Estimating default probabilities for CDO's: a regime switching model

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by

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March 3, 2011

Preface

This dissertation is part of my Final Project for the Master Applied Mathematics, track Financial Engineering, at the University of Twente in Enschede. I started my project in March 2010 at Christofferson, Robb and Company (CRC) in London, UK. I worked on two different projects: the insolvency of DSB Bank and an analysis on liquidation times. I want to thank my supervisor Glenn Blasius of CRC for all his ideas and comments during my time with them.

In September 2010 I started my Master thesis at the University of Twente. Thanks to prof. Bagchi I had a desk at Citadel, were I worked for five months. I want to thank my supervisors Prof. Bagchi and dr. Krystul for all their ideas, comments and guidance.

Last, but not least, I want to thank my parents and my friend Pieter for their support the last years.

Jantine Koebrugge

Abstract

In this paper we estimate default probabilities for synthetic CDO's. These kind of CDO's have a basket of CDS's as underlying portfolio. For pricing purposes and to set tranches for CDO's, default probabilities are important. We adopt the interacting particle system (IPS) approach to compute these kind of rare probabilities. By choosing two different potential functions for the IPS algorithm, we have two approaches. We adopt a structural model with a regime switch in the market volatility for the asset values of the firms. A firm defaults if the asset value falls below a certain threshold D. From the numerical results, we see that the switch in the market volatility and the correlation factor in the underlying portfolio have the biggest impact on the default probabilities in a CDS portfolio.

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Chapter 1

Introduction

1.1 Credit risk

Banks and other financial institutions face several risks in their daily practice. One of these risks is credit risk. Credit risk arises from the possibility that borrowers, bond issuers and counterparties in derivatives transactions may default, see Hull [7]. Banks are required to keep capital for credit risk. Since the late nineties the credit derivative market has developed very fast. Credit derivatives are contracts where the payoff depends on the creditworthiness of one or more companies or countries. By entering into credit derivative contracts, banks and other financial institutions can protect themselves better against credit risk, by selling risks to third parties. In this way they can manage their credit risk portfolio better. According to Hull [7] banks have been the biggest buyers of protection and insurance companies the biggest sellers.
There are two kinds of credit derivative contracts: single-name and multi-name contracts. A single-name credit derivative is a contract on one company or country for which the buyer wants protection. An example of a single-name derivative is a credit default swap (CDS). In this swap we have a buyer of protection and a seller. The buyer pays periodic payments to the seller. In case of a default, the seller will pay off the buyer. To regulate the market for credit default swaps better and to track the spreads, indices are developed. One of these indices is the iTraxx Europe. This is an index of credit default swaps of 125 investment grade European entities, see the website of Markit. For investors it is now more easy to sell and buy credit default swaps.
The second type of credit derivative, the multi-name contract, sells protection for a portfolio of credit risk. An example of multi-name contract is a collaterized debt obligation (CDO). This instrument is structured in such a way that the underlying portfolio is divided into different risk categories, called tranches. The iTraxx indices are used to set standard tranches for CDO's. From the late nineties, the CDO market has grown explosively, see table 1.1. In 2007 the value of credit derivatives such as CDO's decreased dramatically. The reason for this were the large number of defaults in the underlying portfolios. Especially portfolios with subprime mortgages showed a large number of defaults. The reason for this were increasing interest rates and decreasing housing prices (in the U.S.). This was the start of the credit crunch, as we know today. Since the credit crunch there was hardly any market for structured products as CDO's. It appeared that these products are hard to price and especially to classify the risks within the underlying portfolio. Since 2010 the market is increasing again, see table 1.1.

1.2 Credit risk assessment for CDO's

Credit risk is defined as the risk that a counterparty defaults. For pricing and structuring credit derivatives it is very important to estimate the default probability of a counterparty. There are several methods to derive default probabilities. One of these is based on the credit rating

Global CDO Issuance Volume	In bil. $
2000	68.0
2001	78.5
2002	83.1
2003	86.6
2004	157.4
2005	251.3
2006	520.6
2007	481.6
2008	61.9
2009	4.3
2010	8.0

Table 1.1: *Source: Securities Industry and Financial Markets Association*

for countries, banks and all kinds of bonds provided by rating agencies. There are three big rating agencies: *Moody's*, *Standard and Poor's (S&P)* and *Fitch Ratings*. All three use different ratings, see table 1.2. The agencies have created tables with historical default probabilities per rating, see Hull [7] for the Moody's table. The problem with ratings is that they are not adapted very often. For example, if a company is having a hard time, it does not necessarily mean that the rating agencies will downgrade bonds issued by this company. We can also derive default probabilities from bond prices; see Hull [7]. The difference is that these probabilities are risk-neutral and the historical probabilities are real-world probabilities. Instead of using bond prices or historical data to calculate default probabilities, we will use another approach to assess the default of a firm. This approach is based on the equity price of a firm, which can provide more recent information than ratings or bond prices. This is the structural approach, introduced by Merton [10] in 1974.

Moody's	S&P	Fitch	
Aaa	AAA	AAA	Prime
Aa1 Aa2 Aa3	AA+ AA AA-	AA+ AA AA-	High grade
A1 A2 A3	A+ A A-	A+ A A-	Upper medium grade
Baa1 Baa2 Baa3	BBB+ BBB BBB-	BBB+ BBB BBB-	Lower medium grade
Ba1 Ba2 Ba3	BB+ BB BB-	BB+ BB BB-	Non-investment grade speculative
B1 B2 B3	B+ B B-	B+ B B-	Highly speculative
Caa1 Caa2 Caa3	CCC+ CCC CCC-	CCC CCC CCC	Substantial risks

Table 1.2: Ratings per agency

1.3 Structural model

In the structural model we assume that a firm's equity is an option on its assets, where the asset value can be described by a stochastic differential equation like

$$dS_t = (aS_t - C)dt + \sigma_S S_t dW_t, \tag{1.1}$$

Here a is the rate of return of the assets, C the payout of the firm to its shareholders (if applicable), σ_S is the volatility of the assets and W_t is a Wiener process, which has the following conditions, see Björk [1]:

1. $W(0) = 0$;
2. The process W has independent increments, i.e. for $r < s \leq t < u$, $W(s) - W(r)$ and $W(u) - W(t)$ are independent;
3. For $s < t$, $W(t) - W(s) \sim N[0, t-s]$;
4. W has continuous trajectories.

We assume that if the asset value falls below a certain threshold D, the firm defaults. With this approach we both can compute real-world and risk-neutral probabilities. In the case for one firm, the probability of default at time T is specified by

$$P(S_T \leq D).$$

By the assumption of no arbitrage in the asset model described above, we can compute this probability analytically. In Merton's model, the rate of return and volatility of the asset value are assumed to be flat. According to Krystul, Bagchi and Bouman [9], this kind of asset dynamics usually does not give a good approximation of the asset value. So instead of using a flat return and volatility, we will adopt the regime switching approach described in Krystul et al. [9]. Here we allow a switch in the rate of return, the risk-free rate and market volatility. In this way, we can capture the behavior of these parameters better, as we will show in chapter 3.3. We will expand the one-asset regime switching model to a d-asset regime switching model, which we will explain in detail in chapter 3.2. We want to compute probabilities that k firms out of a portfolio of $d = 125$ firms default. We cannot do this analytically and therefore we have to adopt a numerical approximation.

1.4 Numerical approximation

With the structural approach, it is possible to compute the risk-neutral default probability for one firm analytically. The probability of default for k out of d firms is impossible to compute analytically, especially if we allow switches in the asset model parameters. Therefore we need numerical techniques. With Monte Carlo simulation, we can approximate these probabilities. The idea is to simulate the asset portfolio a large number of times, say M, and for each portfolio i count the number of defaults at maturity T:

$$L_i(T) = \sum_{j=1}^{d} \mathbf{1}_{\{\min_{0 \leq t \leq T} S_j(t) \leq D\}}.$$

Then the probability of default for x firms can be approximated by

$$P(k \text{ out of } d \text{ firms default before } T) \approx \frac{1}{M} \sum_{i=1}^{M} \mathbf{1}_{\{L_i(T)=k\}}. \tag{1.2}$$

Intuitively the probability that all d firms default in one year is very small. For the Monte Carlo simulations, we have to take M extremely large to get some results, as is shown in Carmona, Fouque and Vestal [2]. This will be very time-consuming and for k large, the probabilities produced by the Monte Carlo simulation will be zero. So instead of using regular Monte Carlo simulation, we adopt the interacting particle system (IPS) approach described in Del Moral [4] and Carmona et al. [2] to compute these rare default probabilities. In this thesis we will describe two approaches to calculate these probabilities. The first is based on an approach introduced by Carmona et al. [2] and the second approach is based on the PhD dissertation of Krystul [8]. We will run the algorithms for a portfolio of $d = 125$ firms to find default probabilities. We also compare the two methods and see which parameters have the most impact on the default probabilities.

1.5 Structure of the thesis

This thesis is structured as follows. We start by explaining collaterized debt obligations in more detail and specify on which kind of CDO we will focus in this thesis. In chapter 3 we explain more on structural credit models and introduce the regime switching asset model. By observing the market volatility and LIBOR interest rate, we also back up our choice for a switch in the risk-free rate and volatility. By computing risk-neutral hitting probabilities for a one asset model analytically, we show the impact of the risk-free rate and volatility on these probabilities. In chapter 4 we will explain the interacting particle approach in more detail, give the general mathematical description and explain how to use this approach to calculate probabilities. In chapter 5 we will introduce two different approaches of the IPS algorithm to compute default probabilities and explain these algorithms step by step. In the chapters 'Numerical results' and 'Discussion', we report the results of our two algorithms and discuss these results.

Chapter 2

Collaterized debt obligations

In this chapter we will explain collaterized debt obligations and credit default swaps in more detail. We will also specify the kind of CDO for which we will estimate default probabilities.

2.1 Structure of a CDO

A CDO is a credit derivative with an underlying portfolio of credit derivatives, such as bonds, loans or a basket of credit default swaps. The portfolio is bought by a special purpose vehicle (SPV), which is usually registered as a private company (B.V. in The Netherlands). The shares of this company are held by a foundation, which is managed by the company administrator, such as a capital management firm. The underlying portfolio is sorted in tranches with different risk exposure. For example, 85% of the total portfolio value is put in tranche A, 6% in tranche B, 5% in tranche C and the last 4% in tranche D. The SPV issues notes with a rating and return based on the tranches. The total initial value of notes in a tranche equal the percentage of the portfolio value in this tranche. The senior tranches, say tranche A and B have the least risk exposure and therefore will have the highest rating, typically AAA (*S&P* rating). The lowest tranche D, called the equity tranche will absorb the first losses in the portfolio and will have the lowest, or sometimes no, rating. The return on the notes is typically LIBOR plus x basis points, where the basis points are based on the risk exposure of the tranche. The company administrator receives the income on the underlyings and makes sure that the noteholders receive their payments. For an illustration of a CDO, see figure 2.2.
There are two types of CDO's: synthetic and cash CDO's. In this paper we focus on synthetic CDO's. This kind of CDO is based on a portfolio of credit default swaps. Before we can explain synthetic CDO's in more detail, we need to understand how a CDS works.

2.2 Credit default swaps

A credit default swap is a contract that provides insurance against the default of a certain company, see Hull [7]. We refer to the company as reference entity and a default is a credit event. We have a buyer and a seller of this insurance. The buyer makes periodic payments to the seller until maturity or a credit event. The buyer usually makes these payments in arrears, every quarter, every six months or every year. If a credit event occurs, the buyer has the right to sell bonds issued by the reference entity for the face value and the seller agrees to buy these bonds for face value [7]. The face value of a bond is the principal amount that the issuer of the bond (in this case the company) has to repay at maturity. The total face value of the bonds that can be sold is the notional principal of the CDS. The amount a buyer pays per year, as a percentage of the notional value, is known as the spread of the CDS. We will illustrate how a CDS works by an example from Hull [7].

Example CDS
Two parties enter into a five-year CDS contract on March 1, 2006. The notional principal of the CDS is \$100 million and the buyer pays 90 basis points annually for the protection against the default of the reference company, see figure 2.1. In case the company does not default, we buyer will receive no payoff and has to pay $0.009 \times \$100$ million $= \$900,000$ on March 1 of 2007, 2008, 2009, 2010 and 2011.
In the case of a credit event, the buyer has to notify the seller about this event. Suppose this happens on June 1, 2009. Depending on what is specified in the contract, there can by a physical or cash settlement. In the case of a physical settlement, the buyer has the right to sell bonds issued by the reference entity for their face value, that is \$100 million. In case of a cash settlement, an independent calculation agent will conduct a poll of dealers at a pre-designed number of days after the credit event. These dealers have to determine the mid-market value of the cheapest deliverable bond. Usually it is specified in the CDS contract which bonds can be delivered in case of a credit event. By choosing for the cheapest-to-deliver bond, the payoff will be the most, since the seller has to buy these bonds for face value. Suppose the cheapest-to-deliver bond is worth \$35 per \$100 face value. Then the payoff for the buyer will be $(\$100 - \$35) \times 1$ million $= \$65$ million.
In the case of a credit event, the periodic payments from the buyer to the seller stop. But since these payments are made in arrears, a final accrual payment is required. In our case, the buyer pays annually. The credit event is notified at June 1, 2009, so the buyer has to make an accrual payment for three months, that is $1/4 \times \$900,000 = \$225,000$.

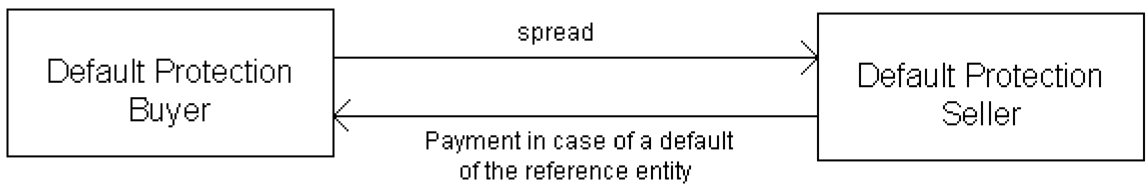

Figure 2.1: Structure of a credit default swap

2.3 Synthetic CDO's

A synthetic CDO has a basket of d credit default swaps as underlying portfolio. The total notional principal of the d credit default swaps is the notional principle of the synthetic CDO. The notional principal is divided into tranches, see figure 2.2. For each tranche notes are issued with a principal equal to the percentage of notional principle in that tranche. For example in tranche A, the principal of the notes issued is 85% of the total notional principle of 1 billion. Investors can buy these notes and in return they will receive periodic payments based on the

risk exposure of the tranche and the total spread income on the CDS's. For tranche A this return is LIBOR plus 50 basis points of the notes principal.
If a company defaults in the underlying portfolio, the seller of the insurance has to pay the notional principle of the CDS to the buyer of the CDS. This notional principle is paid by the SPV to the issuer and this is a first loss in our CDO. For example, a company defaults on which the buyer has a CDS with notional principle of \$20 million. This principle has to be paid and this amount is a first loss for the equity tranche. There will remain $\$40 - \$20 = \$20$ million in the equity tranche. The first 4% of the losses is for the equity tranche. In figure 2.2 the structure of a synthetic CDO is illustrated. Here P&I are the principal and interest payments to the noteholders. With principal we refer to the amount that a noteholder has paid to the SPV. The spread that the sellers of the credit default swaps pay to the issuer, are the premiums the bank arranger pays to the SPV. In case of a credit event, the SPV will pay the protection to the bank arranger, which will payoff the buyer of that CDS.

