Universiteit Twente

Double Effect

Master Thesis

Multiple Interest Rate Curve Bootstrapping

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September 23, 2013

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Chapter 1

Introduction

Before the start of the crisis in 2007, most big financial institutions had a good credit quality and therefore their probability of default was very small. The LIBOR[1] rate is a daily quoted reference rate estimated by leading banks in London that they would be charged if borrowing from other banks. As the default probability of these leading banks could be neglected, the LIBOR rate could be considered as a risk-free rate.

Many interest rate products are based on LIBOR rates. Assuming that these products are traded between two parties that both have a negligible probability of default, we may assume that all cash flows are risk-free. To compute the present value of a risk-free cash flow, we use the prices of risk-free zero-coupon bonds to discount these cash flows. As LIBOR rates can be used as risk-free rates, we may express them in terms of prices of risk-free zero-coupon bonds.

In practice, however, only a few short-dated risk-free zero-coupon bonds are quoted in the market. To avoid this problem, market quotes of several interest rate products are used to obtain a risk-free discount curve corresponding to prices of risk-free zero-coupon bonds. This technique is called bootstrapping and requires an interpolation method.

This means that if we neglect the probability of default of the leading banks on which the LIBOR rate is based, as well as the probability of default of the parties entering into the trade, we are able to price interest rate products based on LIBOR rates using the market quotes of other interest rate products by bootstrapping.

The start of the financial crisis in 2007 resulted in the collapse of many large financial institutions like Lehman Brothers. Banks were taken over by other financial institutions and governments had to stand surety for banks. Many people lost their money as a lot of banks could not meet their liabilities. As a consequence of this crisis, the probability of default of leading banks on which the LIBOR rate is based cannot be neglected anymore. The probability of default of counterparty in a trade also cannot be neglected anymore.

An overnight index swap (OIS) is an interest rate product between two parties; one party pays a fixed rate periodically and one party pays a floating rate periodically. The floating rate is determined by overnight rates[2]. Overnight index swaps are traded with zero present value; the fixed rate is determined by setting the present value at zero. This fixed rate is called swap rate. Overnight index swaps with maturities up to one year usually have one payment at maturity.

The spread between the swap rate of an OIS with one payment at maturity and the LIBOR rate with the same maturity should be zero under the assumption that the LIBOR rate is based on banks with a negligible probability of default. We plotted some LIBOR-OIS spreads in Figure 1.1.

[1] London Interbank Offered Rate.
[2] LIBOR rates with a maturity of one day.

Prior to the start of the crisis in 2007, the LIBOR-OIS spread hovered within 10 bp and after the start of the crisis the spread increased dramatically. Nowadays the spread is stable, but significant, meaning we cannot neglect the probability of default of the banks on which the LIBOR is based and thus we cannot express LIBOR rates in terms of risk-free zero-coupon bonds anymore.

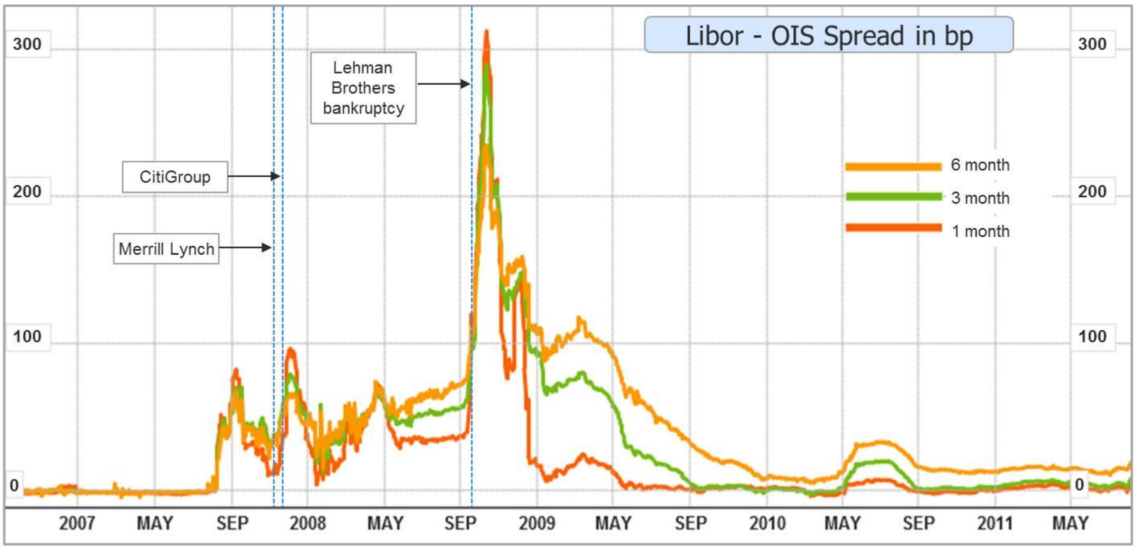

Figure 1.1: Spreads between several LIBOR rates and OIS rates, source: Bloomberg Market Data 2011.

Another consequence of the crisis is that all trades become vulnerable for counterparty risk, i.e. the risk that the counterparty defaults before the maturity of a contract. To get rid of this risk, many companies entered into collateralized contracts.[3] In a collateralized contract the present value of a contract is kept zero during the contract by so-called margin payments of both parties.

We cannot use the price of a risk-free zero-coupon bond to discount a collateral cash flow, as collateral payments need to be paid during the contract. We also cannot express the LIBOR rate in terms of risk-free zero-coupon bonds anymore, as the default probability of the banks on which the LIBOR rate is based is no longer negligible.

The price of a fictive collateral zero-coupon bond is used to discount collateral cash flows and we express the LIBOR rate in terms of fictive risky zero-coupon bonds, depending on the tenor of the index. As these types of bonds are not traded on the market, we need to apply bootstrapping to obtain multiple discount curves corresponding to the prices of zero-coupon bonds.

OIS rates are solely based on collateral zero-coupon bonds and prove to be very helpful instruments to bootstrap the overnight curve corresponding to the prices of collateral zero-coupon bonds. Unfortunately, overnight index swaps in general will have maturities with a maximum of ten years. This means that we will need to bootstrap multiple discount curves simultaneously for maturities over ten years.

To bootstrap the three month discount curve, used to price products depending on the 3 month LIBOR rate, Eurodollar futures are very helpful. These products are very liquid and the market quote of an ED future does not depend on collateral zero-coupon bonds. However, you will need to apply a convexity adjustment to the quote of an ED future in order to implement the future in a bootstrap algorithm. There are several models to approximate the convexity adjustments; some of them take stochastic volatility into account.

Most readers should be familiar with pricing risk-free cash flows using zero-coupon bonds; Appendix A and B will explain this in detail. In Chapter 2, we will explain the connection between LIBOR rates and risk-free zero-coupon bonds as well as the contracts of several interest rate products.

[3]In 2012 83 % of the OTC trade volume is traded collateralized compared to 30 % in 2003 (ISDA Margin Survey).

We will use the pricing formulas of some interest rate products to bootstrap a single discount curve in Chapter 3, using the assumptions made before the start of the crisis. Collateralization and pricing formulas regarding to collateralized products is explained in Chapter 4.

The theory of pricing collateralized products will be used to bootstrap multiple interest rate curves in Chapter 5. Eurodollar futures quotes are used to bootstrap a discount curve corresponding to a tenor of three months. To use Eurodollar futures in our bootstrapping algorithm, we need to compute convexity adjustments. Chapter 6 will give some models to compute the convexity adjustment.

We will end with the conclusion and further research in Chapter 7. Apart from the link between zero-coupon bonds and pricing risk-free cash flows, the appendix will give some conventions and calendars as well as some tests executed. These can be found in Appendix C and D.

Chapter 2

Interest Rate Products

We will explain the contracts of some interest rate products in this chapter and give an example of each contract. This chapter discusses the following topics:

- Preliminaries.
- OTC.
- Deposits.
- Forward Rate Agreement.
- Fixed-Floating Interest Rate Swaps.
- Overnight Index Swaps (OIS).
- Basis Swaps.
- Eurodollar Futures.

The products explained in this chapter will be used throughout the thesis, therefore it is important to fully understand the way these products are defined. In the first section we will explain some preliminaries, which are needed to understand the contracts.

2.1 Preliminaries

In this section we will explain some preliminaries which are needed to fully understand the contracts of the interest rate products in this chapter. We will discuss the following topics this section:

- LIBOR/Euribor Rate.
- Futures Rates.
- Year Fractions.
- Spot Lag.
- Calendar.
- Date Roll Conventions.
- End-Of-Month Convention.
- Reference Rate Conventions.

The dynamics of zero-coupon bonds and pricing formulas of risk-free cash flows can be found in Appendix A and B.

2.1.1 LIBOR/Euribor Rate

The London Interbank Offered Rate (*LIBOR*) and the Euro Interbank Offered Rate (*Euribor*) are daily reference rates based on average interest rates at which banks offer to lend unsecured funds to other banks. The LIBOR is determined by London banks and the Euribor is determined by Eurozone banks. Both LIBOR and Euribor are computed using simple compounding. LIBOR and Euribor rates are published daily for several tenors.

Some interest rate products are based on overnight interest rates. The most used overnight index in Europe is Eonia[1], which is based on overnight interbank transactions in the eurozone. Eonia rates are published on each business day.

In practice, the LIBOR (Euribor) rate is fixed several business days before the start of the accrual period (see Section 2.1). The payment date, on which the swap payments are made may differ to the end of the accrual period. The conventions of LIBOR, Euribor and Eonia rates can be found in Section 2.1.9.

Definition 2.1. *The LIBOR (Euribor) rate, fixing at time $\widetilde{S}$, with accrual period $[S,T]$ and payment time $\widetilde{T}$, is denoted by*

$$L(S,T) := L_{\widetilde{S},\widetilde{T}}(S,T), \tag{2.1}$$

where:

$\widetilde{S}$:	*Fixing time.*
$[S,T]$:	*Accrual period.*
$\widetilde{T}$:	*Payment time.*

Assumption 2.2. *The probability of default of the banks on which the LIBOR and Euribor rates are based can be neglected.*

Theorem 2.3. *The present value of an interest rate $L(S,T)$ is given by*

$$PV(t) = P(t,\widetilde{T})\mathbb{E}_t^{\widetilde{T}}\left(L(S,T)\right). \tag{2.2}$$

Proof. By Theorem B.3 and Definition 2.1. □

Definition 2.4. *We denote the forward LIBOR (Euribor) rate at time t by*

$$\mathbb{E}_t^{\widetilde{T}}\left(L(S,T)\right).$$

By Assumption 2.2 the default probability of the banks on which the rate is based can be neglected. Therefore, we may express the forward LIBOR (Euribor) rate in terms of zero-coupon bonds. We replicate the payoff of a forward LIBOR (Euribor) rate by zero-coupon bonds:

We will write a contract at time $t \leq \widetilde{S}$ which allows us to make an investment of one unit at time S and which gives a deterministic rate of return, determined at time t, over the interval $[S,T]$;

1. At time t sell one zero-coupon bond maturing at S and buy $\frac{P(t,S)}{P(t,T)}$ zero-coupon bonds maturing at T. Our net investment at time t equals zero.
2. At time S we will have to pay one unit.
3. At time T we will receive $\frac{P(t,S)}{P(t,T)}$ units.

This means an investment of one unit at time S will give us a profit of $\frac{P(t,S)}{P(t,T)} - 1$ units. The simple forward rate $\mathbb{E}_t^{\widetilde{T}}\left(L(S,T)\right)$ is then given by the next definition.

Definition 2.5. *Under Assumption 2.2, we express the forward LIBOR (Euribor) rate in terms of risk-free zero-coupon bonds by*

$$\mathbb{E}_t^{\widetilde{T}}\left(L(S,T)\right) = \frac{1}{\tau}\left(\frac{P(t,S)}{P(t,T)} - 1\right), \qquad t \leq \widetilde{S} \leq S \leq T, \tag{2.3}$$

where τ is the year fraction between S and T. Note that $\frac{1}{\tau}\left(\frac{P(t,S)}{P(t,T)} - 1\right)$ is a martingale associated with the numeraire $P(t,T)$ and there will be a small convexity adjustment to the measure associated with the numeraire $P(t,\widetilde{T})$. In practice, this difference turns out to be very small.

[1] Euro OverNight Index Average.

2.1.2 Futures Rates

A futures contract is very much like a forward contract as it is an agreement between two parties on a claim depending on the LIBOR (Euribor) rate at a prespecified time in the future. In Section 2.8 we will explain a Eurodollar futures contract. The difference between the forward rate and the futures rate is called *convexity adjustment*.

Definition 2.6. *We define the futures rate and the convexity adjustment;*

- *The futures rate at time t of a LIBOR (Euribor) rate fixing at time S and maturing at time T is defined by*
$$F(t; S, T). \tag{2.4}$$

- *The convexity adjustment between a futures rate $F(t; S, T)$ and a forward rate $\mathbb{E}_t^{\widetilde{T}}\left(L(t; S, T)\right)$ is defined by*
$$CA(t; S, T) = F(t; S, T) - \mathbb{E}_t^{\widetilde{T}}\left(L(t; S, T)\right). \tag{2.5}$$

2.1.3 Year Fractions

Throughout this thesis we will use the phrase *time* a lot. Unless otherwise stated we assume all times to be in year fractions. The year fraction between two year fractions is just the difference between the two fractions. To compute the year fraction between two dates, we will have to make conventions. The most easy day count convention is the *Actual/365* convention. Using the Actual/365 day count convention the difference between dates D_1 and D_2 is defined by

$$\frac{\#\text{ days between } D_1 \text{ and } D_2}{365}.$$

In Appendix C.1 we explain some more day count conventions.

2.1.4 Spot Lag

For financial institutions, usually there will be a delay between the day at which the rate is fixed and the date at which the accrual period starts. This delay is introduced because many trades are between several parties mostly living in different time-zones.

The delay between those days is called *spot lag* and is generally two business days.

2.1.5 Critical Dates for Interest Rate Swaps

The *trade date* is the date on which the parties agree the terms of a contract. Each payment is fixed at the *fixing date*, accrued over an accrual period, and paid at the *payment date*. The start of the first accrual period is called *effective date* and the last payment is at the *maturity date*.

2.1.6 Calendar

Business days are normally weekdays, with the exception of some holidays. The *calendar* tells us whether a day is a business day or a non-business day. For simplicity reasons, one could use a no holiday calendar as well, meaning every day will be a business day, ignoring weekends. In Appendix C.2 we show some more calendars.

2.1.7 Date Roll Convention

It may occur that a payment date, start date or end date of an accrual period falls on a non-business day. This means that this date has to be shifted to a day which is a business day. We need to make a convention to decide to which business day we should move our original date. This convention is called *date roll convention*. The most easy day count convention is *Following*, which means we shift our date to the next business day. Check Appendix C.3 for more conventions.

2.1.8 End-Of-Month Convention

For swaps, payments usually are periodically. Not every month has the same number of days and some parties may want that each accrual period starts at the end of each month. In case the effective date is at the end of a month, the *end-of-month convention* will set the start of all accrual periods at the last business day of the corresponding month as well. Remark this only holds for monthly payments or periodically payments of multiple months.

