x approximately

$$\eta(x) \approx \frac{3V}{2\left(\frac{e^{\frac{1}{2}\sqrt{V}x}}{2}\right)^2} = \frac{6V}{e^{\sqrt{V}x}} \qquad \text{for} \quad x >> 0 \tag{3.33}$$

So the asymptotic behavior is $e^{-\sqrt{V}x}$. This is the same as for the initial condition of section 3.1 if $\gamma_2 = \gamma_1 V$. Next are some plots. In these plots the dashed line corresponds to the soliton and the solid line to the initial condition of section 3.1, which is the optimal solution of chapter 2. In these pictures I used $V = 0.75$ and, to have the same momentum and asymptotic behavior for both initial conditions, $\gamma_1 = 6V^{3/2} = 2.92$ and $\gamma_2 = \gamma_1 V = 3.90$.

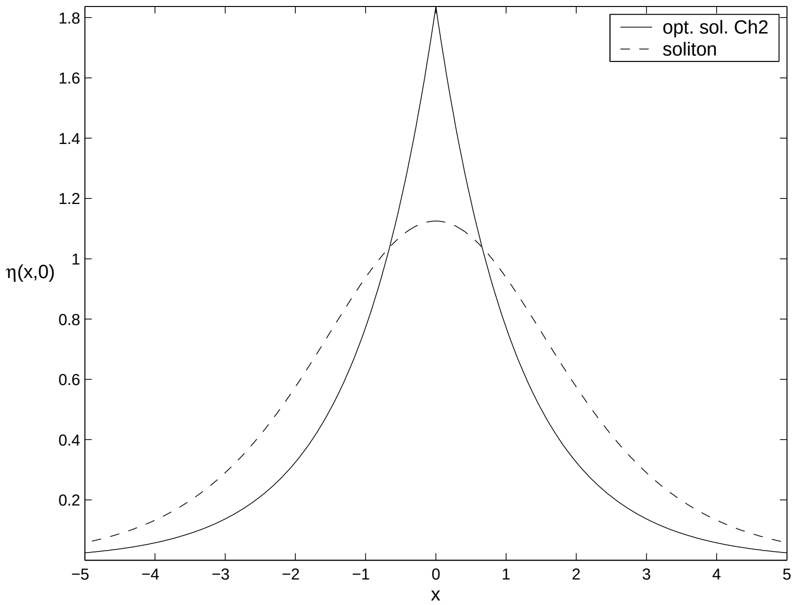

Figure 3.9: Initial condition

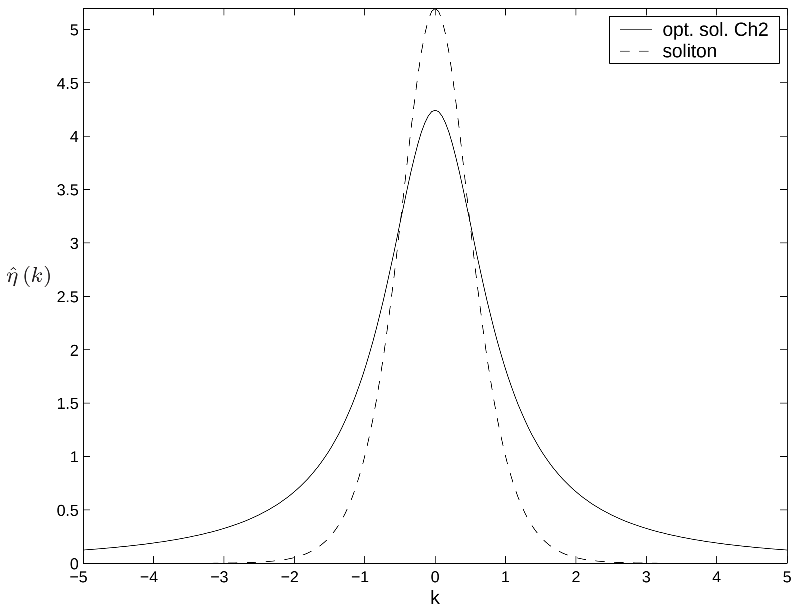

Figure 3.10: Spectrum corresponding to the initial condition in figure 3.9

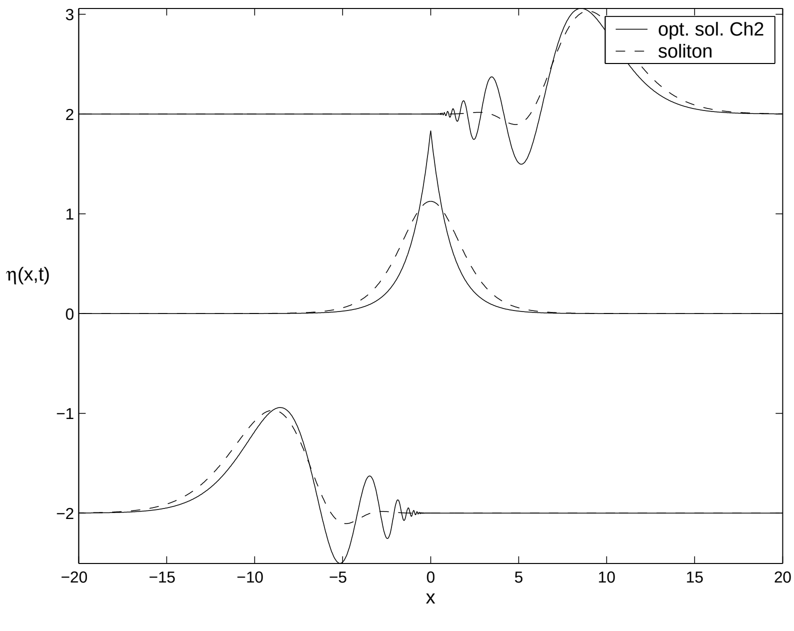

Figure 3.11: $\eta(x, t)$ for $t = -10$ (bottom), $t = 0$ (middle) and $t = 10$ (top)

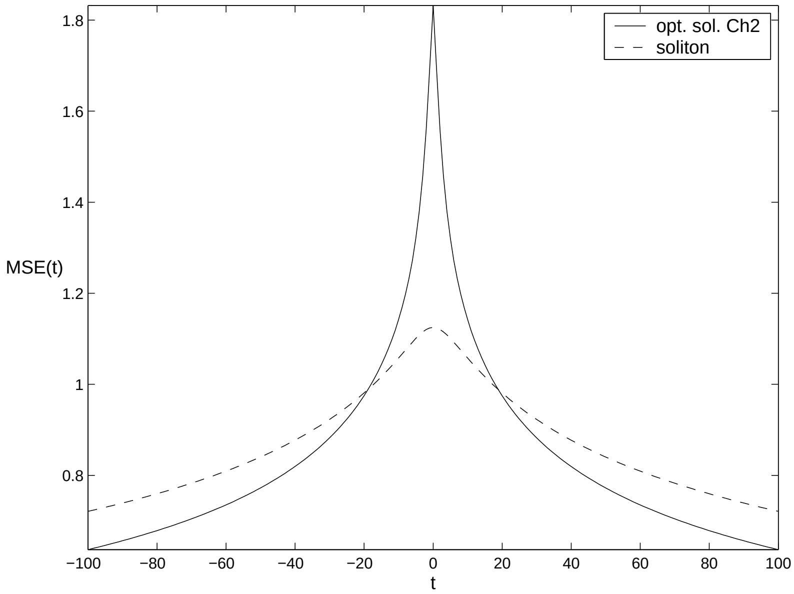

Figure 3.12: Maximal Spatial Envelope

In figure 3.11 you can see that the shape of the soliton changes much slower than the shape of the solution from chapter 2. As a result the MSE has a much slower descent.