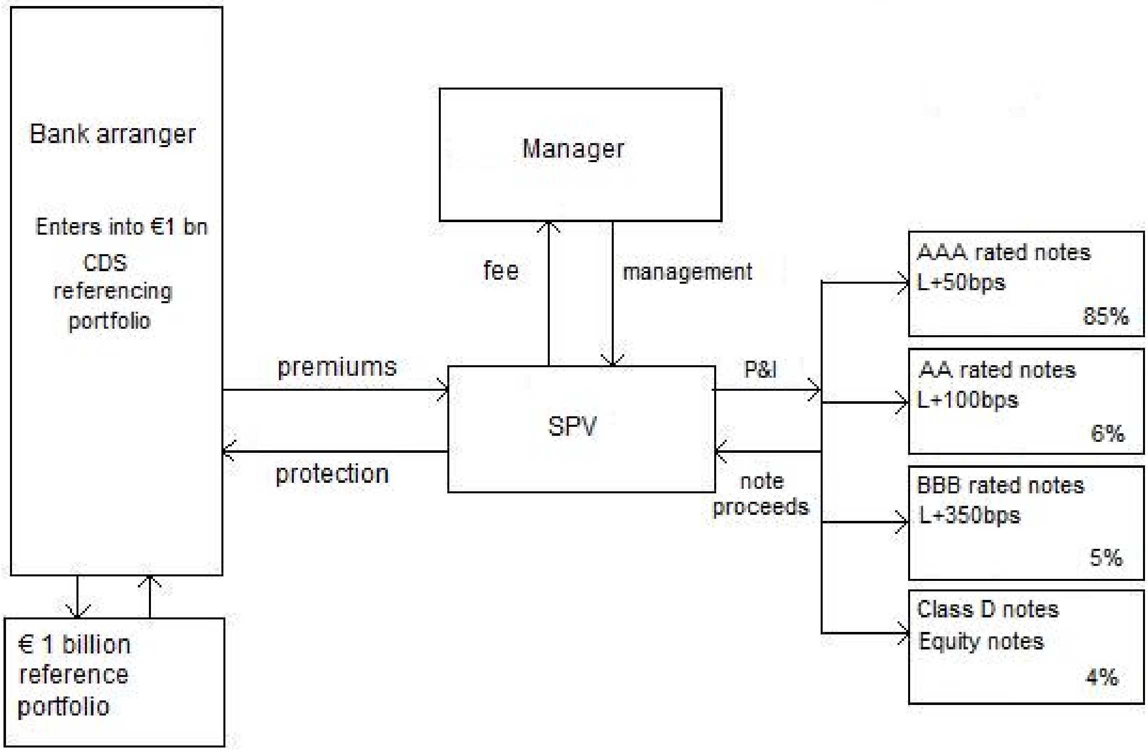

Figure 2.2: Structure of a CDO

2.4 Summary

In this chapter we explained collaterized debt obligations in more detail. A CDO is a credit derivative with an underlying portfolio of loans, bonds or a basket of credit default swaps. This portfolio is divided in tranches with different risk exposure, see figure 2.2. For the principal amount in each tranche, notes are issued. Investors can buy these notes, where the return is based on the risk exposure of the tranche. We will focus on synthetic CDO's, that is a CDO with a basket of credit default swaps as underlying portfolio. A credit default swap is a credit derivative which provides insurance in case a company defaults. The buyer of this insurance has to make periodic payments to the seller until maturity or the default of the company. In case of a default, the seller has to payoff the buyer. The total notional principle of the credit default swaps is the notional principle of the CDO. A default in the underlying portfolio means a loss for the CDO. The lowest tranche will absorb the first losses. These losses will be netted against the principal of that tranche. To price CDO's and to assess the risks, the default probabilities of the underlying firms are important. In this thesis we adopt the structural approach to describe the default of a company and to compute default probabilities. In the next chapter, we will explain this approach in more detail.

Chapter 3

Asset portfolio model

In this section we will describe the structural model approach in more detail and introduce the regime switching model we use for simulations later on. We will back up our choice for the regime switching model by observing the *S&P 500* and LIBOR rate. In the last section, we will show how to calculate default probabilities based on the structural model approach.

3.1 Structural model

The structural model is proposed in 1974 by Merton [10]. He assumed that a firms equity is an option on its assets. Let us define the following parameters:

$$\begin{aligned}
S_t &= \text{value of company's assets at time } t \\
E_t &= \text{value of company's equity at time } t \\
D &= \text{amount of debt to be repaid at time } T \\
\sigma_S &= \text{volatility assets} \\
r &= \text{risk-free rate}
\end{aligned}$$

The dynamics of the assets can be described by a stochastic differential equation like:

$$dS_t = (aS_t - C)dt + \sigma_S S_t dW_t, \tag{3.1}$$

where a is the instantaneous expected rate of return on the firm per unit time, C is the total dollar payouts by the firm per unit time to either its shareholders or liabilities-holders (e.g., dividends or interest payments) if positive, and it is the net dollars received by the firm from new financing if negative and dW is a standard Wiener process, see Merton [10]. Now if at time T the value of the assets is less than the amount of debt, we assume a company defaults. The value of equity is zero in this case. If $S_T < D$, the company can repay its debt and the value of the equity will be $E_T = S_T - D$. We can say that the equity of a firm is a call option on the value of the assets of a company:

$$E_T = \max\left(S_T - D, 0\right). \tag{3.2}$$

We now know that the probability that a company defaults on its debt is

$$P\left(S_T \leq D\right).$$

Under the assumption that there is no arbitrage, we can compute this probability explicitly, see appendix A. In section 3.4 we will use this explicit formula to compute risk-neutral default probabilities. A problem with this way of computing probabilities is that we take into account the asset value at time T, but it may happen that the asset value drops below D before time

T. To avoid this, in our model we will keep track of the minimum value of the assets in interval $[0, T]$.
Another drawback of the Merton model is the assumption that the risk-free rate and the volatility of the assets are flat. With this assumption, the asset value of a firm is usually not approximated very well. Therefore we adapt a model which incorporates switches in the drift, risk-free rate and volatility. We will explain the regime switching model in more detail in the next section.

3.2 Regime switching model

In this section we will first introduce the dynamics of the asset values and the switch incorporated in the model. In section 3.3 we will back up our choice for a regime switch in the model.
Let $\{\theta_t\}$ be a continuous-time Markov chain taking values in

$$\mathbb{M} = \{e_1, e_2, \ldots, e_N\},$$

where $e_i = (0, \ldots, 1, \ldots, 0) \in \mathbb{R}^N$. Assume that the risk-free rate $r(t)$, drift $\mu_i(t)$ and the market volatility factor $\sigma^m(t)$ are depending on this Markov chain, see Krystul et al. [9]. Since we have a d asset values, we will have d drift coefficients μ_i.

$$r(t) = r(t, \theta_t) =< r, \theta_t >, \qquad \mu_i(t) = \mu_i(t, \theta_t) =< \mu_i, \theta_t > \tag{3.3}$$
$$\sigma_t^m = \sigma^m(t, \theta_t) =< \sigma^m, \theta_t > . \tag{3.4}$$

Here $r = (r_1, \ldots, r_N)$, $\mu_i = (\mu_{i1}, \ldots, \mu_{iN})$ and $\sigma^m = (\sigma_1^m, \sigma_2^m, \ldots, \sigma_N^m)$, with $r_j, \sigma_j, \mu_j > 0$ for all $j \in \{1, 2, \ldots, N\}$. Since $\{\theta_t\}$ is a continuous-time Markov chain, the time spent in a certain state e_i is exponentially distributed with mean $1/\lambda_i$. We assume that the asset values of the firms have the following dynamics:

$$dS_i(t) = \mu_i(t)S_i(t)dt + \sigma_i\sigma^m(t)S_i(t)d\bar{W}_i(t) \qquad i \in \{1, \ldots, d\} \tag{3.5}$$
$$P_{\theta_{t+\delta}|\theta_t}(e_j|e_k) = \lambda_{kj}\delta + o(\delta) \qquad k \neq j \tag{3.6}$$

Here the drift $\mu_i(t)$ and the market volatility factor depend on the Markov chain $\{\theta_t\}$ and σ_i refers to the idiosyncratic volatility factor of the firm. $\bar{W}_i(t)$ is a Wiener process. The correlation structure of the Wiener processes is given by

$$d(\bar{W}_i, \bar{W}_j)(t) = \rho_{ij}dt.$$

We assume that $\{\bar{W}_i(t)\}$ and $\{\theta_t\}$ are independent for all t. In the second expression λ_{kj} refers to the switching rate from state k to state j.
For pricing purposes and risk-neutral probabilities we want to have absence of arbitrage in our model. To this end, we need to find a risk-neutral measure $\mathbb{Q}$. Because of the extra source of randomness, the market described by the switching dynamics above is incomplete, therefore we will have infinitely many risk-neutral measures. By using the Girsanov Theorem and the regime switching Esscher transform described in the paper of Elliott, Chan and Siu [5], we choose a risk-neutral measure for our computations. For a proof, see appendix B. Now the multiple asset dynamics under the risk-neutral measure $\mathbb{Q}_\Theta$ are

$$dS_i(t) = r(t)S_i(t)dt + \sigma_i\sigma^m(t)S_i(t)d\bar{W}_i(t) \qquad i \in \{1, \ldots, d\} \tag{3.7}$$
$$P_{\theta_{t+\delta}|\theta_t}(e_j|e_k) = \lambda_{kj}\delta + o(\delta) \qquad k \neq j. \tag{3.8}$$

Now the drift coefficient equals the risk-free rate $r(t)$ for every asset. We will assume that a firm defaults if the asset value drops below a certain barrier $B_i(t)$. This is not necessarily the debt of the firm.

3.3 The choice for a regime switch

To back up our choice for the regime switch in our asset model, we will do some analysis on the *Standard & Poor's 500* data and LIBOR rate data. The *S&P 500* is a capitalization weighted index of the 500 largest U.S.-based companies. The stocks of these companies trade either on the *New York Stock Exchange* or *NASDAQ*. The LIBOR rate is the London Interbank Offered Rate, which is a mean rate for which banks in London loan credit.
We will start with the *S&P 500* to analyze the market volatility. We use the daily prices from January 1980 until December 2010 (source: Freelunch.com). We calculate the sample variance of the log-returns with the following formula:

$$\frac{1}{251}\sum_{i=1}^{252}\left(\ln\left(\frac{x_i}{x_{i-1}}\right)-\frac{1}{252}\sum_{j=1}^{252}\ln\left(\frac{x_j}{x_{j-1}}\right)\right) \tag{3.9}$$

Here x_i is the closing price of the index at day i. We take the average over 252 days, that is one trading year. As we can see in graph 3.1 we can assume that we have two states: one for low

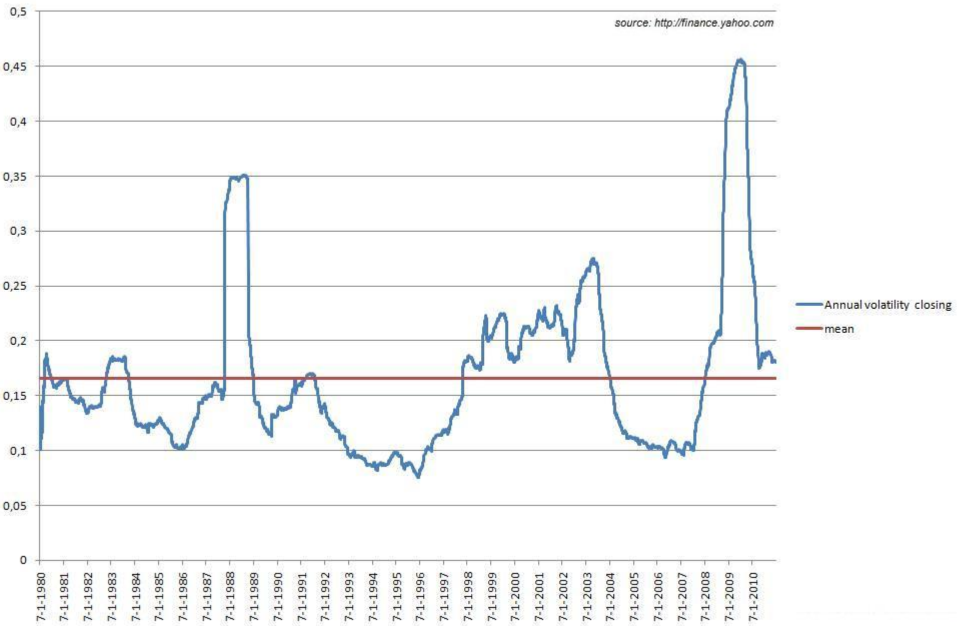

Figure 3.1: Volatility *S&P 500* 1980-2010

or mean volatility, say $\theta_t = e_1$ and one for highly volatile times, say $\theta_t = e_2$. So in our model we have a two-state Markov chain: $\{\theta_t\} \in \{e_1, e_2\}$. Let T_i be the time spend in state e_i. We assume T_i is exponentially distributed with mean $1/\lambda_i$:

$$P(T_i \leq t) = 1 - e^{-\lambda_i t} \quad \text{for } t > 0.$$

We want to know the mean time spent in each state, so we can determine λ_1 and λ_2. To do this we will use an ad-hoc technique, for illustration purposes. As we see in graph 3.1 there are three highly volatile periods. The overall mean volatility is 16.6%. For more statistics, see table 3.1. In total we have 7819 observations over 30 years. Assume that a high volatility is 25% or bigger. The number of observations bigger than 25% is 790. So over 30 years we have a high volatile period of on average three years, that is on average one year out of ten years. Now we can set the switch rate λ_1 from a low or mean volatility to high volatility and λ_2, the switch rate from a high volatility to a low or mean:

$$\lambda_1 = 1/9 \approx 0.11 \tag{3.10}$$
$$\lambda_2 = 1. \tag{3.11}$$

The probability of starting in a highly volatile period is 0.1 and 0.9 for a low or mean volatile period. We also want to know the market volatility factor, that is by which factor the volatility changes if it makes a switch from the low state to the high state. We calculate the mean for the observations with a volatility higher than 25%. This is 34.5%. The overall mean is 16.6%. So in high volatile times, the volatility is on average (approximately) twice as high.

