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Infinite Dimensional System Theory

for Networks of Flows

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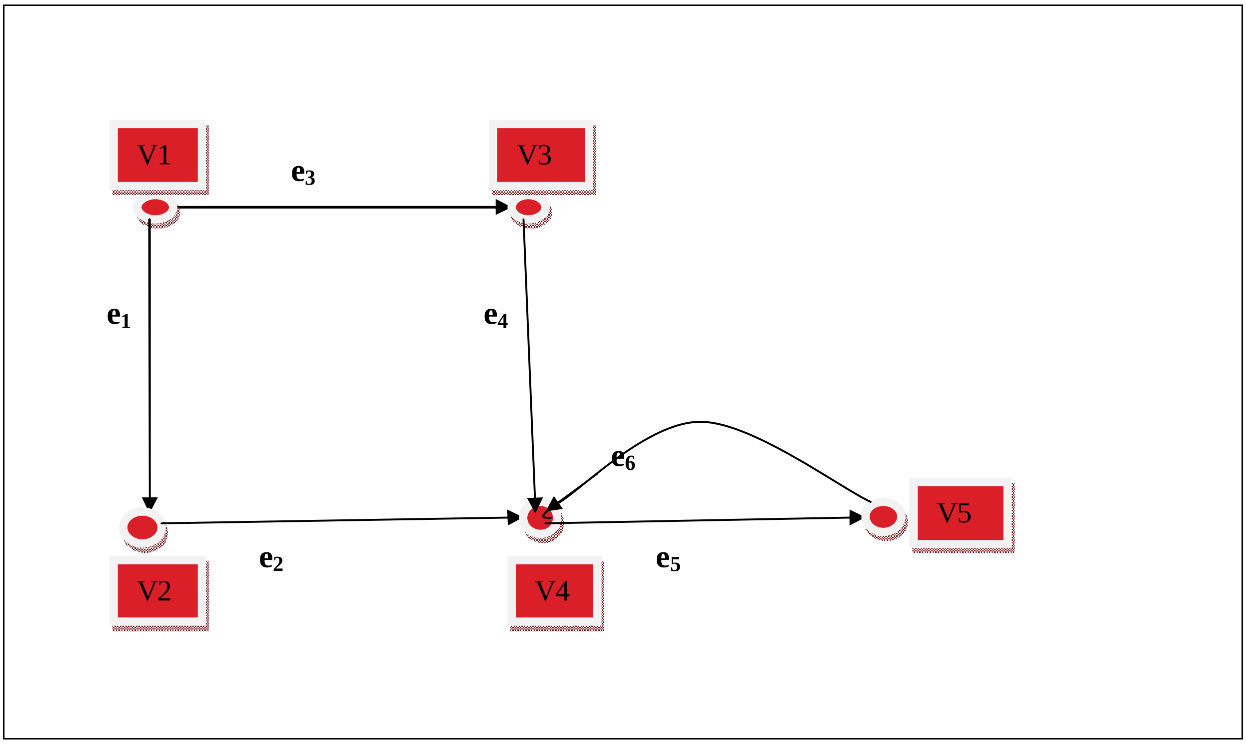
Chapter 1 Introduction

Network flow theory is a mathematical treatment of flows in networks. Network flow is important because it can be used to express a wide variety of different kinds of problems. Different kinds of networks have been studied since many years with both theoretical and application' point of view. Network of flows have endless applications in daily life, some of the examples are electrical power grids, cellular networks, transportation networks, neural networks, financial networks, flow of nutrients and energy between different organizations in a food web, and last but not least is the World-Wide Web.

In this thesis we have considered a network flow problem which is basically motivated from the research article [1]. The description of the problem and the tasks to investigate are as follows:

Description: We consider a closed network such as depicted in the following figure



We assume that the edges all have equal length (one) and that on every edge. The information/material is transported with equal and constant speed; this means that on every edge the following partial differential equation holds

$$\frac{\partial z}{\partial t}(\xi,t) = \frac{\partial}{\partial \xi} z(\xi,t)$$

The arrow in the figure indicates way the information flow. At the vertices no information can be lost. Hence at every vertex the total amount of incoming equals the total amount of out-coming. However, the outflow may not be equal to all connected edges. The outflow is weighted by w_{ij} .

Main tasks:

1) The 1st task is to formulate the partial differential equations together with the relations at the vertices as an abstract differential equation on the state space $L^2\left((0,\ 1);R^n\right)$, where n is the number of edges.
2) The 2nd task is to prove that this (homogeneous) differential equation generates a contraction semigroup on the state space.
3) Add an input at one of the vertices, and show that for every square integrable input function, there exists a unique solution of the inhomogeneous abstract differential equation.
4) As output, the flow at one of the edges is chosen. After the well-posedness question similar to the one above has been answered, the following optimal control problem is studied

$$\min \int_0^\infty \|y(t)\|^2 + \|u(t)\|^2 \, dt$$

The final aim is to solve this optimal control problem. It is expected that by the special structure of the coupled partial differential equations the solution to this problem can be formulated in a matrix equation.

To investigate the above network of flow and to complete the aforementioned tasks we have divided this thesis into five main chapters. Where needed the chapter is further subdivided into different sections.

Chapter 1 (current chapter) is basically introductory chapter, wherein we have explained the importance of network of flows by giving the various applications in real life. The main problem of the thesis is given along with the related tasks which gives clear idea to the reader regarding the later research work in this project.

Chapter 2 is basically preliminary review of the relevant background information [1], [2], [5]. This knowledge is necessary to understand which will help later to solve the problem and to complete the tasks. In this chapter along with various important definitions, theorems we have given some important theorems, lemmas and explain the Port-Hamiltonian and boundary control systems. All the information given and explained in this chapter provides the solid back ground to proceed further in this thesis to complete the tasks.

In Chapter 3 we have discussed two important theorems; the results of these theorems are used later in this thesis at different occasions. The most important part of this chapter is two explanatory examples. In these examples; we have considered two different closed networks of flows (one of these is the same as given in problem description) which hold linear transport equation. Hamiltonian matrices have been created for these networks of flows. On the basis of these matrices we have shown that one of these networks flows generate the contraction semigroup, and for one network the contraction semigroup doesn't hold.

The network for which the contraction semigroup generated is further continued to obtain the inhomogeneous abstract differential equation and finally shown that the classical solution for this network of flow exists.

In Chapter 4 first we studied the relevant literature [2], [4], [9], [11] regarding the boundary control system and port-Hamiltonian system for boundary control problems. Next we add an input to the closed network (for which the contraction semigroup generated) and obtain the inhomogeneous abstract differential equation. Further we showed that the port-Hamiltonian system is boundary control system for this network of flow. Later we proved that the system is well-posed as well, and the unique solution (classical solution) for this network of flow exists.

Chapter 5 is the final concluding chapter, in this chapter we have concluded different tasks which we have done in this thesis. The ultimate results which we have deduced during the research work are given in this chapter.

Chapter 2 Background Information

This chapter describes the background information [1], [2] needed to proceed with our project/thesis. First section is about the semigroup, next section treats information regarding port-Hamiltonian linear systems, and last section provides the description of the boundary control system and boundary observation for port-Hamiltonian system.

2.1 STRONGLY CONTINUOUS SEMIGROUP

In this section; we introduce strongly continuous semigroup (C_0-semigroup) and their generators.

Definition:

Let X be a Hilbert space. $T(t)_{t\geq 0}$ is called a strongly continuous semigroup (C_0- semigroup) if the following holds

1. For all $t \geq 0$, is a bounded linear operator on X, i.e. $T(t) \in L(X)$;
2. $T(0)=I$;
3. $T(t+\tau) = T(t)T(\tau)$ for all $t, \tau \geq 0$
4. For all $x_0 \in X$, we have that $\|T(t)x_0 - x_0\|_X$ converges to zero, when $t \downarrow 0$, i.e., $T(\cdot)$ is strongly continuous at zero.

We call X the state space.

2.1.1 Infinitesimal generator of C_0 - semigroup

Let $T(t)_{t\geq 0}$ be a C_0- semigroup on the Hilbert space X , if the following limit exists

$$\lim_{t\downarrow 0} \frac{T(t)x_0 - x_0}{t}$$

Then we say that x_0 is an element of the domain of A, or we can say $x_0 \in D(A)$, we define Ax_0 as

$$Ax_0 = \lim_{t\downarrow 0} \frac{T(t)x_0 - x_0}{t}$$

and say that A is the infinitesimal generator of the strongly continuous semigroup $T(t)_{t\geq 0}$.

This definition implies that every strongly continuous semigroup has unique generator.

2.1.2 Theorem

Let $T(t)_{t\geq 0}$ be a strongly continuous semigroup on the Hilbert space X with infinitesimal generator A. Then the following results hold:

a. For $x_0 \in D(A)$ and $t \geq 0$ we have $T(t)x_0 \in D(A)$

b. $\frac{d}{dt}\left(T(t)x_0\right) = AT(t)x_0 = T(t)Ax_0$ for $x_0 \in D(A)$, $t \geq 0$

c. $\frac{d^n}{dt^n}\left(T(t)x_0\right) = A^n T(t)x_0 = T(t)A^n x_0$ for $x_0 \in D(A^n)$, $t \geq 0$

d. $T(t)x_0 - x_0 = \int_0^t T(s)Ax_0\, ds$ for $x_0 \in D(A)$

e. $\int_0^t T(s)x\, ds \in D(A)$ and $A\int_0^t T(s)x\, ds = T(t)x - x$ for all $x \in X$ and $D(A)$ is dense in X

f. A is a closed linear operator.

This theorem implies that for $x_0 \in D(A)$ the function x defined by $x(t) = T(t)x_0$ satisfies the abstract differential equation $\dot{x}(t) = Ax(t)$.

2.2 ABSTRACT DIFFERENTIAL EQUATION

In this section; we showed how to rewrite a p.d.e. as an abstract differential equation. Note that for partial differential equations the question of existence and uniqueness of solutions is more difficult than ordinary differential equation.

As the theorem 2.1.2 shows that for $x_0 \in D(A)$ the function $x(t) = T(t)x_0$ is a solution of abstract differential equation

$$\dot{x}(t) = Ax(t) \ , \qquad x(0) = x_0 \qquad \text{..........................} \quad (2.1)$$

2.2.1 Classical solution

A differentiable function $x : [0, \infty] \to X$ is called a classical solution of eq:2.1

If for all $t \geq 0$ we have $x \in D(A)$ and eq:2.1 is satisfied.

2.2.2 Lemma

Let A be the infinitesimal generator of the strongly continuous semigroup $T(t)_{t\geq 0}$, then for every $x_0 \in D(A)$, the map $t \mapsto T(t)x_0$ is the unique classical solution of eq:2.1. This lemma shows that the classical solution is uniquely determined for $x_0 \in D(A)$.

2.3 CONTRACTION SEMIGROUPS

In this section, we studied the other implication and restrict ourselves to infinitesimal generator of contraction semigroup on (separable) Hilbert space and also discussed the dissipative operator.

Definition:

Let $T(t)_{t\geq 0}$ be a C_0 -semigroup on the Hilbert space X , then $T(t)_{t\geq 0}$ is called contraction semigroup, if $\|T(t)\| \leq 1$, for every $t \geq 0$.

2.3.1 Dissipative operator

A linear operator $A: D(A) \subset X \rightarrow X$ is called dissipative if $\mathrm{Re}\langle Ax, x\rangle \leq 0$, $x \in D(A)$

2.3.2 Theorem (Lumer-Phillips Theorem)

Let A be a linear operator with domain $D(A)$ on a Hilbert space X , then A is the infinitesimal generator of a contraction semigroup $T(t)_{t\geq 0}$ on X if and only if A is dissipative and has $\mathrm{ran}(I - A) = X$.

This theorem shows that the generators of contraction semigroup are precisely those dissipative operators, for which $I - A$, is surjective.

2.3.3 Theorem

Let A be a linear, densely defined and closed operator on a Hilbert space X . Then A is the infinitesimal generator of a contraction semigroup $T(t)_{t\geq 0}$ on X if and only if A and A^* are dissipative.

2.4 PORT-HAMILTONIAN SYSTEM

In this section; we studied that how port-Hamiltonian system generates contraction semigroup [2]. Hamiltonian differential equation form an important sub class with ordinary and partial differential equations, these includes their linear and non linear differential equations which appear in many physical models. We restrict ourselves to linear differential equation and normally omit the terms, linear, first order.

2.4.1 Linear, first order port-Hamiltonian system

Definition:

Let $P_1 \in \mathrm{K}^{n\times n}$ be invertible and self-adjoint, let $P_0 \in \mathrm{K}^{n\times n}$ be the skew-adjoint, i.e., $P_0^* = -P_0$ and let $\mathrm{H} \in L^\infty\left(\left[a,b\right]; \mathrm{K}^{n\times n}\right)$ such that $\mathrm{H}(\xi)^* = \mathrm{H}(\xi)$, $mI \le \mathrm{H}(\xi) \le MI$ for a.e. $\xi \in [a,b]$ and constants $m, M > 0$ independent of ξ. The Hilbert space $\mathrm{X} = L^2\left([a,b]; \mathrm{K}^n\right)$ with the inner product space

$$\langle f, g \rangle_{\mathrm{X}} = \frac{1}{2}\int_a^b g(\xi)^* \mathrm{H}(\xi) f(\xi)\, d\xi$$

then the differential equation

$$\frac{\partial x}{\partial t}(\xi,t) = P_1 \frac{\partial}{\partial \xi}\left(\mathrm{H}(\xi)x(\xi,t)\right) + P_0 \frac{\partial}{\partial \xi}\left(\mathrm{H}(\xi)x(\xi,t)\right)$$

is called a linear, first order port-Hamiltonian system.