The soliton is a smooth solution of the constrained problem 3.27. But this problem also has cornered solutions consisting of part of a soliton for $x > 0$ and also for $x < 0$, combined at $x = 0$ to a continuous but non-smooth function. The dotted line in the following plot shows such a function. The solid line is the solution of chapter 2 again. The constants (γ_1, γ_2 and V) again are chosen in a way that the momentum and asymptotic behavior of these two initial conditions are the same.

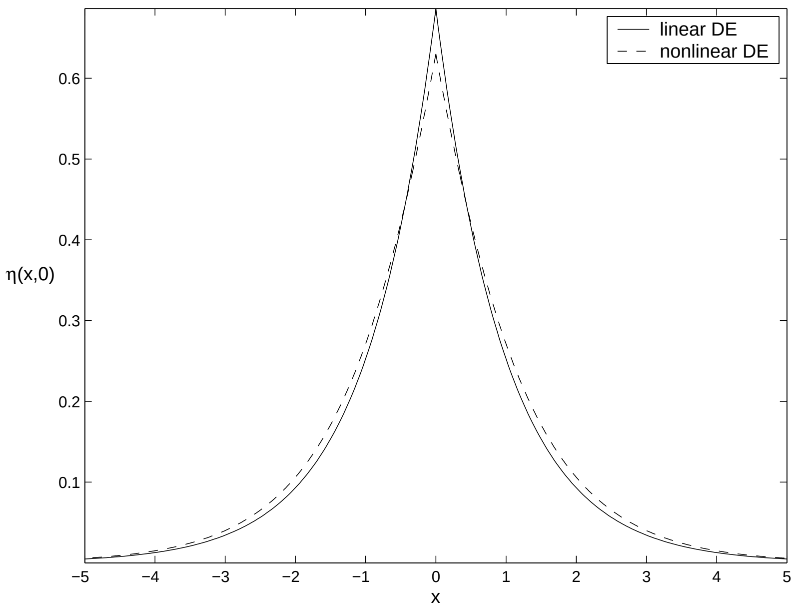

Figure 3.13: Initial condition

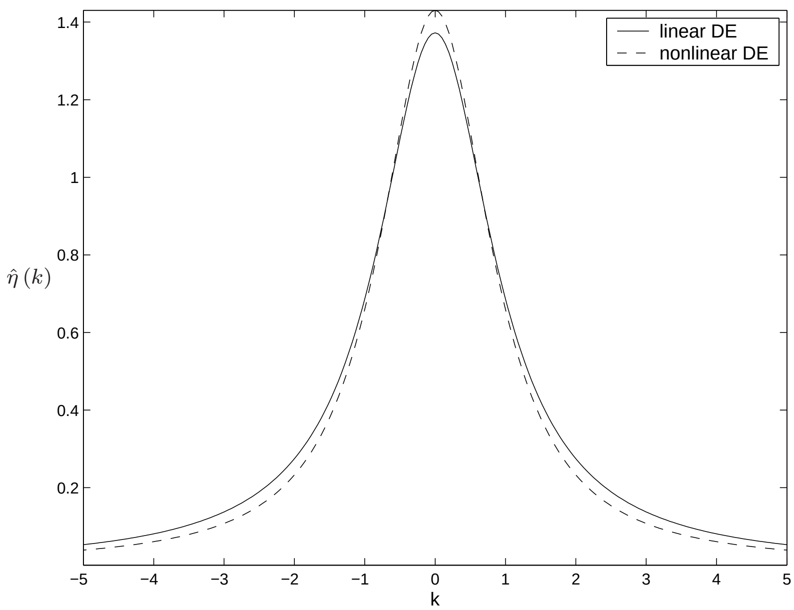

Figure 3.14: Spectrum corresponding to the initial condition in figure 3.13

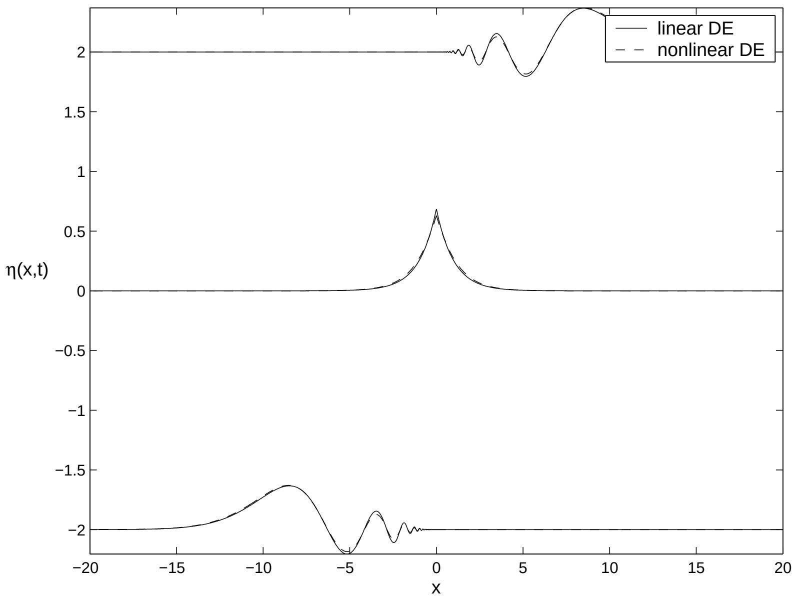

Figure 3.15: $\eta\,(x,t)$ for $t=-10$ (bottom), $t=0$ (middle) and $t=10$ (top)