mean	16.6
median	14.6
max	45.6
min	7.5

Table 3.1: Stats volatility *S&P 500* 1980-2010 in %

For the risk-free rate we analyze the LIBOR rates from January 1987 until December 2010 (source: Yahoo Finance), in total 6081 observations. As we can see in graph 3.2 there are also

Figure 3.2: LIBOR rates 1987-2010

switches in the LIBOR rate. A high interest rate corresponds to a 'good economic period' and a low interest rate to a bad period. In the late eighties the interest rate is high, especially compared to now. Overall there are three big declines in the interest rate: in 1993, 2003 and 2009. In the late nineties the interest rate evolves around 6%. Now the interest rate is still (very) low. In table 3.2 you can find some statistics. The overall mean is 5%. The number of observations higher than the mean value is 3403 out of 6081. This number is rather high, because the first five years, every observation is bigger than 4.98.
We see that the switches in the interest rate graph do not exactly coincide with the switches in the volatility. Therefore we will set switch rates for the interest rate too. We again assume

that there are two states and that the time spend in each state is exponentially distributed with mean $1/\lambda_i^r$. To find the mean time spend in each state, we split the graph into two periods: 1987-1998 and 1999-2010, see graph 3.3. In the first graph we see one period of low interest

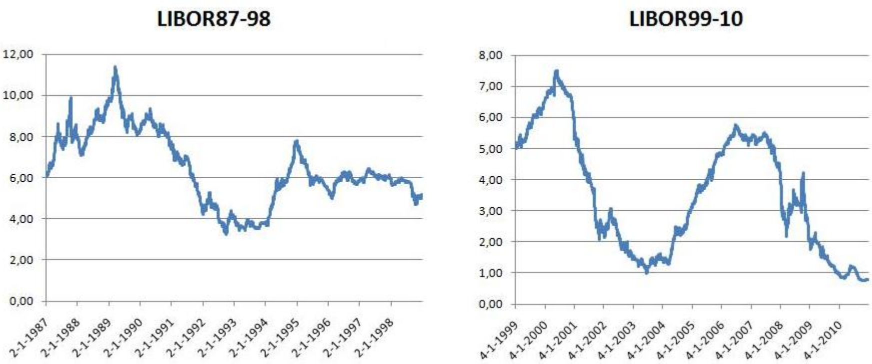

Figure 3.3: LIBOR rates 87-98 and 99-10

and in the second graph we see two lows. The mean values for these graphs are respectively 6.44% and 3.52%. Assume that we have a low interest period if the rate is resp. below 6% or 3%. For the fist period, 1414 out of 3035 days corresponds to a low interest rate period. For the second period, that is 1430 out of 3046 days. Both periods cover 12 years and on average 5.6 years cover a low interest rate period. So the switch rate λ_1^r from a mean to a low interest period and the switch rate λ_2^r are

$$\lambda_1^r = 1/6.4 \approx 0.16 \tag{3.12}$$
$$\lambda_2^r = 1/5.6 \approx 0.18 \tag{3.13}$$

The probability of starting in a low interest rate period is $5.6/12 = 0.47$. Assume that in a normal interest period, we have an interest rate of 5%. The mean for the observations smaller than 6% or 3% in figure 3.3, is respectively 4.79% and 1.74%. Then the average decline is 63% of the overall mean value in a low interest rate period.

mean	4.98
median	5.28
max	11.38
min	0.76
# obs. > 4.98	3403

Table 3.2: Stats LIBOR rate 1987-2010 in %

3.4 Actual probabilities

To see what influence the volatility and the interest rate have on default probabilities, we will compute actual default probabilities for a one asset Merton model described in section 3.1. To compute analytical default probabilities, we assume that the market is free of arbitrage. From Björk [1] we know that a model is free of arbitrage, if the local rate of return equals the risk-free rate. So the dynamics of the asset value will equal:

$$dS_t = rS_tdt + \sigma S_tdW_t \tag{3.14}$$

We assume that the risk-free rate and the volatility are flat. We define $D = 36$ as the barrier to be hit. So if the asset value falls below barrier D, the firm defaults. Now we can compute the probability of default at time T with the following explicit formula:

$$P(S_T \leq D) = \mathcal{N}(d_1) \tag{3.15}$$

Here $\mathcal{N}$ is the standard normal cumulative distribution function and

$$d_1 = \frac{\ln(D/S_0) - \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}.$$

For a derivation of this formula, see appendix A. We substitute the following values:

$$S_0 = 90, D = 36, \text{ and } T = 1.$$

For σ we choose several values between 0.1 and 0.5 and for r we choose 0.05 and 0.01. For the results, see graph 3.4. As we can see a change in σ means a big difference in default probability. For a portfolio of d assets we will expect higher probabilities of default if we incorporate the switch for the volatility in our model. The influence of r is not that big, but we will include the switch for the interest rate in our model later.

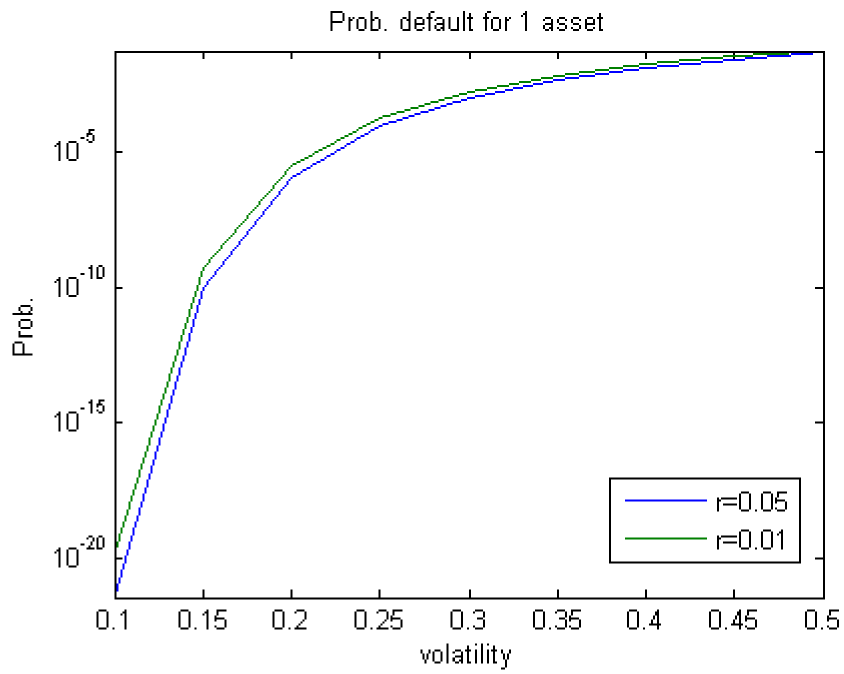

Figure 3.4: Probability of default for one asset

3.5 Summary

In this section we described the structural model and the regime switching multi-asset model. In a structural model we assume that the asset value of a firm evolves according to a stochastic differential equation. If the asset value drops below a certain threshold D, the firm defaults. In the first structural model proposed by Merton [10], it is assumed that the risk-free rate r and the volatility of the asset value are flat. By assuming this, we usually do not estimate the asset value correctly. Therefore we adopt the regime switching model to describe the asset value. In

this model we allow switches in the risk-free rate r and market volatility σ^m. The choice for a switch in these parameters is explained in section 3.3. By observing the volatility in the *S&P 500* and the LIBOR rate, we see that there are usually two states. We set the switch rates for one state to another for both the risk-free rate and market volatility. For a one asset model, we can compute default probabilities analytically. To see what influence the volatility and risk-free rate have on the default probability for one firm, we use the explicit formula in equation 3.15 for calculations. We compute default probabilities for several volatilities and interest rates. We see that the impact of the volatility on the default probability is much bigger than the impact of the risk-free rate.
As we showed in this chapter, we can compute the default probability for one asset analytically, if we assume a flat interest rate and volatility. In our model we allow switches in these parameters and we concern a portfolio of d assets, so analytic computations will be impossible. Therefore we need a numerical technique to compute default probabilities for a portfolio of d firms. In the next chapter we will explain which numerical technique we use to do this.

Chapter 4

Interacting particle system

In this chapter we will explain the interacting particle approach in more detail. We will give a general mathematical description of the IPS approach and explain how we can use it to compute default probabilities for CDO's.

4.1 Motivation

The aim of this thesis is to estimate rare probabilities. Since we have a portfolio of $d = 125$ firms, where the asset models of the firms allow switches in the parameters, we cannot compute the default probabilities analytically. Therefore we have to use numerical approximations. As explained in the introduction we will use the interacting particle approach to compute default probabilities instead of regular Monte Carlo simulation. From section 3.4 we know that the default probability for one asset in one year varies between 5.0^{-2} and 3.0^{-22}. These probabilities are already small, imagine the probability of $d = 125$ firms defaulting in one year. It will be extremely time consuming to do a large number of Monte Carlo simulations and according to Carmona et al. [2] we will not be able to compute these rare probabilities. Therefore we choose the interacting particle system to compute these rare probabilities. Before we start with the mathematical description, we will explain the idea behind the IPS approach.
We start by defining a function $\gamma_n(f)$ which computes expectations (or probabilities) for a suitable function f for every n. We cannot compute this function straightforward, so we need an approximation. By using the interacting particle approach, we are able to approximate γ_n and update it to the next point $n + 1$. We will start with a large number of M particles and in our case, each particle will represent the portfolio of assets. The updating process consists of a mutation and selection stage. During the mutation stage the asset values of the firms will evolve according to the stochastic differential equation defined in chapter 3.2. At the selection stage we resample with replacement M particles and the resampled particles will continue to the next mutation stage. With this updating process we are able to approximate function γ_n at every point n. By using this updating system, we are able to calculate rare probabilities, as we will show later. In the next section we will explain the interacting particle approach mathematically.

4.2 IPS explained mathematically

In this section we will briefly explain the theory on interacting particle systems. This section is based on Del Moral [4] and Carmona et al. [2]. We start by introducing a background Markov chain $X = \{X_n\}_{n\leq 0}$, where X_n takes values in measurable space $(E_n, \mathcal{E}_n)$. By

$$K_n(x_{n-1}, dx_n) = \mathbb{P}(X_n \in dx_n | X_{n-1} = x_{n-1})$$

we denote the transition kernel of Markov chain X_n. In the discrete time case, the corresponding distribution flow η_n, $n \in \mathbb{N}$ is defined by the Feynman-Kac formula:

$$\eta_n(f) = \gamma_n(f)/\gamma_n(1), \tag{4.1}$$

where

$$\gamma_n(f) = \mathbb{E}\left(f(X_n)\prod_{p=0}^{n-1} G_p(X_p)\right). \tag{4.2}$$

Here X_n is the Markov chain introduced above, f is any bounded $\mathcal{E}$-measurable function on E and G_n is a bounded $\mathcal{E}$-measurable potential function on E, such that the normalizing constants η_n are well-defined. We also observe that

$$\gamma_{n+1}(1) = \mathbb{E}\left(\prod_{p=0}^{n} G_p(X_p)\right) = \gamma_n(G_n) = \eta_n(G_n)\gamma_n(1) = \prod_{p=0}^{n} \eta_p(G_p).$$

So we can write

$$\gamma_n(f) = \eta_n(f)\prod_{p=0}^{n-1} \eta_p(G_p). \tag{4.3}$$

We will use these notations later on for computational purposes. The measures η_n satisfy the nonlinear equation:

$$\eta_{n+1} = \Phi_{n+1}(\eta_n) = \Psi_n(\eta_n)K_{n+1} \tag{4.4}$$

Where the mapping $\Phi_{n+1}(\eta) : \mathcal{P}(E_n) \to \mathcal{P}(E_{n+1})$ is defined for any $\eta_n \in \mathcal{P}(E_n)$ and Ψ_n is the Boltzmann-Gibbs transformation from the set $\mathcal{P}(E_n)$ into itself defined by

$$\Psi_n(\eta)(dx) = \frac{G_n(x)}{\eta(G_n)}\eta(dx). \tag{4.5}$$

So we have that

$$\eta_{n+1} = \Phi_{n+1}(\eta_n) = \int_{E_n} \frac{G_n(x)}{\eta(G_n)}\eta(dx)K_{n+1}.$$

It takes a two-step transition to update the measure η_n to the next time point $n+1$. These are respectively the selection and mutation stage:

$$\eta_n \quad \overset{selection}{\longrightarrow} \quad \widehat{\eta}_n = \Psi_n(\eta_n) \quad \overset{mutation}{\longrightarrow} \quad \eta_{n+1} = \Psi_n(\eta_n)K_{n+1} \tag{4.6}$$

Particle methods can be viewed as a kind of stochastic linearization technique of solving non-linear equations in distribution space. We want to approximate the Feynman-Kac flow by the method of interacting particles system. We construct a M-particle Markov Chain $\{\xi_n\}_{n\geq 0}$, where n denotes the time point. This E^M-valued Markov chain can be interpret as

$$\xi_n^{(M)} = (\xi_n^{(M,1)}, \xi_n^{(M,2)}, \ldots, \xi_n^{(M,M)}).$$

The elementary transition probabilities are given by:

$$\mathbb{P}\left(\xi_{n+1}^{(M)} \in d(x^1, \ldots, x^M) | \xi_n^{(M)}\right) = \prod_{i=1}^{M} \Phi_n(\eta_n^M)(dx^i) \tag{4.7}$$

Here η_n^M refers to the empirical distribution of the particles $\xi_n^{(j)}$, for $0 \leq j \leq M$:

$$\eta_n^M = \frac{1}{M}\sum_{j=1}^{M} \delta_{\xi_n^{(j)}}. \tag{4.8}$$

In Del Moral [4] it is proved that $\eta_n^M \to \eta_n$ in distribution for $M \to \infty$ for any time point n. We will approximate the two-step transition from η_n to η_{n+1} with the Markov chain $\xi_n^{(M)}$:

$$\xi_n^{(M)} \in E_n^M \quad \overset{selection}{\longrightarrow} \quad \widehat{\xi}_n^{(M)} \in E_n^M \quad \overset{mutation}{\longrightarrow} \quad \xi_{n+1}^{(M)} \in E_{n+1}^M \tag{4.9}$$

The selection is performed by choosing randomly M particles according to the Boltzmann-Gibbs measure:

$$\begin{aligned}
\Psi_n(\eta_n^M) &= \int_{E_n} \frac{G_n(\xi_n)}{\eta_n^M(G_n)} \eta_n^M(d\xi_n) \\
&= \int_{E_n} G_n(\xi_n)\eta_n^M(d\xi_n) \frac{1}{\int_{E_n} G_n \;\; d\eta_n^M} \\
&= \frac{\frac{1}{M}\sum_{j=1}^M G_n(\xi_n^j)\delta_{\xi_n^j}}{\frac{1}{M}\sum_{k=1}^M G_n(\xi_n^k)} \\
&= \sum_{j=1}^M \frac{G_n(\xi_n^j)}{\sum_{k=1}^M G_n(\xi_n^k)} \delta_{\xi_n^j}
\end{aligned} \tag{4.10}$$

During the mutation stage each particle evolves according to the mutation transition kernel K_{n+1}.
By this particle method we can approximate measure η_n for each n. In equation 4.3 we write γ_n in terms of measure η_n. By choosing an appropriate potential function G and function f, as we will do in the next chapter, we are able to calculate probabilities with this approach.