2.1.9 Reference Rate Conventions

In section 2.1.1 we mentioned the existence of LIBOR, Euribor and Eonia as reference rates. Each index uses its own conventions. In our thesis we will use the USD LIBOR, Eonia and Euribor indices. In Table 2.1 we show the conventions of these indices, the *settlement delay* is a delay of the payments in business days.

	Day count convention	Spot lag	Date roll convention	End-of-month rule	Calendar	Settlement delay
USD LIBOR	Actual/360	Two days	Modified Following	Applies	London	None
Euribor	Actual/360	Two days	Modified Following	Applies	Target	None
Eonia	Actual/360	None	n/a	n/a	Target	None

Table 2.1: Reference rate conventions.

2.2 OTC

Most products treated in this chapter are traded OTC, which stands for *Over The Counter*. OTC trading is done directly between two parties, without any supervision of a clearing house. This means the liquidity of these products is less compared to exchange trading. OTC products also encounter counterparty risk, as this risk is no longer taken care of by the clearing house.

2.3 Deposits

Deposits are standard money market zero-coupon contracts. One party pays a nominal amount to the counterparty at the effective date (trade date plus spot lag) and receives in exchange the nominal amount plus the interest earned over the period at the maturity date. The interest earned over the period is determined by the nominal amount times the fixed rate times the year fraction between the effective date and the maturity date. The fixed rate is determined by the value of an underlying index with the same tenor. An example of a deposits contract is given in Table 2.2.

Deposits Contract	
Trade date	4-4-2013
Maturity date	4-7-2013
Fixed rate	2%
Underlying index	Euribor3M
Notional value	EUR 100
Spot lag	2 days
Day Count Convention	Act/365
Calendar	Monday-to-Friday
Date Roll Convention	Mod. Following
End-Of-Month Convention	not used

Table 2.2: Example of a deposits contract (dates are dd-mm-yyyy)

2.4 Forward Rate Agreement

In a forward rate agreement two parties agree on a single fixed-floating cash flow on a pre-determined date in the future. Both counterparties agree to pay either a fixed or a floating rate to each other. The fixed rate is given in the contract and the floating rate will be an underlying index quoted at the fixing date. The payment will be at the effective date instead of the maturity date and will consist of the difference between the fixed and floating rate times the nominal value of the contract times the year fraction between the effective date and maturity spot date. As the payment is at the start of the period, it will be discounted to the effective date. An example of an FRA-contract is given in Table 2.3. The price of an FRA is zero, the fixed rate is determined such that the present value of the FRA will be zero.

Deposits Contract	
Trade date	4-4-2013
Effective date	15-6-2013
Maturity date	15-9-2013
Fixed rate	2%
Underlying index	Euribor3M
Notional value	EUR 100
Spot lag	2 days
Day Count Convention	Act/365
Calendar	Monday-to-Friday
Date Roll Convention	Mod. Following
End-Of-Month Convention	not used

Table 2.3: Example of an FRA-contract (dates are dd-mm-yyyy)

2.5 Fixed-Floating Interest Rate Swaps

A fixed-floating IRS consists of two counterparties. Both counterparties agree to pay either a fixed or floating rate. The fixed or floating rate is multiplied by a notional amount (say EUR 100) and an accrual factor. The accrual factor denotes the year fraction between the payment and the preceding payment, given by the appropriate day count convention.

For both legs, the frequency of the payments, the spot lag, a calendar and some conventions are defined in the contract. The contract also includes the trade date, effective date and maturity date. An example of a contract is given in Table 2.4. The price of a fixed-floating IRS is zero, the fixed rate is determined such that the present value of the swap will be zero.

IRS Contract		
Trade date	4-4-2013	
Effective date	15-6-2013	
Maturity date swap	15-6-2014	
Fixed rate	2%	
Underlying index	Euribor3M	
Notional value	EUR 100	
	Fixed leg	Floating leg
Frequency	Semi-annual	Quarterly
Spot Lag	2 days	2 days
Day Count Convention	Act/365	Act/365
Calendar	Monday-to-Friday	Monday-to-Friday
Date Roll Convention	Mod. Following	Mod. Following
End-Of-Month Convention	not used	not used

Table 2.4: Example of an IRS contract (dates are dd-mm-yyyy)

2.6 Overnight Index Swap (OIS)

Overnight index swaps play an important role in defining the discount curve. Overnight index swaps usually have maturities shorter than two year. The swaps with maturities no longer than a year normally have one payment at maturity, while swaps with a maturity over a year normally have yearly payments. Overnight index swaps also have a floating leg and a fixed leg. The floating rate is determined by several overnight rates, the first overnight rate is taken at the effective date and the last rate is taken at the last business day prior to the maturity date. The payment for both legs is defined by the floating resp. fixed rate times the year fraction between the spot date and maturity date. Both payments are paid at the same time and both legs use the same conventions. The price of an OIS is zero, the fixed rate is determined such that the present value of the swap will be zero.

Assuming we pay fixed and receive floating, the payment at maturity plus settlement delay will be

$$\text{payment}(\widetilde{T}_i) = \tau_i(\overline{L}_i - k), \tag{2.6}$$

where:

τ_i:	Accrual factor (year fraction) of i'th leg.[2]
$[S_i, T_i]$:	Accrual period of i'th payment.
$\widetilde{T}_i$:	Payment time of i'th payment.
k:	Fixed rate.

The compounded overnight floating rate $\overline{L}_i$ is defined by

$$\overline{L}_i = \frac{1}{\tau_i}\left(\prod_{j=0}^{K_i-1}\Big(1 + \alpha(t_{i,j}, t_{i,j+1})L(t_{i,j}, t_{i,j+1})\Big) - 1\right), \tag{2.7}$$

where:

$\{t_{i,j}\}_{j=0}^{K_i}$:	The collection of all business days in the accrual period $[S_i, T_i]$.[3]
$\alpha(t_{i,j}, t_{i,j+1})$:	Year fraction between business days $t_{i,j}$ and $t_{i,j+1}$.
$L(t_{i,j}, t_{i,j+1})$:	Overnight interest rate at $t_{i,j}$.[4]

An example of an OIS contract is given in Table 2.5.

OIS Contract	
Trade date	4-4-2013
Effective date	15-6-2013
Maturity date swap	15-6-2014
Fixed rate	2%
Overnight index	Eonia
Settlement delay	1 day
Notional value	EUR 1000
Frequency	Annual
Spot Lag	2 days
Day Count Convention	Act/360
Calendar	Monday-to-Friday
Date Roll Convention	Mod. Following
End-Of-Month Convention	not used

Table 2.5: Example of an OIS contract (dates are dd-mm-yyyy)

2.7 Basis Swaps

Like fixed-floating IRS and overnight index swaps, basis swaps also have two counterparties. One party agrees to pay a floating rate with a long tenor and the other party agrees to pay a floating rate with short tenor plus a fixed spread. This spread should be positive. An example of a basis swap contract is given in Table 2.6. The price of a basis swap is zero, the fixed rate is determined such that the present value of the basis swap will be zero.

[2] $\tau_i = \tau(T_{i-1}, T_i)$.
[3] $S_i = t_{i,0} < \ldots < t_{i,K_i} = T_i$.
[4] $L(t_{i,j}, t_{i,j+1}) := L_{t_{i,j}, t_{i,j+1}}(t_{i,j}, t_{i,j+1})$.

Basis Swap Contract		
Trade date	4-4-2013	
Effective date	15-6-2013	
Maturity date swap	15-6-2014	
Notional value	EUR 1000	
Spread	2%	
	Leg A	Leg B
Index	Euribor1M	Euribor3M
Frequency	Monthly	Quarterly
Spot Lag	2 days	2 days
Day Count Convention	Act/360	Act/360
Calendar	Monday-to-Friday	Monday-to-Friday
Date Roll Convention	Mod. Following	Mod. Following
End-Of-Month Convention	not used	not used

Table 2.6: Example of a Basis Swap Contract (dates are dd-mm-yyyy)

2.8 Eurodollar Futures

Eurodollars are deposits denominated in US dollars at banks outside the United States. There is no connection between the Euro currency, the euro- prefix is derived from the fact that the first dollars outside the United States were traded in Europe. Similar to Eurodollar, the terms EuroSwiss and EuroYen are used for denoting deposits of Swiss Francs resp. Japanese Yens outside Switzerland resp. Japan. We will focus on the futures market of the Eurodollars.
Financial futures are traded on an exchange. This is different compared to interest rate products discussed in this chapter, as these products are traded OTC. This means futures have the benefit of a clearing house and there will be no influence of counterparty risk. Eurodollar futures are traded on several exchanges, like the NYSE Liffe and the CME.

Eurodollar futures are daily collateralized (see Section 4.1), meaning both parties will have to pay a margin on each business day. Eurodollar futures are based on the USD LIBOR rates, with a tenor of three months. ED futures contracts expire on the third Wednesday of several months, namely March, June, September and December for the next five years plus one additional expiration plus the two nearest months that are not in the March, June, September, December cycle. This means there is a total of 23 ED futures contracts available on the market. The last trading day of a contract is the second business day prior to the underlying start date. An example of an ED futures contract is given in Table 2.7. The price of an ED futures contract is zero.

Eurodollar Futures Contract	
Trade date	4-4-2013
Underlying start date	17-4-2013
Underlying end date	17-7-2013
Quote	99.825%
Underlying index	USD LIBOR3M
Notional value	USD 1,000,000
Margin payment per change of 1 bp	USD 25

Table 2.7: Example of an ED futures contract (dates are dd-mm-yyyy)

One may notice there is no calendar, date roll convention or end-of-month convention given in the contract. This is because the underlying start date and underlying end date of a contract will be on weekdays and there is no possibility that a United States bank holiday will be on the same day as the underlying start or end date. Therefore we don't need to worry about calendar or date-roll conventions.

The quote of an ED futures contract is defined by one minus the implied futures rate:

$$\text{Quote} = 1 - F(t; S, T), \tag{2.8}$$

where $F(t; S, T)$ is the implied futures rate on time t of an underlying index fixing at time S and maturing at time T.

The margin payment per change of 1 bp will give us the accrual factor used for ED futures. A quote change of 1 bp means a change of 1 bp in the implied futures rate. This would result in a payment of $1,000,000 \cdot 0.0001 \cdot \alpha$ USD, where α is the accrual factor. This means a margin payment per change of 1 bp of USD 25 corresponds to an accrual factor of 0.25 for an ED futures contract, independent of the number of days in the accrual period. This is also the reason there is no day count convention given in the contract.

Chapter 3

Single Curve Approach

Before the start of the crisis in 2007, the default probability of big banks could be neglected. Therefore Assumption 2.2 held and interest rate products could be priced using the prices of risk-free zero-coupon bonds, as explained in Section 2.1.1. In practice, however, only a few short-dated risk-free zero-coupon bonds are quoted in the market. We will use the market quotes of several interest rate products to obtain a discount curve. We may use this discount curve to price new interest rate products or to discount cash flows. The technique to obtain this curve is called *bootstrapping*.

To perform bootstrapping and to price new interest rate products, we need to explain the pricing formulas of some interest rate products. We will explain the pricing formulas of several interest rate products, which contracts are explained in Chapter 2.

We will start by explaining the pricing formulas of some interest rate products and will continue by bootstrapping a single discount curve, using some market quotes of interest rate products. This chapter discusses the following topics:

- Pricing Interest Rate Products.
- Single Curve Bootstrapping.

3.1 Pricing Interest Rate Products.

In this section, we will explain the pricing formulas of several interest rate products. To verify our formulas, we performed some tests to price some products. These tests can be found in Appendix D. This section discusses the following topics:

- Pricing Fixed-Floating IRS.
- Pricing an OIS.
- Pricing a Basis Swap.

3.1.1 Pricing Fixed-Floating IRS

In Section 2.5 we explained the contract of a fixed-floating IRS. In order to price such a swap, we use Theorem 2.3 to compute the present value of all the payments. We assume we pay fixed and receive floating and thus the present value of the swap is given by

$$PV(t) = \sum_{i=1}^{N} \alpha_i \mathbb{E}_t^{\widetilde{T}_i} \left(L(S_i, T_i) \right) P(t, \widetilde{T}_i) - k \sum_{j=1}^{M} \tau_j P(t, t_j), \tag{3.1}$$

where:

α_i, τ_i:	Accrual factors (year fractions) of i'th floating resp. fixed leg.[1]
N, M:	Number of floating resp. fixed payments.
$\widetilde{T}_i$, t_i:	Time of i'th floating resp. fixed payment.
$\mathbb{E}_t^{\widetilde{T}_i} \left(L(S_i, T_i) \right)$:	The forward Euribor rate.[2]
k:	Fixed rate.

We use Definition 2.5 to express the forward LIBOR (Euribor) rate in terms of zero-coupon bonds:

$$\mathbb{E}_t^{\widetilde{T}_i}\left(L(S_i,T_i)\right) = \frac{1}{\widetilde{\alpha}_i}\left(\frac{P(t,S_i)}{P(t,T_i)} - 1\right), \tag{3.2}$$

where $\widetilde{\alpha}_i$ is the accrual factor (year fraction) between S_i and T_i.[3]

The market only trades interest rate swaps with zero present value, the fixed rate k is determined by setting the present value at zero. This fixed rate is called the swap rate or par rate. Using (3.1), (3.2) and setting $PV(t)$ to zero gives us the swap rate k:

$$k = \frac{\sum_{i=1}^N \alpha_i \mathbb{E}_t^{\widetilde{T}_i}\left(L(S_i,T_i)\right) P(t,\widetilde{T}_i)}{\sum_{j=1}^M \tau_j P(t,t_j)} = \frac{\sum_{i=1}^N \frac{\alpha_i}{\widetilde{\alpha}_i}\left(\frac{P(t,S_i)}{P(t,T_i)} - 1\right) P(t,\widetilde{T}_i)}{\sum_{j=1}^M \tau_j P(t,t_j)}. \tag{3.3}$$

We used our open source software OpenGamma to compute the present value of a fixed-floating IRS. The results can be found in Appendix D.1.

3.1.2 Pricing an OIS

We will assume that there is no settlement delay, i.e. the end date of the accrual period is the same date as the payment date.[4] We will assume as well that the accrual periods will fit to eachother.[5] We assume we pay fixed and receive floating and thus the present value of the swap is given by

$$PV(t) = \sum_{i=1}^N \tau_i \mathbb{E}_t^{T_i}\left(\overline{L}_i - k\right) P(t,T_i). \tag{3.4}$$

where:

N: Number of payments.