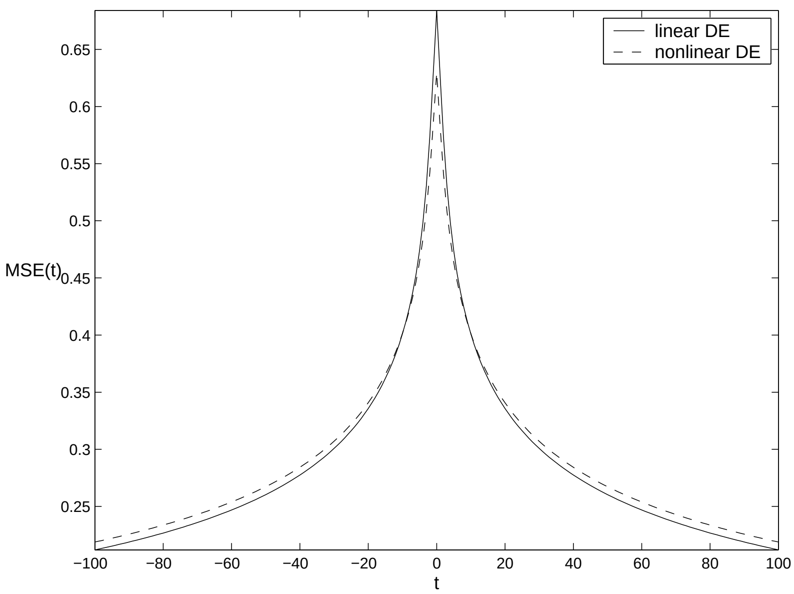

Figure 3.16: Maximal Spatial Envelope

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Appendix A

In this appendix I'll show how to get equation (2.36) in chapter 2. Which means I'll show how to express $\int (\partial_t \eta(t))^2 \, dt$ in the Fourier transform $\hat{\eta}$. First we need to express the derivative of η in the Fourier transform $\hat{\eta}$.

$$\begin{aligned} \partial_t \eta(t) &= \partial_t \left(\frac{1}{2\pi} \int \hat{\eta}(\omega) e^{i\omega t} d\omega \right) \\ &= \frac{1}{2\pi} \int i\omega \hat{\eta}(\omega) e^{i\omega t} d\omega \end{aligned} \tag{A.1}$$

This equation states that $i\omega\hat{\eta}(\omega)$ is the Fourier transform of $\partial_t \eta(t)$.

$$i\omega\hat{\eta}(\omega) = \int \partial_t \eta(t) e^{-i\omega t} dt \tag{A.2}$$