2.4.2 Generation of contraction semigroup:

We applied some general results to the port-Hamiltonian systems, i.e., we considered partial differential equation of the form

$$\frac{\partial x}{\partial t}(\xi,t) = P_1 \frac{\partial}{\partial \xi}\left(\mathrm{H}(\xi)x(\xi,t)\right) + P_0 \mathrm{H}(\xi)x(\xi,t) \quad \text{.............................} \quad (2.2)$$

We want to characterize (homogenous) boundary conditions such that eq: 2.2 possess a unique solution. For that, we write the p.d.e. as the abstract ordinary differential equation (in the absence of spatial dependence)

$$\frac{dx}{dt}(t) = P_1 \frac{\partial}{\partial \xi}\big(\mathrm{H}(\xi)\, x(t,\xi)\big) + P_0 \mathrm{H}(\xi)\, x(t)$$

we consider the operator

$$A_0 x = P_1 \frac{d}{d\xi}(\mathrm{H}\, x) + P_0 (\mathrm{H}\, x)$$

on the state space

$$X = L^2\big([a,b]\,;\, \mathbb{K}^n\big)$$

with inner product

$$\langle f, g \rangle_X = \frac{1}{2}\int_a^b g(\xi)^* \mathrm{H}(\xi)\, f(\xi)\, d\xi$$

and domain

$$D(A_0) = \{\, x \in X \,/\, \mathrm{H}\, x \in H^1\big([a,b]\,;\, \mathbb{K}^n\big) \,\}$$

Here $H^1\big([a,b]\,;\,\mathbb{K}^n\big)$ is the vector space of all functions from $[a,b]$ to $\mathbb{K}^n$, which are square integrable , absolutely continuous and the derivative is again square integrable , that is

$$H^1\big([a,b]\,;\,\mathbb{K}^n\big) = \Big\{\, f \in L^2\big([a,b]\,;\,\mathbb{K}^n\big) \,\big|\, f \text{ is absolutely continuous and } \frac{df}{d\xi} \in L^2\big([a,b]\,;\,\mathbb{K}^n\big) \,\Big\}$$

.

Here A_0 is the maximal domain. In order to guarantee that eq: 2.2 possess unique solution we have to add boundary conditions. It is better to formulate boundary conditions in the boundary effort and boundary flow, which are defined as

$$e_\partial = \frac{1}{\sqrt{2}}\big((\mathrm{H}\, x)(b) + (\mathrm{H}\, x)(a)\big)\ , \qquad f_\partial = \frac{1}{\sqrt{2}}\big(P_1 (\mathrm{H}\, x)(b) - P_1 (\mathrm{H}\, x)(a)\big)$$

respectively.

As the boundary flow is determined by $\mathrm{H}x$ and not by x. Therefore we formulate the boundary conditions in $\mathrm{H}x$ variable. So we consider the boundary conditions

$$\tilde{W}_B \begin{bmatrix} \mathrm{H}(b)\,x(b,t) \\ \mathrm{H}(a)\,x(a,t) \end{bmatrix} = 0 \quad , \qquad t \geq 0 \quad \text{.......................... (2.3)}$$

To formulate the boundary conditions directly in x or $\mathrm{H}x$ at $\xi = a$ and $\xi = b$ is not the best choice for characterizing generators of contraction semigroups. It is better to formulate them in the boundary effort and boundary flow. We write this as matrix vector product, i.e.

$$\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = R_0 \begin{bmatrix} (\mathrm{H}\,x)(b) \\ (\mathrm{H}\,x)(a) \end{bmatrix} \quad \text{................................ (2.4)}$$

with $R_0 \in \mathrm{K}^{2n \times 2n}$ defined as

$$R_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix}$$

here the matrix R_0 is invertible, we can write any condition which is formulated in $(\mathrm{H}x)(b)$ and $(\mathrm{H}x)(a)$ into an equivalent condition which is formulated in f_∂ and e_∂ . Using eq: 2.4, we write the boundary condition eq:2.3 (equivalently) as

$$W_B \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} = 0$$

where $W_B = \tilde{W}_B R_0^{-1}$. Thus we study the operator

$$Ax = P_1 \frac{d}{d\xi}(\mathrm{H}\,x) + P_0(\mathrm{H}\,x) \quad \text{.............................. (2.5)}$$

with domain

$$D(A) = \left\{ x \in L^2\left([a,b];\mathrm{K}^n\right) \mid \mathrm{H}x \in H^1\left([a,b];\mathrm{K}^n\right),\ W_B \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = 0 \right\} \quad \text{............ (2.6)}$$

2.4.2.1 Theorem:

Consider the operator A defined in eq:2.5 and eq:2.6 associated to a port-Hamiltonian system, that is, the assumption of definition 2.4.1 are satisfied. Furthermore, W_B, or equivalently $\tilde{W}_B$, is an $n \times 2n$ matrix of rank n. Then the following statements are equivalent.

1. A is the infinitesimal generator of a contraction semigroup on X
2. $\mathrm{Re}\langle Ax, x\rangle \le 0$ for every $x \in D(A)$
3. $W_B \Sigma W_B^* \ge 0$

this theorem characterizes the matrices W_B for which the operator A with domain eq: 2.6 generates a contraction semigroup.

Proof: The proof of this theorem is given in [2].

2.5 INHOMOGENEOUS ABSTRACT DIFFERENTIAL EQUATION

If A is the infinitesimal generator of a C_0 - semigroup $T(t)_{t\ge 0}$, then the classical solution of the abstract homogeneous Cauchy initial value problem

$$\dot{x}(t) = Ax(t) \ , \ t \ge 0 \ , \ x(0) = x_0 \in D(A)$$

is given by $x(t) = T(t)x_0$. We consider the abstract inhomogeneous Cauchy problem is of the form

$$\dot{x}(t) = Ax(t) + f(t) \ , \ t \ge 0 \quad ; \quad x(0) = x_0$$

here we assume that f is continuous differentiable i-e $f \in C([0,\tau];X)$. The above equation is also called an abstract evolution equation or abstract differential equation.

2.6 PORT-HAMILTONIAN SYSTEM AS BOUNDARY CONTROL SYSTEM:

We add a boundary control to a port-Hamiltonian system and showed that the assumptions of a boundary control system are satisfied. The port-Hamiltonian system with control is given by

$$\frac{\partial x}{\partial t}(\xi,t) = P_1 \frac{\partial}{\partial \xi}\left(\mathrm{H}(\xi)x(\xi,t)\right) + P_0\left(\mathrm{H}(\xi)x(\xi,t)\right) \ldots\ldots\ldots\ldots\ldots (2.7)$$

$$u(t) = W_{B,1}\begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots \ (2.8)$$

$$0 = W_{B,2}\begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} \quad \text{..............................} \ (2.9)$$

We make the following assumptions.

2.6.1 Assumption

- $P_1 \in K^{n\times n}$ is invertible and self-adjoint.
- $\mathrm{H} \in L^\infty\left([a,b];\mathrm{K}^{n\times n}\right)$, $\mathrm{H}(\xi)$ is self-adjoint for a.e. $\xi \in [a,b]$ and there exist $M, m > 0$ such that $mI \le \mathrm{H}(\xi) \le M\,I$ for a.e. $\xi \in [a,b]$
- $W_B = \begin{bmatrix} W_{B,1} \\ W_{B,2} \end{bmatrix} \in \mathrm{K}^{n\times 2n}$ has full rank.

We can write the port-Hamiltonian system eq: 2.7—2.9 as boundary control system

$$\dot{x}(t) = U\,x(t)\ , \qquad x(0) = x_0$$
$$B\,x(t) = u(t)$$

By defining

$$U\,x = P_1 \frac{\partial}{\partial \xi}(\mathrm{H}\,x) + P_0(\mathrm{H}\,x) \quad \text{..................................} \ (2.10)$$

$$D\,(U) = \left\{ x \in L^2\left([a,b];K^n\right) \mid \mathrm{H}x \in H^1\left([a,b];K^n\right), W_{B,2}\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = 0 \right\} \quad \text{.........} \ (2.11)$$

$$Bx = W_{B,1}\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix}, \quad \text{..........................} \ (2.12)$$

$$D(B) = D(U) \quad \text{................................} \ (2.13)$$

we choose the Hilbert space $X = L^2\left([a,b];K^n\right)$, with the inner product space

$$\langle f, g \rangle_{\mathrm{X}} = \frac{1}{2}\int_a^b f(\xi)^* \mathrm{H}(\xi)\, g(\xi)\, d\xi$$

as the state space. The input space U equals K^m , where m is the number of rows of $W_{B,1}$ .We are now in the position to show that the controlled port- Hamiltonian system is indeed a boundary control system. For this, we have the following theorem:

2.6.2 Theorem

If the operator

$$Ax = P_1 \frac{\partial}{\partial \xi}(\mathrm{H}\,x) + P_0(\mathrm{H}\,x) \qquad \text{.................................... (2.14)}$$

With domain

$$D(U) = \left\{ x \in X \mid \mathrm{H}x \in H^1\left([a,b]\,;\,K^n\right),\ \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} \in \ker \begin{bmatrix} W_{B,1} \\ W_{B,2} \end{bmatrix} \right\} \text{......... (2.15)}$$

generates a $C_0 -$ semigroup on X ,then the system eq:2.7-2.9 is a boundary control system on X.

Remark:

An essential condition in the above theorem is that A given by (2.14) with the domain (2.15) generates a $C_0 -$ semigroup . Theorem 2.4.2.1 and assumption 2.6.1 imply that this holds in particular when $P_0^* = -P_0$ and $W_B \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} W_B^* \geq 0$. Since the term $P_0\mathrm{H}$ can be seen as is a bounded perturbation of eq:(2.10) with $P_0 = 0$, theorem 2.4.2.1 showed that A given by eq:2.14 with domain eq:2.15 generates a $C_0 -$ semigroup when $W_B \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} W_B^* \geq 0$.

The above theorem 2.6.2 is useful for proving the bounded semigroup when the operator generates the contraction semigroup.

2.7 FLOWS IN NETWORKS

We consider a finite network, modelled by a simple directed graph [1] , [3]. We denote $V = \{v_1, \ldots, v_n\}$ the set of vertices and $E = \{e_1, \ldots, e_n\}$ the set of (directed) edges of the graph. The edges are parameterizes on the interval [0, 1]. The vertex $e_j(0)$ is thus called the head and the vertex $e_j(1)$ the tail of the edge $e_j \in E$. The edge e_j is an incoming edge for the vertex v_i if $v_i = e_j(0)$ holds, and it called an outgoing edge for v_i if $v_i = e_j(1)$ holds.

We assume that in every vertex there is at least one incoming as well as at least one outgoing edge.

We will use the following graph matrices to describe the structure of the network.

2.7.1 Definitions

i. The outgoing incidence matrix $\Phi^{-} = \left(\varphi_{ij}^{-}\right)_{n\times m}$ has entries

$$\varphi_{ij}^{-} = \begin{cases} 1, & v_i \text{ tail of } e_j \\ 0, & \text{else}; \end{cases}$$

ii. The outgoing incidence matrix $\Phi^{+} = \left(\varphi_{ij}^{+}\right)_{n\times m}$ has entries

$$\varphi_{ij}^{+} = \begin{cases} 1, & v_i \text{ head of } e_j \\ 0, & \text{else}; \end{cases}$$

iii. The weighted outgoing incidence matrix is $\Phi_w^{-} = \left(\omega_{ij}^{-}\right)_{n\times m}$, where

$0 \le \omega_{ij}^{-} \le 1$ satisfy $\omega_{ij}^{-} = 0 \Leftrightarrow \varphi_{ij}^{-} = 0$ and $\sum_{j=1}^{m} \omega_{ij}^{-} = 1$ for all $i = 1, \ldots, n$.

iv. The weighted adjacency matrix is defined as $\mathrm{A} = \left(a_{ik}\right)_{n\times n} = \Phi^{+}\left(\Phi_w^{-}\right)^T$.

v. The weighted adjacency matrix of the line graph is defined as

$$\mathrm{B} = \left(b_{ik}\right)_{m\times m} = \left(\Phi_w^{-}\right)^T \Phi^{+}.$$

Remark:

Both adjacency matrices A and B are column stochastic, hence

$$\|\mathrm{A}\|_1 = 1 \text{ and } \|\mathrm{B}\|_1 = 1.$$

Furthermore, the relation $\left(\Phi_w^{-}\right)^T \mathrm{A} = \mathrm{B}\left(\Phi_w^{-}\right)^T$ holds.

Chapter 3 Explanatory Examples

In this chapter we discussed how different research questions, as stated in the research aim, can be answered and how the objectives can be fulfilled. Basically we have treated two different networks of flows (as two examples) in this chapter. In order to validate theories obtained from the previous research studies, which have been summarized in the previous chapter, and subsequently apply them to these two examples.

3.1 THEOREMS

3.1.1 Theorem

If A generates contraction semigroup $T_{t\geq 0}$ with respect to $\langle . , . \rangle_{\mathrm{H}}$ then A generates bounded semigroup $T_{t\geq 0}$ with respect to $\langle . , . \rangle$.

where we have

$$\langle f , g \rangle = \int_a^b g(\xi)^* f(\xi)\, d\xi \qquad \text{(Natural norm)}$$

and

$$\langle f , g \rangle_{\mathrm{H}} = \int_a^b g(\xi)^* \mathrm{H}(\xi) f(\xi)\, d\xi \quad ; \quad mI \leq \mathrm{H}(\xi) \leq M\, I$$

Proof:

Given: A generates Contraction semigroup with respect to $\langle . , . \rangle_{\mathrm{H}}$ means that we have

- T be a C_0-semigroup with respect to $\langle . , . \rangle_{\mathrm{H}}$ (so it holds all the conditions of C_0 -semigroup).
- $\| T(\mathrm{t}) x \|_{\mathrm{H}} \leq \| x \|_{\mathrm{H}} \quad ; \ \forall \ \mathrm{t} \geq 0, \ x \in \mathrm{L}^2(a,b)$

To Prove: A generates bounded semigroup T with respect to $\langle . , . \rangle$

To prove A generates semigroup ; first we need to show that both $\langle . , . \rangle_{\mathrm{H}}$ and $\langle . , . \rangle$ are equivalent. for this we have

$$\|x\|^2_{\mathrm{H}} = \langle x, x\rangle_{\mathrm{H}} \quad ; \quad mI \le \mathrm{H}(\xi) \le \mathrm{MI}$$

and

$$\langle x, x\rangle_{\mathrm{H}} \le \langle x, \mathrm{MI}x\rangle$$

$$\Rightarrow \quad \|x\|^2_{\mathrm{H}} \le \mathrm{M}\|x\|^2 \quad \ldots\ldots. (i)$$

Also

$$\langle x, x\rangle_{\mathrm{H}} \ge \langle x, m\mathrm{I}x\rangle$$

$$\Rightarrow \quad \|x\|^2_{\mathrm{H}} \ge m\|x\|^2 \quad \ldots\ldots\ldots\ldots\ldots (ii)$$

Combining (i) and (ii) ,we get

$$m\|x\|^2 \le \|x\|_{\mathrm{H}}^2 \le \mathrm{M}\|x\|^2 \text{ (which is the definition of equivalent norms)}$$

Thus both norms $\|.\|_{\mathrm{H}}$ and $\|.\|$ are equivalent. so A holds all the conditions of semigroup with respect to $\langle . , .\rangle$.