4.3 Summary

In this chapter we explained the IPS approach in more detail, by describing it mathematically and show how we can use it for estimating probabilities. We want to approximate function γ_n, which is defined as

$$\gamma_n(f) = \mathbb{E}\left(f(X_n) \prod_{p=0}^{n-1} G_p(X_p) \right),$$

and

$$\eta_n(f) = \gamma_n(f)/\gamma_n(1).$$

By choosing function f and G appropriately, we can compute probabilities with function γ_n. We can rewrite function γ_n in terms of measure η_n, see equation 4.2. We can approximate η_n for every n with the interacting particle approach. We start with a large number of M particles. The approach consists of a mutation and selection stage. During the mutation stage the particles will evolve according to the transition kernel of the underlying Markov chain. At the selection stage the particles are resampled according to the Boltmann-Gibbs measure, see equation 4.10. By using this mutation and selection scheme, we can update measure η_n to the next point $n + 1$. In this way we can approximate γ_n at every time point n, so we are able to compute probabilities at every point n.
Our goal is to compute default probabilities for CDO's. In the next chapter we describe two approaches to compute these kind of probabilities. Each particle will represent the portfolio of assets. The asset values evolve according to the regime switching model described in chapter 3.2. In each approach we choose a suitable potential function G and function f, so that we can approximate the default probabilities with function γ_n.

Chapter 5

Two IPS algorithms

In this chapter we will introduce two approaches to compute default probabilities for CDO's. The first approach is based on Carmona et al. [2]. In this paper default probabilities are computed for a portfolio of 125 names. Each particle is a portfolio of 125 assets. During the mutation stage, the assets evolve according to the regime switching model introduced in chapter 3.2. We also keep track of the minimum asset value during the mutation stage. The selection is done at fixed points in time. We will resample with replacement M particles at each selection stage. The weights for resampling are based on a sophisticated potential function G. This function attaches a weight to each asset in the portfolio if the minimum value of this asset has declined with respect to the minimum of this asset at the prior selection. The weight for the particle is computed by summing the asset weights and the particle weight is normalized by dividing it by the sum of total particle weights. At maturity we count the number of defaults in each particle, so that we can approximate the probabilities of number of defaults within a portfolio. In figure 5.2 we illustrate this particle method. We start with five particles, each having its own color. The particles are resampled at each barrier n, according to its weight. We will explain this approach in more detail in the next section.
Since the selection in the latter approach is done at fixed points in time, the second approach is based on a different selection procedure. At first we wanted to resample if x assets within a particle default. Since the probability of one asset defaulting within a year is already small, this approach is not realistic. Therefore we look at the total asset portfolio value and resample if this portfolio value hits barriers smaller than the initial value. Note that the portfolio asset value is not the same as the portfolio value of a basket of credit default swaps. The mutation stage is the same as for the first approach. At the selection stages, only the particles that hit the barrier are taken into account. The other particles are killed, i.e. get zero weight. We attach a simple weight to each particle that hits a barrier: $1/M$, where M is the number of particles. This weight is normalized by dividing it by the sum of total weights. At each barrier we count the number of particles that hit this barrier, and by dividing this number by M, we have the hitting probability for that barrier, given that the particle has hit the prior barrier. The target barrier is defined as the sum of default thresholds for the assets. The hitting probability of this last barrier is computed by multiplying all the conditional hitting probabilities of the higher barriers. We also keep track of the minimum asset values in a particle, so we can compute the number of defaults in each particle at time T. In figure 5.2 we illustrate this particle method. We start with five particles, each having its own color. If a particle does not reach a barrier D, it is not resampled anymore.
In the next two sections, the two approaches are explained in more detail. We show which potential function G we choose and how we use the formulas in chapter 4 to compute default and hitting probabilities in our models. We also show how to incorporate the regime switching asset model as described in chapter 3.

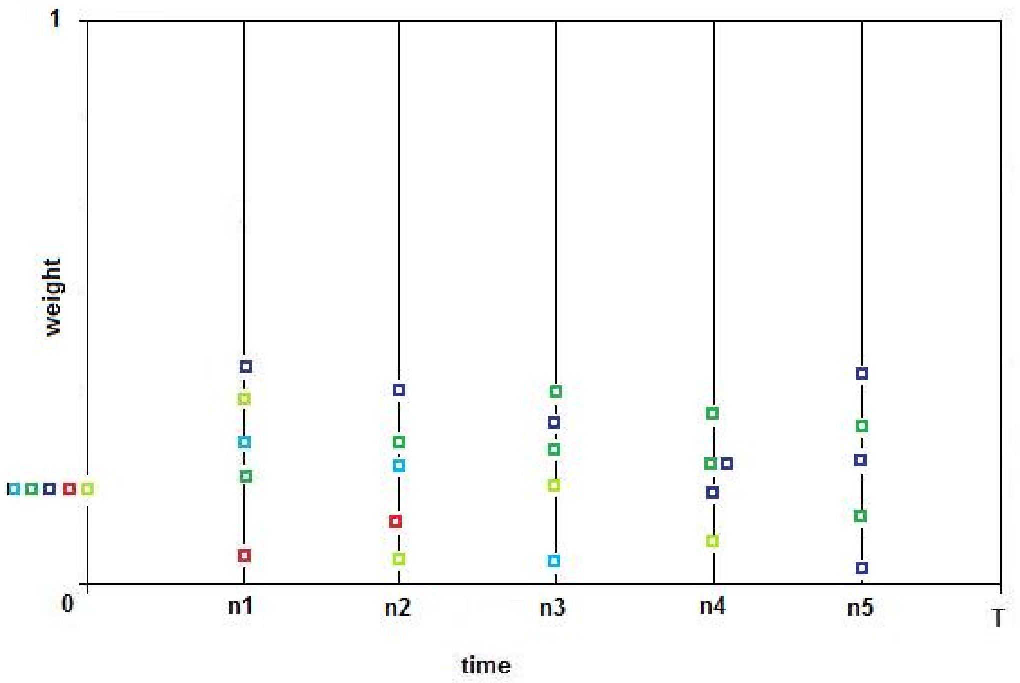

Figure 5.1: IPS approach 1

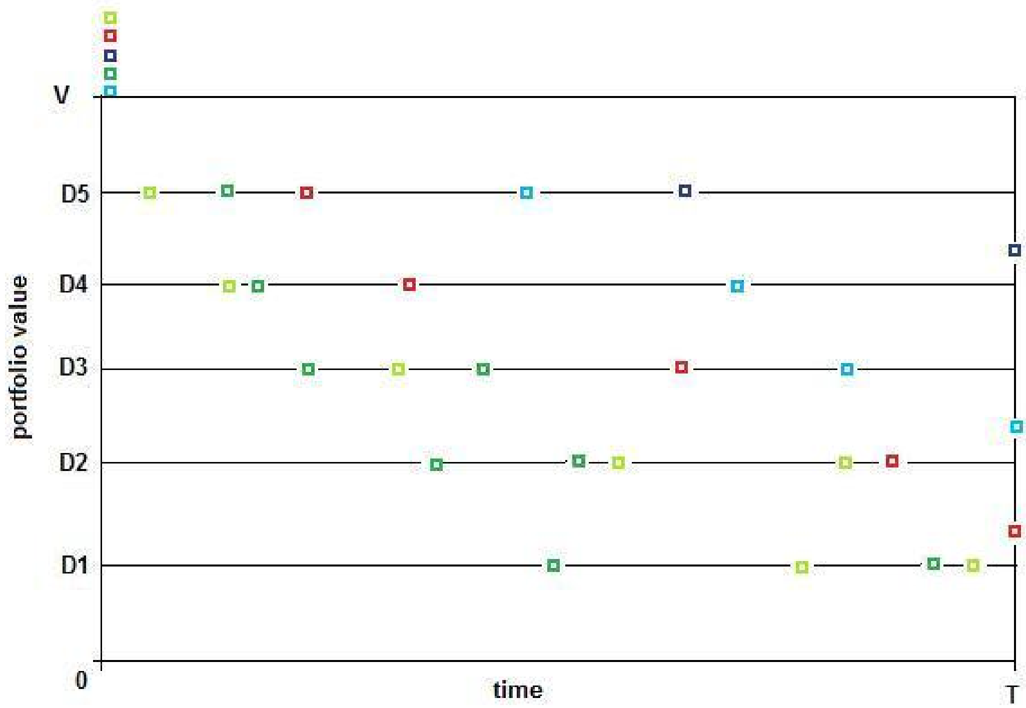

Figure 5.2: IPS approach 2

5.1 First approach

For this approach we will use the IPS method explained in chapter 4 to compute probabilities of the number of defaults in a portfolio. The portfolio will consist of $d = 125$ firms. We introduce Markov chain

$$Y_n = (X_0, \ldots, X_n).$$

This Markov chain contains the history of X_n up to time point n, with transition kernel $M_n(y_{n-1}, dy_n)$.
A firm defaults if the asset value falls below a certain threshold $B_i(t)$. The time of default τ_i for firm i is then defined by

$$\tau_i = \inf\{t : S_i(t) \leq B_i(t)\}.$$

We need to define the portfolio loss function in order to find the probabilities of default. Let $L(T)$ be the number of defaults at maturity T:

$$L(T) = \sum_{i=1}^{N} 1_{\{\tau_i \leq T\}}.$$

We would like to know the probability distribution of loss function $L(T)$:

$$\mathbb{P}(L(T) = k) = p_k(T) \qquad k = 0, \ldots, d. \tag{5.1}$$

To find this probability distribution we recall the Feynman-Kac expectations. For Markov chain Y_n the Feynman-Kac expectations are defined by

$$\gamma_n(f) = \mathbb{E}\left(f(Y_n) \prod_{p=0}^{n-1} G_p(Y_p) \right). \tag{5.2}$$

Let $G_p^- = 1/G_p$, so we have that:

$$\begin{aligned}
\mathbb{E}\left[f(Y_n)\right] &= \mathbb{E}\left[f(Y_n) \prod_{p=1}^{n-1} G_p^-(Y_p) \times \prod_{p=1}^{n-1} G_p(Y_p) \right] \\
&= \gamma_n\left(f \prod_{p=1}^{n-1} G_p^- \right) \\
&= \eta_n\left(f \prod_{p=1}^{n-1} G_p^- \right) \prod_{p=1}^{n-1} \eta_p(G_p).
\end{aligned}$$

Where the latter expression follows from equation (4.3).
For the numerical computations we need to discretize the time variable t in the portfolio asset model and the time of default τ. We will have n selection stages, or resampling times, up to maturity T. We divide the time interval $[0, T]$ into n equal intervals with length $T/n = \Delta t$. We approximate the asset value dynamics by the Euler method, see Glasserman [6]. The Markov chain $\{X_p\}_{p \geq 0}$ will contain the asset values at the starting point $p = 0$, at the resampling points $p = 1, \ldots, n-1$ and the overall minimum asset value up to the resampling time point p:

$$X_p = \left((S_i(p\Delta t))_{1 \leq i \leq d}, (\min_{0 \leq m \leq p} S_i(m\Delta t))_{1 \leq i \leq d} \right). \tag{5.3}$$

In line with Carmona et al. [2] we choose potential function

$$G(Y_p) = \exp[-\alpha(V(X_p) - V(X_{p-1}))], \tag{5.4}$$

where

$$V(X_p) = \sum_{i=1}^{d} \log \left(\min_{0 \le m \le p} S_i(m\Delta t) \right). \tag{5.5}$$

So we have

$$G(Y_p) = \exp \left[-\alpha \sum_{i=1}^{d} \log \frac{\min_{0 \le m \le p} S_i(m\Delta t)}{\min_{0 \le m \le p-1} S_i(m\Delta t)} \right], \tag{5.6}$$

Here α is a free parameter and $\alpha > 0$. From equation 5.6 we see that for different choices of α we get different weights for the particles. According to Carmona et al. [2], different choices for α will result in default probabilities for different sets of k. So in order to get default probabilities for $0 \le k \le 125$, we need to change α.
With the potential function in equation 5.6, the likelihood of resampling particles with firms having a bigger decline in asset value, increases. In this way, the probability of default of a large number of firms increases. We will now describe the algorithm for the first approach step by step.

5.1.1 Algorithm 1

Initialization
We will start with M particles:

$$\xi_0^{(j)} = \Big((S_i(0))_{1 \le i \le d}, (S_i(0))_{1 \le i \le d} \Big) \quad \text{for } j = 1, \ldots, M. \tag{5.7}$$

And the corresponding switching vector is

$$\hat{\nu}^{(j)} = (\hat{\nu}_q)_{1 \le q \le q_T}.$$

Here $\hat{\nu}_q$ represents a switch time. For each particle j, we generate a vector of switch times. In section 3 we computed the switch rates for the market volatility. The probability of starting in a low or mean volatile period is 0.9 and 0.1 for a highly volatile period. By generating uniform random variables, we can calculate the time spent in each state. If we start in a low volatile state, we calculate the switch times as follows:

$$\begin{aligned} \hat{\nu}_1 &= -1/\lambda_1 \log(U) \\ \hat{\nu}_2 &= \hat{\nu}_1 - 1/\lambda_2 \log(U) \\ \hat{\nu}_3 &= \hat{\nu}_2 - 1/\lambda_1 \log(U) \\ &\textit{etc.} \end{aligned}$$

Here $U \sim Unif[0,1]$ and λ_1, λ_2 are the switch rates for the volatility given by equations (3.10) and (3.11). Later we will incorporate the switch the in interest rate in the same manner.

Mutation
During the n intervals $[(p-1)\Delta t, p\Delta t]$, we approximate the asset values by an Euler scheme. Since the Wiener processes $\{\bar{W}_i\}_{\{1 \le i \le d\}}$ of the assets are correlated, we define a $d \times d$ matrix Σ by

$$\Sigma_{ij}(t) = \sigma(t)^2 \sigma_i \sigma_j \rho_{ij}.$$

Then $(\sigma_1 \sigma(t) \bar{W}_1, \ldots, \sigma_d \sigma(t) \bar{W}_d)$ can be represented as $A(t)W(t)$, where $W(t)$ is a standard Wiener process in $\mathbb{R}^d$ and A is any matrix satisfying $AA^\intercal = \Sigma$. To find this matrix A we use the Cholesky decomposition, see appendix C and Glasserman [6].