Multiplication of equation (A.1) with $\partial_t \eta(t)$ yields

$$(\partial_t \eta(t))^2 = \frac{1}{2\pi} \int i\omega\hat{\eta}(\omega) \partial_t \eta(t) e^{i\omega t} d\omega \tag{A.3}$$

Now take the integral over t on both sides of this equation

$$\begin{aligned} \int (\partial_t \eta(t))^2 dt &= \int \frac{1}{2\pi} \int i\omega\hat{\eta}(\omega) \partial_t \eta(t) e^{i\omega t} d\omega dt \\ &= \frac{1}{2\pi} \int i\omega\hat{\eta}(\omega) \left(\int \partial_t \eta(t) e^{i\omega t} dt \right) d\omega \end{aligned} \tag{A.4}$$

The part between the brackets is the complex conjugate of the right hand side of equation (A.2). Therefore

$$\begin{aligned} \int (\partial_t \eta(t))^2 dt &= \frac{1}{2\pi} \int i\omega\hat{\eta}(\omega) \overline{i\omega\hat{\eta}(\omega)} d\omega \\ &= \frac{1}{2\pi} \int \omega^2 |\hat{\eta}(\omega)|^2 d\omega \end{aligned} \tag{A.5}$$

Appendix B

In section 2.4 I found periodic solutions of the differential equation (2.46)

$$\sigma\delta\left(x-x^*\right)=2\lambda_1\eta\left(x\right)-2\lambda_2\partial_{xx}\eta\left(x\right) \tag{B.1}$$

where I assumed that λ_1 and λ_2 have the same sign. Now I will try to find solutions when λ_1 and λ_2 have a different sign.

Similar to section 2.4 I can choose $x^*=0$, which means that the solution $\eta\left(x\right)$ attains it's maximum at $x=0$. I prescribe the period to be of length X. In the interior of the interval $[0,X]$ the differential equation reduces to

$$2\lambda_1\eta\left(x\right)-2\lambda_2\partial_{xx}\eta\left(x\right)=0 \tag{B.2}$$

The solution of this equation is

$$\eta\left(x\right)=c_1\cos\left(sx\right)+c_2\sin\left(sx\right) \tag{B.3}$$

where $s=\sqrt{-\lambda_1/\lambda_2}$. I can use three equations to find the values for the constants c_1, c_2 and s: one from the periodicity and two from the constraints.

$$\eta\left(0\right)=\eta\left(X\right) \tag{B.4}$$

$$\int_0^X\eta^2\left(x\right)dx=\gamma_1 \tag{B.5}$$

$$\int_0^X\left(\partial_x\eta\left(x\right)\right)^2dx=\gamma_2 \tag{B.6}$$

From the first equation (B.4) we get $c_1=c2\frac{\sin(sX)}{1-\cos(sX)}$. Using this expression for c_1 and the second equation (B.5) it follows that $c_2=\pm\sqrt{\gamma_1 s\frac{1-\cos(sX)}{sX+sin(sX)}}$. Now the third equation (B.6) reduces to

$$\gamma_2=\gamma_1 s^2\frac{sX-\sin\left(sX\right)}{sX+\sin\left(sX\right)} \tag{B.7}$$

I can't find an explicit expression for s from equation (B.7), but If I choose a value for X, I can make a plot of s versus $\frac{\gamma_2}{\gamma_1}$. In the following plot I used $X=\pi$. Other values of X result in a similar plot.