Now to prove the boundedness; we have

$$\|\mathrm{T(t)}x\|_{\mathrm{H}} \le \|x\|_{\mathrm{H}} \qquad \text{(As } A \text{ generates contraction semigroup)}$$

then we have

$$\|T(\mathrm{t})x\|^2 \le \frac{1}{m}\|T(t)x\|^2_{\mathrm{H}} \le \frac{1}{m}\|x\|^2_{\mathrm{H}} \le \frac{M}{m}\|x\|^2$$

$$\Rightarrow \quad \|T(\mathrm{t})x\| \le \mathrm{C}\|x\| \qquad (\because \mathrm{C} = \frac{M}{m})$$

The above expression shows the boundedness. Hence proved that A generates bounded semigroup with respect to $\langle . , .\rangle$.

3.1.2 Theorem

Let Q be a given (square) matrix. There exists a self-adjoint matrix 'H' that satisfying

$mI \le \mathrm{H} \le \mathrm{M}I$ and $Q^*\mathrm{H}Q - \mathrm{H} \le 0$ if and only if the following two conditions are satisfied.

1. The eigen values of Q satisfy $|\lambda| \le 1$.
2. The eigen values of Q lying on the unit circle.

Proof:

Given: H is constant and $Q^{*}\mathrm{H}Q - \mathrm{H} \le 0$ ....(i) holds.

To prove: conditions (1) and (2) should be satisfy.

1- consider λ be eigen value of Q with the corresponding eigen vector

$$v = \begin{pmatrix} v_1 \\ \cdot \\ \cdot \\ \cdot \\ v_n \end{pmatrix}$$

then v satisfies the matrix equation $Qv = \lambda v$, so we need to show that λ lies in unit circle. so the (i) will become

$$v^{*}Q^{*}\mathrm{H}Qv - v^{*}\mathrm{H}v \le 0$$

$$\Leftrightarrow \qquad |\lambda|^{2} v^{*}\mathrm{H}v - v^{*}\mathrm{H}v \le 0$$

$$\Leftrightarrow \qquad (|\lambda|^{2} - 1)v^{*}\mathrm{H}v \le 0$$

$$\Leftrightarrow \qquad |\lambda| \le 1$$

Hence condition 1 is satisfied.

2- Assume that Q has the eigen values not on the unit circle then there does not exist H such that

$$Q^{*}\mathrm{H}Q - \mathrm{H} \le 0 \qquad \text{holds}$$

Hence it is given that (i) holds. So eigen values of Q lying on the unit circle.

Conversely:

Given: The eigen values of Q satisfy $|\lambda| \le 1$.

The eigen values of Q lying on the unit circle.

To prove: H is constant and $Q^{*}\mathrm{H}Q - \mathrm{H} \le 0$ holds.

We split the Hilbert space $X = X_1 + X_2$. Where $X_1 = span\{v_k\}$; v_k is the corresponding eigen vector of generalized eigen values $|\lambda| < 1$ and $X_2 = span\{v_k\}$; v_k is eigen vector with $|\lambda| = 1$.

consider $X = X_1$

here Q has the eigen values not on the the unit circle, so H does not satisfy equation (i).

Assume that there exist H such that

$$\mathrm{H} = \sum_{n=0}^{\infty} \left(Q^*\right)^n Q^n$$

This satisfies

$$Q^* \mathrm{H} Q - \mathrm{H} = -\mathrm{I}$$

As Q has the Jordan block structure so it is not diagonalizable. It also has generalized eigen values.

Now consider $X = X_2$

$$\text{Then} \quad Q = \begin{pmatrix} \lambda_1 & . & . & . & 0 \\ . & . & & & . \\ . & & . & & . \\ . & & & . & 0 \\ 0 & . & . & 0 & \lambda_n \end{pmatrix}$$

here Q is transition/stochastic matrix.

$$Q^* Q = \begin{pmatrix} \lambda_1^* & . & . & . & 0 \\ . & . & & & . \\ . & & . & & . \\ . & & & . & 0 \\ 0 & . & . & 0 & \lambda_n^* \end{pmatrix} \begin{pmatrix} \lambda_1 & . & . & . & 0 \\ . & . & & & . \\ . & & . & & . \\ . & & & . & 0. \\ 0 & . & . & 0 & \lambda_n \end{pmatrix} = I$$

We obtain $Q^* Q = I$. here the eigen values of Q lies on the unit circle.

This mean that $Q^* I Q - I \leq 0$ holds only when $\mathrm{H} = I$ (constant valued matrix).

Hence $Q^* \mathrm{H} Q - \mathrm{H} \leq 0$. This completes the prove.

3.2 EXAMPLE 1

We consider a network such as depicted in Figure 1.

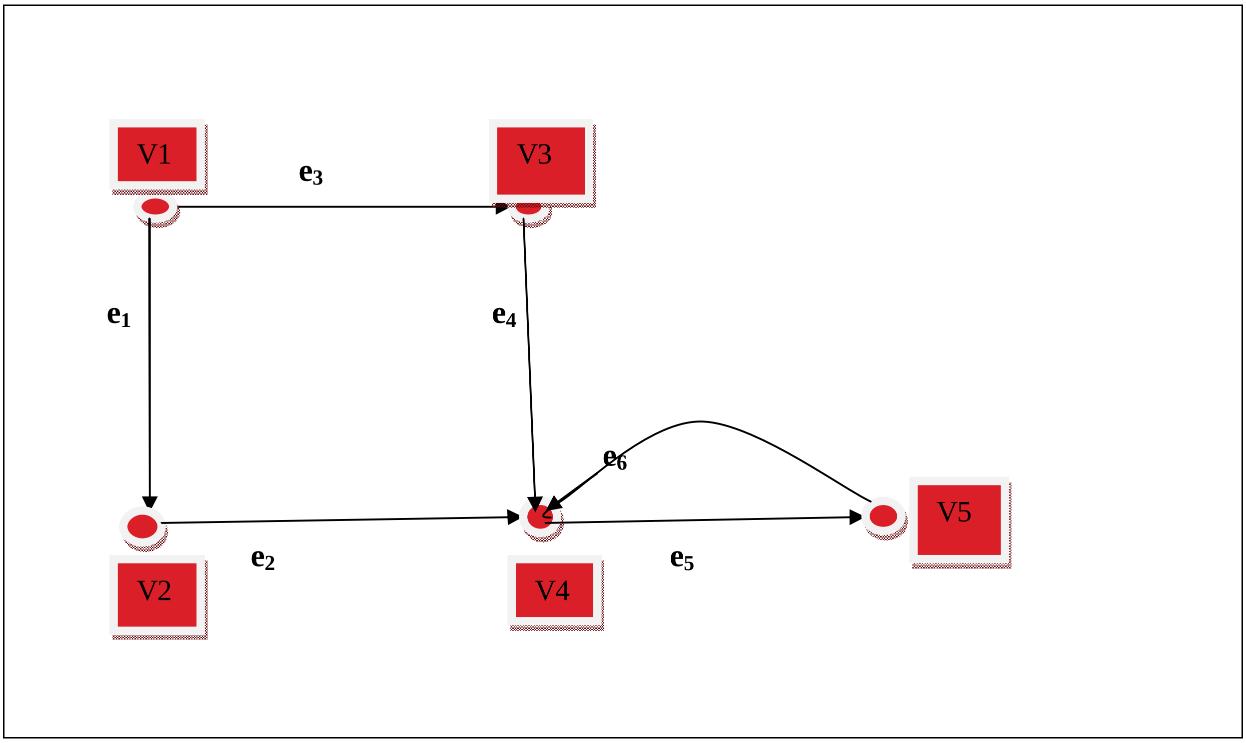


Figure 1

We assume that the edges all have equal length (one) and that on every edge the information/material is transported with equal and constant speed. This means that on every edge the following partial differential equation holds

$$\frac{\partial z_j}{\partial t}(\xi,t) = \frac{\partial z_j}{\partial \xi}(\xi,t) \quad ; \qquad \xi \in [0,1] \quad , \quad t \geq 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3.1)$$

Boundary condition: $\varphi_{ij}^{-} z_j(t,1) = \omega_{ij}^{-} \sum_{k=1}^{m} \varphi_{ik}^{+} z_k(t,0)$

Initial condition: $z_j(0,\xi) = f_j(\xi)$

The arrow in the figure indicates way the information flow. At the vertices no information can be lost. Hence at every vertex the total amount of incoming equals the total amount of out-coming. However, the outflow may not be equal to all connected edges. The outflow is weighted by ω_{ij} .

3.2.1 Methodology

1. Formulate the partial differential equations together with the relations at the vertices as an abstract differential equation on the state space $L^2\left([0,1];\Re^n\right)$ where n is the number of edges.
2. To prove that this (homogeneous) differential equation generates a contraction semigroup on the state space.
3. Add an input at one of the vertices, and show that for every square integrable input function, there exists a unique solution of the inhomogeneous abstract differential equation.
4. As output the flow at one of the edges is chosen. After the well-posedness question similar to the one above has been answered, the following optimal control problem is studied

$$\min \int_0^\infty \|y(t)\|^2 + \|u(t)\|^2 \, dt$$

The final aim is to solve this optimal control problem. It is expected that by the special structure of the coupled partial differential equations the solution to this problem can be formulated in a matrix equation.

- Abstract Differential Equation:

The given p.d.e. eq:3.1 can be written in abstract differential equation as

$$\begin{cases} \dot{z}(t) = Az(t) \ , & t \geq 0 \\ z(0) = f \end{cases}$$

with the state space

$$X := L^2\left([0,1] \, ; C^n \right)$$

we define the operator

$$A := diag\left(\frac{d}{d\xi}\right)_{j=1,\dots,m}$$

with domain $D(A)=\left\{z\in \mathrm{H}^1\left([0,1],C^m\right) \mid z(1)\in rg\left(\varphi_\omega^-\right)^T \wedge \varphi_\omega^- z(1)=\varphi_\omega^+ z(0)\right\}$

In the domain of A, the first condition $z(1)\in rg\left(\varphi_\omega^-\right)^T$ means that in every vertex the total incoming flow is distributed in (given) weighted proportions to the outgoing edges. the second condition $\varphi_\omega^- z(1)=\varphi_\omega^+ z(0)$ is the Kirchhoff's law in each vertex.

- Generation of Contraction semigroup:

Consider the linear transport equation

$$\frac{\partial z}{\partial t}(\xi,t)=\frac{\partial}{\partial \xi}(\mathrm{H}(\xi)\ z(\xi,t)) \quad \text{...} \quad (3.2)$$

In the given p.d.e eq:3.1 we have;

$$P_1 = Identity\,matrix \quad , \quad \mathrm{H}(\xi) = Identity\,matrix \quad \text{and} \quad P_0 = 0$$

with Boundary conditions on vertices(from flow diagram):

$$\left.\begin{cases} z_1(1)=0 \\ z_2(1)=z_1(0) \\ z_3(1)=0 \\ z_4(1)=z_3(0) \\ z_5(1)=z_2(0)+z_4(0)+z_6(0) \\ z_6(1)=z_5(0) \end{cases}\right\} \text{..........................}(3.3)$$

and we have the operator

$$A_0 z = P_1\frac{d}{d\xi}(\mathrm{H}z)+P_0(\mathrm{H}z)$$

with domain

$$D(A_0)=\left\{z\in X/\mathrm{H}z\in H^1\left([a,b];\mathrm{K}^n\right)\right\}$$

Here $H^1\left([a,b];\mathrm{K}^n\right)$ is the vector space of all functions from $[a,b]$ to K^n, which are square integrable , absolutely continuous and the derivative is again square integrable , that is

$$H^1\left([0,1];C^n\right)=\left\{\ f\in L^2\left([0,1];C^n\right) \mid f \text{ is absolutely continuous and } \frac{df}{d\xi}\in L^2\left([0,1];C^n\right)\ \right\}.$$

- Formulation of Boundary Conditions:

In order to formulate the boundary conditions into boundary effort and boundary flow which are (in our example1)

$$e_\partial = \frac{1}{\sqrt{2}}\big((\mathrm{H}\, z)(1) + (\mathrm{H}\, z)(0) \big) \quad ,$$

$$f_\partial = \frac{1}{\sqrt{2}}\big(P_1 (\mathrm{H}\, z)(1) - P_1 (\mathrm{H}\, z)(0) \big)$$

here $P_1 = \text{Identity matrix}$ and

$\mathrm{H}(\xi) = \text{Identity matrix}$

The boundary variables e_∂ and f_∂ will be

$$e_\partial = \frac{1}{\sqrt{2}} \left(\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_1(1) \\ z_2(1) \\ z_3(1) \\ z_4(1) \\ z_5(1) \\ z_6(1) \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_1(0) \\ z_2(0) \\ z_3(0) \\ z_4(0) \\ z_5(0) \\ z_6(0) \end{bmatrix} \right)$$

finally we obtained

$$e_\partial = \frac{1}{\sqrt{2}} \begin{pmatrix} z_1(1) + z_1(0) \\ z_2(1) + z_2(0) \\ z_3(1) + z_3(0) \\ z_4(1) + z_4(0) \\ z_5(1) + z_5(0) \\ z_6(1) + z_6(0) \end{pmatrix}$$

and

$$f_\partial = \frac{1}{\sqrt{2}}\left(\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}\begin{bmatrix}z_1(1)\\z_2(1)\\z_3(1)\\z_4(1)\\z_5(1)\\z_6(1)\end{bmatrix}\right) + \left(\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}\begin{bmatrix}z_1(0)\\z_2(0)\\z_3(0)\\z_4(0)\\z_5(0)\\z_6(1)\end{bmatrix}\right)$$

hence we get

$$f_\partial = \frac{1}{\sqrt{2}}\begin{pmatrix}z_1(1)-z_1(0)\\z_2(1)-z_2(0)\\z_3(1)-z_3(0)\\z_4(1)-z_4(0)\\z_5(1)-z_5(0)\\z_6(1)-z_6(0)\end{pmatrix}$$

As we have the boundary conditions as follows:

$$\begin{bmatrix}0\\0\\0\\0\\0\\0\end{bmatrix} = \begin{bmatrix}z_1(1)\\z_2(1)-z_1(0)\\z_3(1)\\z_4(1)-z_3(0)\\z_5(1)-z_2(0)-z_4(0)-z_6(0)\\z_6(1)-z_5(0)\end{bmatrix}$$

we can write the above conditions as

$$\tilde{W}_B\begin{bmatrix}\mathrm{H}(1)\ z(1,t)\\\mathrm{H}(1)\ z(0,t)\end{bmatrix} = 0 \quad , \qquad t \geq 0$$

where

$$\tilde{W}_B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

the boundary conditions on vertices can be written in terms of boundary effort and boundary flow as

$$W_B \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} = 0$$

where $W_B = \tilde{W}_B R_0^{-1}$; and $R_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix}$

$$W_B = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Now we need to prove one of the condition of the theorem 2.4.2.1

$$W_B \Sigma W_B^* \geq 0$$

$$\Sigma = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

so

$$W_B \Sigma W_B^* = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It is clear that eigen values of the above matrix are not positive, hence the condition of the theorem 2.4.2.1does not hold.