We rewrite $\overline{L}_i$, using $P(t,T) = P(t,S)P(S,T)$ and Definition 2.5 :

$$\begin{aligned}
\overline{L}_i &= \frac{1}{\tau_i}\left(\prod_{j=0}^{K_i-1}\Big(1+\alpha(t_{i,j},t_{i,j+1})L(t_{i,j},t_{i,j+1})\Big) - 1\right) \\
&= \frac{1}{\tau_i}\left(\prod_{j=0}^{K_i-1}\left(\frac{1}{P(t_{i,j},t_{i,j+1})}\right) - 1\right) \\
&= \frac{1}{\tau_i}\left(\frac{1}{P(t_{i,0},t_{i,K_i})} - 1\right) \\
&= \frac{1}{\tau_i}\left(\frac{1}{P(T_{i-1},T_i)} - 1\right).
\end{aligned} \tag{3.5}$$

Now we have

$$\tau_i \mathbb{E}_t^{T_i}(\overline{L}_i - k)P(t,T_i) = P(t,T_{i-1}) - P(t,T_i) - \tau_i k P(t,T_i) \tag{3.6}$$

Now, filling in (3.6) in (3.4) gives us

$$\begin{aligned}
PV(t) &= \sum_{i=1}^N \Big(P(t,T_{i-1}) - P(t,T_i)\Big) - k\sum_{i=1}^N \tau_i P(t,T_i) \\
&= P(t,T_0) - P(t,T_N) - k\sum_{i=1}^N \tau_i P(t,T_i).
\end{aligned} \tag{3.7}$$

[1] $\alpha_i = \alpha(\widetilde{T}_{i-1},\widetilde{T}_i)$ and $\tau_i = \tau(t_{i-1},t_i)$.

[2] $[S_i,T_i]$ is the accrual period of the index. Normally $S_i = \widetilde{T}_{i-1}$ and T_i^I is S_i^I shifted forward by the tenor, adjusted by date roll conventions.

[3] $\widetilde{\alpha}_i = \widetilde{\alpha}(S_i,T_i)$.

[4] I.e. we use $\widetilde{T}_i = T_i$ for the notation in Section 2.6.

[5] I.e. we use $S_i = T_{i-1}$ for the notation in Section 2.6.

For overnight index swaps also holds that the market only shows swap rates, assuming the present value of the swap to be zero. Therefore we are able to compute the OIS rate k by setting (3.7) to zero:

$$k = \frac{P(t,T_0) - P(t,T_N)}{\sum_{i=1}^{N} \tau_i P(t,T_i)}. \tag{3.8}$$

We used our open source software OpenGamma to compute the present value of an OIS. The results can be found in Appendix D.2.

3.1.3 Pricing a Basis Swap

We assume we pay a floating rate with a short tenor plus a fixed spread and receive a floating rate with a long tenor. "A" denotes the leg paying the rate with short tenor plus fixed spread and "B" denotes the leg paying the rate with long tenor. Then the present value of our swap is given by

$$PV(t) = \sum_{i=1}^{N} \alpha_i \mathbb{E}_t^{\widetilde{T}_i^B}(L(S_i^B, T_i^B))P(t,\widetilde{T}_i^B) - \sum_{j=1}^{M} \tau_j \left(\mathbb{E}_t^{\widetilde{T}_i^A}(L(S_j^A, T_j^A)) + s\right) P(t,\widetilde{T}_j^A), \tag{3.9}$$

where:

α_i, τ_i:	Accrual factors (year fractions) of i'th payment of leg B resp. leg A.[6]
N, M:	Number of payments of leg B resp. leg A.
$\widetilde{T}_i^B$, $\widetilde{T}_i^A$:	Time of i'th payment of leg B resp. leg A.
$\mathbb{E}_t^{\widetilde{T}_i^B}(L(S_i^B, T_i^B))$:	The forward LIBOR (Euribor) rate of leg B.[7]
$\mathbb{E}_t^{\widetilde{T}_i^A}(L(S_i^A, T_i^A))$:	The forward LIBOR (Euribor) rate of leg A.[8]
s:	Fixed spread.

We use Definition 2.5 to express the forward LIBOR (Euribor) rates in terms of zero-coupon bonds:

$$\mathbb{E}_t^{\widetilde{T}_i^A}(L(S_i^A, T_i^A)) = \frac{1}{\widetilde{\tau}}\left(\frac{P(t,S_i^A)}{P(t,T_i^A)} - 1\right), \tag{3.10}$$

$$\mathbb{E}_t^{\widetilde{T}_i^B}(L(S_i^B, T_i^B)) = \frac{1}{\widetilde{\tau}}\left(\frac{P(t,S_i^B)}{P(t,T_i^B)} - 1\right). \tag{3.11}$$

where:

$\widetilde{\tau}_i$, $\widetilde{\alpha}_i$:	Accrual factors (year fractions) between S_i^A and T_i^A resp. S_i^B and T_i^B.[9]

Market quotes for basis swaps are given in spreads, assuming the present value of the swap to be zero. Using (3.9), (3.10), (3.11) and setting $PV(t)$ to zero will give us the spread s:

$$\begin{aligned} s &= \frac{\sum_{i=1}^{N} \alpha_i \mathbb{E}_t^{\widetilde{T}_i^B}(L(S_i^B, T_i^B))P(t,\widetilde{T}_i^B) - \sum_{j=1}^{M} \tau_j \mathbb{E}_t^{\widetilde{T}_i^A}(L(S_j^A, T_j^A))P(t,\widetilde{T}_j^A)}{\sum_{j=1}^{M} \tau_j P(t,\widetilde{T}_j^A)} \\ &= \frac{\sum_{i=1}^{N} \frac{\alpha_i}{\widetilde{\alpha}_i}\left(\frac{P(t,S_i^B)}{P(t,T_i^B)} - 1\right) P(t,\widetilde{T}_i^B) - \sum_{j=1}^{M} \frac{\tau_j}{\widetilde{\tau}_j}\left(\frac{P(t,S_j^A)}{P(t,T_j^A)} - 1\right) P(t,\widetilde{T}_j^A)}{\sum_{j=1}^{M} \tau_j P(t,\widetilde{T}_j^A)}. \end{aligned} \tag{3.12}$$

We used our open source software OpenGamma to compute the present value of a basis swap. The results can be found in Appendix D.3.

[6] $\alpha_i = \alpha(\widetilde{T}_{i-1}^B, \widetilde{T}_i^B)$ and $\tau_i = \tau(\widetilde{T}_{i-1}^A, \widetilde{T}_i^A)$.

[7] $[S_i^B, T_i^B]$ is the accrual period of the index of leg B. Normally $S_i^B = \widetilde{T}_{i-1}^B$ and T_i^B is S_i^B shifted forward by the tenor, adjusted by date roll conventions.

[8] $[S_i^A, T_i^A]$ is the accrual period of the index of leg A. Normally $S_i^A = \widetilde{T}_{i-1}^A$ and T_i^A is S_i^A shifted forward by the tenor, adjusted by date roll conventions.

[9] $\widetilde{\tau}_i = \widetilde{\tau}(S_i^A, T_i^A)$ and $\widetilde{\alpha}_i = \widetilde{\alpha}(S_i^B, T_i^B)$.

3.2 Single Curve Bootstrapping

There are only a finite set of securities quoted in the market. Therefore we cannot use a continuous curve as input to our system, as this would result in a finite number of equations having infinitely unknowns, making our system unsolvable. We want to make this system solvable and still obtain a continuous curve. Therefore, we will use as many nodal points as market quotes available on the market and use an interpolation method to obtain a continuous discount curve. Our system is solved if we have a discount curve, constructed by interpolating nodal points, which makes the present value of all interest rate products equal to zero.

This section discusses the following topics:

- Interpolation Techniques.
- Bootstrapping.
- Forwards.

We will start by defining some interpolation techniques in Section 3.2.1. We will apply these techniques in Section 3.2.2 to an example to bootstrap a discount curve. In Section 3.2.3, we will use this curve to obtain forward LIBOR (Euribor) rates, which can be used to price new interest rate products.

3.2.1 Interpolation Techniques

In this section, we will explain two different interpolation techniques, namely *linear interpolation* and *natural cubic spline interpolation*. The price of a risk-free zero-coupon bond in terms of continuous compounded yield over the same period can be found in Appendix A. We will use our discount curve as risk-free zero-coupon bond price. We use the same notation as Appendix A, this means we will denote the discount curve by $P(\cdot,\cdot)$ and the yield curve by $R(\cdot,\cdot)$. We use Lemma A.4, which expresses the relation between the discount curve and yield curve by

$$P(S,T) = \exp\{-R(S,T)(T-S)\}, \qquad S \leq T, \tag{3.13}$$

where S and T are year fractions. We will define $P(t)$ and $R(t)$ by

$$P(t) \equiv P(0,t) \tag{3.14}$$

$$R(t) \equiv R(0,t). \tag{3.15}$$

Instead of interpolating nodal points of the discount curve $P(\cdot,\cdot)$, we will apply interpolation techniques on $R(\cdot,\cdot)$ and $\log\{P(S,T)\}$. We will convert these curves to $P(\cdot,\cdot)$ and use them as input in our system. The reason we interpolate on $R(\cdot,\cdot)$ and $\log\{P(S,T)\}$ is because this is applied more common and we want to compare our results in the next sections with an example of [1, Ch.6], in which they use the same techniques.

We divided this section in three subsections:

- Linear Interpolation on the Yield Curve.
- Linear Interpolation on the Log Discount Curve.
- Natural Cubic Spline Interpolation.

Linear Interpolation on Yield Curve

We apply linear interpolation between nodal points and flat extrapolation outside nodal points. Assume we have a collection of nodal points $\{T_i\}_{i=1}^N$ with nodal values $\{R(T_i)\}_{i=1}^N$. We will draw a straight line between two successive nodal points, i.e.:

$$R(t) = (t - T_i)\frac{R(T_{i+1}) - R(T_i)}{T_{i+1} - T_i} + R(T_i), \qquad t \in [T_i, T_{i+1}). \tag{3.16}$$

Points outside the nodal points will be flat extrapolated:

$$R(t) = R(T_1), \qquad t < T_1 \tag{3.17}$$

$$R(t) = R(T_N), \qquad t \geq T_N. \tag{3.18}$$

We will convert $R(\cdot)$ to $P(\cdot)$ by (3.13), which we will use in our system.

Linear Interpolation on Log Discount Curve

We applied linear interpolation between nodal points and flat extrapolation outside nodal points.

Again, assume we have a collection of nodal points $\{T_i\}_{i=1}^N$, this time with nodal values $\{\log P(T_i)\}_{i=1}^N$. We will draw a straight line between two successive points in the collection $\{\ln P(T_i)\}_{i=1}^N$ and points outside the nodal points will be flat extrapolated:

$$\log P(t) = (t - T_i)\frac{\log P(T_{i+1}) - \log P(T_i)}{T_{i+1} - T_i} + \log P(T_i), \qquad t \in [T_i, T_{i+1}) \tag{3.19}$$

$$\log P(t) = \log P(T_1), \qquad t < T_1 \tag{3.20}$$

$$\log P(t) = \log P(T_N), \qquad t \geq T_N. \tag{3.21}$$

The yield curve will be constructed by converting the log discount curve to yield curve by

$$R(t) = \frac{\log P(t)}{-t}. \tag{3.22}$$

The discount curve which we will use in our system is obtained by

$$P(t) = \exp\{\log\{P(t)\}\}. \tag{3.23}$$

Natural Cubic Spline Interpolation

This technique assumes the curve to be a cubic spline. A cubic spline is a smooth polynomial function of degree three that is piecewise-defined.

Assume we have a collection of nodal points $\{x_i\}_{i=1}^N$ with nodal values $\{y_i\}_{i=1}^N$. We denote the function between point i and point $i+1$ by $f_i(x)$. The cubic spline $f(x)$ is then defined by

$$f(x) = f_i(x), \qquad x \in [x_i, x_{i+1}). \tag{3.24}$$

$f_i(x)$ is a function of degree three, therefore we may express it as

$$f_i(x) = \alpha_i x^3 + \beta_i x^2 + \gamma_i x + \delta_i. \tag{3.25}$$

In order for $f_i(x)$ to pass through (x_i, y_i) and (x_{i+1}, y_{i+1}) we must have

$$\begin{aligned} f_i(x_i) &= y_i \\ f_i(x_{i+1}) &= y_{i+1}. \end{aligned} \tag{3.26}$$

The cubic spline needs to be smooth; this means $f(x)$ needs to have derivatives of all orders. By (3.24), (3.25) and (3.26) this reduces to

$$\begin{aligned} f_i'(x_{i+1}) &= f_{i+1}'(x_{i+1}) \\ f_i''(x_{i+1}) &= f_{i+1}''(x_{i+1}). \end{aligned} \tag{3.27}$$

Requiring the cubic spline to turn into a straight line at x_1 and x_N, i.e.

$$f_1''(x_1) = 0 \tag{3.28}$$

$$f_{N-1}''(x_N) = 0, \tag{3.29}$$

will make the system solvable. We used OpenGamma to solve this system and compute $f(x)$ for $x \in [x_1, x_N]$.

We applied this to $R(\cdot)$ and to $\log\{P(\cdot)\}$ and obtained the yield and discount curves by (3.13), (3.22) and (3.23). We used flat left extrapolation and linear right extrapolation on the yield curve.

3.2.2 Bootstrapping

We will explain the principle of bootstrapping by showing an example. We will follow the same example as [1, Ch.6]; we will use the market quotes of fixed-floating IRS, these are shown in Table 3.1. In [1, Ch.6] they do not explain which conventions they used. We did use our own conventions, shown in the contract in Table 3.2.

Maturity (Years)	Swap Par Rate
1	4.20%
2	4.30%
3	4.70%
5	5.40%
7	5.70%
10	6.00%
12	6.10%
15	5.90%
20	5.60%
25	5.55%

Table 3.1: Swap rates [1, Ch6]

IRS Contract		
Trade date	13-3-2013	
Effective date	13-3-2013	
Notional value	EUR 100	
	Fixed leg	Floating leg
Frequency	Semi-annual	Semi-annual
Spot Lag	0 days	0 days
Day Count Convention	Act/365	Act/365
Calendar	No Holiday	No Holiday
Date Roll Convention	n/a	n/a
End-Of-Month Convention	not used	not used

Table 3.2: IRS contract used for bootstrapping using the swap rates in Table 3.1 (dates are dd-mm-yyyy)

Recall by (3.1) and (3.2) that under the assumption that LIBOR (Euribor) rates are risk-free[10], we may express the present value of a fixed-floating IRS by

$$PV(t) = \sum_{i=1}^{N} \frac{\alpha_i}{\widetilde{\alpha}_i} \left(\frac{P(t,S_i)}{P(t,T_i)} - 1 \right) P(t,\widetilde{T}_i) - k \sum_{j=1}^{M} \tau_j P(t,t_j). \tag{3.30}$$

We use ten products to bootstrap the discount curve; this means we will have a system of ten equations. The market quotes of the fixed-floating IRS are quoted with zero present value. So we will need to solve the following system of equations, where we use $t = 0$

$$\text{for } l = 1 \dots 10 \qquad 0 = \sum_{i=1}^{N_l} \frac{\alpha_{l,i}}{\widetilde{\alpha}_{l,i}} \left(\frac{P(S_{l,i})}{P(T_{l,i})} - 1 \right) P(\widetilde{T}_{l,i}) - k_l \sum_{j=1}^{M_l} \tau_{l,j} P(t_{l,j}), \tag{3.31}$$

where:

l:	Denotes the l'th swap of Table 3.1.
k_l:	Swap rate of the l'th swap.
N_l, M_l:	Number of payments of the floating resp. fixed leg of the l'th swap.
$\alpha_{l,i}$, $\widetilde{\alpha}_{l,i}$, $\tau_{l,i}$:	Accrual factors (year fractions) of the l'th swap.[11]
$\widetilde{T}_{l,i}$, $t_{l,i}$:	Time of the i'th floating resp. fixed payment of the l'th swap.
$[S_{l,i}, T_{l,i}]$:	Accrual period of the i'th floating payment of the l'th swap.