Now we can use the following algorithm to simulate our asset dynamics for time t_k, $k = 0, \dots, m-1$:

$$S_i(t_{k+1}) = S_i(t_k)\exp\{r(t_{k+1})\delta t + \sqrt{\delta t}\sum_{j=1}^{d} A_{ij}(t_{k+1})Z_{k+1,j}\} \text{ for } i = 1, \dots, d. \tag{5.8}$$

Here and $\delta t = t_k - t_{k-1} << \Delta t$, $Z_k = (Z_{k1}, \dots, Z_{kd}) \sim \mathcal{N}(0, I)$ and $Z_1, \dots, Z_m$ are independent.

Selection
After each mutation stage, we have a selection stage. We resample by choosing independently M particles according to the Boltzmann-Gibbs measure:

$$\sum_{j=1}^{M} \frac{G_n(\xi_n^{(j)})}{\sum_{k=1}^{M} G_n(\xi_n^{(k)})}\delta_{\xi_n^{(j)}} \quad = \quad \sum_{j=1}^{M} \frac{\exp[-\alpha(V(\xi_p^{(j)}) - V(\xi_{p-1}^{(j)}))]}{\sum_{k=1}^{M}\exp[-\alpha(V(\xi_p^{(k)}) - V(\xi_{p-1}^{(k)}))]}\delta_{(\xi_{p-1}^{(j)}, \xi_p^{(j)})} \tag{5.9}$$

We will refer to the resampled particles as $\check{\xi}_p^{(j)}$ for time point p. These particles will continue to the mutation stage and after interval Δt we resample again and continue until we reach maturity T, i.e. $p = n$. Note that we remember the minimum value of the assets over the whole time interval.

Maturity
At maturity we want to compute the estimator $\hat{p}_k(T)$ for $\mathbb{P}(L(T) = k) = p_k(T)$. We know that

$$\mathbb{P}(L(T) = k) = \mathbb{E}\left[1_{\{L(T)=k\}}\right] \tag{5.10}$$

We can approximate the distribution of the number of losses up to time T by counting the number of defaults in each particle:

$$\hat{f}(\xi_n^{(j)}) = \sum_{i=1}^{d} \mathbf{1}\{\min_{0\le m\le n} S_i(m\Delta t) \le B_i(n\Delta t)\} \qquad \text{for } j = 1, \dots, M \tag{5.11}$$

If we substitute for $f(Y_n)$ in equation (5.3), $\mathbf{1}_{\{L(T)=k\}}) \approx \mathbf{1}_{\{\hat{f}(\xi_n^{(j)})=k\}}$, we have that

$$\begin{aligned} \mathbb{E}[f(Y_n)] &= \mathbb{E}\left[\mathbf{1}_{\{\hat{f}(\xi_n^{(j)})=k\}}\right] \\ &\approx \eta_n^M\left(\mathbf{1}_{\{\hat{f}(\xi_n^{(j)})=k\}}\prod_{p=1}^{n-1} G_p^-\right)\prod_{p=1}^{n-1}\eta_p^M(G_p) \end{aligned}$$

So we have that the estimator for $\mathbb{P}(L(T) = k)$ is

$$\hat{p}_k(T) = \left[\frac{1}{M}\sum_{k=1}^{M}\mathbf{1}_{\{\hat{f}(\xi_n^{(j)})=k\}}\exp\left[\alpha\left(V(\xi_{n-1}^{(j)}) - V(\xi_0)\right)\right]\right] \times \left[\prod_{p=1}^{n-1}\frac{1}{M}\sum_{j=1}^{M}\exp[-\alpha(V(\xi_p^{(j)}) - V(\xi_{p-1}^{(j)}))]\right]. \tag{5.12}$$

The second expression follows from

$$\begin{aligned} \prod_{p=1}^{n-1} G_p^- &= \prod_{p=1}^{n-1}\exp[\alpha(V(\xi_p^{(j)}) - V(\xi_{p-1}^{(j)}))] \\ &= \exp\left[\alpha\left(V(\xi_1^{(j)}) - V(\xi_0^{(j)})\right)\right]\cdot\exp\left[\alpha\left(V(\xi_2^{(j)}) - V(\xi_1^{(j)})\right)\right]\cdots\exp\left[\alpha\left(V(\xi_{n-1}^{(j)}) - V(\xi_{n-2}^{(j)})\right)\right] \\ &= \exp\left[\alpha\left(V(\xi_{n-1}^{(j)}) - V(\xi_0^{(j)})\right)\right]. \end{aligned}$$

5.2 Second approach

The second approach is based on Krystul [8]. We now take the total portfolio asset value V into account:

$$V(t) = \sum_{i=1}^{d} S_i(t). \tag{5.13}$$

We define $n+1$ barriers $D_n < D_{n-1} < \ldots < D_0 < V(0)$. Define the first passage time of a barrier by:

$$\tau_p = \inf\{t \geq 0 : V(t) \leq D_p\} \quad \text{for } p = 0, \ldots, n. \tag{5.14}$$

We want to compute the probabilities that the portfolio value V hits the barriers before T, and we are especially interested in the hitting probability of the target value D_n:

$$\mathbb{P}(\tau_n \leq T) = \mathbb{E}\left[\mathbf{1}_{\{\tau_n \leq T\}}\right] = \mathbb{E}\left[\prod_{p=0}^{n} \mathbf{1}_{\{\tau_p \leq T\}}\right]. \tag{5.15}$$

The latter equation follows from

$$\mathbf{1}_{\{\tau_n \leq T\}} = 1 \Leftrightarrow \mathbf{1}_{\{\tau_0 \leq T\}} = \ldots = \mathbf{1}_{\{\tau_{n-1} \leq T\}} = 1.$$

To approximate these probabilities, we will apply the IPS with resampling points defined as the barriers $D_n < D_{n-1} < \ldots < D_0$. We recall the Feynman-Kac equation (4.2)

$$\gamma_n(f) = \mathbb{E}\left(f(X_n) \prod_{p=0}^{n-1} G_p(X_p)\right). \tag{5.16}$$

If we choose potential function

$$G_p(X_p) = \mathbf{1}_{\{V(\tau_p) \leq D_p\}} \quad \text{for } p = 0, \ldots, n. \tag{5.17}$$

and $f(X_n) = \mathbf{1}_{\{\tau_n \leq T\}}$, we see that

$$\gamma_n\left(\mathbf{1}_{\{\tau_n \leq T\}}\right) = \mathbb{E}\left[\prod_{p=0}^{n} \mathbf{1}_{\{\tau_p < T\}}\right]. \tag{5.18}$$

From equation (4.3) we know that we can rewrite the latter equation in terms of η_n

$$\gamma_n\left(\mathbf{1}_{\{\tau_n \leq T\}}\right) = \eta_n(\mathbf{1}_{\{\tau_n \leq T\}}) \prod_{p=0}^{n-1} \eta_p(\mathbf{1}_{\{V(\tau_p) \leq D_p\}}). \tag{5.19}$$

We will use this formula for the computation of the hitting probabilities. For the simulation we need to discretize the time variable t and approximate η_n by η_n^M. We divide interval $[0, T]$ into equal intervals of length δt. In this setting Markov chain $\{X_p\}_{p \geq 0}$ is defined as

$$X_p = \left((S_i(\tau_p^M))_{(1 \leq i \leq d)}, (\min_{0 \leq m \leq p} S_i(m\delta t))_{1 \leq i \leq d}, V(\tau_p^M)\right). \tag{5.20}$$

Here τ_p^M is the discrete hitting time for barrier D_p. As in the first approach, we also keep track of the minimum value of the assets. In the next section we will describe the algorithm step by step.

5.2.1 Algorithm 2

Initialization
We start with M particles:

$$\xi_0^{(j)} = \left((S_i(0))_{(1\leq i\leq d)}, V(0)\right).$$

And the corresponding switch times:

$$\hat{\nu}^{(j)} = (\hat{\nu}_q)_{1\leq q\leq q_T}.$$

Here the switch times are generated in the same way as in IPS1.
Mutation
The asset values S_i will be approximated by an Euler scheme with time step δt, as in IPS1. For each particle $\xi^{(j)}$ a vector of switch times will be generated. If a particle hits barrier D_p, we remember the hitting time τ_p^j and continue with the selection stage. If a particle does not hit a barrier before T it is killed, i.e. it will have weight zero. We also keep track of the minimum value of the assets.
Selection
We resample M particles with replacement according to the Boltzmann-Gibbs measure:

$$\sum_{j=1}^{M} \frac{G_n(\xi_n^{(j)})}{\sum_{k=1}^{M} G_n(\xi_n^{(k)})}\delta_{\xi_n^j} = \sum_{j=1}^{M} \frac{\mathbf{1}_{\{V^{(j)}(\tau_p^j)\leq D_p\}}}{\sum_{k=1}^{M}\mathbf{1}_{\{V^{(k)}(\tau_p^k)\leq D_p\}}}\delta_{\xi_p^{(j)}} \tag{5.21}$$

Denote the number of particles that hit barrier D_p by I_p. Then we assign a weight $\frac{1}{I_p}$ to each particle for resampling. The particles that do not reach barrier D_p are killed. We refer to the resampled particles as $\check{\xi}_p^{(j)}$ and continue with the mutation at time $\check{\tau}_p^j$, the corresponding hitting time of barrier D_p for particle j.

Maturity
At maturity we compute the estimator for $\mathbb{P}(\tau_n < T)$. We keep track of the hitting probabilities by computing γ_p^M for each barrier:

$$\begin{aligned}
\gamma_n^M\left(\mathbf{1}_{\{\tau_n\leq T\}}\right) &= \eta_n^M\left(\mathbf{1}_{\{\tau_n\leq T\}}\right)\prod_{p=0}^{n-1}\eta_p^M\left(\mathbf{1}_{\{V(\tau_p)\leq D_p\}}\right)\\
&= \prod_{p=0}^{n}\left(\frac{1}{M}\sum_{j=1}^{M}\mathbf{1}_{\{V^{(j)}(\tau_p)\leq D_p\}}\right).
\end{aligned}$$

At each barrier we count the number of particles left and divide this by the total number of particles M. By multiplying these probabilities at each point p, we can compute the hitting probabilities for barriers $D_0, D_1, \ldots, D_n$. We also count the defaults in each particle, so we have a default probability for that number of defaults.

5.3 Summary

In this chapter we described two ways to compute default probabilities for a portfolio of $d = 125$ names. In both approaches, the asset value is described by the dynamics in equation 3.8. During the mutation stage we approximate these dynamics by an Euler scheme. The first approach is based on Carmona et al. [2], for which a sophisticated potential function, see equation 5.6, determines the selection procedure, i.e. the resampling of the particles. The resampling is done

at fixed points in time. By keeping track of the minimum values of the assets, we know whether a firm has defaulted before or at maturity T. With this approach we can compute default probabilities for x firms within a portfolio. The second approach is based on Krystul [8]. For this approach we look at the total portfolio asset value $V(t) = \sum_{i=1}^{d} S_i(t)$. The resampling points are barriers $D_n < D_{n-1} < \ldots < D_0 < V(0)$. If a particle hits a barrier before maturity T, the resampling weight attached to such a particle is $1/M$, where M is the total number of particles. If a particle does not hit a barrier before T, the particle is killed, i.e. the weight will be zero. The resampling weight for each particle is normalized by dividing it by the total sum of weights. By keeping track of the minimum value of the assets, we can count the number of defaults within a portfolio at time T.
In the next chapter we will simulate both approaches for several choices of parameters T, risk-free rate r, number of particles M and correlation factor ρ. We will discuss these numerical results in chapter 7.

Chapter 6

Numerical results

In this section we will run the algorithms above and discuss the outcomes. All computations are done in C++ (Microsoft Visual Studio 2008). Before we start with the portfolio of 125 assets, we will check whether our two algorithms give the correct results.

6.1 Hitting probabilities for one asset

As we know from section 3.4 we can compute default probabilities for one asset analytically. To check whether the probabilities computed by our models are in line with the actual probabilities, we will compute hitting probabilities for one asset with algorithm 1 and 2. Because in both approaches we track the minimum value of the asset, we have to compute analytically

$$P(0,T) = P\left(\min_{0\le t\le T} S_1(t) \le B\right). \tag{6.1}$$

We can compute this probability with the following explicit formula, see Carmona et al. [2]:

$$\begin{aligned} P(0,T) &= 1 - \left(\mathcal{N}\left(d_2^+\right) - \left(\frac{S(0)}{B}\right)^p \mathcal{N}\left(d_2^-\right)\right) \\ &\text{where} \\ d_2^\pm &= \frac{\pm\ln\left(S_1(0)/B\right) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \\ p &= 1 - \frac{2r}{\sigma^2}. \end{aligned}$$

We will compute hitting probabilities for $B = 85, 80, \ldots, 15$. For analytical probability computations we cannot incorporate the switch in the market volatility, so we fix the volatility at 0.25. The other parameters used can be found in table 6.1. For the number of particles M we choose $20,000$. For each barrier we will run 20 simulations and take the average. We choose $n = 20$ intervals for IPS1, so we resample 20 times. The resampling points for IPS2 are the 15 barriers. To show the power of IPS, we will also compute the hitting probabilities with Monte Carlo simulation. We will simulate the asset value $20,000$ times and for every barrier B we will count the number of assets that hit this barrier and divide it by the total number of simulations. In this way, we can approximate the hitting probabilities. We will do 20 Monte Carlo simulations and take the average for each barrier. In figure 6.1 the results can be found. As you can see, we have Monte Carlo probabilities up to barrier $B = 30$. For both IPS approaches we have estimations up to barrier $B = 15$. In order to compute these hitting probabilities with Monte Carlo, we need a lot more simulations than $20,000$. The values for IPS2 can be found in one run. For IPS1 we need to run more than one time, because we need to change the barrier values. We also need to change α, otherwise we cannot find the probabilities for the lower barriers. For the higher barriers we can choose $\alpha = 1$, but for the lower barriers, we need to choose a bigger α, so that the weight for resampling is adjusted. This is probably also necessary in the multiple asset case.