Now for verification purpose we will investigate the other condition $\mathrm{Re}\langle Az, z\rangle \le 0$ of theorem 2.4.2.1

$$\begin{aligned} \mathrm{Re}\langle Az, z\rangle &= \int_0^1 \left(\frac{dz}{d\xi}\bar{z}(\xi) + z(\xi)\frac{d\bar{z}}{d\xi} \right) d\xi \\ &= \frac{1}{2}\Big[z(\xi)\bar{z}(\xi) \Big]_0^1 \\ &= \frac{1}{2}\Big[|z(\xi)| \Big]_0^1 \\ &= \frac{1}{2}\left(z(1)^2 - z(0)^2 \right) \end{aligned}$$

or we can write it as

$$\begin{aligned} \mathrm{Re}\langle Az, z\rangle = {} & \left[z_1(1)^2 - z_1(0)^2 \right] + \left[z_2(1)^2 - z_2(0)^2 \right] + \left[z_3(1)^2 - z_3(0)^2 \right] + \\ & \left[z_4(1)^2 - z_4(0)^2 \right] + \left[z_5(1)^2 - z_5(0)^2 \right] + \left[z_6(1)^2 - z_6(0)^2 \right] \end{aligned}$$

by using the boundary conditions on vertices (from the flow diagram), we have

$$\begin{aligned} \mathrm{Re}\langle Az, z\rangle = {} & \left[0 - z_1(0)^2 \right] + \left[z_1(0)^2 - z_2(0)^2 \right] + \left[0 - z_3(0)^2 \right] \\ & \left[z_3(0)^2 - z_4(0)^2 \right] + \left[\left(z_2(0) + z_4(0) + z_6(0) \right)^2 - z_5(0)^2 \right] + \left[z_5(0)^2 - z_6(0)^2 \right] \end{aligned}$$

finally we obtained

$$\mathrm{Re}\langle Az, z\rangle = 2z_2(0)z_4(0) + 2z_4(0)z_6(0) + 2z_6(0)z_2(0)$$

Hence it is clear that the real part is not positive. So we conclude here, $\mathrm{H}(\xi) = \text{Identity matrix}$ is not a good choice for generating the contraction semigroup. Also theorem 2.4.2.1 does not hold. Hence for this network flow; we cannot generate contraction semigroup by choosing $\mathrm{H}(\xi) = \text{Identity matrix}$

Now we need to find such constant valued Hamiltonian diagonal matrix, which can satisfy theorem 2.4.2.1.

3.2.2 Formulation of Hamiltonian (diagonal) matrix

We need to find Hamiltonian which will satisfy the conditions of the theorem 2.4.2.1, consider the constant valued Hamiltonian matrix is in the form

$$\mathrm{H} = \begin{pmatrix} \gamma_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0\gamma_6 \end{pmatrix}$$

we already defined the operator as

$$A = diag\frac{\partial}{\partial \xi}$$

So

$$\operatorname{Re}\langle Az, z\rangle = \int_0^1 \left(\begin{bmatrix} \frac{\partial z_1}{\partial \xi} \\ . \\ . \\ . \\ \frac{\partial z_6}{\partial \xi} \end{bmatrix}^T \begin{bmatrix} \gamma_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma_6 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{bmatrix} + \right.$$

$$\left[z_1 \quad z_2 \quad z_3 \quad z_4 \quad z_5 \quad z_6 \right] \begin{bmatrix} \gamma_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma_6 \end{bmatrix} \begin{bmatrix} \frac{\partial z_1}{\partial \xi} \\ . \\ . \\ . \\ \frac{\partial z_6}{\partial \xi} \end{bmatrix}^T \Bigg) d\xi$$

$$= \int_0^1 \frac{\partial}{\partial \xi} \left[\gamma_1 z_1(\xi)^2 + \quad . \quad . \quad . + \gamma_6 z_6(\xi)^2 \right] d\xi$$

$$= \gamma_1 z_1(\xi)^2 + \quad . \quad . \quad . + \gamma_6 z_6(\xi)^2 \Big|_0^1$$

$$= \gamma_1 \left[z_1(1)^2 - z_1(0)^2 \right] + \gamma_2 \left[z_2(1)^2 - z_2(0)^2 \right] + \gamma_3 \left[z_3(1)^2 - z_3(0)^2 \right]$$
$$\gamma_4 \left[z_4(1)^2 - z_4(0)^2 \right] + \gamma_5 \left[z_5(1)^2 - z_5(0)^2 \right] + \gamma_6 \left[z_6(1)^2 - z_6(0)^2 \right]$$

Now by using the conditions on vertices (from the flow diagram), we get

$$= \gamma_1 \left[0 - z_1(0)^2 \right] + \gamma_2 \left[z_1(0)^2 - z_2(0)^2 \right] + \gamma_3 \left[0 - z_3(0)^2 \right]$$
$$\gamma_4 \left[z_3(0)^2 - z_4(0)^2 \right] + \gamma_5 \left[\left(z_2(0) + z_4(0) + z_6(0) \right)^2 - z_5(0)^2 \right] + \gamma_6 \left[z_5(0)^2 - z_6(0)^2 \right]$$

Finally we obtained

$$\operatorname{Re}\langle Az, z \rangle = z_1(0)^2 \left[-\gamma_1 + \gamma_2 \right] + z_2(0)^2 \left[-\gamma_2 + \gamma_5 \right] + z_3(0)^2 \left[-\gamma_3 + \gamma_4 \right] + z_4(0)^2 \left[-\gamma_4 + \gamma_5 \right] +$$
$$z_5(0)^2 \left[-\gamma_5 + \gamma_6 \right] + z_6(0)^2 \left[-\gamma_6 + \gamma_5 \right] + 2\gamma_5 z_2(0) z_4(0) + 2\gamma_5 z_4(0) z_6(0) + 2\gamma_5 z_2(0) z_6(0)$$

To prove $\operatorname{Re}\langle Az, z \rangle \leq 0$ we need to prove the following:

$$\left(-\gamma_1 + \gamma_2 \right) \leq 0$$
$$\left(-\gamma_3 + \gamma_4 \right) \leq 0$$
$$\left(-\gamma_5 + \gamma_6 \right) \leq 0$$
$$\begin{pmatrix} -\gamma_2 + \gamma_5 & \gamma_5 & \gamma_5 \\ \gamma_5 & -\gamma_4 + \gamma_5 & \gamma_5 \\ \gamma_5 & \gamma_5 & -\gamma_6 + \gamma_5 \end{pmatrix} \leq 0$$

There are no such constants values of $\gamma's$ to satisfy the above inequalities, because H becomes a Jordan block structure. Hence there is no such Hamiltonian, in the form of diagonal matrix, which satisfies the theorem 2.4.2.1.

So we can't move further with this example for other two parts of methodology. Now we are only interested to find the constant valued Hamiltonian matrix, which is not a diagonal matrix. To find such Hamiltonian which satisfy $\operatorname{Re} < Az, z > \leq 0$, we need to formulate Hamiltonian matrix; by using theorem 3.1.2.

3.2.3 Formulation of Hamiltonian (non-diagonal) matrix:

As we have the boundary conditions eq: 3.3 on vertices (from the flow diagram: Figure 1)

Now we can write the above BCs from network flow as:

$$z(1) = Q\,z(0)$$

$$\begin{pmatrix} \mathbf{z_1(1)} \\ \mathbf{z_2(1)} \\ \mathbf{z_3(1)} \\ \mathbf{z_4(1)} \\ \mathbf{z_5(1)} \\ \mathbf{z_6(1)} \end{pmatrix} = \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \end{bmatrix}}_{Q} \begin{pmatrix} \mathbf{z_1(0)} \\ \mathbf{z_2(0)} \\ \mathbf{z_3(0)} \\ \mathbf{z_4(0)} \\ \mathbf{z_5(0)} \\ \mathbf{z_6(0)} \end{pmatrix}$$

here Q is transition matrix.

$$\text{Eigen values of} \quad Q = \begin{pmatrix} +1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \leq 1$$

and

$$\text{generalized eigen vectors} \quad V=\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 & 0 & -1 \\ -1 & 1 & -1 & 0 & -1 & 0 \end{bmatrix}$$

now consider

$$\mathrm{H}=V^{*}V=\begin{bmatrix} 2 & 0 & 1 & -1 & 1 & -1 \\ 0 & 2 & -1 & -1 & -1 & -1 \\ 1 & -1 & 2 & 0 & 1 & 0 \\ -1 & -1 & 0 & 2 & 0 & 1 \\ 1 & -1 & 1 & 0 & 2 & 0 \\ -1 & -1 & 0 & 1 & 0 & 2 \end{bmatrix}$$

so

$\mathrm{H}^{-1}=(V^{-1})(V^{-1})^{*}$; Therefore invertible, $eig(\mathrm{H})=\left\{\frac{5-\sqrt{17}}{2},\frac{5+\sqrt{17}}{2},1\right\}$ (all multiplicity 2)

$\mathrm{H}^{*}=(V^{*})(V^{*})^{*}=V^{*}V=\mathrm{H};\ \therefore\ \mathrm{H}$ is self adjoint

now

We need to show that $Q^{*}\,\mathrm{H}\,Q-\mathrm{H}\leq 0$

$$Q^{*}\,\mathrm{H}\,Q-\mathrm{H}=\begin{bmatrix} 0 & -1 & -2 & 0 & -2 & 0 \\ -1 & 0 & 1 & 3 & 1 & 3 \\ -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Has eigen values $\{-4.1,-2.8,-1,0,0,2.5,5.3\}\not\leq 0$

$\therefore\ Q^{*}\,\mathrm{H}\,Q-\mathrm{H}\not\leq 0$

The Jordan canonical form of Q is

$$D=V^{-1}QV$$

and

$$D = \begin{bmatrix} \mathbf{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

So we need to show that

$$D^* I\, D - I \leq 0$$

$$D^* I\, D - I = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$\text{eigen values } (D^* I\, D - I) = \begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \leq 0$$

so

$$D^* I\, D - I \leq 0$$

Hence ‘A’ generates the Contraction semigroup.

As we suppose $V^* \mathrm{H} V = I$

then

$$\mathrm{H} = \left(V^*\right)^{-1} I\, V^{-1}$$

$$H = \begin{bmatrix} \mathbf{3/2} & \mathbf{0} & \mathbf{1/2} & \mathbf{0} & \mathbf{1/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{3/2} & \mathbf{0} & \mathbf{1/2} & \mathbf{0} & \mathbf{1/2} \\ \mathbf{1/2} & \mathbf{0} & \mathbf{3/2} & \mathbf{0} & \mathbf{1/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{1/2} & \mathbf{0} & \mathbf{3/2} & \mathbf{0} & \mathbf{1/2} \\ \mathbf{1/2} & \mathbf{0} & \mathbf{1/2} & \mathbf{0} & \mathbf{1/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{1/2} & \mathbf{0} & \mathbf{1/2} & \mathbf{0} & \mathbf{1/2} \end{bmatrix}$$

or

$$H = \frac{3}{2}\begin{bmatrix} 1 & \mathbf{0} & \mathbf{1}/3 & \mathbf{0} & \mathbf{1}/3 & \mathbf{0} \\ \mathbf{0} & 1 & \mathbf{0} & \mathbf{1}/3 & \mathbf{0} & \mathbf{1}/3 \\ \mathbf{1}/3 & \mathbf{0} & 1 & \mathbf{0} & \mathbf{1}/3 & \mathbf{0} \\ \mathbf{0} & \mathbf{1}/3 & \mathbf{0} & 1 & \mathbf{0} & \mathbf{1}/3 \\ \mathbf{1}/3 & \mathbf{0} & \mathbf{1}/3 & \mathbf{0} & \mathbf{1}/3 & \mathbf{0} \\ \mathbf{0} & \mathbf{1}/3 & \mathbf{0} & \mathbf{1}/3 & \mathbf{0} & \mathbf{1}/3 \end{bmatrix}$$

with this Hamiltonian, when we calculate $\therefore\ Q^* H Q - H$;

We get;

$$Q^* H Q - H = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{-2/3} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-2/3} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

now

$$\text{Eigen values } (Q^* H Q - H) = \begin{pmatrix} -2/3 \\ -2/3 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \leq 0$$

so

$(Q^* H Q - H) \leq 0$

&

$$P_1 = \frac{3}{2}\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{0} & \mathbf{-1} & \mathbf{0} & \mathbf{4} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} & \mathbf{0} & \mathbf{-1} & \mathbf{0} & \mathbf{4} \end{bmatrix}$$

hence the following Hamiltonian matrix satisfied theorem 2.4.2.1

$$\mathrm{H} = \begin{bmatrix} \frac{3}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{3}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{3}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{3}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

So for this non-diagonal Hamiltonian matrix H, A is the infinitesimal generator of contraction semi group on X.

EXAMPLE 2

Consider the following closed network of flow [1].