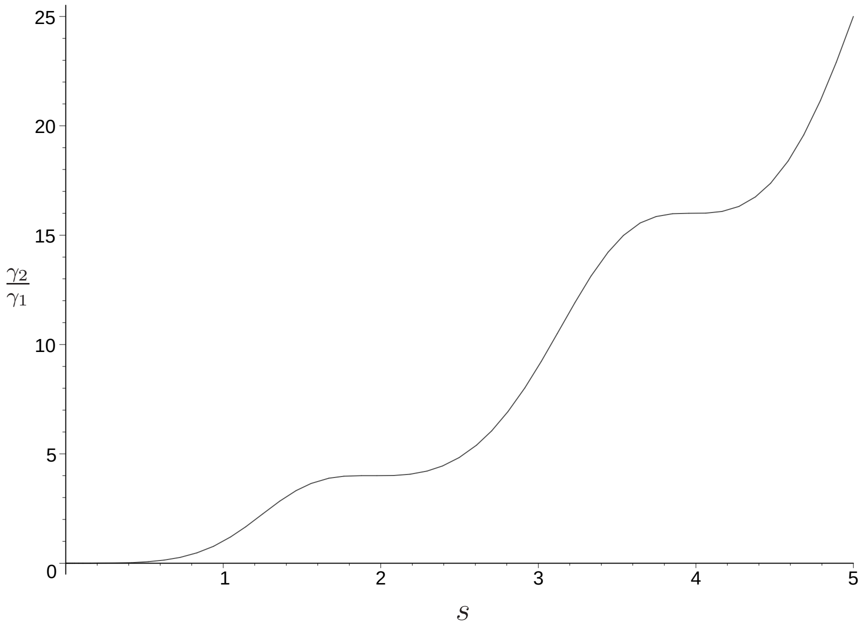

Figure B.1: Relation between s and $\frac{\gamma_2}{\gamma_1}$

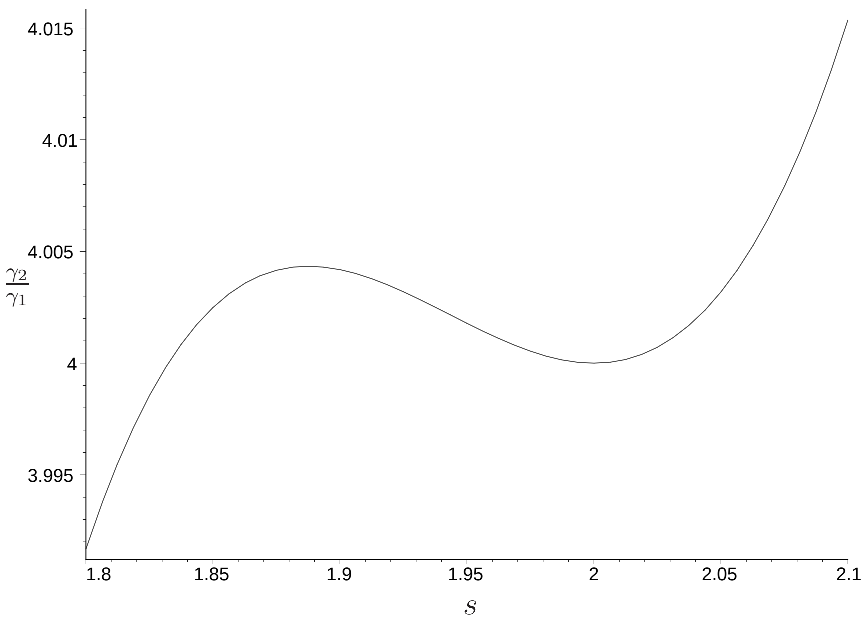

Figure B.2: Relation between s and $\frac{\gamma_2}{\gamma_1}$, zoomed in on the first bump in figure B.1

As you can see in this plot, for some values of $\frac{\gamma_2}{\gamma_1}$, s can have two or three different values.

Now I want to plot the solution $\eta(x)$. If I take $X = \pi$, $\gamma_1 = 1$ and $s = 1$ then also $\gamma_2 = 1$. Then the solution becomes $\eta(x) = \pm\sqrt{\frac{2}{\pi}}\sin(x)$ with $x \in [0, \pi]$. The solution with the minus-sign gives a maximum at $x = 0$, as we stated at the beginning of this chapter.