S_0	r	δt	M	# sims
90	0.05	10^{-3}	$20,000$	20

Table 6.1: Parameters IPS

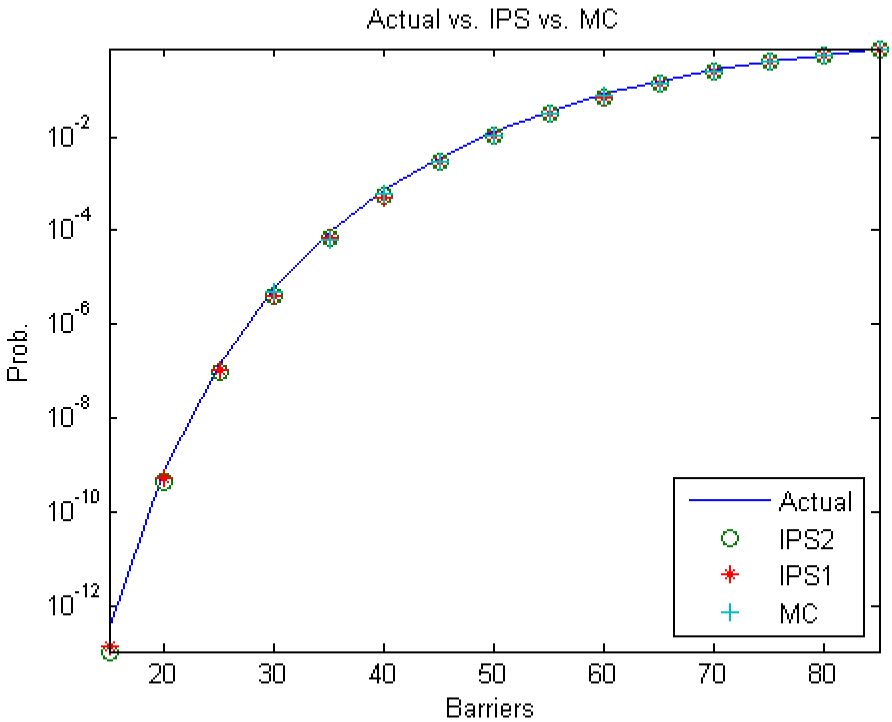

Figure 6.1: Probabilities of default for one asset

6.2 Portfolio of 125 assets

We start with the first approach, IPS1. For the number of assets we choose $d = 125$. For all assets we choose the same initial value of 90 and the default barrier for each asset will be fixed at 36, that is 40% of the initial value, as in Carmona et al. [2]. The correlation factor ρ_{ij} for the Wiener processes will be fixed for all assets, that is $\rho_{ij} = \rho$. We will run the algorithm with the parameters in table 6.2. Here n is the number of intervals for resampling. For every

$S_i(0)$	r	ρ	n	δt	B_i
90	0.05	0.1	20	10^{-3}	36

Table 6.2: Parameters IPS1

α we will compute the probabilities with estimator (5.12) for all $k \in \{0, 1, \ldots, 125\}$. To get reliable estimates for all k, we will vary α in such a way that we find probabilities for every k. For each α we will simulate 20 times and take the average. For the final estimate we take the maximum value over all α's for each $k \in \{0, 1, \ldots, 125\}$. In figure 6.2 you see the default probability estimates for $T = 1, 2, 3$ and 4 years.
For the second IPS approach we will use the same parameters for the asset portfolio model. The resampling barriers will be expressed as a percentage of the portfolio value. We set 18 barriers, where the last barrier is 4500, that is 40% of the total initial portfolio value:

$$\begin{aligned} B_0 &= 0.95V \\ B_i &= 0.95B_{i-1} \quad \text{for } 1 \leq i \leq 16 \\ B_{17} &= 4500. \end{aligned}$$

Again we simulate 20 times and take the average to get reliable probability estimates. In figure 6.3 you can find the results for for $T = 1, 2, 3$ and 4 years. We also keep track of the number of assets with a value smaller than 36, see table 6.3. Even though the total asset portfolio value is 40% of its initial value, the number of defaults is much smaller than 125. This means that the value of individual assets in a portfolio should be much smaller than 36.

T	1	2	3	4
$\# < 36$	71	85	90	92

Table 6.3: Number of defaults

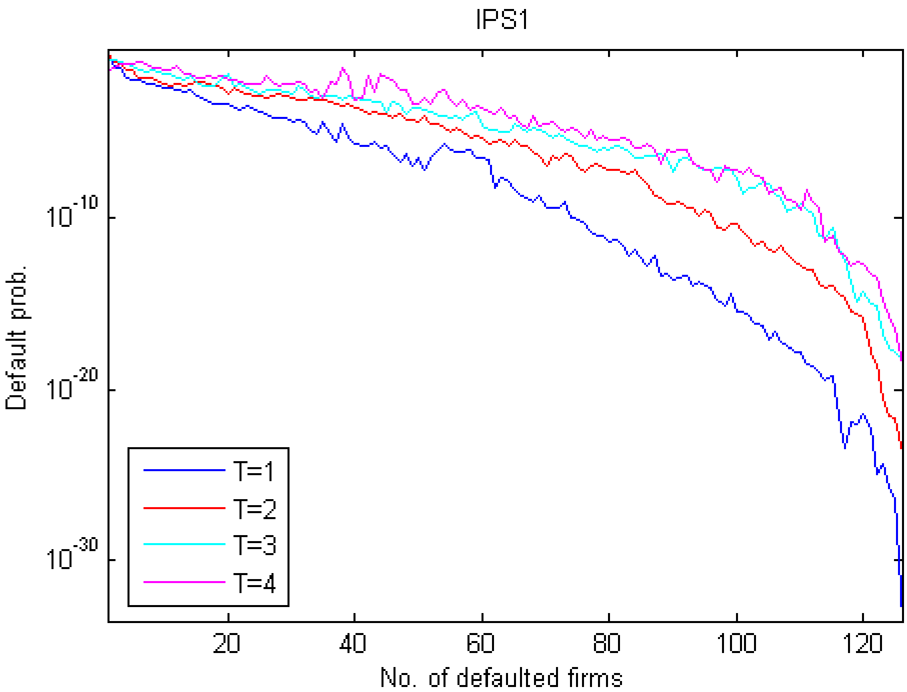

Figure 6.2: Probability of default for 1-125 assets

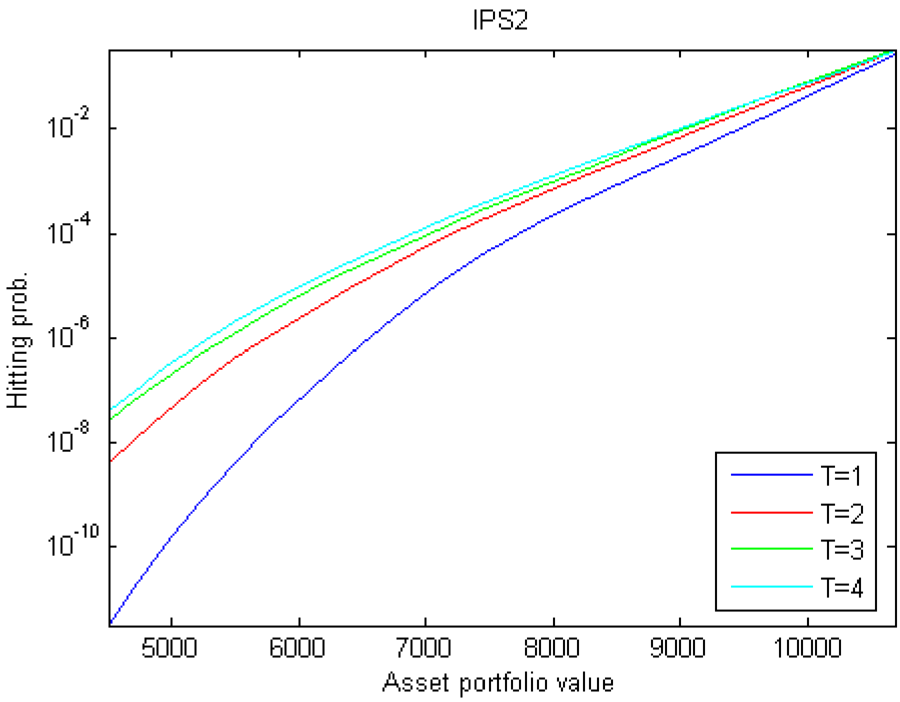

Figure 6.3: Hitting probabilities for portfolio value

6.3 Changing the number of particles

For both approaches we chose $M = 200$ particles. We know that the empirical measure η_n^M converges in distribution to η_n for $M \to \infty$. Therefore we now take $M = 1000$ particles for our simulations and see whether there are differences. For the results, see figures 6.4 and 6.5. We see that there are hardly any differences, so we will choose $M = 200$ for the rest of the simulations, that will save us time. That we only need 200 particles to get reliable estimations proves the power of the IPS approach.

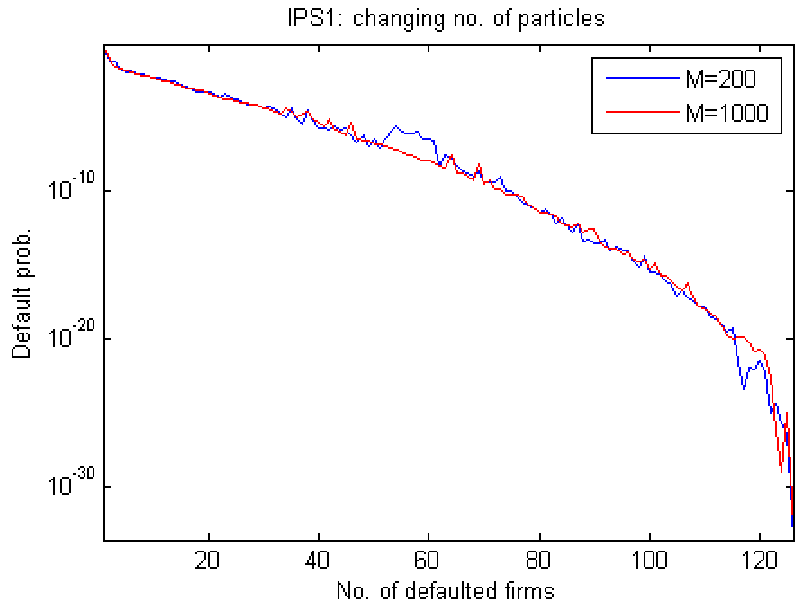

Figure 6.4: IPS1: changing the no. of particles

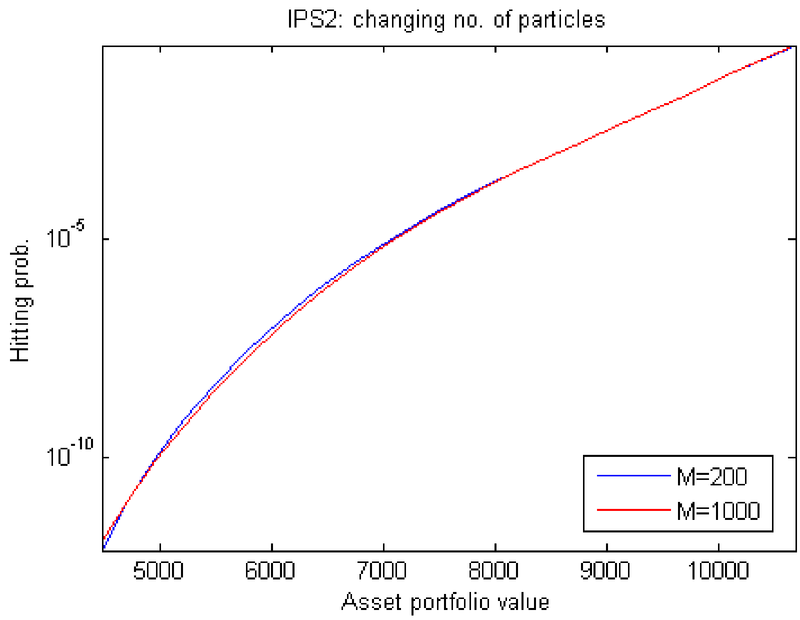

Figure 6.5: IPS2: changing the no. of particles

6.4 Regular asset model

To see the influence of the switch in the volatility, we run both programs with a flat volatility rate. Now the market volatility factor is 1, as in a low or mean volatile period. Again we have the same parameters as in table 6.2. For the results, see figures 6.6 and 6.7. For IPS1 we see that we have default probabilities up to 78 defaults out of 125. For IPS2 we see that 6736 is the last barrier that is been hit, but our target barrier is 4500, 40% of the initial value. The next barrier is 6399. Apparently the difference between the barriers is too big, because the particles are not able to hit barriers below 6736. Therefore we choose 45 barriers and make the difference between the lower barriers smaller, see the second graph in figure 6.7. Now the last barrier is 6406, so we improved the model. In the third graph we change the number of barriers to 60, see figure 6.7. Again the the model is improved; the last barrier is 5747, that is 54% of the initial portfolio asset value.