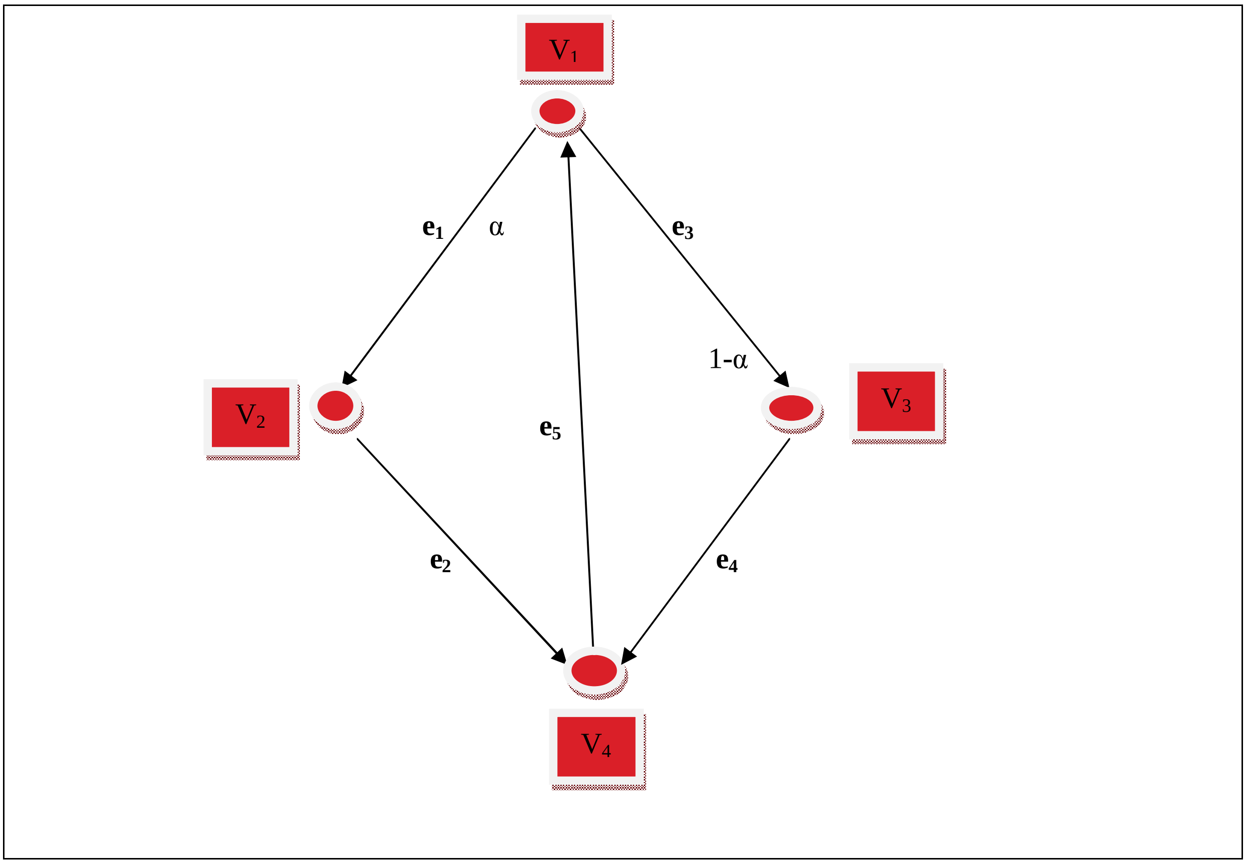


Figure 2

Assumptions:

- Material flowing with constant speed c_j on each edge e_j with no friction or loss.
- In each node v_i of the network the material is redistributed according to certain weight ω_{ij}.
- Take the weights for all $0 < \alpha < 1$ on the edges.

Simplified Physical model:

- This situation can be described by a system of linear transport equation on the edges

$$\frac{\partial}{\partial t} z_j(\xi,t) = c_j \frac{\partial}{\partial \xi} z_j(\xi,t)$$

- Initial condition: $z_j(\xi,0) = h_j$

- Boundary conditions at vertices

Outgoing flow on edge $\left(e_j\right) = \left(\omega_{ij}^{-} \sum \right)$ incoming flows into vertex v_i .

So from the flow diagram; we have the following conditions

$$\left\{\begin{array}{l} z_1(1) = \alpha\, z_5(0) \\ z_2(1) = z_1(0) \\ z_3(1) = (1-\alpha)\, z_5(0) \qquad ; \qquad (\text{ choose } \alpha = \frac{1}{2}) \\ z_4(1) = z_3(0) \\ z_5(1) = z_2(0) + z_4(0) \end{array}\right\} \text{.........(3.4)}$$

Typical phenomena of flows in networks:

The mass distribution on edges e_1 and e_3 will always satisfy the relation $\dfrac{\alpha}{1-\alpha}$.

3.3.1 Methodology

The methodology remains the same as given in section 3.2.1 for this network

Abstract Differential Equation:

$$\begin{aligned} \dot{z}(t) &= Az(t) \quad , \quad t \geq 0 \\ z(0) &= f \end{aligned}$$

on state space $X = L^2\left([0,1];C^m\right)$

define operator $A = diag\left(\dfrac{d}{d\xi}\right)_{j=1,\ldots,m}$

with domain $D(A) = \left\{ z \in H^1\left([0,1]\,,\, C^m\right) \mid z\,(1) \in rg\left(\varphi_\omega^-\right)^T \wedge \varphi_\omega^- z\,(1) = \varphi_\omega^+ z\,(0) \right\}$

Generation of Contraction Semigroup:

Consider the partial differential equation

$$\frac{\partial z}{\partial t}(\xi,t)=\frac{\partial}{\partial \xi}z(\xi,t)$$

here we have

$P_1 = Identity\ matrix$, $\mathrm{H}(\xi) = Identity\ matrix$ and $P_0 = 0$

so the partial differential equation will be

$$\frac{\partial z}{\partial t}(\xi,t)=\frac{\partial}{\partial \xi}\left(\mathrm{H}(\xi)\,z(\xi,t)\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3.5)$$

The corresponding abstract differential equation of the above p.d.e is

$$\frac{dz}{dt}(t)=P_1\frac{d}{d\xi}\left(\mathrm{H}\,z\,(t)\right)$$

consider the operator

$$Az=P_1\frac{d}{d\xi}\left(\mathrm{H}\,z\,(t)\right)$$

on the state space $X = L^2\left([0,1]\,;\,C^n\right)$

with the inner product

$$\langle f,g\rangle_X=\frac{1}{2}\int_0^1 g(\xi)^*\,\mathrm{H}(\xi)\,f(\xi)\;d\xi$$

and the domain

$$D(A)=\{\;z\in X\,/\,\mathrm{H}\,z\in H^1\left([0,1]\,;\,C^n\right)\;\}$$

here $H^1\left([0,1]\,;\,C^n\right)$ is the vector space of all functions from $[0,1]$ to C^n, which are square integrable , absolutely continuous and the derivative is again square integrable , that is

$$H^1\left([0,1]\,;\,C^n\right)=\{\;f\in L^2\left([0,1]\,;\,C^n\right)|\;f\text{ is absolutely continuous and }\frac{df}{d\xi}\in L^2\left([0,1]\,;\,C^n\right)\;\}.$$

Formulation of Boundary Conditions:

To formulate the boundary conditions from the flow diagram in the boundary variables e_∂ and f_∂. We have the boundary effort

$$e_\partial = \frac{1}{\sqrt{2}} \begin{pmatrix} z_1(1) + z_1(0) \\ z_2(1) + z_2(0) \\ z_3(1) + z_3(0) \\ z_4(1) + z_4(0) \\ z_5(1) + z_5(0) \end{pmatrix}$$

and boundary flow is:

$$f_\partial = \frac{1}{\sqrt{2}} \begin{pmatrix} z_1(1) - z_1(0) \\ z_2(1) - z_2(0) \\ z_3(1) - z_3(0) \\ z_4(1) - z_4(0) \\ z_5(1) - z_5(0) \end{pmatrix}$$

we have the conditions in the vertices (from the flow diagram: Figure 2) (eq:3.4), we can write the boundary conditions in the following form

$$\tilde{W}_B \begin{bmatrix} \mathrm{H}(1)\ z(1,t) \\ \mathrm{H}(1)\ z(0,t) \end{bmatrix} = 0 \quad , \qquad t \geq 0$$

so the boundary conditions will become

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & -1 & 0 \end{bmatrix}}_{\tilde{W}_B} \begin{bmatrix} z(1) \\ z(0) \end{bmatrix} = 0$$

now the boundary conditions in terms of boundary effort and boundary flow can be defined as

$$W_B \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} = 0$$

where $W_B = \widetilde{W}_B R_0^{-1}$; and $R_0 = \frac{1}{\sqrt{2}}\begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix}$

so , we get

$$W_B = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

To prove the contraction semigroup; by using the theorem 2.4.2.1, we need to show that:

$$W_B \Sigma W_B^* \geq 0$$

where

$$\Sigma = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

Now (using Matlab) , we obtain

$$W_B \Sigma W_B^* = \begin{bmatrix} 0.375 & 0 & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{4} & 0 & 0.375 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} \end{bmatrix}$$

To check the positivity of the above matrix the eigen values must be positive.

eigen values of $W_B \Sigma W_B^*$ are

$$\begin{pmatrix} -\frac{1}{2} \\ 0 \\ 0 \\ \frac{1}{4} \\ \frac{1}{2} \end{pmatrix} \not\geq 0$$

Hence the theorem 2.4.2.1 does not satisfy.

now for verification purpose we will check the other condition of the theorem 2.4.2.1 i-e $\operatorname{Re}\langle Az, z\rangle \leq 0$

$$\begin{aligned} \operatorname{Re}\langle Az, z\rangle &= \int_0^1 \left(\frac{dz}{d\xi}\bar{z}(\xi) + z(\xi)\frac{d\bar{z}}{d\xi} \right) d\xi \\ &= \frac{1}{2}\left[z(\xi)\bar{z}(\xi)\right]_0^1 \\ &= \frac{1}{2}\left[\left|z(\xi)\right|\right]_0^1 \\ &= \frac{1}{2}\left(z(1)^2 - z(0)^2\right) \end{aligned}$$

by using the conditions on vertices from the flow diagram, finally we get

$$\operatorname{Re}\langle Az, z\rangle = -\frac{1}{4}z_5(0)^2 + z_2(0)z_4(0)$$

The above expression shows that the condition $\operatorname{Re}\langle Az, z\rangle \leq 0$ doesn't hold.

thus theorem 2.4.2.1 does not satisfy; so we cannot obtain the contraction semigroup on $L^2\left([0,1]; C^n\right)$ when $\mathrm{H}(\xi) = \textit{Identity matrix}$ and $P_1 = \textit{Identity matrix}$.

now we need to find the constant valued Hamiltonian diagonal matrix; which satisfies theorem 2.4.2.1.

3.3.2 Formulation of diagonal Hamiltonian matrix

Consider the Hamiltonian is in the following form;

$$H = \begin{pmatrix} \gamma_1 & 0 & 0 & 0 & 0 \\ 0 & \gamma_2 & 0 & 0 & 0 \\ 0 & 0 & \gamma_3 & 0 & 0 \\ 0 & 0 & 0 & \gamma_4 & 0 \\ 0 & 0 & 0 & 0 & \gamma_5 \end{pmatrix}$$

as we have the operator

$$A = diag\frac{\partial}{\partial \xi}$$

let's find the second condition of theorem 2.4.2.1; which is

$$\operatorname{Re}\langle Az, z\rangle \le 0$$

and we know

$$\operatorname{Re}\langle Az, z\rangle = \int_0^1 \begin{pmatrix} \frac{\partial z_1}{\partial \xi} \\ . \\ . \\ . \\ \frac{\partial z_5}{\partial \xi} \end{pmatrix} \begin{pmatrix} \gamma_1 & 0 & 0 & 0 & 0 \\ 0 & \gamma_2 & 0 & 0 & 0 \\ 0 & 0 & \gamma_3 & 0 & 0 \\ 0 & 0 & 0 & \gamma_4 & 0 \\ 0 & 0 & 0 & 0 & \gamma_5 \end{pmatrix} \begin{pmatrix} z_1 \\ . \\ . \\ . \\ z_5 \end{pmatrix}^T + \begin{pmatrix} z_1 & . & . & . & z_5 \end{pmatrix} \begin{pmatrix} \gamma_1 & 0 & 0 & 0 & 0 \\ 0 & \gamma_2 & 0 & 0 & 0 \\ 0 & 0 & \gamma_3 & 0 & 0 \\ 0 & 0 & 0 & \gamma_4 & 0 \\ 0 & 0 & 0 & 0 & \gamma_5 \end{pmatrix} \begin{pmatrix} \frac{\partial z_1}{\partial \xi} \\ . \\ . \\ . \\ \frac{\partial z_5}{\partial \xi} \end{pmatrix}^T d\xi$$

$$= \int_0^1 \frac{\partial}{\partial \xi}[\gamma_1 z_1(\xi) + \ . \ . \ . + \gamma_5 z_5(\xi)]\, d\xi$$

$$= [\gamma_1 z_1(\xi) + \ . \ . \ . + \gamma_5 z_5(\xi)]_0^1$$

$$= \gamma_1[z_1(1)^2 - z_1(0)^2] + \gamma_2[z_2(1)^2 - z_2(0)^2] + \gamma_3[z_3(1)^2 - z_3(0)^2]$$
$$+ \gamma_4[z_4(1)^2 - z_4(0)^2] + \gamma_5[z_5(1)^2 - z_5(0)^2]$$

here by using the conditions on vertices; we get

$$\operatorname{Re}\langle Az, z\rangle = z_1(0)^2(-\gamma_1 + \gamma_2) + z_2(0)^2(-\gamma_2 + \gamma_5) + z_3(0)^2(-\gamma_3 + \gamma_4)$$
$$+ z_4(0)^2(-\gamma_4 + \gamma_5) + z_5(0)^2\left(\frac{1}{4}\gamma_1 + \frac{1}{4}\gamma_3 - \gamma_5\right) + 2\gamma_5 z_2(0) z_4(0)$$

To prove $\operatorname{Re}\langle Az, z\rangle \le 0$; we need to prove the following:

$$\left\{\begin{array}{l} \left(-\gamma_1+\gamma_2\right) \leq 0 \\ \left(-\gamma_3+\gamma_4\right) \leq 0 \\ \left(\frac{1}{4}\gamma_1+\frac{1}{4}\gamma_3-\gamma_5\right) \leq 0 \\ \begin{pmatrix} -\gamma_2+\gamma_5 & \gamma_5 \\ \gamma_5 & -\gamma_4+\gamma_5 \end{pmatrix} \leq 0 \end{array}\right\}$$

To prove the above inequalities; we use here the Sylvester‘s criterion by choosing

$$\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 1 \quad \& \quad \gamma_5 = \frac{1}{2}$$

hence from this trick we finally obtain the new Hamiltonian as

$$\mathrm{H} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

So the second condition of the theorem 2.4.2.1, $\mathrm{Re}\langle Az, z\rangle \leq 0$ with the above new Hamiltonian satisfied.