To solve this system, we will use ten nodal points and an interpolation method to construct a discount curve $P(\cdot)$. Now, we may express $P(t)$ as a function using ten nodal points as input:

$$P(t) = f(t; n_1, \dots, n_{10}), \tag{3.32}$$

where $n_1, \dots, n_{10}$ are the values of our nodal points.

[10] I.e. the default probability of banks on which the LIBOR (Euribor) rates are based can be neglected.

Now, our system of equations (3.31) becomes solvable and will be

$$\text{for } l = 1 \ldots 10 \qquad 0 = \sum_{i=1}^{N_l} \frac{\alpha_{l,i}}{\widetilde{\alpha}_{l,i}} \left(\frac{f(S_{l,i}; n_1, \ldots, n_{10})}{f(T_{l,i}; n_1, \ldots, n_{10})} - 1 \right) f(\widetilde{T}_{l,i}; n_1, \ldots, n_{10}) - k_l \sum_{j=1}^{M_l} \tau_{l,j} f(t_{l,j}; n_1, \ldots, n_{10}). \tag{3.33}$$

We solved this system using OpenGamma using linear interpolation on the yield curve and on the log discount curve, as discussed in Section 3.2.1 and 3.2.1, where we used *Actual/Actual ISDA* as day count convention to determine the year fractions of $S_{l,i}$, $T_{l,i}$ and $t_{l,i}$. We plotted the yield curves $R(t) = \frac{\log P(t)}{-t}$ in Figure 3.1 and 3.2.

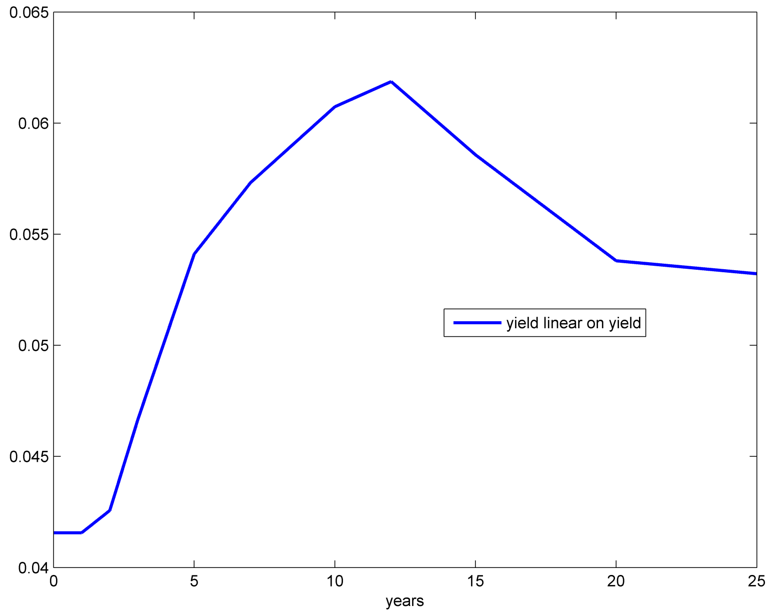

Figure 3.1: Plot of yield curve bootstrapped using linear interpolation on the yield curve, see Section 3.2.1. Swap rates in Table 3.1.

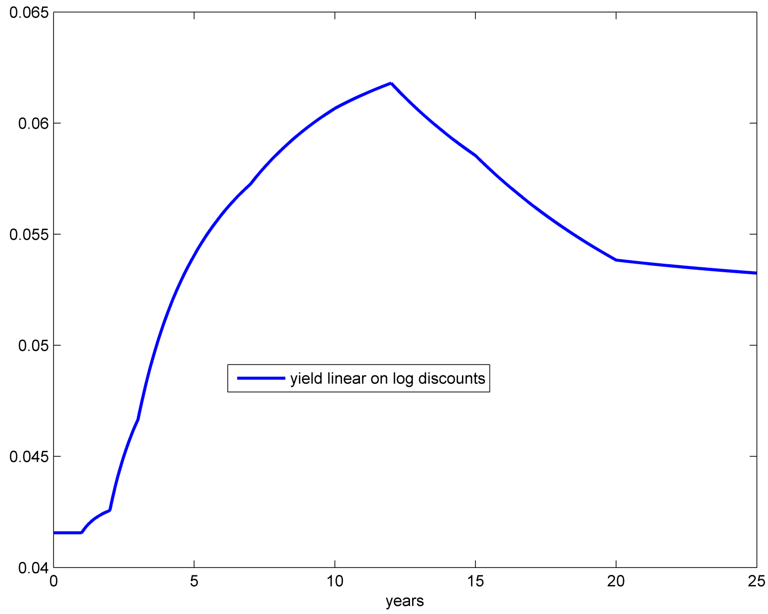

Figure 3.2: Plot of yield curve bootstrapped using linear interpolation on the log discounts, see Section 3.2.1. Swap rates in Table 3.1.

Both curves don't seem to differ a lot. However, in the next section we will see that the forward rates constructed by these yield curves will differ.

3.2.3 Forwards

We will use the yield curve bootstrapped in the previous section to compute forward LIBOR (Euribor) rates.

We defined the forward LIBOR (Euribor) rates at $t = 0$ by

$$\mathbb{E}_0^{\widetilde{T}}\left(L(S,T)\right) = \frac{1}{\tau}\left(\frac{P(S)}{P(T)} - 1\right). \tag{3.34}$$

The instantaneous forward rate is defined in Definition A.3, which is in fact a forward rate over an infinitesimal interval. The instantaneous forward rate over the interval $[0,t]$ is given by

$$f(t) = f(0,t) = -\frac{\partial \log P(t)}{\partial t} = R(t) + \frac{\partial R(t)}{\partial t}t. \tag{3.35}$$

We plotted the instantaneous forward rates of the yield curves of the last section in Figure 3.3 and 3.4.

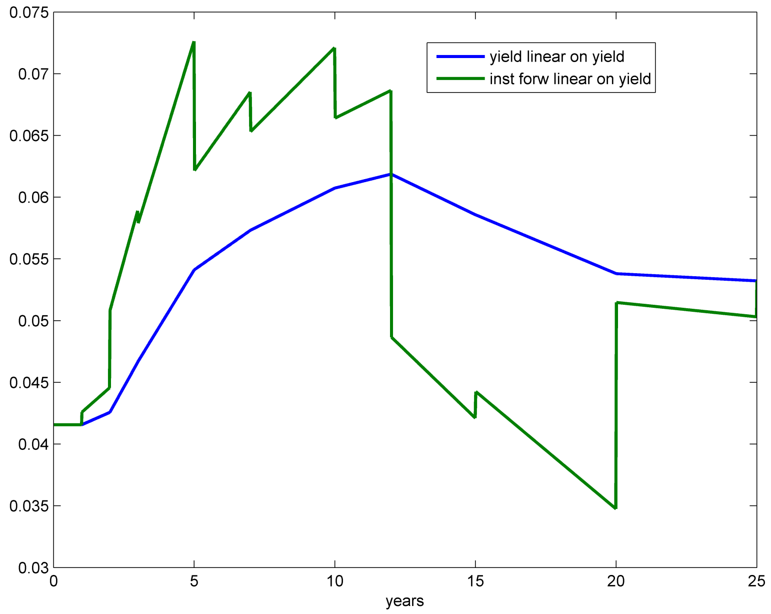

Figure 3.3: Instantaneous forward curve of yield curve in Figure 3.1.

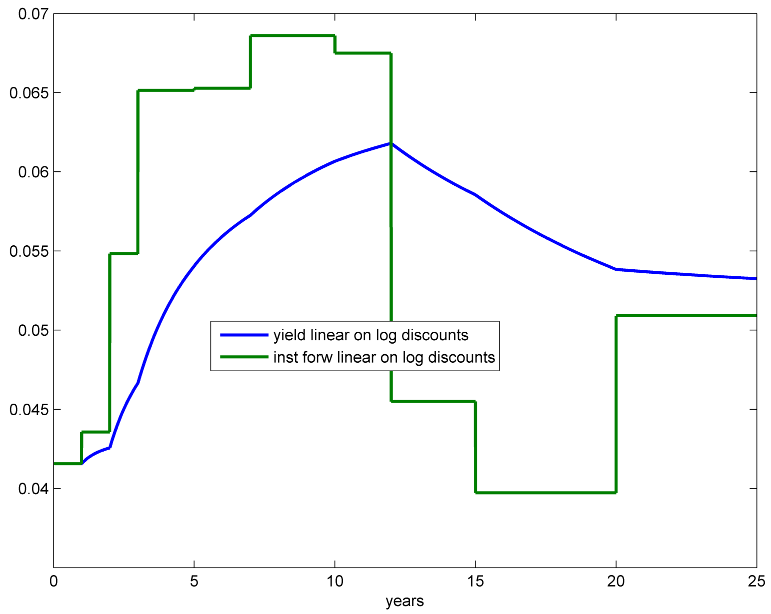

Figure 3.4: Instantaneous forward curve of yield curve in Figure 3.2.

One may notice Figure 3.4 has piecewise flat instantaneous forward rates. We will show this is the case when you construct yield curves by linear interpolating on log discounts.

$$\frac{d\log P(t)}{dt} = -\frac{dR(t)\cdot t}{dt} = -\frac{dR(t)}{dt}\cdot t - R(t) \tag{3.36}$$

$$\Longrightarrow$$

$$\frac{dR(t)}{dt} = \frac{-\frac{d\log P(t)}{dt} - R(t)}{t} = \frac{-\frac{\log P(T_{i+1}) - \log P(T_i)}{T_{i+1} - T_i} - R(t)}{t} = \frac{\frac{R(T_{i+1}) \cdot T_{i+1} - R(T_i) \cdot T_i}{T_{i+1} - T_i} - R(t)}{t}. \tag{3.37}$$

Using (3.35) and (3.37) we have

$$f(t) = \frac{R(T_{i+1}) \cdot T_{i+1} - R(T_i) \cdot T_i}{T_{i+1} - T_i}, \qquad t \in [T_i, T_{i+1}), \tag{3.38}$$

which declares the piecewise flat forward rates.

The instantaneous forward rates in Figure 3.3 and 3.4 are discontinuous. From an economic perspective, such discontinuous forward curves are unrealistic. We will use natural cubic spline interpolation as discussed in Section 3.2.1 to avoid discontinuous forward curves. The reason the instantaneous forward curves of the natural cubic spline interpolation method are continuous is because the first derivative of the cubic spline is continuous. We only have a small discontinuity around one year, which arises because we use flat extrapolation on the yield curve.[12]

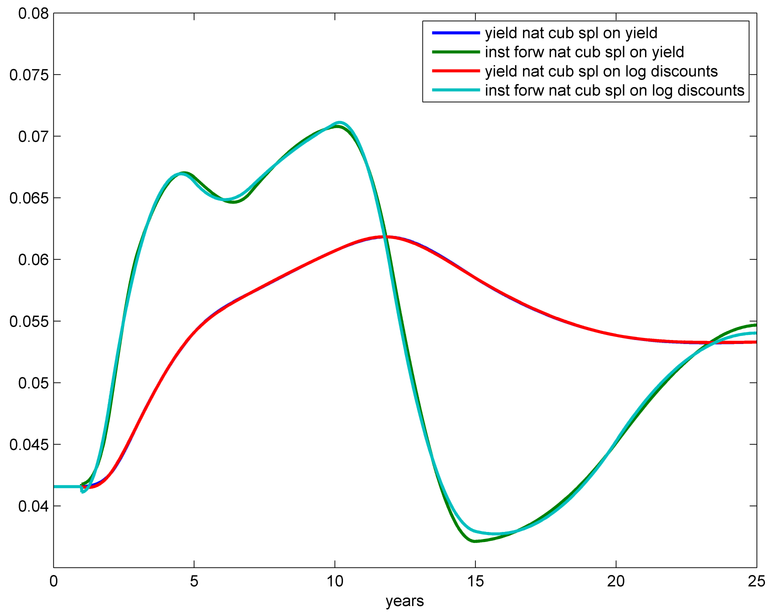

Figure 3.5: Instantaneous forward curves and yield curves constructed by natural cubic spline interpolation. Swap rates in Table 3.1.

[12]We use flat extrapolation because we have little information about the first part of the curve and other extrapolation methods may have extreme values on the first part of the curve.

Chapter 4

Multi Curve Approach

The LIBOR (Euribor) rate is based on the world's most creditworthy banks' interbank deposit rates. Before the start of the crisis in 2007, the credit quality of these banks was so good that the default probability of these banks could be neglected. Therefore, the payoff of a LIBOR (Euribor) rate could be seen as a deterministic return and we could express the forward LIBOR (Euribor) rate in terms of zero-coupon bonds, like we did in Definition 2.5:

$$\mathbb{E}_t^{\widetilde{T}}\left(L(S,T)\right) = \frac{1}{\tau}\left(\frac{P(t,S)}{P(t,T)} - 1\right).$$

The start of the financial crisis in 2007 resulted in the collapse of large financial institutions and the bailout of banks by their national government. The collapse of large financial institutions resulted in a decreased credit quality of banks. Banks which credit quality was very good before the start of the crisis suddenly became vulnerable. Even the most creditworthy banks saw their credit quality decrease, which resulted in a significant default probability. This means we can no longer see the LIBOR (Euribor) rate as a deterministic return and we can no longer use Definition 2.5. In Figure 1.1, we showed a plot of the LIBOR-OIS spread between 2007 and 2011. We used the LIBOR with tenors shorter than one year. Overnight index swaps with maturities up to one year usually have one single payment at maturity, with both legs having the same payment dates and day count conventions, which makes it possible to express the OIS rate in terms of risk-free zero-coupon bonds (see (3.8)):

$$k = \frac{1}{\tau}\left(\frac{P(t,T_0)}{P(t,T_1)} - 1\right).$$

One can see that the OIS rate should be equal to the forward LIBOR (Euribor) rate if we use Definition 2.5, which is allowed if we neglect the default probability of the banks on which the LIBOR rate is based. Prior to the start of the crisis in 2007 the LIBOR - OIS spread hovered within 10 bp and after the start of the crisis the spread increased dramatically. Nowadays the spread is stable, but significant, meaning we cannot neglect the default probability of the banks on which the LIBOR rate is based. Therefore, we can no longer express the forward LIBOR (Euribor) rate by Definition 2.5.

In [7], they show that the discount factors obtained by LIBOR (Euribor) rates remain appropriate for *unsecured trades* between financial firms with the same credit quality as the LIBOR (Euribor) banks. This makes it very hard to valuate cash flows, as you need to know the credit quality of your counterparty and its credit quality should be the same as the credit quality of banks on which a reference rate is based.