We can extend the solution $\eta(x) = -\sqrt{\frac{2}{\pi}} \sin(x)$ on the interval $[0, \pi]$ to the entire real line, which gives the following plot:

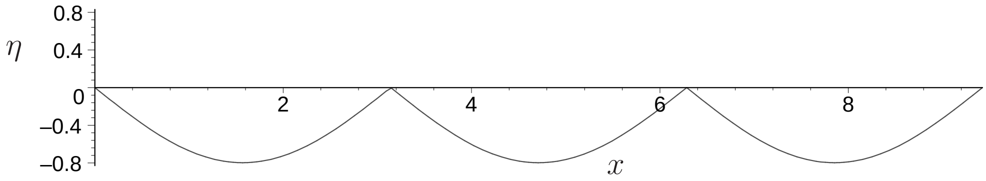

Figure B.3: Solution $\eta(x)$ if $X = \pi$, $\gamma_1 = 1$ and $\gamma_2 = 1$.

The solution with the plus-sign has a minimum at $x = 0$, but the maximum at $x = \frac{\pi}{2}$ is higher than the maximum in the previous plot.

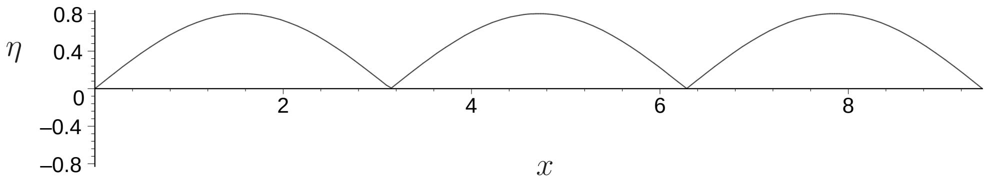

Figure B.4: Solution with the plus-sign if $X = \pi$, $\gamma_1 = 1$ and $\gamma_2 = 1$.

In the next plot I use $s = 3$, so $\gamma_2 = 9$. In this case the solution with the minus-sign is $\eta(x) = -\sqrt{\frac{2}{\pi}} \sin(3x)$.

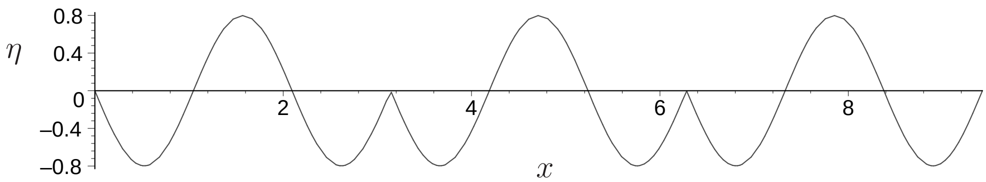

Figure B.5: Solution $\eta(x)$ if $X = \pi$, $\gamma_1 = 1$ and $\gamma_2 = 9$.

As you can see, this solution has a local maximum at $x = 0$, but it's global maximum is attained at some time $x \neq 0$. If I want larger values of γ_2, I have to choose larger values for s and more local extrema will appear in one period.

The next plot shows the maximum $\eta(x)$ as a function of γ_2. The solid line corresponds to the solution $\eta(x)$ with a minimum in $x = 0$ and the dotted line (which is the same as the solid line for higher values of γ_2) corresponds to the solution with a maximum in $x = 0$.