Now for confirmation purpose we will check the other condition of the theorem 2.4.2.1; which is $W_B \Sigma W_B^* \geq 0$

For this, first we write the boundary conditions in terms of new variables (boundary effort and boundary flow) e_∂ and f_∂ :

$$e_{\partial} = \frac{1}{\sqrt{2}}\left(\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} z_1(1) \\ z_2(1) \\ z_3(1) \\ z_4(1) \\ z_5(1) \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} z_1(0) \\ z_2(0) \\ z_3(0) \\ z_4(0) \\ z_5(0) \end{bmatrix}\right)$$

and

$$e_{\partial} = \frac{1}{\sqrt{2}}\begin{pmatrix} z_1(1) + z_1(0) \\ z_2(1) + z_2(0) \\ z_3(1) + z_3(0) \\ z_4(1) + z_4(0) \\ \frac{1}{2}z_5(1) + \frac{1}{2}z_5(0) \end{pmatrix}$$

for the Boundary flow;

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

so we have

$$f_{\partial} = \frac{1}{\sqrt{2}}\left(\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} z_1(1) \\ z_2(1) \\ z_3(1) \\ z_4(1) \\ z_5(1) \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} z_1(0) \\ z_2(0) \\ z_3(0) \\ z_4(0) \\ z_5(0) \end{bmatrix}\right)$$

and finally we get

$$f_\partial = \frac{1}{\sqrt{2}} \begin{pmatrix} z_1(1) - z_1(0) \\ z_2(1) - z_2(0) \\ z_3(1) - z_3(0) \\ z_4(1) - z_4(0) \\ z_5(1) - z_5(0) \end{pmatrix}$$

now we have the boundary conditions

$$\tilde{W}_B \begin{bmatrix} \mathrm{H}(1)\ z(1,t) \\ \mathrm{H}(1)\ z(0,t) \end{bmatrix} = 0 \quad , \qquad t \geq 0$$

here

$$\tilde{W}_B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & -1 & 0 & -1 & 0 \end{bmatrix}$$

and

$$W_B = \tilde{W}_B\, R_0^{-1}$$

$$W_B = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 1 \end{bmatrix}$$

So we check $W_B \Sigma W_B{}^* \geq 0$

$$W_B \Sigma W_B{}^* = \begin{bmatrix} \frac{1}{4} & 0 & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

To prove the above matrix is positive , we need to show that the eigen values are positive.

so

$$Eigen\ values(W_B \Sigma W_B{}^*) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \geq 0$$

Hence the theorem 2.4.2.1 is satisfied, so operator A is the infinitesimal generator of contraction semigroup on X with

$$\mathrm{H} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

and

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

Chapter 4 Boundary Control System

In this chapter we are in particular interested in systems with a control at the boundary of their spatial domain. We have shown that these systems have well-defined solutions provided the input is sufficiently smooth. In first section we discussed port-Hamiltonian as boundary control system and describe some background information and theorems [2], which we needed later in this chapter. In final section we studied the well-posedness and also used the known result to show that the system is well-posed if and only if A is infinitesimal generator of a C_0- semigroup [2].

Definition (Boundary Control System):

Consider the following control system

$\dot{x}(t) = \mathfrak{U}\ x(t)$, $x(0) = x_0$

$x(t) = u(t)$

where

$\mathfrak{U} : D(\mathfrak{U}) \subset X \to X$ is linear , the control function u takes values in the Hilbert space U and boundary operators $\mathfrak{B}$: $D(\mathfrak{B}) \subset X \to U$ is linear & satisfies $D(\mathfrak{U}) \subset D(\mathfrak{B})$. This system is a boundary control system if the following holds:

1. The operator $A : D(A) \to X$ with $D(A) = D(\mathfrak{U}) \cap \ker(\mathfrak{B})$ and

   $A(x) = \mathfrak{U} x$; for $x \in D(A)$ is the infinitesimal generator of C_0-semigroup $\left(T(t)\right)_{t\geq 0}$ on X

2. There exists an operator B ∈ L(U,X) s.t for all $u \in$ U we have $Bu \in D(\mathfrak{U})$, $\mathfrak{U}B \in L(U,X)$ and $\mathfrak{B}Bu = u$, $u \in$ U.

4.1 PORT-HAMILTONIAN SYSTEMS AS BOUNDARY CONTROL SYSTEMS

We add a boundary control to a port-Hamiltonian system; we will show that the assumption of a boundary control system is satisfied. The port-Hamiltonian system with control is given by

$$\frac{\partial x}{\partial t}(\xi,t) = P_1 \frac{\partial}{\partial \xi}\left(\mathcal{H}(\xi)x(\xi,t)\right) + P_0\left(\mathcal{H}(\xi)x(\xi,t)\right) \quad \text{.......................} (4.1)$$

$$u(t) = W_{B,1}\begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} \quad \text{..............................} \quad (4.2)$$

$$0 = W_{B,2}\begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} \quad \text{................................} \quad (4.3)$$

We make the following assumptions.

4.1.1 Assumptions

- $P_1 \in K^{n\times n}$ is invertible and self-adjoint.
- $\mathrm{H} \in L^\infty([a,b]; \mathrm{K}^{n\times n})$, $\mathrm{H}(\xi)$ is self-adjoint for a.e. $\xi \in [a,b]$ and there exist $M, m > 0$ such that $mI \le \mathrm{H}(\xi) \le M\,I$ for a.e. $\xi \in [a,b]$
- $W_B = \begin{bmatrix} W_{B,1} \\ W_{B,2} \end{bmatrix} \in \mathrm{K}^{n\times 2n}$ has full rank.

We can write the port-Hamiltonian system eq: 4.1—4.3 as boundary control system

$$\dot{x}(t) = U\,x(t)\ , \qquad x(0) = x_0$$
$$B\,x(t) = u(t)$$

By defining

$$U\,x = P_1 \frac{\partial}{\partial \xi}(\mathrm{H}\,x) + P_0(\mathrm{H}\,x) \quad \text{...................................} \quad (4.4)$$

$$D\,(U) = \left\{ x \in L^2([a,b]; K^n) \mid \mathrm{H}x \in H^1([a,b]; K^n),\ W_{B,2}\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = 0 \right\} \quad \text{.........} \quad (4.5)$$

$$Bx = W_{B,1}\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix}, \quad \text{..........................} \quad (4.6)$$

$$D(B) = D(U) \quad \text{.................................} \quad (4.7)$$

We choose the Hilbert space $X = L^2([a,b]; K^n)$, with the inner product space

$$\langle f, g \rangle_X = \frac{1}{2}\int_a^b f(\xi)^* \mathrm{H}(\xi)\, g(\xi)\, d\xi$$

as the state space and $u(t)$ is a smooth control function. The input space U equals K^m , where m is the number of rows of $W_{B,1}$. We are now in the position to show that the controlled port- Hamiltonian system is indeed a boundary control system.

4.1.2 Theorem

If the operator

$$Ax = P_1 \frac{\partial}{\partial \xi}(\mathcal{H} x) + P_0 (\mathcal{H} x) \qquad \text{..................................} \ (4.8)$$

With domain

$$D\,(U) = \left\{ x \in X \mid \mathcal{H}x \in H^1\left([a,b]\,;\, K^n\right)\,,\, \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} \in \ker \begin{bmatrix} W_{B,1} \\ W_{B,2} \end{bmatrix} \right\} \text{.........} \ (4.9)$$

generates a C_0 – semigroup on X ,then the system eq: 4.1-4.3 is a boundary control system on X.

Remark:

An essential condition in the above theorem is that A given by eq: 4.8 with the domain eq: 4.9 generates a C_0 – semigroup . Theorem 4.1.2 and assumptions 4.1.1 imply that this holds in particular when $P_0^* = -P_0$ and $W_B \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} W_B^* \geq 0$. Since the term $P_0\mathcal{H}$ can be seen as is a bounded perturbation of (4.4) with $P_0 = 0$, Theorem 4.1.2 shows that A given by eq: 4.8 with domain eq: 4.9 generates a C_0 – semigroup when $W_B \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} W_B^* \geq 0$.

4.1.3 Boundary observations for port-Hamiltonian System

We use the conditions on the boundary observation which guarantee that a certain balance equation is satisfied, which is important. The standard Hamiltonian system with boundary control and boundary observation is given by

$$\dot{x}(t) = P_1 \frac{\partial}{\partial \xi}(\mathrm{H}\, x(t)) + P_0(\mathrm{H}\, x(t)) \quad (4.10)$$

$$(input) \qquad u(t) = W_B \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} \quad (4.11)$$

$$(output) \qquad y(t) = W_C \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} \quad (4.12)$$

It is assumed that P_1, H and W_B satisfy the conditions of assumptions 4.1.1 . The output equation is formulated very similar to the control equation. So we assume that the output space $Y = K^k$ and thus W_C is a matrix of size $k \times 2n$.. Since we want the output to be independent, we assume that W_C has full rank.

4.1.4 Theorem

Consider the system eq: 4.10-4.12, satisfying assumptions 4.1.1, $W_C \in K^{k \times 2n}$ and $\begin{bmatrix} W_B \\ W_C \end{bmatrix} \in K^{(k+n)\times 2n}$ having full rank. Assume that the operator A defined by eq: 4.8 and eq: 4.9 generates C_0 – semigroup on X. then for every $u \in C^2([a,b]; K^n)$, $\mathrm{H}x(0) \in H^1([a,b]; K^n)$ and $u(0) = W_B \begin{bmatrix} f_\partial(0) \\ e_\partial(0) \end{bmatrix}$, the system eq; 4.10-4.12 has a unique (classical) solution, with $\mathrm{H}x(t) \in H^1([a,b]; K^n)$, $t \geq 0$, and the output y is continuous.

Furthermore, if additionally $P_0^* = -P_0$ and $k = n$, then the following balance equation is satisfied for every $t \geq 0$

$$\frac{d}{dt}\|x(t)\|_X^2 = \frac{1}{2}\begin{bmatrix} u^*(t) & y^*(t) \end{bmatrix} P_{W_B, W_C} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}.$$

Now we continue with example 0, which explains the port-Hamiltonian system as boundary control system.

4.1.5 Example

Consider the controlled transport equation

$$\frac{\partial z}{\partial t} = \frac{\partial}{\partial \xi}\{\mathrm{H}(\xi) z(\xi, t)\}$$

We add the input $u(t)$ at vertex v_3. Now we will discuss the following network (Figure 3) as boundary control system and boundary observation.

first we consider the corresponding inhomogeneous differential equation

$$\dot{z}(t) = Az(t) + u(t) \qquad ; z(0) = z_0$$

here $u(t)$ is smooth input function.

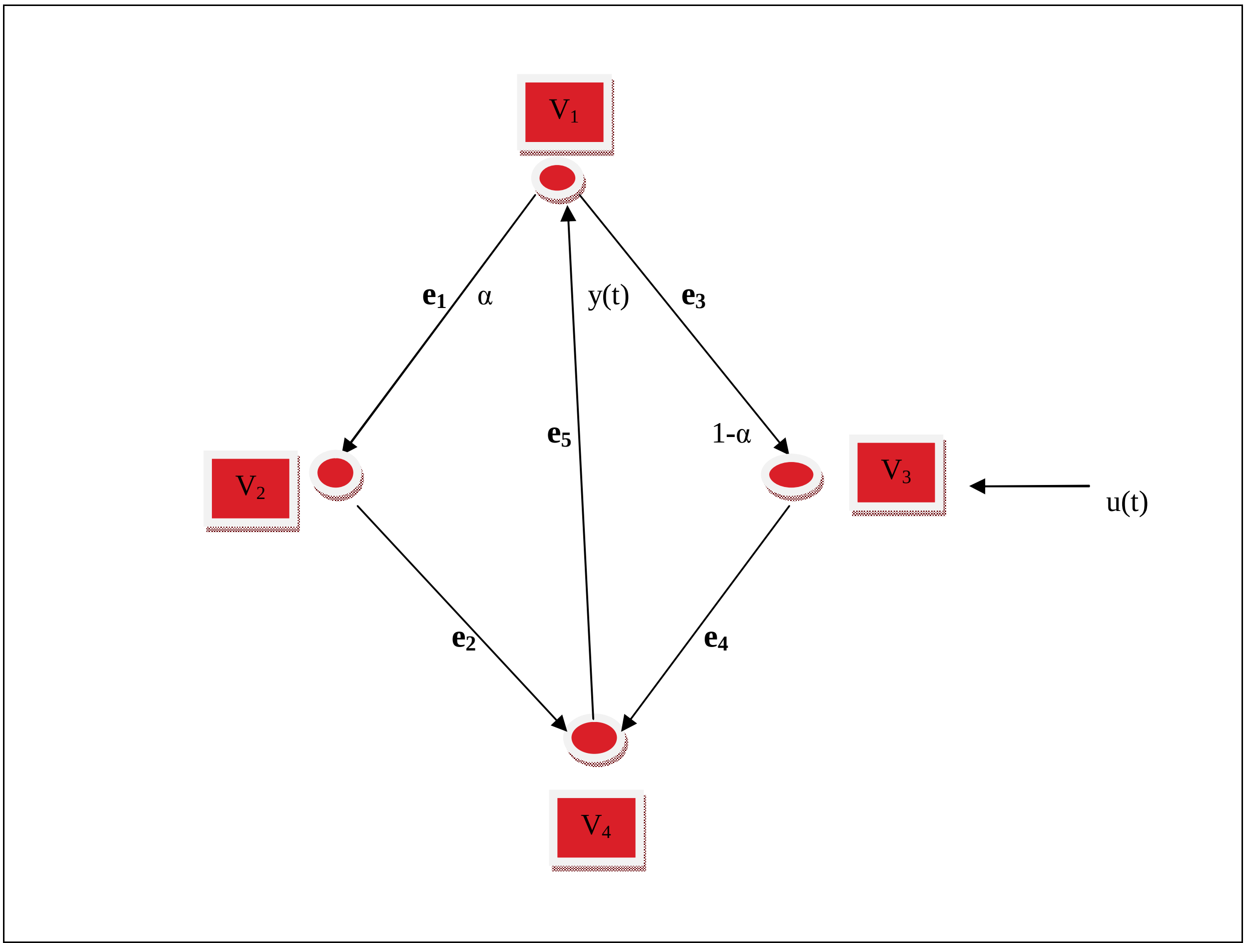


Figure 3

here

$$P_1 = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{2} \end{bmatrix}$$

&

$$\mathrm{H} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1/2} \end{bmatrix}$$

where $P_0 = 0$

now we write the port-Hamiltonian system as control boundary system in the following way:

we have the Boundary Conditions on vertices from the flow diagram as

$$z_1(1) = \frac{1}{2} z_5(0)$$
$$z_3(1) = \frac{1}{2} z_5(0)$$
$$z_2(1) = z_1(0)$$
$$z_4(1) = z_3(0) + u(t)$$
$$z_5(1) = z_2(0) + z_4(0)$$

We can write it as;

$$z_4(1) - z_3(0) = u(t)$$
$$z_1(1) - \frac{1}{2} z_5(0) = 0$$
$$z_2(1) - z_1(0) = 0$$
$$z_3(1) - \frac{1}{2} z_5(0) = 0$$
$$z_5(1) - z_2(0) - z_4(0) = 0$$

Now to find $W_{B,1}$:

$$u(t) = z_4(1) - z_3(0)$$

or

$$u(t) = \underbrace{\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}}_{\tilde{W}_{B,1}} \begin{pmatrix} \mathrm{Hz}(1) \\ \mathrm{Hz}(0) \end{pmatrix}$$

where

$$W_{B,1} = \tilde{W}_{B,1} R_0^{-1}$$

here

$$R_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} P_1 & -P_1 \\ I & I \end{pmatrix}, \quad here \;\; P_1 = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{2} \end{bmatrix}$$