The decreased credit quality of many banks and companies increases the risk that your counterparty defaults. To get rid of counterparty risk, people started to trade *collateralized contracts.*

In this chapter we will explain the meaning of collateralized contracts and we will explain pricing formulas for collateralized cash flows. These pricing formulas will be used to obtain pricing formulas for collateralized interest rate products. This chapter discusses the following topics:

- Collateralization.
- Pricing under Collateralization.
- Forward Measure under Collateralization.

- Forward Rate under Collateralization.
- Pricing Interest Rate Products under Collateralization.

4.1 Collateralization

Many interest rate products are traded OTC and therefore do not have any supervision of a clearing house. This means that the increasing probability of default by the counterparty has huge consequences. As most of the products are traded for a high amount of money, it will mean a big loss when the counterparty is not able to perform their payments.

To get rid of this risk, many companies entered into collateralized contracts.[1] In a collateralized contract both parties agree to pay collateral payments on some *settlement dates* prior to the trade date. On a settlement date the firm receives the collateral from the counterparty when the present value of the contract has increased compared to the previous settlement date. The firm needs to pay the margin, called *collateral rate* on the outstanding collateral to the payer. As most contracts are daily collateralized, meaning that each business day is a settlement date, the collateral rate is commonly determined by an overnight rate like Eonia.

By trading collateralized contracts, firms decrease the risk that the counterparty defaults before maturity. Pricing collateralized cash flows, however, requires a new pricing method. In the next section, we will show how to price collateralized cash flows.

4.2 Pricing under Collateralization

To price collateralized cash flows, we use the method discussed in [13]. The collateralized derivative contracts require collateral posting, as default insurance, along with its remuneration. These effects cannot be neglected. As a consequence, we have to consider the cash flows associated to the collateral margining procedure. This means the value of a collateralized security behaves as a dividend paying asset, and its discounted price is no longer a martingale. So

$$h(t) \neq B_t \mathbb{E}_t^Q \left(\frac{h(T)}{B_T} \right), \tag{4.1}$$

where B_t is the risk-free rate bank account, $h(t)$ is the value of the security and Q is the risk-neutral measure associated with the numeraire B_t. We will assume the products are fully collateralized and neglect the effects of a possible default of a counterparty between two settlements and we will use continuous collateral rate to denote the collateral payments.

We try to find a self-financing collateral account $V(t)$ under the new framework. Unless the fact that r_t and the related bank account B_t are no longer observable, we will still use the theory based on them and we will show that the final equations will only depend on market observables. We assume we may invest the posted collateral with the risk-free interest rate but need to pay the collateral rate, the process of the collateral account is given by

$$dV(s) = r(s)V(s)ds - c(s)V(s)ds + a(s)dh(s), \tag{4.2}$$

where:

$r(s)$:	The risk-free rate at time s.
$c(s)$:	The collateral rate at time s.
$h(s)$:	The value of the derivative at time s which matures at time T with cash flow $h(T)$.
$a(s)$:	Number of positions of the derivative at time s.

By (4.2) we get

$$V(T) = e^{\int_t^T r(u)-c(u)du} V(t) + \int_t^T e^{\int_s^T r(u)-c(u)du} a(s)dh(s). \tag{4.3}$$

We adopt the trading strategy specified by

$$\begin{aligned} V(t) &= h(t) \\ a(s) &= e^{\int_t^s y(u)du}. \end{aligned} \tag{4.4}$$

[1]In 2012 83% of the OTC trade volume is traded collateralized compared to 30 % in 2003 (ISDA Margin Survey).

Which gives us

$$V(T) = e^{\int_t^T r(s)-c(s)ds} h(T). \tag{4.5}$$

Now, we are able to compute the present value of the underlying derivative using the risk-neutral measure Q associated with the numeraire $e^{\int_t^T r(s)ds}$:

$$\begin{aligned} h(t) &= \mathbb{E}_t^Q \left(\frac{e^{\int_t^T r(s)-c(s)ds} h(T)}{e^{\int_t^T r(s)ds}} \right) \\ &= \mathbb{E}_t^Q \left(e^{-\int_t^T c(s)ds} h(T) \right). \end{aligned} \tag{4.6}$$

Definition 4.1. *We denote the price of a collateral zero-coupon bond by*

$$P^c(t,T) = \mathbb{E}_t^Q \left(e^{-\int_t^T c(s)ds} \right). \tag{4.7}$$

We will use the price of a collateral zero-coupon bond as numeraire to introduce the forward measure under collateralization.

4.3 Forward Measure

By (4.6), we know that $h(T)e^{\int_t^T c(s)ds}$ is a Q-martingale. Therefore, we may use the Radon-Nikodym theorem to define a new forward measure T^c, associated with the numeraire $P^c(t,T)$. The Radon-Nikodym derivative is given by

$$\varsigma(t) = \mathbb{E}_t^Q \left(\frac{dT^c}{dQ} \right) = \frac{P^c(t,T)/P^c(0,T)}{e^{\int_t^t c(s)ds} / e^{\int_t^0 c(s)ds}} = \frac{P^c(t,T) \cdot e^{-\int_0^t c(s)ds}}{P^c(0,T)}. \tag{4.8}$$

Therefore, we may write

$$P^c(t,T)\mathbb{E}_t^{T^c}(h(T)) = \mathbb{E}_t^{T^c}(P^c(t,T)h(T)) = \mathbb{E}_t^Q \left(P^c(t,T)h(T)\frac{\varsigma(T)}{\varsigma(t)} \right) = \mathbb{E}_t^Q \left(e^{-\int_t^T c(s)ds} h(T) \right). \tag{4.9}$$

This brings us to the pricing expression under collateralization:

Definition 4.2. *The price of a collateralized derivative at t which matures at T with the cash flow $h(T)$ is given by*

$$h(t) = P^c(t,T)\mathbb{E}_t^{T^c}(h(T)), \tag{4.10}$$

where $P^c(t,T) = \mathbb{E}_t^Q \left(e^{-\int_t^T c(s)ds} \right)$ and T^c is the forward measure associated with the numeraire $P^c(t,T)$.

We have defined a formula to discount collateralized cash flows. We want to price collateralized interest rate products, therefore we need a formula to discount LIBOR (Euribor) rates, as most interest rate products are still based on LIBOR (Euribor) rates.

4.4 Forward Rate

Definition 4.2 gives us the present value of a derivative in terms of the forward measure. For several interest rate products, the payoff will be $L(S,T)$ at $\widetilde{T}$, see Chapter 2 and 3. This means the present value of this LIBOR (Euribor) rate will be

$$PV(t) = P^c(t,\widetilde{T})\mathbb{E}_t^{\widetilde{T}^c}(L(S,T)). \tag{4.11}$$

In Section 3.2 we used Definition 2.5 to bootstrap the discount curve using quotes of interest rate products. This curve can be used to discount cash flows as well as computing forward rates. In this chapter, we showed this method no longer works.

Having found a definition to discount collateralized cash flows (Definition 4.2), we still want to find a convenient expression of collateralized forward rates for bootstrapping. We will follow the method explained in [11]. We start by defining a so-called *risky zero-coupon bond*:

Definition 4.3. *We denote the price of a risky zero-coupon bond, corresponding to the risk of the LIBOR (Euribor) rate $L(S,T)$ with tenor τ by*

$$P^\tau(S,T). \tag{4.12}$$

In [11], they state that we may express collateralized forward rates in terms of risky zero-coupon bonds by

Definition 4.4. *We assume that the collateralized forward LIBOR (Euribor) rate* $\mathbb{E}_t^{\widetilde{T}^c}(L(S,T))$ *will depend on risky zero-coupon bonds by*

$$\mathbb{E}_t^{\widetilde{T}^c}(L(S,T)) = \frac{1}{\widetilde{\alpha}}\left(\frac{P^\tau(t,S)}{P^\tau(t,T)} - 1\right), \tag{4.13}$$

where τ *is the tenor of the LIBOR (Euribor) rate and* $\widetilde{\alpha}$ *is the year fraction between* S *and* T*. Note that* $L(S,T)$ *is not a perfect martingale under the collateral forward measure* $\widetilde{T}^c$*. We do use this expression as it proves to be very useful to bootstrap (pseudo) discount curves, which can be used to price interest rate products.*

4.5 Pricing Interest Rate Products under Collateralization

We explained the contracts of some interest rate products in Chapter 2. We did price some products and we bootstrapped the discount curve in Chapter 3, which could be used to price new interest rate products.

In this chapter we explained that most products became collateralized after the start of the crisis. This means we can no longer use the single curve approach to price interest rate products. In this section we will use multiple curves to price some collateralized interest rate products. We will use these pricing formulas in Chapter 5 to bootstrap multiple discount curves, which can be used to price new collateralized interest rate products.

4.5.1 Pricing Collateralized Overnight Index Swaps

We explained the contract of an overnight index swap in Section 2.6. A collateralized overnight index swap will have multiple margin payments before maturity. Like in [13], we will assume that our OIS is perfectly collateralized with zero threshold, and approximate the daily compounding with continuous compounding. Like in Section 3.1.2 we neglect the settlement delay and we assume that the accrual periods fit. In this case, the collateral payments between t and T are given by

$$e^{\int_t^T c(s)ds} = \prod_{i=0}^{K-1}\Big(1 + \alpha(t_i, t_{i+1})L(t_i, t_{i+1})\Big), \tag{4.14}$$

where:

$\{t_i\}_{i=0}^K$:	The collection of all business days in the accrual period $[t,T]$.[2]
$\alpha(t_i, t_{i+1})$:	Year fraction between business days t_i and t_{i+1}.
$L(t_i, t_{i+1})$:	Overnight interest rate at t_i.

In Section 2.6 we explained the cash flows of overnight index swaps. We will use (4.14) and Definition 4.1 to compute the present value of overnight index swaps:

$$\begin{aligned}
PV(t) &= \sum_{i=1}^N \tau_i \mathbb{E}_t^Q\left(e^{-\int_t^{T_i} c(s)ds}\left(\overline{L}_i - k\right)\right) && (4.15)\\
&= \sum_{i=1}^N \mathbb{E}_t^Q\left(e^{-\int_t^{T_{i-1}} c(s)ds} - e^{-\int_t^{T_i} c(s)ds}\right) - k\sum_{i=1}^N \tau_i \mathbb{E}_t^Q\left(e^{-\int_t^{T_i} c(s)ds}\right) && (4.16)\\
&= \sum_{i=1}^N \left(P^c(t,T_{i-1}) - P^c(t,T_i)\right) - k\sum_{i=1}^N \tau_i P^c(t,T_i) && (4.17)\\
&= P^c(t,T_0) - P^c(t,T_N) - k\sum_{i=1}^N \tau_i P^c(t,T_i). && (4.18)
\end{aligned}$$

This means, if we know the curve $P^c(\cdot,\cdot)$ we are able to price collateralized overnight index swaps. As overnight index swaps are quoted with their OIS rate k, assuming the present value of the swap to be zero, one can give the swap rate by converting (4.18):

$$k = \frac{P^c(t,T_0) - P^c(t,T_N)}{\sum_{i=1}^N \tau_i P^c(t,T_i)}. \tag{4.19}$$

[2] $t = t_0 < \ldots < t_K = T$.

One may notice that the pricing formula for a collateralized OIS is the same as an uncollateralized OIS, using the pricing framework used before the start of the crisis (see Section 3.1.2), with just a different discount curve. We performed a test in which we priced a collateralized overnight index swap using OpenGamma; this test can be found in Appendix D.2.

4.5.2 Pricing Collateralized Fixed-Floating Interest Rate Swaps

In Section 2.5 we explained the cash flows of a fixed-floating interest rate swap. We did price a fixed-floating IRS earlier this thesis in Section 3.1.1. However, in Chapter 4 we introduced collateralized products. We use Definition 4.1 and 4.2 to compute the present value of collateralized fixed-floating interest rate swaps:

$$\begin{aligned} PV(t) &= \sum_{i=1}^{N} \alpha_i \mathbb{E}_t^Q \left(e^{-\int_t^{\widetilde{T}_i} c(s)ds} L(S_i, T_i) \right) - k \sum_{j=1}^{M} \tau_j \mathbb{E}_t^Q \left(e^{-\int_t^{t_j} c(s)ds} \right) && (4.20) \\ &= \sum_{i=1}^{N} \alpha_i P^c(t, \widetilde{T}_i) \mathbb{E}_t^{\widetilde{T}_i^c} \left(L(S_i, T_i) \right) - k \sum_{j=1}^{M} \tau_j P^c(t, t_j), && (4.21) \end{aligned}$$

where:

α_i, τ_i:	Accrual factors (year fractions) of i'th floating resp. fixed leg.[3]
N, M:	Number of floating resp. fixed payments.
$\widetilde{T}_i$, t_i:	Time of i'th floating resp. fixed payment.
$\mathbb{E}_t^{\widetilde{T}_i^c}\left(L(S_i, T_i)\right)$:	The collateralized forward LIBOR (Euribor) rate.[4]
k:	Fixed rate.

We use Definition 4.4 to express the collateralized forward rate in terms of risky zero-coupon bonds:

$$\mathbb{E}_t^{\widetilde{T}^c}\left(L(S_i, T_i)\right) = \frac{1}{\widetilde{\alpha}_i}\left(\frac{P^\tau(t, S_i)}{P^\tau(t, T_i)} - 1\right),$$

where $\widetilde{\alpha}_i$ is the accrual factor (year fraction) between S_i and T_i[5] and τ is the tenor of the LIBOR (Euribor) rate $L(S_i, T_i)$.

As fixed-floating interest rate swaps are quoted with their swap rate k, assuming the present value of the swap to be zero, one can give the swap rate by converting (4.21):

$$k = \frac{\sum_{i=1}^{N} \alpha_i P^c(t, \widetilde{T}_i) \mathbb{E}_t^{\widetilde{T}_i^c}\left(L(S_i, T_i)\right)}{\sum_{j=1}^{M} \tau_j P^c(t, t_j)} = \frac{\sum_{i=1}^{N} \frac{\alpha_i}{\widetilde{\alpha}_i} P^c(t, \widetilde{T}_i) \left(\frac{P^\tau(t,S_i)}{P^\tau(t,T_i)} - 1\right)}{\sum_{j=1}^{M} \tau_j P^c(t, t_j)}. \qquad (4.22)$$

We performed a test in which we priced a collateralized fixed-floating interest rate swap using OpenGamma. This test can be found in Appendix D.4.