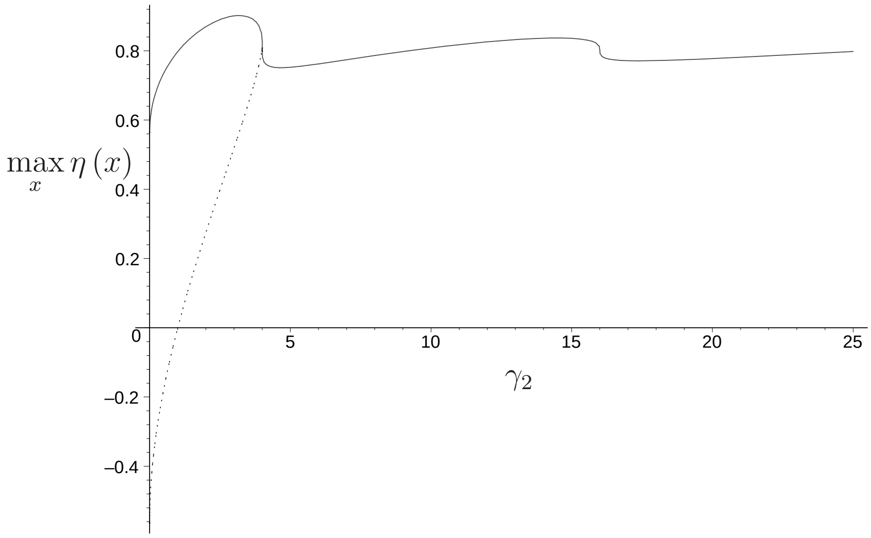

Figure B.6: The maximum of the solution $\eta(x)$ versus γ_2. The solid line corresponds with the solution with a plus-sign, the dotted one with a minus-sign.

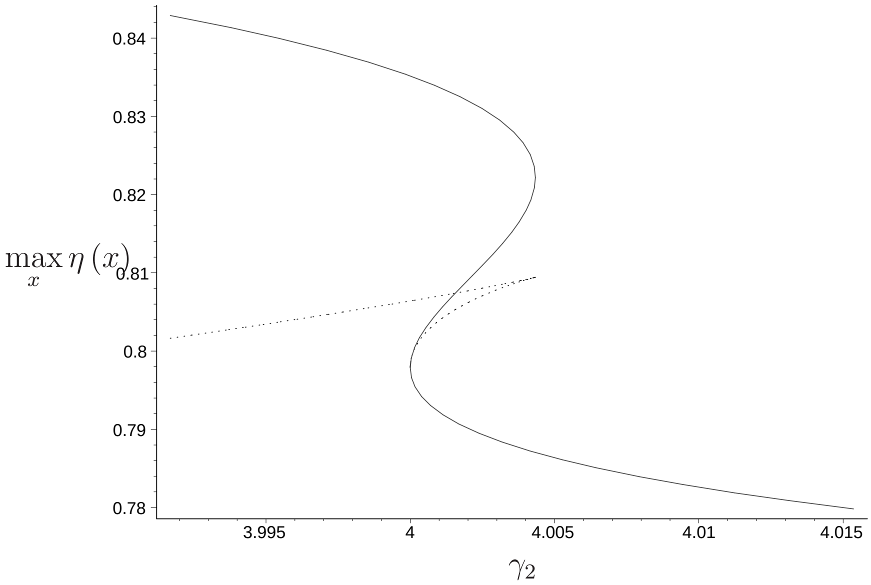

Figure B.7: The maximum of the solution $\eta(x)$ versus γ_2 zoomed in on the first bump in the previous picture

If the solution $\eta(x)$ I discussed in this section is the solution of the constrained variational problem (2.3), the line in the previous plot is the value-function $V(\gamma_2)$ of that constrained variational problem (with $\gamma_1 = 1$). This value function must be increasing, but the line in the plot is not. Therefore the solution I found in this section is not the solution of the constrained variational problem (2.3)

$$\max_{\eta}\left\{\max_{x}\eta\left(x\right)\middle|\int\eta^{2}\left(x\right)dx=\gamma_{1},\int\left(\partial_{x}\eta\left(x\right)\right)^{2}dx=\gamma_{2}\right\}\tag{B.8}$$

but it is a solution of

$$\operatorname*{crit}_{\eta}\left\{\max_{x}\eta\left(x\right)\middle|\int\eta^{2}\left(x\right)dx=\gamma_{1},\int\left(\partial_{x}\eta\left(x\right)\right)^{2}dx=\gamma_{2}\right\}\tag{B.9}$$

Besides, whatever values I choose for γ_1 and γ_2, the solution in section 2.4 has always a higher maximum than the solution in this appendix.