So

$$W_{B,1} = \begin{pmatrix} 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 & -1/2 & 1/2 & 0 \end{pmatrix}$$

hence

$$u(t) = \begin{pmatrix} 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 & -1/2 & 1/2 & 0 \end{pmatrix} \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix}$$

where $$\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = \begin{pmatrix} (\mathrm{Hz})(1) - (\mathrm{Hz})(0) \\ (\mathrm{Hz})(1) + (\mathrm{Hz})(0) \end{pmatrix}$$

To find $W_{B,2}$:

we have

$$\begin{aligned} &z_1(1) - \frac{1}{2} z_5(0) = 0 \\ &z_2(1) - z_1(0) = 0 \\ &z_3(1) - \frac{1}{2} z_5(0) = 0 \\ &z_5(1) - z_2(0) - z_4(0) = 0 \end{aligned}$$

now

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 \end{pmatrix}}_{\tilde{W}_{B,2}} \begin{pmatrix} Hz(1) \\ Hz(0) \end{pmatrix}$$

&

$$W_{B,2} = \tilde{W}_{B,2} R_0^{-1}$$

$$W_{B,2} = \begin{pmatrix} 1/2 & 0 & 0 & 0 & 1/8 & 1/2 & 0 & 0 & 0 & -1/2 \\ 1/2 & 1/2 & 0 & 0 & 0 & -1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/8 & 0 & 0 & 1/2 & 0 & -1/2 \\ 0 & 1/2 & 0 & 1 & 0 & 0 & -1/2 & 0 & 0 & 0 \end{pmatrix}$$

so

$$0 = \begin{pmatrix} 1/2 & 0 & 0 & 0 & 1/8 & 1/2 & 0 & 0 & 0 & -1/2 \\ 1/2 & 1/2 & 0 & 0 & 0 & -1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/8 & 0 & 0 & 1/2 & 0 & -1/2 \\ 0 & 1/2 & 0 & 1 & 0 & 0 & -1/2 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix}$$

now

$$W_B = \begin{pmatrix} W_{B,1} \\ W_{B,2} \end{pmatrix}$$

$$W_B = \begin{pmatrix} 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 & -1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 & 0 & 1/8 & 1/2 & 0 & 0 & 0 & -1/2 \\ 1/2 & 1/2 & 0 & 0 & 0 & -1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/8 & 0 & 0 & 1/2 & 0 & -1/2 \\ 0 & 1/2 & 0 & 1 & 0 & 0 & -1/2 & 0 & 0 & 0 \end{pmatrix}$$

here W_B is full rank matrix.

Hence all the assumptions 4.1.1are satisfied. So we can write the port-Hamiltonian system eq:4.1-4.3 as a boundary control system.

Now the standard Hamiltonian system with the boundary control and boundary observation is given by :

$$\dot{z}(t) = Az(t) \quad ; \quad z(0) = z_0$$
$$Bz(t) = u(t) \quad \text{where}\, u \in C\big([a,b];\mathrm{X}\big)$$

by defining

$$\mathrm{Az} = \frac{\partial}{\partial \xi}(Hz)$$
$$D(A) = \left\{ z \in L^2([0,1]) : \mathbb{C}^n \,|\, \mathrm{Hz} \in H^1([a,b]); W_{B,2}\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = 0 \right\}$$
$$Bz = W_{B,1}\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix}$$
$$D(B) = D(A)$$

We choose the Hilbert space $X = L^2([0,1];\mathbb{C}^n)$ with the inner product

$$\langle f, g \rangle = \frac{1}{2}\int_0^1 f(\xi)^* \,\mathrm{H}(\xi)\, g(\xi)\; d\xi$$

and

$$\dot{z}(t) = \frac{\partial}{\partial \xi}\big(\mathrm{Hz}(t)\big)$$
$$\text{smooth input} \quad u(t) = W_B\begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix}$$
$$\text{smooth output} \quad \mathrm{y(t)=W_C}\begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix}$$
$$y(t) = z_5(0)$$

now

$$\tilde{W}_C = \begin{pmatrix} 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
$$W_C = \tilde{W}_C R_0^{-1}$$

where

$$R_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} P_1 & -P_1 \\ I & I \end{pmatrix}, \quad and \quad P_1 = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{2} \end{bmatrix}$$

so

$$W_C = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

So finally we have P_1, H, & W_B which satisfies the conditions 4.1.1

now W_C is full rank matrix, also $\begin{pmatrix} W_B \\ W_C \end{pmatrix}$ has full rank.

So we have

$$\begin{pmatrix} W_B \\ W_C \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1/2} & \mathbf{1/2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-1/2} & \mathbf{1/2} & \mathbf{0} \\ \mathbf{1/2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1/8} & \mathbf{1/2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-1/2} \\ \mathbf{1/2} & \mathbf{1/2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-1/2} & \mathbf{1/2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1/2} & \mathbf{0} & \mathbf{1/8} & \mathbf{0} & \mathbf{0} & \mathbf{1/2} & \mathbf{0} & \mathbf{-1/2} \\ \mathbf{0} & \mathbf{1/2} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{-1/2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

has full rank.

Hence we have shown that port-Hamiltonian system for this network of flow (Figure 3) is boundary control system. We also studied the boundary observation for this network of flow.

Finally we need to show that the system is well posed and hence A generates C_0-semigroup. In the following, first we discuss some background information [2] of well-posedness for port-Hamiltonian system and then prove that the given port-Hamiltonian system is well-posed and A is the infinitesimal generator of the C_0-semigroup on X.

Definition (well-posed):

Consider the system

$\dot{x}(t) = \mathfrak{A}\ x(t)$, $\qquad x(0) = x_0$

$\mathfrak{B}\ x(t) = u(t)$

$\mathfrak{C}\, x(t) = y(t)$

satisfying the following assumptions ;

1. The operators $\mathfrak{U}$: D($\mathfrak{U}$)⊂ X→X , $\mathfrak{B}$: D($\mathfrak{B}$) ⊂ X→U and $\mathfrak{C}$: D($\mathfrak{U}$)⊂ X→Y are linear operators , D($\mathfrak{U}$) ⊂ D($\mathfrak{B}$) & X, U, Y are Hilbert spaces.
2. The operator A:D(A) →X with D(A)= D($\mathfrak{U}$)∩ ker ($\mathfrak{B}$)and

$$Ax = \mathfrak{U}\, x \qquad \text{for} \quad x \in D\,(A)$$

is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on X

3. There exists an operator B ∈ L(U,X) s.t for all u ∈U we have Bu∈ D($\mathfrak{U}$), $\mathfrak{U}$B∈ L(U,X) and

$$\mathfrak{B}\mathrm{B}u = u\,, \quad u \in \mathrm{U}.$$

4. The operator $\mathfrak{C}$ is bounded from the domain of A to Y. Here D(A) is equipped with the grapg norm.

We call this system is well-posed if there exists a τ>0 and $m_\tau \geq 0$ s.t for all

$x_0 \in$ D ($\mathfrak{U}$) and $u \in C^2\left([0,\tau];\mathrm{U}\right)$ with $u(0) = \mathfrak{B}\, x_0$

We have

$$\|x(\tau)\|_X^2 + \int_0^\tau \|y(t)\|^2 dt \leq m_\tau \left(\|x_0\|_X^2 + \int_0^\tau \|u(t)\|^2 dt \right).$$

In general it is not easy to show that a boundary control system is well-posed. However there is a special class of system for which well-posedness can easily be proved.

4.2 WELL-POSEDNESS FOR PORT-HAMILTONIAN SYSTEMS

The port-Hamiltonian systems with boundary control and boundary observation, is of the following form

$$\frac{\partial x}{\partial t}(\xi,t) = P_1 \frac{\partial}{\partial \xi}\left(\mathrm{H}(\xi)\, x(\xi,\, t)\right) + P_0\left(\mathrm{H}(\xi)\, x(\xi,\, t)\right) \text{.......................} \quad (4.13)$$

$$u(t) = W_{B,1}\begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} \quad .... \quad (4.14)$$

$$0 = W_{B,2}\begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} \quad .... \quad (4.15)$$

$$y(t) = W_C\begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} \quad .... \quad (4.16)$$

We assume that P_1, H and $W_B = \begin{bmatrix} W_{B,1} \\ W_{B,2} \end{bmatrix}$ satisfy assumptions 4.1.1, thus in particular, for a.e. $\xi \in [a,b]$, $\mathrm{H}(\xi)$ is self-adjoint $n \times n$ matrix satisfying $0 < mI \le \mathrm{H}(\xi) \le M\,I$.

Furthermore , $W_B = \begin{bmatrix} W_{B,1} \\ W_{B,2} \end{bmatrix}$ is a full rank matrix of size $n \times 2n$. We assume that $W_{B,1}$ is a $m \times 2n$ matrix. The state space is given by the Hilbert space $X = L^2\left([a,b]\,;\,K^n\right)$ with the inner product

$$\langle f, g \rangle_X = \frac{1}{2}\int_a^b f(\xi)^* \mathrm{H}(\xi)\, g(\xi)\, d\xi \quad .... \quad (4.17)$$

4.2.1 Theorem

Consider the port-Hamiltonian system eq: 4.13-4.16 and assume that the conditions of assumptions 4.1.1. are satisfied. Furthermore, we assume that the multiplication operator $P_1\mathrm{H}$ can be written as

$$P_1\mathrm{H}(\xi) = S^{-1}(\xi)\Delta(\xi)S(\xi) \quad , \qquad \xi \in [a,b]$$

where Δ is diagonal matrix-valued function, S is a matrix valued function and both Δ and S are continuously differentiable on $[a,b]$,

$$rank\begin{bmatrix} W_{B,1} \\ W_{B,2} \\ W_C \end{bmatrix} = n + rank\,(W_C)$$

Let X be the Hilbert space $L^2\left([a,b];K^n\right)$ with inner product eq:4.17 . If the operator A corresponding to the homogenous p.d.e., $u \equiv 0$, generates a C_0 – semigroup on X , then the system eq: 4.13-4.16 is regular, and thus in particular well-posed.

Remark:

If A generates a C_0 – semigroup on X , then the system possesses classical solutions for smooth inputs and initial conditions. Well-posedness implies that there exist solutions for every initial condition and every square integrable input.

4.2.2 Lemma

Let P_1 and H satisfy the conditions of theorem 4.2.1,then Δ can be written as

$$\Delta = \begin{bmatrix} \Lambda & 0 \\ 0 & \Theta \end{bmatrix}$$

Where Λ is a diagonal real matrix-valued function, with (strictly) positive function on the diagonal, and Θ is a diagonal real matrix-valued function, with (strictly) negative functions on the diagonal.

Remark:

In theorem 4.2.1 we asserted that under some weak conditions every port-Hamiltonian system is well- posed provided that corresponding homogenous equation generates a strongly continuous semigroup.

4.2.3 P_1H Diagonal

We now investigate when the p.d.e. with the control and observation at the boundary is well-posed. We discussed in theorem 4.2.1 the situation that P_1H is diagonal, i.e. when $S = I$ thus we consider the following diagonal port-Hamiltonian system

$$\frac{\partial}{\partial t}\begin{bmatrix} x_+(\xi,t) \\ x_+(\xi,t) \end{bmatrix} = \frac{\partial}{\partial \xi}\left(\begin{bmatrix} \Lambda(\xi) & 0 \\ 0 & \Theta(\xi) \end{bmatrix}\begin{bmatrix} x_+(\xi,t) \\ x_+(\xi,t) \end{bmatrix}\right) \quad \text{.....................................} \quad (4.18)$$

where for every $\xi \in [a,b]$, $\Lambda(\xi)$ is a diagonal (real) matrix, with positive numbers on the diagonal, and $\Theta(\xi)$ is a diagonal (real) matrix, with negative numbers on the diagonal. Furthermore, we assume that Λ and Θ are continuously differentiable and that $\begin{bmatrix} \Lambda(\xi) & 0 \\ 0 & \Theta(\xi) \end{bmatrix}$ is an $n \times n$-matrix. As eq: 4.18 is a port-Hamiltonian system with $\mathrm{H} = \begin{bmatrix} \Lambda & 0 \\ 0 & \Theta \end{bmatrix}$ and $P_1 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$.

The following boundary control and observation are of interest

$$u_s(t) = \begin{bmatrix} \Lambda(b)\, x_+(b,t) \\ \Theta(a)\, x_-(a,t) \end{bmatrix}, \qquad \text{.............................} \quad (4.19)$$

$$y_s(t) = \begin{bmatrix} \Lambda(a)\, x_+(a,t) \\ \Theta(b)\, x_-(b,t) \end{bmatrix}. \qquad \text{.............................} \quad (4.20)$$

4.2.4 Theorem

Consider the p.d.e. (eq: 4.18) with boundary control u_s and boundary observation y_s as defined in eq: 4.19 and eq: 4.20 respectively.

The system defined by eq: 4.18-4.20 is well-posed and regular.

We equip the port-Hamiltonian system eq: 4.18 with a new set of inputs and outputs. The new input u_s is of the form

$$u_s = \mathrm{K}\, u_s(t) + Q\, y_s(t) \qquad \text{..........................} \quad (4.21)$$

where K and Q are two square $n \times n$-matrices with $[\mathrm{K} \quad Q]$ of rank n. The new output is of the form

$$y_s = O_1\, u_s(t) + O_2\, y_s(t) \qquad \text{..........................} \quad (4.22)$$

Where O_1 and O_2 are $k \times n$-matrices. For system eq: 4.18 with input u and output y, we have the following results:

- If K is invertible, then the system eq: 4.18, eq: 4.21 and eq: 4.22 is well-posed.
- If K is not invertible, then the operator A_K defined as

$$A_K \begin{bmatrix} g_+(\xi) \\ g_-(\xi) \end{bmatrix} = \frac{\partial}{\partial \xi}\left(\begin{bmatrix} \Lambda(\xi) & 0 \\ 0 & \Theta(\xi) \end{bmatrix} \begin{bmatrix} g_+(\xi) \\ g_-(\xi) \end{bmatrix} \right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (4.23)$$

with domain

$$D(A_K) = \left\{ \begin{bmatrix} g_+(\xi) \\ g_-(\xi) \end{bmatrix} \in H^1\left([a,b],\, K^n\right) \mid \right.$$

$$\left. \mathrm{K} \begin{bmatrix} \Lambda(b) g_+(b) \\ \Theta(a) g_-(a) \end{bmatrix} + Q \begin{bmatrix} \Lambda(a) g_+(a) \\ \Theta(b) g_-(b) \end{bmatrix} = 0 \right\} \quad \ldots\ldots\ldots \ (4.24)$$

does not generate a C_0 – semigroup on $L^2\left([a,b]\,;\, K^n\right)$.