4.5.3 Pricing Collateralized Basis Swaps

In Section 2.7 we explained the cash flows of basis swaps. We did price a basis swap earlier this thesis in Section 3.1.3. Now, we will price a collateralized basis swap. We use Definition 4.1 and 4.2 to compute the present value of a collateralized basis swap. We assume we pay a floating rate with a short tenor plus a fixed spread and receive a floating rate with a long tenor. "A" denotes the leg paying the rate with short tenor plus fixed spread and "B" denotes the leg paying the rate with long tenor. Then the present value of our swap is given by

$$\begin{aligned} PV(t) &= \sum_{i=1}^{N} \alpha_i \mathbb{E}_t^Q \left(e^{-\int_t^{\widetilde{T}_i^B} c(s)ds} L(S_i^B, T_i^B) \right) - \sum_{j=1}^{M} \tau_j \mathbb{E}_t^Q \left(e^{-\int_t^{\widetilde{T}_j^A} c(s)ds} \left(L(S_j^A, T_j^A) + s \right) \right) && (4.23) \\ &= \sum_{i=1}^{N} \alpha_i P^c(t, \widetilde{T}_i^B) \mathbb{E}_t^{\widetilde{T}_i^{c,B}} \left(L(S_i^B, T_i^B) \right) - \sum_{j=1}^{M} \tau_j P^c(t, \widetilde{T}_j^A) \left(\mathbb{E}_t^{\widetilde{T}_j^{c,A}} \left(L(S_j^A, T_j^A) \right) + s \right), && (4.24) \end{aligned}$$

where:

[3] $\alpha_i = \alpha(\widetilde{T}_{i-1}, \widetilde{T}_i)$ and $\tau_i = \tau(t_{i-1}, t_i)$.

[4] $[S_i, T_i]$ is the accrual period of the index. Normally $S_i = \widetilde{T}_{i-1}$ and T_i^I is S_i^I shifted forward by the tenor, adjusted by date roll conventions.

[5] $\widetilde{\alpha}_i = \widetilde{\alpha}(S_i, T_i)$.

α_i, τ_i: Accrual factors (year fractions) of i'th payment of leg B resp. leg A.[6]
N, M: Number of payments of leg B resp. leg A.
$\widetilde{T}_i^B$, $\widetilde{T}_i^A$: Time of i'th payment of leg B resp. leg A.
$\mathbb{E}_t^{\widetilde{T}_i^{c,B}}(L(S_i^B, T_i^B))$: The collateralized forward LIBOR (Euribor) rate of leg B.[7]
$\mathbb{E}_t^{\widetilde{T}_i^{c,A}}(L(S_i^A, T_i^A))$: The collateralized forward LIBOR (Euribor) rate of leg A.[8]
s: Fixed spread.

We use Definition 4.4 to express the collateral forward LIBOR (Euribor) rate in terms of risky zero-coupon bonds:

$$\mathbb{E}_t^{\widetilde{T}_i^{c,A}}\left(L(S_i^A, T_i^A)\right) = \frac{1}{\widetilde{\tau}_i}\left(\frac{P^{\tau_A}(t, S_i^A)}{P^{\tau_A}(t, T_i^A)} - 1\right),$$

$$\mathbb{E}_t^{\widetilde{T}_i^{c,B}}\left(L(S_i^B, T_i^B)\right) = \frac{1}{\widetilde{\alpha}_i}\left(\frac{P^{\tau_B}(t, S_i^B)}{P^{\tau_B}(t, T_i^B)} - 1\right),$$

where:

$\widetilde{\tau}_i$, $\widetilde{\alpha}_i$: Accrual factors (year fractions) between S_i^A and T_i^A resp. S_i^B and T_i^B.[9]
τ_A, τ_B: Tenor of LIBOR (Euribor) rate $L(S_i^A, T_i^A)$ resp. $L(S_i^B, T_i^B)$.

As basis swaps are quoted with their basis spread s, assuming the present value of the swap to be zero, one can give the basis spread by converting (4.24):

$$\begin{aligned} s &= \frac{\sum_{i=1}^N \alpha_i P^c(t, \widetilde{T}_i^B)\mathbb{E}_t^{\widetilde{T}_i^{c,B}}\left(L(S_i^B, T_i^B)\right) - \sum_{j=1}^M \tau_j P^c(t, \widetilde{T}_j^A)\mathbb{E}_t^{\widetilde{T}_j^{c,A}}\left(L(S_j^A, T_j^A)\right)}{\sum_{j=1}^M \tau_j P^c(t, \widetilde{T}_j^A)} \\ &= \frac{\sum_{i=1}^N \frac{\alpha_i}{\widetilde{\alpha}_i} P^c(t, \widetilde{T}_i^B)\left(\frac{P^{\tau_B}(t,S_i^B)}{P^{\tau_B}(t,T_i^B)} - 1\right) - \sum_{j=1}^M \frac{\tau_j}{\widetilde{\tau}_j} P^c(t, \widetilde{T}_j^A)\left(\frac{P^{\tau_A}(t,S_j^A)}{P^{\tau_A}(t,T_j^A)} - 1\right)}{\sum_{j=1}^M \tau_j P^c(t, \widetilde{T}_j^A)}. \end{aligned} \tag{4.25}$$

We performed a test in which we priced a collateralized basis swap using OpenGamma. This test can be found in Appendix D.5.

In the next chapter we will use the pricing formulas of this chapter to bootstrap multiple (pseudo) discount curves, which can be used to price collateralized interest rate products.

[6] $\alpha_i = \alpha(\widetilde{T}_{i-1}^B, \widetilde{T}_i^B)$ and $\tau_i = \tau(\widetilde{T}_{i-1}^A, \widetilde{T}_i^A)$.

[7] $[S_i^B, T_i^B]$ is the accrual period of the index of leg B. Normally $S_i^B = \widetilde{T}_{i-1}^B$ and T_i^B is S_i^B shifted forward by the tenor, adjusted by date roll conventions.

[8] $[S_i^A, T_i^A]$ is the accrual period of the index of leg A. Normally $S_i^A = \widetilde{T}_{i-1}^A$ and T_i^A is S_i^A shifted forward by the tenor, adjusted by date roll conventions.

[9] $\widetilde{\tau}_i = \widetilde{\tau}(S_i^A, T_i^A)$ and $\widetilde{\alpha}_i = \widetilde{\alpha}(S_i^B, T_i^B)$.

Chapter 5

Multi Curve Bootstrapping

In Chapter 3 we showed we are able to bootstrap a discount curve by market quotes of interest rate products. This discount curve could be used to compute risk-free zero-coupon bond prices. Before the start of the crisis, we showed we may use these risk-free zero-coupon bond prices to price new interest rate products.

In Chapter 4 we showed most interest rate products traded these days are collateralized, which can be priced using collateral zero-coupon bonds and risky zero-coupon bonds corresponding to the risk of the LIBOR (Euribor) rate. In this chapter we will use the market quotes of several interest rate products to obtain the curve corresponding to collateral zero-coupon bonds and the curve corresponding to risky zero-coupon bonds. These curves can be used to price new interest rate products. We will denote the yield curve corresponding to a risky zero-coupon bond by its tenor and we denote the yield curve corresponding to the collateral zero-coupon bond by *overnight yield curve*.[1]

In this chapter, we used different techniques to bootstrap multiple discount curves. This chapter discusses the following topics:

- One by One Bootstrapping.
- Simultaneous Bootstrapping.
  - Converted to One by One.
  - Simultaneous Bootstrapping using Basis Swaps.
  - Simultaneous Bootstrapping using Eurodollar Futures.

5.1 One by One Bootstrapping

Assuming there are sufficient products available on the market, we are able to bootstrap multiple curves one by one. Say we want to bootstrap curves which have a maximum length of thirty years. That means there need to be products available on the market with maturities varying from zero to thirty years for each tenor.

We will start with bootstrapping the overnight (*risk-free*) yield curve using overnight index swaps, which is the single curve bootstrapping we did earlier on in Section 3.2. Having bootstrapped the overnight yield curve, we are now able to bootstrap curves with longer tenors.
To bootstrap a curve with a longer tenor, we need products which can be priced using just the forward index curve and the overnight curve. Examples of such products are fixed-floating interest rate swaps and OIS-floating swaps. As we already know the overnight curve, we are able to bootstrap the forward index curve.
We are also able to price forward index curves with different tenors using the same technique. We may also bootstrap curves with different tenors using basis swaps, as we already have bootstrapped a forward index curve.

Interest rate products are traded over the counter, which means that quotes of interest rate products are not quoted on a stock exchange. We need a ticker of Bloomberg or Reuters to obtain the quotes for interest rate

[1]The collateral zero-coupon bond price corresponds to a risky zero-coupon bond with a very short tenor, as the default probability can be neglected. The overnight default probability is very small and can be neglected.

products. As using such a ticker is quite expensive, we use quotes of an article ([11]) to bootstrap (pseudo) discount curves. These quotes are of 11 December 2012.

In [11], they use synthetic market quotes[2] of Eonia swaps with maturities over 10 years. We will need these quotes if we want to bootstrap our curves one by one. Therefore, we will use these synthetic quotes in this section. In the next section we won't use these synthetic quotes.

We will bootstrap the overnight yield curve using some of the instruments of [11, Figure 25], we won't use the forward starting swaps to keep things simple (we will use Eurodollar futures in Section 5.2.3). The instruments we used are given in Table 5.1. The day count convention is *Actual/360* and the payment frequency for swaps having a maturity over one year is one year. The swaps with maturities of 15, 18 and 21 Months have the broken period at the beginning.

Maturity	OIS rate	Start Date	End Date
One Day	0.0400%	13-12-2012	14-12-2012
One Week	0.0700%	13-12-2012	20-12-2012
Two Weeks	0.0690%	13-12-2012	27-12-2012
Three Weeks	0.0780%	13-12-2012	3-1-2013
One Month	0.0740%	13-12-2012	14-1-2013
15 Months	0.0020%	13-12-2012	13-3-2014
18 Months	0.0080%	13-12-2012	13-6-2014
21 Months	0.0210%	13-12-2012	15-9-2014
2 Years	0.0360%	13-12-2012	15-12-2014
3 Years	0.1270%	13-12-2012	14-12-2015
4 Years	0.2740%	13-12-2012	13-12-2016
5 Years	0.4560%	13-12-2012	13-12-2017
6 Years	0.6470%	13-12-2012	13-12-2018
7 Years	0.8270%	13-12-2012	13-12-2019
8 Years	0.9960%	13-12-2012	14-12-2020
9 Years	1.1470%	13-12-2012	13-12-2021
10 Years	1.2800%	13-12-2012	13-12-2022
11 Years	1.4040%	13-12-2012	13-12-2023
12 Years	1.5160%	13-12-2012	13-12-2024
15 Years	1.7640%	13-12-2012	13-12-2027
20 Years	1.9390%	13-12-2012	13-12-2032
25 Years	2.0030%	13-12-2012	14-12-2037
30 Years	2.0380%	13-12-2012	15-12-2042

Table 5.1: Eonia Swaps used for bootstrapping overnight yield curve. Trade date is 11-12-2012.

We performed single curve bootstrapping as discussed in Section 3.2 using linear interpolation on the log discount curve. We used *Actual/Actual ISDA* as day count convention to determine the year fraction used for discounting.
We also bootstrapped the overnight yield curve using linear interpolation on the yield curve. Both are shown together with their yield curves in Figure 5.1.
Finally we bootstrapped the overnight yield curve using natural cubic spline interpolation on both the yield curves and log discounts. Plots are in Figure 5.2.

[2]Quotes that are not directly quoted on the market, but are built from different market quotes.

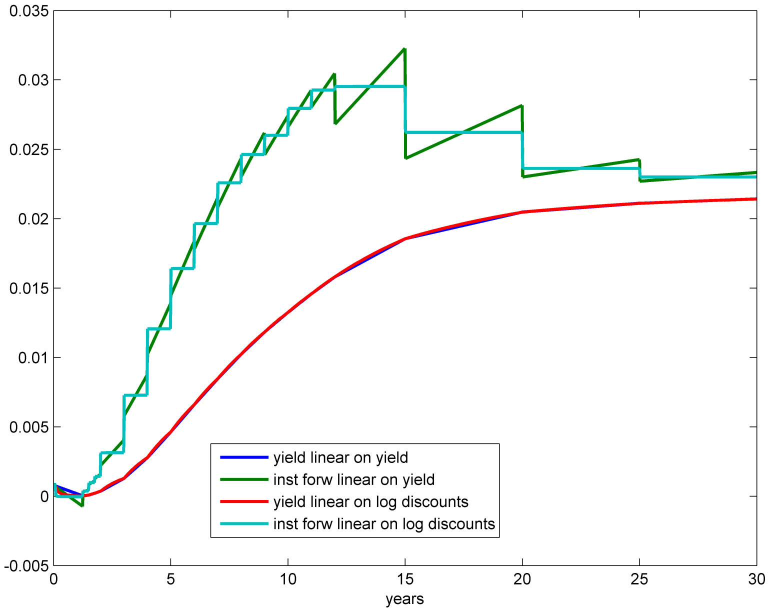

Figure 5.1: Yield and Forward curves. Swap rates in Table 5.1.

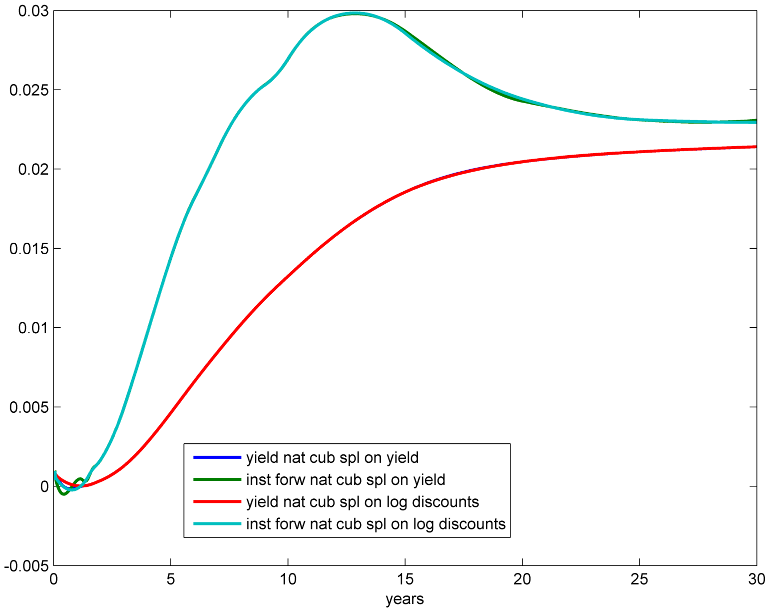

Figure 5.2: Yield and Forward curves. Swap rates in Table 5.1.

In [11, Figure 26] they show a plot of the overnight FRA rate bootstrapped using linear interpolation on log discounts, which should be the same as our curve in Figure 5.1.
We continued by bootstrapping the 1M yield curve using several instruments of [11, Figure 27], we did not use the synthetic quotes with short maturities. Our instruments used are shown in Table 5.2. Again, the day count convention is *Actual/360*. The payment frequency for both legs is 1 Month.