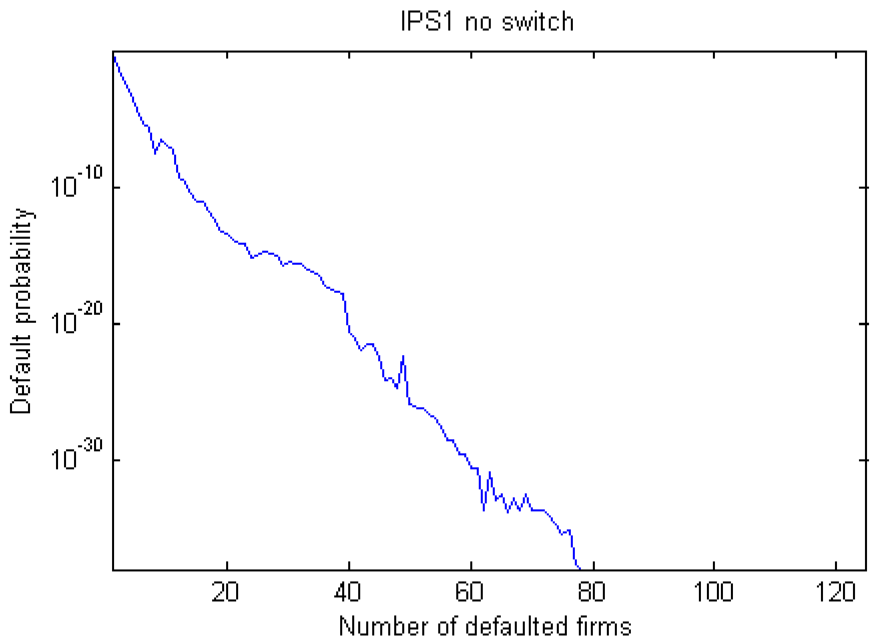

Figure 6.6: IPS1 no switch

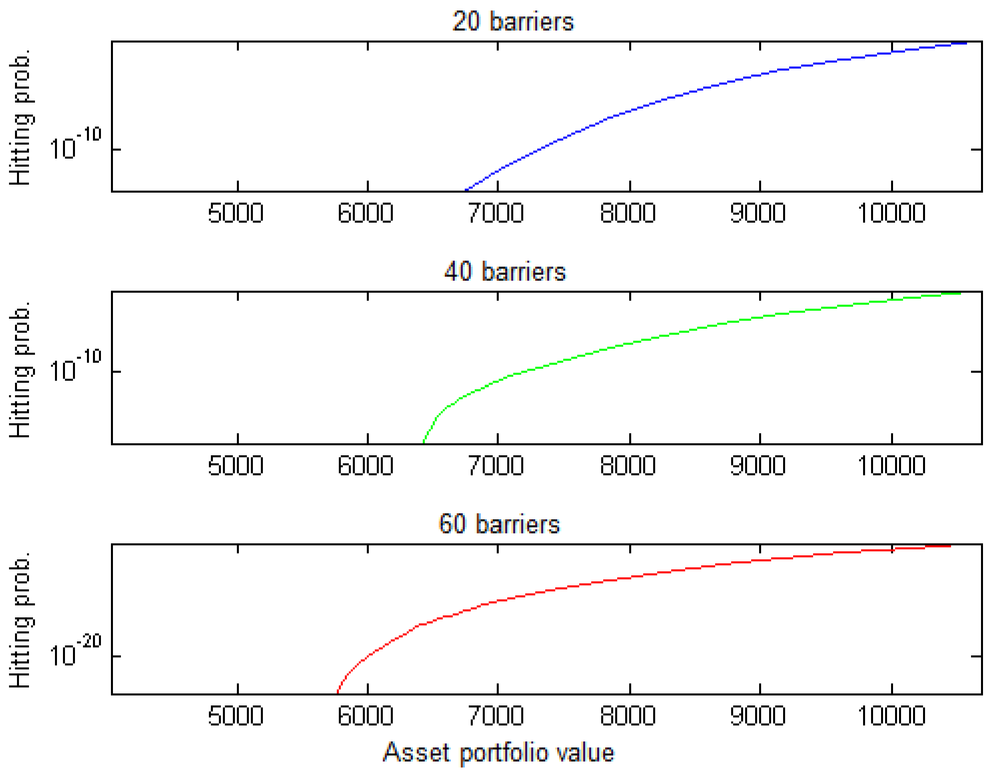

Figure 6.7: IPS2 no switch

6.5 Incorporating a switch in the risk-free rate r

Now we incorporate a switch in the risk-free rate and the market volatility. To this end, we incorporate a second switching vector in our model, based on the λ^r's found in section 3.3. Based on our findings in section 3.4 we do not expect that much influence on the default probabilities. For the results, see figures 6.8 and 6.9. As we expected, the switch in the risk-free rate does not have that much impact on the probabilities, actually there is no difference with a flat risk-free rate at all.

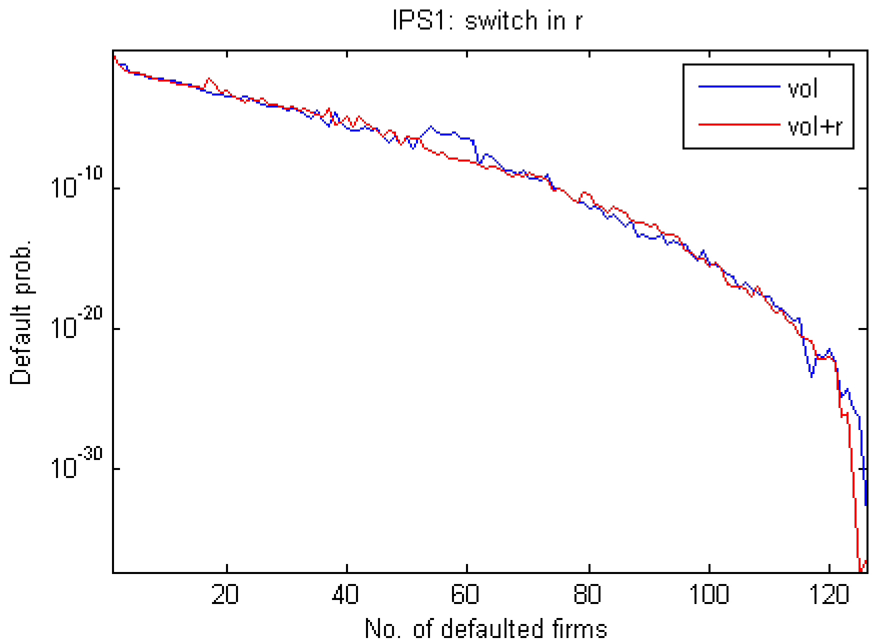

Figure 6.8: IPS1 switch in r

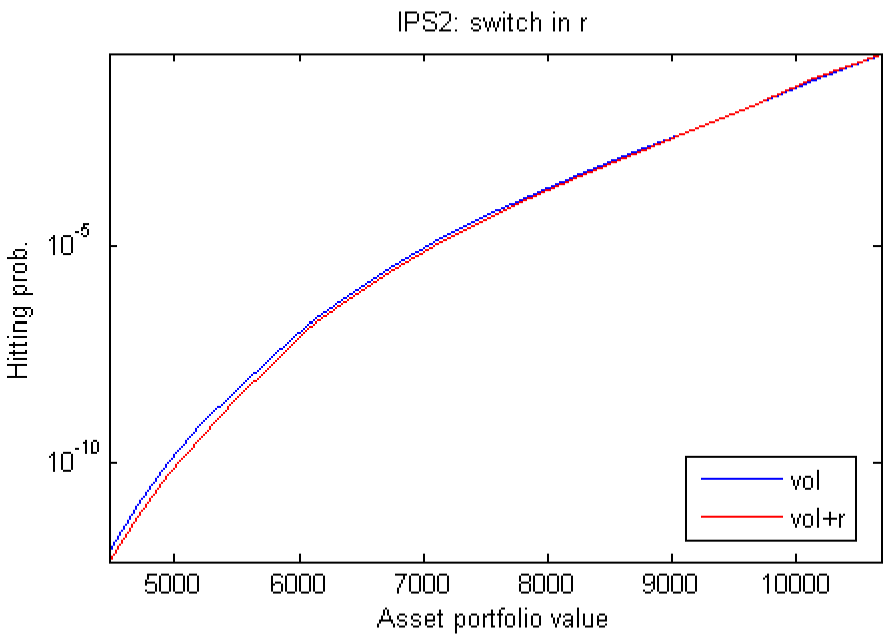

Figure 6.9: IPS2 switch in r

6.6 Changing the correlation factor

To see the influence of the correlation within the model, we change the correlation factor ρ. We will run the models for $\rho = 0$ and $\rho = 0.5$. For the results, see figures 6.10, 6.11, 6.12 and 6.13. For IPS1 we see big differences if we change ρ. For no correlation in the portfolio we can compute default probabilities up to 40 out of 125 firms. For $\rho = 0.5$, we see that the probability that 125 firms default in one year increases dramatically from 2.9^{-34} to 2.9^{-8}. For the second approach we see that for $\rho = 0$ the lowest barrier that is been hit is 9579, the next barrier to hit is 9483. If we set the barriers as in equation (6.2), we do not have hits at all. For $\rho = 0.5$ we see that the probability of hitting threshold 4500 increases to 6.2^{-5}.

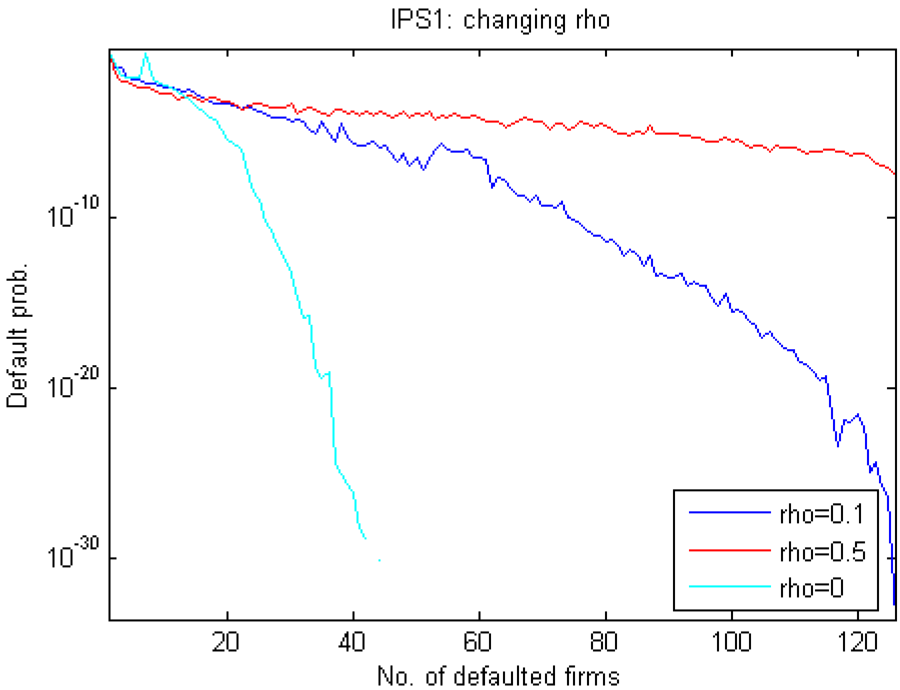

Figure 6.10: IPS1: changing rho

6.7 Summary

In this chapter we demonstrated our two models computing default and hitting probabilities. We changed maturity T, the number of particles M, incorporated a switch in the risk-free rate and changed correlation factor ρ, to see the influence of the parameters. In the next chapter we will discuss our findings and conclude with the most important findings.

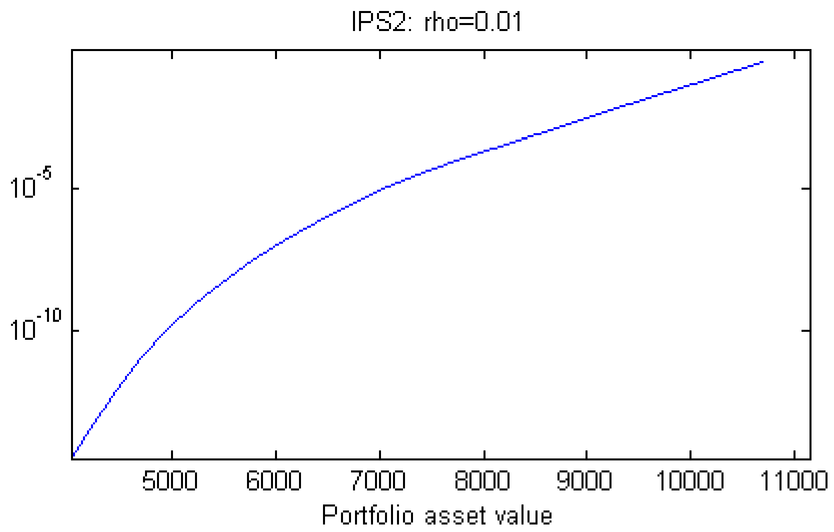

Figure 6.11: IPS2 rho=0.1

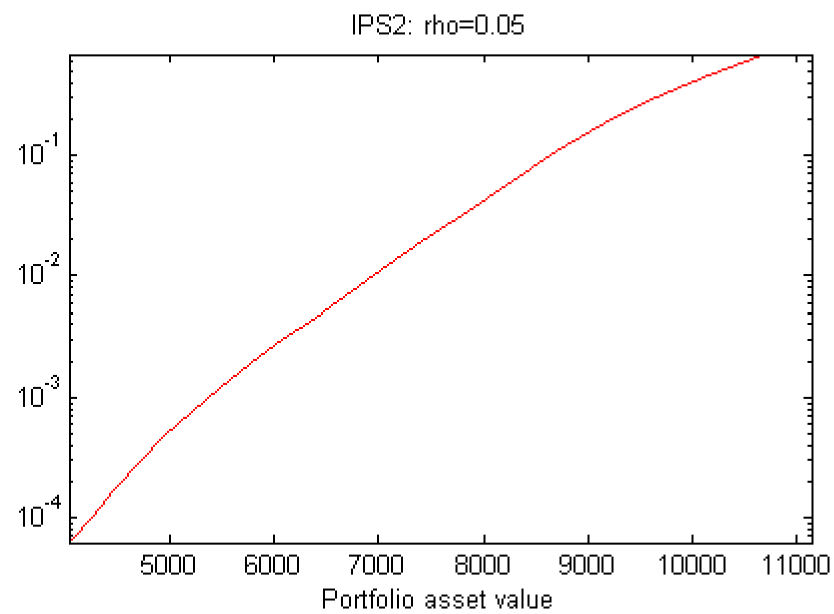

Figure 6.12: IPS2 rho=0.5

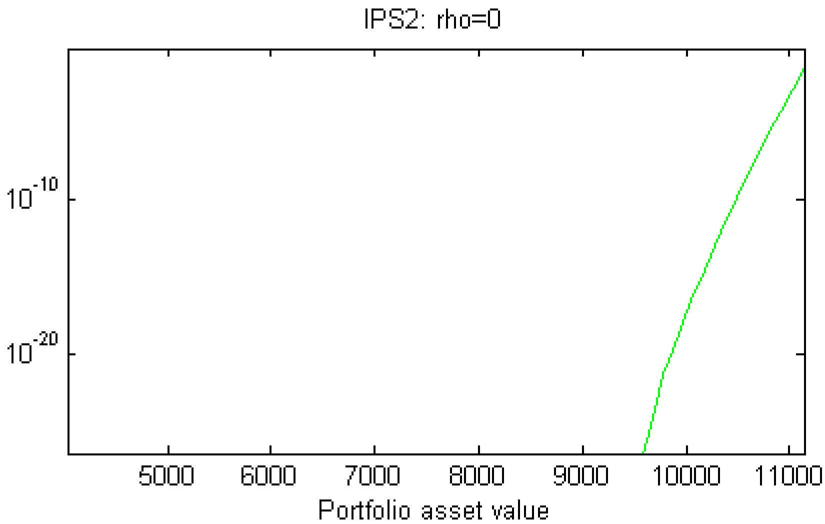

Figure 6.13: IPS2 rho=0

Chapter 7

Discussion and conclusion

In this chapter we will discuss the results obtained and will do suggestions for further research. We conclude with our most important findings based on the discussion.