The part 2 implies that the homogenous p.d.e. does not have a well-defined solution, when K is not invertible.

Proof: The proof of this theorem is given in [2]

Now we will continue example 4.1.5 and show that A generates C_0-semigroup by showing that the system is well-posed.

4.2.5 Example

Consider the Port-Hamiltonian System with boundary control and boundary observations will be

$$\frac{\partial z}{\partial t}(\xi) = \frac{\partial}{\partial \xi}\{\mathrm{H}(\xi) z(\xi,t)\} \quad (P_1 = Identity\, matrix \,\&\, P_0 = o)$$

With the boundary conditions on vertices (from the flow diagram: Figure 3)

$$
\begin{aligned}
& z_1(1) = \frac{1}{2} z_5(0) \\
& z_2(1) = z_1(0) \\
& z_3(1) = \frac{1}{2} z_5(0) \\
\textit{input} \quad & z_4(1) = z_3(0) + u(t) \\
& z_5(1) = z_2(0) + z_4(0) \\
& \& \\
\textit{output} \quad & y(t) = z_5(0)
\end{aligned}
$$

input will become

$$u(t) = W_{B,1} \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix}$$

$$\text{where} \quad \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} z(1,t) - z(0.t) \\ z(1,t) + z(0,t) \end{pmatrix}$$

&

$$W_{B,1} = \tilde{W}_{B,1} R_0^{-1};$$

$$R_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} P_1 & -P_1 \\ I & I \end{pmatrix}$$

hence

$$u(t) = \underbrace{\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}}_{\tilde{W}_{B,1}} \begin{pmatrix} \mathrm{H}z(1) \\ \mathrm{H}z(0) \end{pmatrix}$$

here

$$W_{B,1} = \tilde{W}_{B,1} R_0^{-1},$$

so

$$W_{B,1} = \begin{pmatrix} 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 & -1/2 & 1/2 & 0 \end{pmatrix}$$

hence the input will become

$$u(t) = \begin{pmatrix} 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 & -1/2 & 1/2 & 0 \end{pmatrix} \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix}$$

now

$$0 = W_{B,2}\begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix}$$

where

$$W_{B,2} = \tilde{W}_{B,2} R_0^{-1}$$

and

$$\begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} = \underbrace{\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-1/2} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-1/2} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{0} & \mathbf{-1} & \mathbf{0} \end{bmatrix}}_{\tilde{\mathbf{W}}_{\mathbf{B,2}}} \begin{pmatrix} \mathrm{Hz}(1) \\ \mathrm{Hz}(0) \end{pmatrix}$$

so

$$W_{B,2} = \begin{pmatrix} \mathbf{1/2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1/4} & \mathbf{1/2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-1/4} \\ \mathbf{1/2} & \mathbf{1/2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-1/2} & \mathbf{1/2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1/2} & \mathbf{0} & \mathbf{1/4} & \mathbf{0} & \mathbf{0} & \mathbf{1/2} & \mathbf{0} & \mathbf{-1/4} \\ \mathbf{0} & \mathbf{1/2} & \mathbf{0} & \mathbf{1/2} & \mathbf{1/2} & \mathbf{0} & \mathbf{-1/2} & \mathbf{0} & \mathbf{-1/2} & \mathbf{1/2} \end{pmatrix}$$

$$0 = \begin{pmatrix} \mathbf{1/2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1/4} & \mathbf{1/2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-1/4} \\ \mathbf{1/2} & \mathbf{1/2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-1/2} & \mathbf{1/2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1/2} & \mathbf{0} & \mathbf{1/4} & \mathbf{0} & \mathbf{0} & \mathbf{1/2} & \mathbf{0} & \mathbf{-1/4} \\ \mathbf{0} & \mathbf{1/2} & \mathbf{0} & \mathbf{1/2} & \mathbf{1/2} & \mathbf{0} & \mathbf{-1/2} & \mathbf{0} & \mathbf{-1/2} & \mathbf{1/2} \end{pmatrix} \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix}$$

as we have ; (from BC's)

$z_1(1) = \frac{1}{2} z_s(0)$

$$z_1(1) = \frac{1}{2} y(t) \ \ (\text{since } y(t) = z_5(0))$$

or

$2z_1(1) = y(t)$

&

$$z_3(1) = \frac{1}{2} y(t)$$

or

$$2z_3(1) = y(t)$$

so

$$y(t) = \underbrace{\begin{pmatrix} 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{\tilde{W}_c} \begin{pmatrix} \mathrm{Hz}(1) \\ \mathrm{Hz}(0) \end{pmatrix}$$

&

$$W_c = \tilde{W}_c R_0^{-1}$$

so

$$W_c = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

hence

$$y(t) = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix}$$

finally we have the port-Hamiltonian system with boundary control & boundary observation as ;

$$\frac{\partial z}{\partial t} = \frac{\partial}{\partial \xi} \{ \mathrm{H}(\xi) z(\xi, t) \} \text{.................(4.25)}$$

$$u(t) = \begin{pmatrix} 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 & -1/2 & 1/2 & 0 \end{pmatrix} \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} \text{...............(4.26)}$$

$$0 = \begin{pmatrix} \mathbf{1/2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1/4} & \mathbf{1/2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-1/4} \\ \mathbf{1/2} & \mathbf{1/2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-1/2} & \mathbf{1/2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1/2} & \mathbf{0} & \mathbf{1/4} & \mathbf{0} & \mathbf{0} & \mathbf{1/2} & \mathbf{0} & \mathbf{-1/4} \\ \mathbf{0} & \mathbf{1/2} & \mathbf{0} & \mathbf{1/2} & \mathbf{1/2} & \mathbf{0} & \mathbf{-1/2} & \mathbf{0} & \mathbf{-1/2} & \mathbf{1/2} \end{pmatrix} \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} \text{....(4.27)}$$

&

$$y(t) = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} \text{............................(4.28)}$$

So we have

$$W_B = \begin{pmatrix} W_{B,1} \\ W_{B,2} \end{pmatrix}$$

which is

$$W_B = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1/2} & \mathbf{1/2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-1/2} & \mathbf{1/2} & \mathbf{0} \\ \mathbf{1/2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1/8} & \mathbf{1/2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-1/4} \\ \mathbf{1/2} & \mathbf{1/2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-1/2} & \mathbf{1/2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1/2} & \mathbf{0} & \mathbf{1/8} & \mathbf{0} & \mathbf{0} & \mathbf{1/2} & \mathbf{0} & \mathbf{-1} \\ \mathbf{0} & \mathbf{1/2} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{-1/2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

Furthermore, P_1 , H & $W_B = \begin{pmatrix} W_{B,1} \\ W_{B,2} \end{pmatrix}$ satisfy the assumptions (4.2.1) and $W_B = \begin{pmatrix} W_{B,1} \\ W_{B,2} \end{pmatrix}$ is a full rank matrix of size 5x10.

- To prove that the system eq: 4.25-4.28 is well-posed; we will use the theorem 4.2.1. and lemma 4.2.2.

As we have discussed that if the operator A corresponding to the homogenous p.d.e i-e u=0 generates a C_0 semigroup on X , then the system eq: 4.25-4.28 is a regular & thus in particular well-posed.

Let P_1 & H satisfy the conditions of theorem 4.2.1, and according to the lemma 4.2.2 we can write Δ as

$$\Delta = \begin{pmatrix} \Lambda & 0 \\ 0 & \Theta \end{pmatrix}$$

In our case;

$P_1\mathrm{H}$ is a diagonal matrix & its eigen values all are strictly positive, which are

$$P_1\mathrm{H} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 1 \end{bmatrix}$$

the eigen values of $P_1\mathrm{H} = \Delta = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

finally we have

$$\Delta = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 1 \end{bmatrix}$$

$P_1\mathrm{H}$ Diagonal:

$P_1\mathrm{H}$ is diagonal ;so we can write $S = I$;

Thus we consider the following diagonal port Hamiltonian system;

$$\frac{\partial z}{\partial t}(\xi,t) = \frac{\partial}{\partial \xi}\left(\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 1 \end{bmatrix} z(\xi,t) \right) \text{............(4.29)}$$

For every $\xi \in [0,1]$, the above is a port Hamiltonian system with $\mathrm{H} = I$ & $P_1 = I$

The following boundary control & observation are of interest:

$u_s(t) = [\Delta\, z(b,t)]$..................(4.30)

&

$y_s(t) = [\Delta\, z(a,t)]$....................(4.31)

According to theorem 4.2.4; we have for homogenous p.d.e. $u(t) = 0$ and the boundary conditions eq: 3.4 (from the flow diagram: Figure 3)

we have

$$u_s(t)=\begin{bmatrix}\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&1\end{bmatrix}\begin{pmatrix}z_1(1)\\z_2(1)\\z_3(1)\\z_4(1)\\z_5(1)\end{pmatrix}=\begin{pmatrix}z_1(1)\\z_2(1)\\z_3(1)\\z_4(1)\\z_5(1)\end{pmatrix}$$

$$y_s(t)=\begin{pmatrix}z_1(0)\\z_2(0)\\z_3(0)\\z_4(0)\\z_5(0)\end{pmatrix}$$

now we have the new input of the form:

$$0=u_s(t)=\underbrace{\begin{bmatrix}\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&1\end{bmatrix}}_{K}\begin{pmatrix}\mathbf{z_1(1)}\\\mathbf{z_2(1)}\\\mathbf{z_3(1)}\\\mathbf{z_4(1)}\\\mathbf{z_5(1)}\end{pmatrix}+\underbrace{\begin{bmatrix}\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{-1/2}\\\mathbf{0}&\mathbf{-1}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{-1/2}\\\mathbf{0}&\mathbf{0}&\mathbf{-1}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{-1}&\mathbf{0}&\mathbf{-1}&\mathbf{0}\end{bmatrix}}_{Q}\begin{pmatrix}\mathbf{z_1(0)}\\\mathbf{z_2(0)}\\\mathbf{z_3(0)}\\\mathbf{z_4(0)}\\\mathbf{z_5(0)}\end{pmatrix}$$

where K and Q are two square 5×5 matrices with $[K \quad Q]$ of rank 5.

when we have

$$\mathrm{H}=\begin{bmatrix}\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&1/2\end{bmatrix},\ P_1=\begin{bmatrix}\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&2\end{bmatrix}$$

$$u(t)=\underbrace{\begin{pmatrix}0&0&0&1&0&0&0&-1&0&0\end{pmatrix}}_{\tilde{W}_{B,1}}\begin{pmatrix}\mathrm{H}z(1)\\\mathrm{H}z(0)\end{pmatrix}$$

so

$$W_{B,1}=\begin{bmatrix}0&0&1/2&1/2&0&0&0&-1/2&1/2&0\end{bmatrix}$$

$$0=W_{B,2}\begin{pmatrix}f_\partial(t)\\e_\partial(t)\end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & -1 & 0 \end{pmatrix}}_{\tilde{W}_{B,2}} \begin{pmatrix} Hz(1) \\ Hz(0) \end{pmatrix}$$

$$W_{B,2} = \begin{pmatrix} 1/2 & 0 & 0 & 0 & 1/4 & 1/2 & 0 & 0 & 0 & -1/2 \\ 1/2 & 1/2 & 0 & 0 & 0 & -1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/4 & 0 & 0 & 1/2 & 0 & -1/2 \\ 0 & 1/2 & 0 & 1/2 & 1/2 & 0 & -1/2 & 0 & -1/2 & 1/2 \end{pmatrix}$$

$$0 = \begin{pmatrix} 1/2 & 0 & 0 & 0 & 1/4 & 1/2 & 0 & 0 & 0 & -1/2 \\ 1/2 & 1/2 & 0 & 0 & 0 & -1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/4 & 0 & 0 & 1/2 & 0 & -1/2 \\ 0 & 1/2 & 0 & 1/2 & 1/2 & 0 & -1/2 & 0 & -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix}$$

$$y(t) = \underbrace{\begin{pmatrix} 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{\tilde{W}_c} \begin{pmatrix} Hz(1) \\ Hz(0) \end{pmatrix}$$

and

$$W_c = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Hence

$$y(t) = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix}$$

As K is invertible, then the system eq: 4.29 - 4.31 is well posed (According to the theorem 4.2.3). Hence The operator 'A' corresponding to the homogenous p.d.e i-e $u = 0$ generates a C_0-semigroup on $X = L^2([0,1]; K^n)$.

Chapter 5 Conclusion

In this thesis, we discussed the linear transport equation for different networks of flows. We were focused on our main tasks as discussed in Chapter 1 . First we formulated the abstract differential equation then we were interested to generate the contraction semigroup by using the port-Hamiltonian system with the state space $X = L^2([0,1];C^n)$. Although in the start of research work we were strict to the network flow as discussed in Chapter 1, but later we have chosen two networks of flows in this thesis vis-à-vis Figure 1 and Figure 2. The main reason to add one more network of flow is that we were unable to reach the conclusion on the basis of only one network flow.

We deduced the following results:

i. For first network flow (see Figure 1); we cannot generate the contraction semigroup by choosing Identity matrix as Hamiltonian. We also concluded that there does not exist any diagonal Hamiltonian matrix which generates contraction semigroup for this network of flow. Hence classical solution for this network of flow doesn´t exists.

ii. For second network flow (See Figure 2); we also could not construct contraction semigroup by using Hamiltonian = Identity matrix. Thus there does not exist unique solution. So we formulated the new diagonal constant valued Hamiltonian matrix, which generates the contraction semigroup and hence classical solution for this network exists.

Finally we concluded that we cannot generate the contraction semigroup by using Hamiltonian = Identity matrix with the inner product space

$$\langle f , g \rangle_X = \frac{1}{2}\int_0^1 f(\xi)^* H(\xi)\, g(\xi)\, d\xi$$

Hence we can say that A is not infinitesimal generator of C_o - Semigroup with the state space $X = L^2([0,1];C^n)$ for both networks of flows as discussed in Chapter 3.

In Chapter 4 , we were interested to find the classical solution for second network of flow (for the first network classical solution doesn´t exists). For this, we have shown port-Hamiltonian system as boundary control system. Next we studied the well-posedness of the boundary control system. At the end we came to the conclusion that the System is well-posed and ‘A’ is infinitesimal generator of C_o - Semigroup, and with the help of some theorems in Chapter 4 we showed that classical solution for this boundary control System exist.

As we formulated the diagonal Hamiltonian matrix for both network of flows, to obtain the contraction semigroup. So we were not able to obtain the good balance equation. Hence it is not possible to minimize the boundary control problem and to formulate the solution in matrix equation.

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