Maturity	Swap rate	Start Date	End Date
1 Month	0.1100%	13-12-2012	14-1-2013
2 Months	0.1060%	13-12-2012	13-2-2013
3 Months	0.0960%	13-12-2012	13-3-2013
4 Months	0.0850%	13-12-2012	15-4-2013
5 Months	0.0790%	13-12-2012	13-5-2013
6 Months	0.0750%	13-12-2012	13-6-2013
7 Months	0.0710%	13-12-2012	15-7-2013
8 Months	0.0690%	13-12-2012	13-8-2013
9 Months	0.0660%	13-12-2012	13-9-2013
10 Months	0.0650%	13-12-2012	14-10-2013
11 Months	0.0640%	13-12-2012	13-11-2013
12 Months	0.0630%	13-12-2012	13-12-2013
2 Years	0.0980%	13-12-2012	15-12-2014
3 Years	0.1860%	13-12-2012	14-12-2015
4 Years	0.3300%	13-12-2012	13-12-2016
5 Years	0.5120%	13-12-2012	13-12-2017
6 Years	0.7040%	13-12-2012	13-12-2018
7 Years	0.8870%	13-12-2012	13-12-2019
8 Years	1.0580%	13-12-2012	14-12-2020
9 Years	1.2110%	13-12-2012	13-12-2021
10 Years	1.3470%	13-12-2012	13-12-2022
11 Years	1.4700%	13-12-2012	13-12-2023
12 Years	1.5810%	13-12-2012	13-12-2024
15 Years	1.8260%	13-12-2012	13-12-2027
20 Years	1.9980%	13-12-2012	13-12-2032
25 Years	2.0590%	13-12-2012	14-12-2037
30 Years	2.0930%	13-12-2012	15-12-2042
35 Years	2.1320%	13-12-2012	13-12-2047
40 Years	2.1850%	13-12-2012	13-12-2052
50 Years	2.5800%	13-12-2012	13-12-2062
60 Years	2.3000%	13-12-2012	13-12-2072

Table 5.2: 1M Euribor Swaps used for bootstrapping 1M yield curve. Trade date is 11-12-2012.

To bootstrap the 1M yield curve, knowing the overnight yield curve, we have to solve a system of equations similar to (3.31). We have used a different number of swaps and we make a distinction between the curve to determine the forward rate and the curve used for discounting. The system of equations we need to solve now are given in (5.1).

$$\text{for } l = 1 \dots 31 \qquad 0 = \sum_{i=1}^{N_l} \frac{\alpha_{l,i}}{\widetilde{\alpha}_{l,i}} \left(\frac{P^{1\mathrm{M}}(0, S_{l,i})}{P^{1\mathrm{M}}(0, T_{l,i})} - 1 \right) P^c(0, \widetilde{T}_{l,i}) - k_l \sum_{i=1}^{M_l} \tau_{l,i} P^c(0, t_{l,i}), \tag{5.1}$$

where $P^{1\mathrm{M}}(\cdot,\cdot)$ denotes the discount curve of the 1M yield curve. Other functions and variables are explained in (3.31).

As we already know $P^c(0,t) \equiv \exp\{-R^c(t) \cdot t\}$ from bootstrapping the overnight yield curve, we only have to make assumptions for $P^{1\mathrm{M}}(0,\cdot)$. We have 31 equations, meaning we have to find 31 nodal points to construct the 1M yield curve. This will make all functions $P^{1\mathrm{M}}(0,\cdot)$ depend on 31 points and we may convert (5.1) into (5.3). We will express $P^{1\mathrm{M}}(t) \equiv P^{1\mathrm{M}}(0,t)$ as a function using 31 nodal points as input:

$$P^{1\mathrm{M}}(t) = f(t; n_1, \dots n_{31}), \tag{5.2}$$

where $n_1, \dots n_{31}$ are the values of our nodal points.

$$\text{for } l = 1 \dots 31 \qquad 0 = \sum_{i=1}^{N_l} \frac{\alpha_{l,i}}{\widetilde{\alpha}_{l,i}} \left(\frac{f(S_{l,i}; n_1, \dots, n_{31})}{f(T_{l,i}; n_1, \dots, n_{31})} - 1 \right) P^c(0, \widetilde{T}_{l,i}) - k_l \sum_{i=1}^{M_l} \tau_{l,i} P^c(0, t_{l,i}). \tag{5.3}$$

We bootstrapped the 1M yield in different ways. We used the discount factor obtained by the overnight yield curve bootstrapped earlier for discounting cash flows. We used *Actual/Actual ISDA* as day count convention to determine the year fraction for discounting.

We used linear interpolating and natural cubic spline interpolating on the yield curve or log discount curve, where we used the overnight curve obtained by the same interpolating method. The bootstrapped yield curves and their forward curves are shown in Figure 5.3 and 5.4.

We used synthetic quotes for maturities over 30 years and used linear extrapolation on our overnight yield curve to bootstrap the one month yield curve. Therefore the curves include a high level of uncertainty after thirty years. In [11] they bootstrapped the one month forward curve as well; this plot is shown in [11, Figure 28].

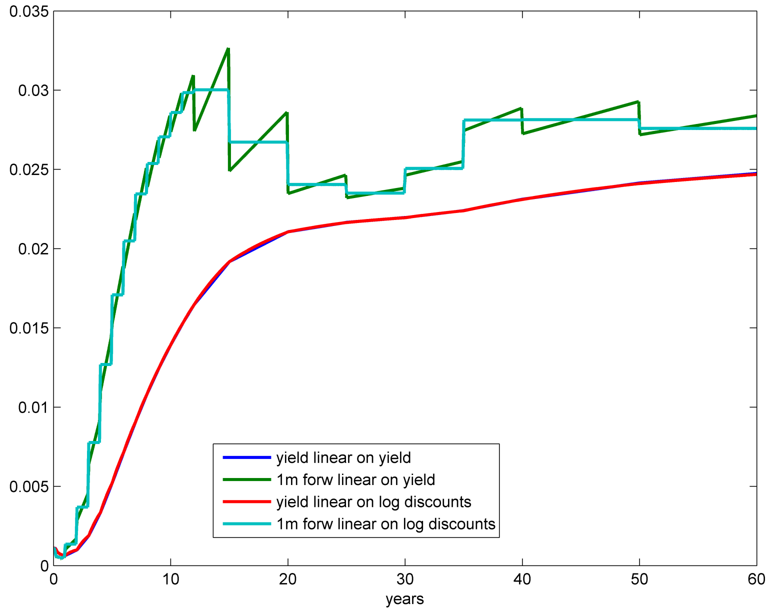

Figure 5.3: One Month Yield and One Month Forward Curves using Linear Interpolation. Swap rates in Table 5.2.

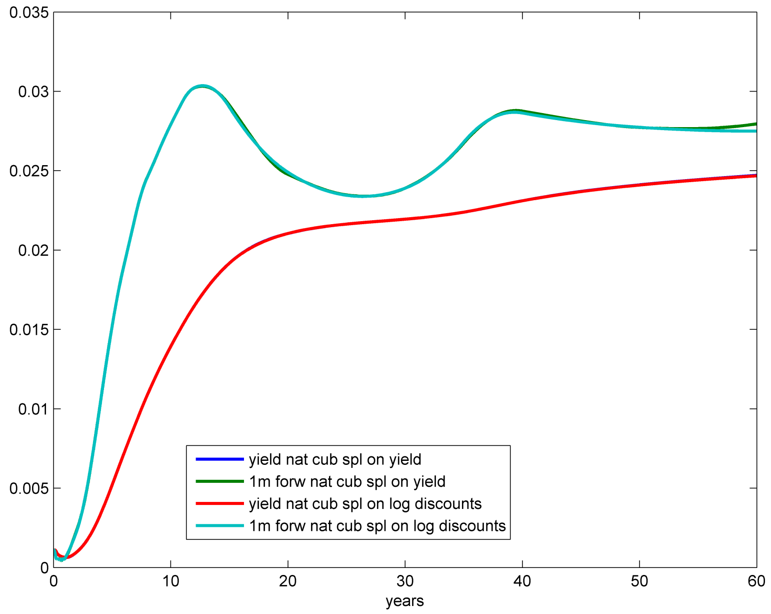

Figure 5.4: One Month Yield and One Month Forward Curves using Natural Cubic Spline Interpolation. Swap rates in Table 5.2.

5.2 Simultaneous Bootstrapping

In practice unfortunately, there are not sufficient products available on the market to bootstrap curves one by one. Overnight index swaps for example will usually have maturities with a maximum of ten years (the quotes used in the previous section were synthetic quotes). This means we cannot bootstrap the overnight yield curve for more than ten years using single curve bootstrapping. We will need simultaneous bootstrapping in this case. In practice fixed-floating interest rate swaps and basis swaps between overnight rates and forward rates will have longer maturities up to thirty years. We will use these swaps to bootstrap the overnight yield curve and the forward index yield curve simultaneously.

We will start by converting the simultaneous bootstrapping problem to a one by one bootstrapping problem using Eoina swaps and fixed-floating IRS. Thereafter we will use basis swaps to apply simultaneous bootstrapping. We will also use Eurodollar futures to bootstrap the three month forward curve.

5.2.1 Converted to One by One

We started by converting the problem into a one by one bootstrapping problem. We used the Eonia swaps and 1M Euribor fixed-floating swaps of the previous section, with the exception of the synthetic quotes. The products we will use are shown in Table 5.3 and the 1M Euribor swaps of Table 5.4. We did not include any basis swaps.

Maturity	OIS rate	Start Date	End Date
One Day	0.0400%	13-12-2012	14-12-2012
One Week	0.0700%	13-12-2012	20-12-2012
Two Weeks	0.0690%	13-12-2012	27-12-2012
Three Weeks	0.0780%	13-12-2012	3-1-2013
One Month	0.0740%	13-12-2012	14-1-2013
15 Months	0.0020%	13-12-2012	13-3-2014
18 Months	0.0080%	13-12-2012	13-6-2014
21 Months	0.0210%	13-12-2012	15-9-2014
2 Years	0.0360%	13-12-2012	15-12-2014
3 Years	0.1270%	13-12-2012	14-12-2015
4 Years	0.2740%	13-12-2012	13-12-2016
5 Years	0.4560%	13-12-2012	13-12-2017
6 Years	0.6470%	13-12-2012	13-12-2018
7 Years	0.8270%	13-12-2012	13-12-2019
8 Years	0.9960%	13-12-2012	14-12-2020
9 Years	1.1470%	13-12-2012	13-12-2021
10 Years	1.2800%	13-12-2012	13-12-2022

Table 5.3: Eonia Swaps used for bootstrapping overnight yield curve.[3] Trade date is 11-12-2012.

Maturity	Swap rate	Start Date	End Date
1 Month	0.1100%	13-12-2012	14-1-2013
2 Months	0.1060%	13-12-2012	13-2-2013
3 Months	0.0960%	13-12-2012	13-3-2013
4 Months	0.0850%	13-12-2012	15-4-2013
5 Months	0.0790%	13-12-2012	13-5-2013
6 Months	0.0750%	13-12-2012	13-6-2013
7 Months	0.0710%	13-12-2012	15-7-2013
8 Months	0.0690%	13-12-2012	13-8-2013
9 Months	0.0660%	13-12-2012	13-9-2013
10 Months	0.0650%	13-12-2012	14-10-2013
11 Months	0.0640%	13-12-2012	13-11-2013
12 Months	0.0630%	13-12-2012	13-12-2013
2 Years	0.0980%	13-12-2012	15-12-2014
3 Years	0.1860%	13-12-2012	14-12-2015
4 Years	0.3300%	13-12-2012	13-12-2016
5 Years	0.5120%	13-12-2012	13-12-2017
6 Years	0.7040%	13-12-2012	13-12-2018
7 Years	0.8870%	13-12-2012	13-12-2019
8 Years	1.0580%	13-12-2012	14-12-2020
9 Years	1.2110%	13-12-2012	13-12-2021
10 Years	1.3470%	13-12-2012	13-12-2022
11 Years	1.4700%	13-12-2012	13-12-2023
12 Years	1.5810%	13-12-2012	13-12-2024
15 Years	1.8260%	13-12-2012	13-12-2027
20 Years	1.9980%	13-12-2012	13-12-2032
25 Years	2.0590%	13-12-2012	14-12-2037
30 Years	2.0930%	13-12-2012	15-12-2042

Table 5.4: 1M Euribor Swaps used for bootstrapping 1M yield curve.[4] Trade date is 11-12-2012.

We began with bootstrapping the overnight yield curve up to ten years, using overnight index swaps. We used the structure of Chapter 3.

[3]Day count convention is *Actual/360* and the payment frequency is one year, with the broken period at the beginning.
[4]Day count convention is *Actual/360* for both legs and the payment frequency is one month for both legs.

Now, we will bootstrap the 1M yield curve and extend the overnight yield curve using the 27 1M Euribor swaps of Table 5.4. The system of equations we need to solve is given in (5.4).

$$\text{for } l = 1 \dots 27 \qquad 0 = \sum_{i=1}^{N_l} \frac{\alpha_{l,i}}{\widetilde{\alpha}_{l,i}} \left(\frac{P^{1\text{M}}(0, S_{l,i})}{P^{1\text{M}}(0, T_{l,i})} - 1 \right) P^c(0, \widetilde{T}_{l,i}) - k_l \sum_{i=1}^{M_l} \tau_{l,i} P^c(0, t_{l,i}), \tag{5.4}$$

where:

l:	Denotes the l'th swap of Table 5.4.
k_l:	Swap rate of the l'th swap.
N_l, M_l:	Number of payments of the floating resp. fixed leg of the l'th swap.
$\alpha_{l,i}$, $\widetilde{\alpha}_{l,i}$, $\tau_{l,i}$:	Accrual factors (year fractions) of the l'th swap.
$\widetilde{T}_{l,i}$, $t_{l,i}$:	Time of the i'th floating resp. fixed payment of the l'th swap.
$[S_{l,i}, T_{l,i}]$:	Accrual period of the i'th floating payment of the l'th swap.

We have 27 equations, this means we need 27 nodal points to construct the 1M yield curve and extend the overnight yield curve. We will use the 27 maturities as nodal points for constructing the 1M yield curve, using an interpolation method. We do know the overnight yield curve up to τ (ten years). We will fix a spread s between the 1M yield curve at τ, constructed by 27 nodal points, and the overnight yield curve at τ:

$$s = R^{1\text{M}}(\tau) - R^c(\tau). \tag{5.5}$$

The extended overnight yield curve will be constructed by the 1M yield curve minus the spread. Denoting the complete overnight yield curve by $\bar{R}^c(t)$, this means

$$\bar{R}^c(t) = \begin{cases} R^c(t) & \text{for } t \leq \tau \\ R^{1\text{M}}(t) - s & \text{for } t > \tau \end{cases} \tag{5.6}$$

Using the relation $P^c(t) \equiv \exp\{-\bar{R}^c(t) \cdot t\}$, will make us able to convert (5.4) into (5.9). Again, we will express $P^{1\text{M}}(t) \equiv P^{1\text{M}}(0,t)$ as a function of nodal points:

$$P^{1\text{M}}(t) = f(t; n_1, \dots n_{27}), \tag{5.7}$$

where $n_1, \dots, n_{27}$ are the values of our nodal points. We will express $P^c(t)$ for $t > \tau$ by a different function of the same nodal points:

$$P^c(t) = g(t; n_1, \dots, n_{27}). \tag{5.8}$$

$$\begin{aligned} \text{for } l = 1 \dots 27 \qquad 0 = {} & \sum_{i=1}^{N_l} \frac{\alpha_{l,i}}{\widetilde{\alpha}_{l,i}} \left(\frac{f(S_{l,i}; n_1, \dots, n_{27})}{f(T_{l,i}; n_1, \dots, n_{27})} - 1 \right) \left(\mathbb{1}_{\{\widetilde{T}_{l,i} \leq \tau\}} P^c(0, \widetilde{T}_{l,i}) + \mathbb{1}_{\{\widetilde{T}_{l,i} > \tau\}} g(t_{l,i}; n_1, \dots, n_{27}) \right) \\ & - k_l \sum_{i=1}^{M_l} \tau_{l,i} \left(\mathbb{1}_{\{t_{l,i} \leq \tau\}} P^c(0, t_{l,i}) + \mathbb{1}_{\{t_{l,i} > \tau\}} g(t_{l,i}; n_1, \dots, n_{27}) \right). \end{aligned} \tag{5.9}$$

We bootstrapped the overnight yield curve and the one month yield curve using natural cubic spline interpolation on log discounts, we plotted these curves together with their forwards in Figure 5.5. We used *Actual/Actual ISDA* as day count convention to determine the year fractions used for discounting.