7.1 Discussion of the results

We start by comparing the IPS algorithm with regular Monte Carlo estimation. As we see from chapter 6.1, IPS can calculate rare probabilities up to 10^{-13} with $M = 20,000$ particles. If we want to compute probabilities like that with Monte Carlo estimation, we would need 10^{+12} simulations, which is extremely time-consuming and perhaps not even possible.
We continue with comparing the two algorithms. The difference in the approaches is the way of resampling, i.e. the choice for potential function G. This choice determines the outcomes. The first approach gives default probabilities for x firms within the portfolio and the second approach computes hitting probabilities of the barriers defined and tallies the number of defaults at maturity. It depends on the kind of analysis that you want to do, which approach to use. The first approach gives you default probabilities which can be used for setting tranches, or for pricing CDO's. The potential function in the first approach is more sophisticated than in the second approach. A drawback of this function is that you have to run the algorithm for several α's to get reliable estimates. For the second approach, only one run can provide us with results, which is much faster. This approach can be used if you want to do scenario analysis for your portfolio. As we see from our results, this approach captures the impact of changing parameters as well as the first approach.
A drawback of both approaches is that you leave actual defaulted firms in the portfolio. In a CDO, the notional amount of the CDS in question has to be paid off, which means a loss for the CDO. In our case, we leave these firms in, so they may have influence on the resampling for both approaches. We actually see this in the second approach. Even though we reach the threshold $125 \times 36 = 4500$, the number of defaults is not equal to 125. An interesting subject for further research is how to capture firms that already defaulted before T in this model. One way of doing this is to freeze the asset value if it drops below the default barrier.
As we can see in figures 6.2 and 6.3 the probabilities of default are very small, so the likelihood of such an event is almost zero. Therefore it is interesting to change the parameters in our models and to see what the difference in probabilities is. We changed maturity T, the volatility factor, the risk-free rate r and the correlation factor ρ. We start with increasing the maturity T. Intuitively we expect the probability of default to go up, since firms have more time to default. We see this in both figures 6.2 and 6.3. Especially the difference between $T = 1$ and $T = 2$ is big. If we increase the maturity to $T = 3$ or $T = 4$ years, the difference declines, so one can expect some limit for T large.
For default probabilities in the asset portfolio model without any regime switch, see figures 6.6 and 6.7. We see that the probabilities for both approaches are very small, we actually need

to adapt the second approach to get some results. If we incorporate the switch in the market volatility, we see a big difference in the default probabilities, see figures 6.2 and 6.3. For IPS1 we have results up to 125 assets and for IPS2 we reach the target barrier 4500. That is a big difference with the regular asset model.
From section 3.4 we expect that a change in the risk-free rate r does not have a big impact on the default probabilities. If we incorporate a switch in the risk-free rate, we see that the results approve this view. There is actually no influence at all, see figures 6.8 and 6.9. So for our asset portfolio model, the risk-free rate has no impact on the default or hitting probabilities.
The last parameter we discuss is ρ, the correlation factor. Intuitively, when we have an economically bad period, one would expect more defaults, since firms within the same sector, but also outside their own sectors, are depending on each other. We expect the correlation factor to increase. At first we choose a fixed correlation for all assets of 0.1. In section 6.6 we change the correlation factor to 0 and 0.5. We see that the correlation has a big impact on the default probabilities. If we have no correlation in our IPS1 model, the default probabilities decrease dramatically and do not give results for defaults bigger than 44 out of 125. For IPS2 we need to adapt our barriers to get some results. Also here the probabilities decrease dramatically; we do not hit barriers below a portfolio value of 9579. As we said, we expect the correlation to go up in economically bad times. Therefore we change the correlation in both models to 0.5. And indeed, as we see in figures 6.10 and 6.12 the probabilities make a big jump up. This suggests that the correlation within an asset model as described above has a huge impact on default probabilities. For the ease of computation, we choose ρ to be the same for every two firms. But in our regime switching model, you have the possibility of setting ρ_{ij} for each company i and j. We did not do any statistical analysis on these portfolio correlations. This research is beyond the scope of this thesis. A suggestion for further research is to analyze correlations within portfolio asset models and, if necessary, incorporate a regime switch for correlation too.

7.2 Conclusion

From our discussion above we can draw the following conclusions. At first the IPS approach is capable of estimating rare probabilities with only 200 particles. For regular Monte Carlo estimation, this is impossible. Even if we want to compute hitting probabilities in a one asset model, we see that the IPS approach does a much better job than the Monte Carlo simulations. The regime switch in the volatility approaches real-world models better and regime switching asset models as described in this thesis have a huge impact on default probabilities. By using the IPS approach, we can estimate rare probabilities and show the impact of the several parameters T, σ, r and ρ of the portfolio asset model on these rare probabilities. The switch in the volatility and correlation within the portfolio have the most impact on default probabilities. These probabilities are extremely important for pricing CDO's and especially for setting tranches. By knowing the influence of the parameters in a structural model as ours, we know how important it is to estimate these parameters correctly. By doing scenario analysis as we did, we can assess the risks in a portfolio of credit risk better.

Appendix A

Merton model

We know that under the assumption of no arbitrage, we have that

$$S_T = S_0 \exp\left\{ \left(r - \frac{1}{2}\sigma^2 \right) T + \sigma W(T) \right\},$$

see Björk [1]. We know that $W(T)$ is a Wiener process having distribution $N(0,T)$, so if we write

$$S_T = S_0 e^Z,$$

we have that $Z \sim N\left(\left(r - \frac{1}{2}\sigma^2 \right) T, \sigma^2 T \right)$. Now we can find the default probability:

$$\begin{aligned}
\mathbb{P}\left(S_T \leq D\right) &= \mathbb{P}\left(S_0 e^Z \leq D\right) \\
&= \mathbb{P}\left(Z \leq \ln\left(D/S_0\right)\right) \\
&= \int_{-\infty}^{\ln(B/S_0)} f(z) dz. \qquad \text{(A.1)}
\end{aligned}$$

Here $f(z)$ is the density function of normal random variable Z. If we standardize random variable Z, we can rewrite the latter equation to a more easy one. Refer to the mean and variance of Z respectively as μ_Z and σ_Z^2. Then

$$\frac{Z - \mu_Z}{\sigma_Z} \sim N(0,1) \qquad \text{(A.2)}$$

If we rewrite the boundaries in equation A.1 we have that the default probability equals

$$\mathbb{P}\left(S_T \leq D\right) = \mathcal{N}(d_1), \qquad \text{(A.3)}$$

where

$$d_1 = \frac{\ln\left(D/S_0\right) - \left(r - \frac{1}{2}\sigma^2 \right) T}{\sigma\sqrt{T}},$$

and $\mathcal{N}$ is the cumulative standard normal distribution.

Appendix B

Proof change of measure

We have the following dynamics for the asset prices:

$$dS_i(t) = \mu_i(t)S_i(t)dt + \sigma_i\sigma^m(t)S_i(t)d\bar{W}_i(t), \quad i \in \{1, \ldots, d\}. \tag{B.1}$$

where the correlation of the Wiener processes is given by

$$d\bar{W}_i(t) \cdot d\bar{W}_j(t) = \rho_{ij}dt.$$

We can write the dynamics in the form

$$dS_i(t) = \mu_i(t)S_i(t)dt + S_i(t)\sum_{j=1}^{d} A_{ij}(t)dW_j(t).$$

Where $(W_1(t), \ldots, W_d(t))^\intercal$ are independent Wiener processes and A is a lower triangular matrix, the solution for $AA^\intercal = \Sigma$. Now we can write the dynamics in vector form:

$$dS(t) = D[S(t)]\mu(t)dt + D[S(t)]A(t)dW(t).$$

Where $\mu(t)^\intercal = (\mu_1(t), \ldots, \mu_d(t))$, $W(t)^\intercal = (W_1(t), \ldots, W_d(t))$ and $D[x]$ is the diagonal matrix:

$$D[x] \quad = \quad \begin{pmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_d \end{pmatrix}.$$

From Björk [1] we know that the absence of arbitrage is equivalent to the existence of an equivalent martingale measure $\mathbb{Q}$ under which the discounted stock prices are martingales. Based on the Girsanov Theorem in Björk [1], we define a likelihood process L by

$$dL_t = \varphi(t)L_t dW_t^{\mathbb{P}}, \quad L_0 = 1. \tag{B.2}$$

where $\varphi(t)$ is a $(d \times 1)$ matrix: $\varphi^\intercal(t) = (\varphi_1(t), \ldots, \varphi_d(t))$. The solution for equation (B.2) is

$$L_t = \exp\left\{\int_0^t \varphi(s)dW_s^{\mathbb{P}} - \frac{1}{2}\int_0^t \|\varphi(s)\|^2 ds\right\}, \tag{B.3}$$

where $\|\cdot\|$ is the Eucledian norm. Now define the candidate martingale measure by $d\mathbb{Q} = L_t d\mathbb{P}$, so we can write

$$dW_t^{\mathbb{P}} = \varphi(t)dt + dW_t^{\mathbb{Q}}.$$

Since our model is incomplete, there are infinitely many equivalent martingale measures. In Elliott et al. [5] they want to find a risk-neutral measure for the single asset case with a regime switch. They choose a risk-neutral measure according to the regime switching Esscher transform, where the choice is verified by the minimal entropy martingale measure. In this single asset case, the likelihood process is of the form:

$$\frac{d\mathbb{Q}_\vartheta}{d\mathbb{P}}|_{\mathcal{G}_t} = \exp\left\{\int_0^t \vartheta_s\sigma_s dB_j(s) - \frac{1}{2}\int_0^t (\vartheta_s\sigma_s)^2\, ds\right\}. \tag{B.4}$$

Here $\vartheta(t)$ is defined as

$$\vartheta(t,\theta_t) =< \vartheta, \theta_t >,$$

with $\vartheta = (\vartheta_1, \ldots, \vartheta_N)$ and σ_t is the volatility of the single asset dynamics. For our multidimensional case, assume that $\varphi(t)$ in equation (B.3) is of the form $A(t)\Theta(t)$, where $\Theta(t)$ is a $(d \times 1)$ matrix and the entries are defined by

$$\Theta_i(t,\theta_t) =< \Theta_i, \theta_t > \quad \text{for } \; i = 1, \ldots, d,$$

with $\Theta_i = (\Theta_{i1}, \ldots, \Theta_{iN})$.

Let $B(t) = -B(t)r(t)dt$ be the dynamics of the discount process. We know that under $\mathbb{Q}$, the discounted price process $S(t)$ is a martingale. We can write:

$$\begin{aligned}
d\left[B(t)S(t)\right] &= S(t)dB(t) + B(t)dS(t) + dB(t)dS(t) \\
&= -B(t)r(t)S(t)dt + B(t)\left(D[S(t)]\mu(t)dt + D[S(t)]A(t)dW^{\mathbb{P}}(t)\right) \\
&\quad + (-B(t)r(t)dt)\left(D[S(t)]\mu(t)dt + D[S(t)]A(t)dW^{\mathbb{P}}(t)\right) \\
&= \begin{pmatrix} -B(t)r(t)S_1(t) \\ -B(t)r(t)S_2(t) \\ \vdots \\ -B(t)r(t)S_d(t) \end{pmatrix} dt + \begin{pmatrix} B(t)\mu_1(t)S_1(t) \\ B(t)\mu_2(t)S_2(t) \\ \vdots \\ B(t)\mu_d(t)S_d(t) \end{pmatrix} dt + B(t)D[S(t)]A(t)dW^{\mathbb{P}}(t) \\
&= B(t)D[S(t)]\left(\mu(t) - R(t)\right)dt + B(t)D[S(t)]A(t)dW^{\mathbb{P}}(t) \\
&= B(t)D[S(t)]\left(\mu(t) - R(t)\right)dt + B(t)D[S(t)]A(t)\left(\varphi(t)dt + dW^{\mathbb{Q}}(t)\right) \\
&= B(t)D[S(t)]\left(\mu(t) - R(t) + A(t)\varphi(t)\right)dt + B(t)D[S(t)]A(t)dW^{\mathbb{Q}}(t)
\end{aligned}$$

Here $R(t)^\intercal = (r(t), r(t), \ldots, r(t))$ is the $(d \times 1)$ vector of risk-free rates. Now $\mathbb{Q}$ is a martingale measure if and only if

$$d\left[B(t)S(t)\right] = B(t)D[S(t)]A(t)dW^{\mathbb{Q}}(t),$$

that is

$$\mu(t) - R(t) + A(t)\varphi(t) = 0.$$

If we substitute $\varphi(t) = A(t)\Theta(t)$, we have

$$\mu(t) - R(t) + A(t)A(t)\Theta(t) = 0.$$

We can only solve for $\Theta(t)$ if the matrix $A(t)$ is invertible. The matrix $A(t)$ is a lower triangular matrix. Such a matrix is invertible if and only if the diagonal entries are nonzero, see .... Since the entries of the volatility matrix Σ are always bigger than zero, we will have only nonzero entries on the diagonal of matrix $A(t)$. Now we can solve for Θ:

$$\Theta(t) = [A(t)]^{-1}[A(t)]^{-1}[R(t) - \mu(t)].$$

Under the risk-neutral martingale measure $\mathbb{Q}_\Theta$ we have the following asset dynamics:

$$dS(t) = D[S(t)]R(t)dt + D[S(t)]A(t)dW^{\mathbb{Q}}(t).$$

Appendix C

Cholesky decomposition

This section is based on Glasserman [6]. We want to solve $AA^\intercal = \Sigma$ for A. We have that

$$\begin{pmatrix} A_{11} & & & \\ A_{21} & A_{22} & & \\ \vdots & \vdots & \ddots & \\ A_{d1} & A_{d2} & \cdots & A_{dd} \end{pmatrix} \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{d1} \\ & A_{22} & \cdots & A_{d2} \\ & & \ddots & \vdots \\ & & & A_{dd} \end{pmatrix} = \begin{pmatrix} \Sigma_{11} & \Sigma_{21} & \cdots & \Sigma_{d1} \\ \Sigma_{21} & \Sigma_{22} & \cdots & \Sigma_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{d1} & \Sigma_{d2} & \cdots & \Sigma_{dd} \end{pmatrix}$$

By multiplying the entries of the first two matrices, we have the following system of equations:

$$\begin{aligned} A_{11}^2 &= \Sigma_{11} \\ A_{21}A_{11} &= \Sigma_{21} \\ &\vdots \\ A_{d1}A_{11} &= \Sigma_{d1} \\ A_{21}^2 + A_{22}^2 &= \Sigma_{22} \\ &\vdots \\ A_{d1}^2 + \ldots + A_{dd}^2 &= \Sigma_{dd} \end{aligned}$$

Or:

$$\begin{aligned} \Sigma_{ij} &= \sum_{k=1}^{j} A_{ik}A_{jk}, \quad \text{for } j \leq i \\ &= \sum_{k=1}^{j-1} A_{ik}A_{jk} + A_{ij}A_{jj}, \quad \text{for } j < i \\ \Sigma_{ii} &= \sum_{k=1}^{i-1} A_{ik}^2 + A_{ii}. \end{aligned}$$

We want to solve for A, so we get

$$A_{ij} = \left(\Sigma_{ij} - \sum_{k=1}^{j-1} A_{ik}A_{jk} \right) / A_{jj}$$

and

$$A_{ii} = \sqrt{\Sigma_{ii} - \sum_{k=1}^{i-1} A_{ik}^2}.$$

Now we can solve for A with a simple recursion, starting with $A_{11} = \sqrt{\Sigma_{11}}$.

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