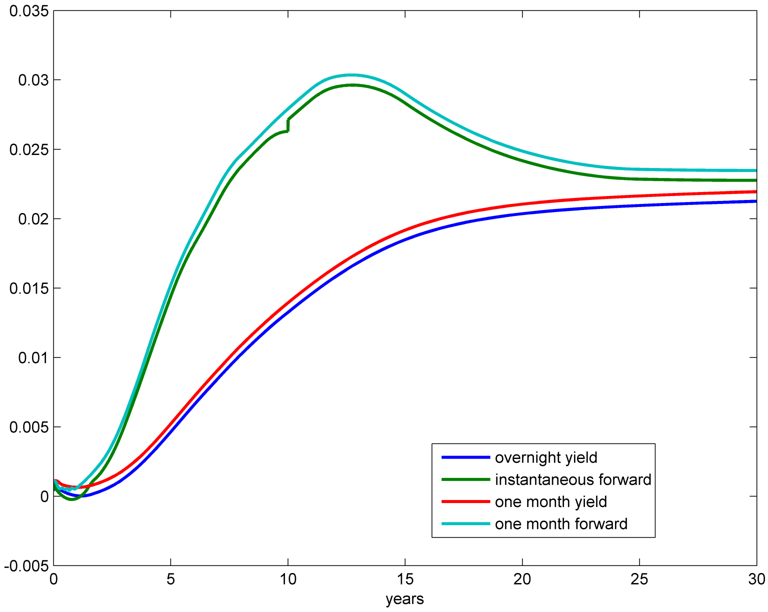

Figure 5.5: Overnight and One Month Yield and Forward Curves using Natural Cubic Spline Interpolation on the Log Discounts. Swap rates in Table 5.3 and 5.4.

One can see that the instantaneous forward curve of the overnight yield curve in Figure 5.5 has a discontinuity at ten years. This is caused by a discontinuity in the derivative of the yield curve. This discontinuity in the derivative of the yield curve arises because we use natural cubic spline interpolation on log discounts to construct the overnight yield curve of the first ten years, which requires a boundary condition of $\frac{\partial^2 \log P(t)}{\partial t^2} = -\frac{\partial f(t)}{\partial t} = 0$ (see Definition A.3 and Section 3.2.1).

This discontinuity in the instantaneous forward curve of the overnight yield curve is not realistic from an economic perspective. To avoid this problem, we will use basis swaps to execute simultaneous bootstrapping in the next subsection.

5.2.2 Simultaneous Bootstrapping using Basis Swaps

In this subsection, we will use OIS-1M basis swaps with maturities of ten to thirty years to bootstrap curves simultaneously. These basis swaps are shown in Table 5.5. Actually, these are synthetic products as we couldn't find real quotes. However, we chose to use these products because we want to demonstrate the principle of simultaneous bootstrapping. The day count convention is *Actual/360* for both legs, the payment frequency for the overnight leg is one year and the payment frequency of the one month leg is one month.

Maturity	Basis Spread	Start Date	End Date
11 Years	0.066%	13-12-2012	13-12-2023
12 Years	0.065%	13-12-2012	13-12-2024
15 Years	0.062%	13-12-2012	13-12-2027
20 Years	0.059%	13-12-2012	13-12-2032
25 Years	0.056%	13-12-2012	14-12-2037
30 Years	0.055%	13-12-2012	15-12-2042

Table 5.5: Basis Swaps used for Bootstrapping Overnight and One Month Yield Curve. Trade date is 11-12-2013.

We will bootstrap the curves simultaneously, using overnight swaps of Table 5.3, the 1M fixed-floating swaps of Table 5.4 and the basis swaps of Table 5.5. This will give us a system of equations given in (5.10).

$$
\begin{aligned}
&\text{for } l = 1 \dots 17 && 0 = \sum_{i=1}^{N} \frac{\alpha_{l,i}}{\widetilde{\alpha}_{l,i}} \left(\frac{P^c(0, S_{l,i})}{P^c(0, T_{l,i})} - 1 \right) P^c(0, \widetilde{T}_{l,i}) - k_l \sum_{i=1}^{M} \tau_{l,i} P^c(0, t_{l,i}), \\
&\text{for } p = 1 \dots 27 && 0 = \sum_{i=1}^{N^*} \frac{\alpha^*_{p,i}}{\widetilde{\alpha}^*_{p,i}} \left(\frac{P^{1\mathrm{M}}(0, S^*_{p,i})}{P^{1\mathrm{M}}(0, T^*_{p,i})} - 1 \right) P^c(0, \widetilde{T}^*_{p,i}) - k^*_p \sum_{i=1}^{M^*} \tau^*_{p,i} P^c(0, t^*_{p,i}), \\
&\text{for } q = 1 \dots 6 && 0 = \sum_{i=1}^{N^{**}} \frac{\alpha^{**}_{q,i}}{\widetilde{\alpha}^{**}_{q,i}} \left(\frac{P^{1\mathrm{M}}(0, S^{**,B}_{q,i})}{P^{1\mathrm{M}}(0, T^{**,B}_{q,i})} - 1 \right) P^c(0, \widetilde{T}^{**,B}_{q,i}) \\
& && \quad - \sum_{i=1}^{M^{**}} \beta^{**}_{q,i} \left(\frac{1}{\widetilde{\beta}^{**}_{q,i}} \left(\frac{P^c(0, S^{**,A}_{q,i})}{P^c(0, T^{**,A}_{q,i})} - 1 \right) + s^{**}_q \right) P^c(0, \widetilde{T}^{**,A}_{q,i}),
\end{aligned}
\tag{5.10}
$$

where:

l:	Denotes the l'th swap of Table 5.3.
p:	Denotes the p'th swap of Table 5.4.
q:	Denotes the q'th swap of Table 5.5.
k_l:	Swap rate of the l'th swap of Table 5.3.
k^*_p:	Swap rate of the p'th swap of Table 5.4.
s^{**}_q:	Basis spread of the q'th swap of Table 5.5.
N_l, M_l:	Number of payments of the floating resp. fixed leg of the l'th swap of Table 5.3.
N^*_p, M^*_p:	Number of payments of the floating resp. fixed leg of the p'th swap of Table 5.4.
N^{**}_q, M^{**}_q:	Number of payments of leg B resp. leg A of the q'th swap of Table 5.5.
$\alpha_{l,i}$, $\widetilde{\alpha}_{l,i}$, $\tau_{l,i}$:	Accrual factors (year fractions) of the l'th swap of Table 5.3.
$\alpha^*_{p,i}$, $\widetilde{\alpha}^*_{p,i}$, $\tau^*_{p,i}$:	Accrual factors (year fractions) of the p'th swap of Table 5.4.
$\alpha^{**}_{q,i}$, $\widetilde{\alpha}^{**}_{q,i}$, $\beta^{**}_{q,i}$, $\widetilde{\beta}^{**}_{q,i}$:	Accrual factors (year fractions) of the q'th swap of Table 5.5.
$\widetilde{T}_{l,i}$, $t_{l,i}$:	Time of the i'th floating resp. fixed payment of the l'th swap of Table 5.3.
$\widetilde{T}^*_{p,i}$, $t^*_{p,i}$:	Time of the i'th floating resp. fixed payment of the p'th swap of Table 5.4.
$\widetilde{T}^{**,A}_{l,i}$, $\widetilde{T}^{**,A}_{l,i}$:	Time of the i'th payment of leg A resp. leg B of the q'th swap of Table 5.5.
$[S_{l,i}, T_{l,i}]$:	Accrual period of the i'th floating payment of the l'th swap of Table 5.3.
$[S^*_{p,i}, T^*_{p,i}]$:	Accrual period of the i'th floating payment of the p'th swap of Table 5.4.
$[S^{**,A}_{q,i}, T^{**,A}_{q,i}]$, $[S^{**,B}_{q,i}, T^{**,B}_{q,i}]$:	Accrual period of the i'th payment of leg A resp. leg B of the q'th swap of Table 5.5.

We can make the bootstrapping problem quite complicated if we insist on $P^{1\mathrm{M}}(\cdot)$ and $P^c(\cdot)$ both being smooth functions. We will use a simpler idea, suggested in [1, Ch.6]. The $P^{1\mathrm{M}}(\cdot)$ (pseudo) discount curve will be used to project forward rates, so the yields and forward rates implied by $P^{1\mathrm{M}}(\cdot)$ should ideally be smooth. The smoothness requirements of the discount curve $P^c(\cdot)$, however, are significantly lower, as we do not need to differentiate this curve to produce forward LIBOR (Euribor) rates.
We will use natural cubic spline interpolation on the log discount curve of $P^{1\mathrm{M}}(\cdot)$, by using the maturities of the fixed-floating IRS as nodal points. We will use the remaining nodal points to construct a spread function, which will be used to construct the discount curve $P^c(\cdot)$.
We will represent $P^c(\cdot)$ as a multiplicative spread to $P^{1\mathrm{M}}(\cdot)$. We assume there will be a piecewise-constant spread function $\eta(\cdot)$ between the instantaneous forward rates of both curves:

$$
f^c(t) = f^{1\mathrm{M}}(t) + \eta(t), \tag{5.11}
$$

where $f^c(\cdot)$ is the instantaneous forward curve of the overnight yield curve and $f^{1\mathrm{M}}(\cdot)$ is the instantaneous forward curve of the one month yield curve.

We will use the definition $P(t) \equiv \exp\{-\int_0^t f(u)du\}$ to represent $P^{1\mathrm{M}}(\cdot)$ in terms of $P^c(\cdot)$ and $\eta(\cdot)$:

$$
P^c(t) = P^{1\mathrm{M}}(t) e^{-\int_0^t \eta(u)du}, \qquad t \geq 0. \tag{5.12}
$$

We have $17 + 27 + 6 = 50$ equations and therefore we have to use 50 nodal points to construct the yield curves. We will use the maturities of the fixed-floating swaps to construct the one month yield curve, using an interpolation method. We will use the maturities of the overnight index swaps and basis swaps to construct the piecewise-constant spread function $\eta(\cdot)$. Denoting the nodal points of the spread function by $\{T_i\}_{i=1}^N$ with nodal values $\{\eta_{T_i}\}_{i=1}^N$, we will construct $\eta(\cdot)$ by

$$\eta(t) = \eta_{T_i}, \qquad t \in (T_{i-1}, T_i] \tag{5.13}$$

$$\eta(t) = \eta_{T_1}, \qquad t \leq T_1 \tag{5.14}$$

$$\eta(t) = \eta_{T_N}, \qquad t > T_N. \tag{5.15}$$

We express $P^{1\mathrm{M}}(t)$ as a function of nodal points, using an interpolation method:

$$P^{1\mathrm{M}}(t) = f(t; n_1, \ldots, n_{27}), \tag{5.16}$$

where $n_1, \ldots, n_{27}$ are the values of the nodal points. Using (5.12), we are able to express $P^c(t)$ similarly by

$$P^c(t) = f(t; n_1, \ldots, n_{27}) g(t; m_1, \ldots m_{23}), \tag{5.17}$$

where $m_1, \ldots, m_{23}$ are the nodal values used to construct the spread function $\eta(\cdot)$.
We used $g(t; \cdot) = e^{-\int_0^t \eta(u)du}$.

Now we have constructed the $P^c(\cdot)$ and $P^{1\mathrm{M}}(\cdot)$ curves using 50 nodal points, we are able to convert (5.10) into (5.18).

$$\begin{aligned}
\text{for } l = 1 \ldots 17 \qquad 0 = &\sum_{i=1}^{N} \frac{\alpha_{l,i}}{\widetilde{\alpha}_{l,i}} \left(\frac{f(S_{l,i}; n_1, \ldots n_{27}) g(S_{l,i}; m_1, \ldots, m_{23})}{f(T_{l,i}; n_1, \ldots n_{27}) g(T_{l,i}; m_1, \ldots, m_{23})} - 1 \right) f(\widetilde{T}_{l,i}; n_1, \ldots n_{27}) g(\widetilde{T}_{l,i}; m_1, \ldots, m_{23}) \\
&-k_l \sum_{i=1}^{M} \tau_{l,i} f(t_{l,i}; n_1, \ldots n_{27}) g(t_{l,i}; m_1, \ldots, m_{23}), \\
\text{for } p = 1 \ldots 27 \qquad 0 = &\sum_{i=1}^{N^*} \frac{\alpha^*_{p,i}}{\widetilde{\alpha}^*_{p,i}} \left(\frac{f(S^*_{p,i}; n_1, \ldots n_{27})}{f(T^*_{p,i}; n_1, \ldots n_{27})} - 1 \right) f(\widetilde{T}^*_{p,i}; n_1, \ldots n_{27}) g(\widetilde{T}^*_{p,i}; m_1, \ldots, m_{23}) \\
&-k^*_p \sum_{i=1}^{M^*} \tau^*_{p,i} f(t^*_{p,i}; n_1, \ldots n_{27}) g(t^*_{p,i}; m_1, \ldots, m_{23}), \\
\text{for } q = 1 \ldots 6 \qquad 0 = &\sum_{i=1}^{N^{**}} \frac{\alpha^{**}_{q,i}}{\widetilde{\alpha}^{**}_{q,i}} \left(\frac{f(S^{**,B}_{q,i}; n_1, \ldots n_{27})}{f(T^{**,B}_{q,i}; n_1, \ldots n_{27})} - 1 \right) f(\widetilde{T}^{**,B}_{q,i}; n_1, \ldots n_{27}) g(\widetilde{T}^{**,B}_{q,i}; m_1, \ldots, m_{23}) \\
&- \sum_{i=1}^{M^{**}} \beta^{**}_{q,i} \left(\frac{1}{\widetilde{\beta}^{**}_{q,i}} \left(\frac{f(S^{**,A}_{q,i}; n_1, \ldots n_{27}) g(S^{**,A}_{q,i}; m_1, \ldots m_{23})}{f(T^{**,A}_{q,i}; n_1, \ldots n_{27}) g(T^{**,A}_{q,i}; m_1, \ldots m_{23})} - 1 \right) + s^{**}_q \right) \\
&\times f(\widetilde{T}^{**,A}_{q,i}; n_1, \ldots n_{27}) g(\widetilde{T}^{**,A}_{q,i}; m_1, \ldots m_{23}),
\end{aligned} \tag{5.18}$$

We bootstrapped the overnight yield curve and the one month yield curve using natural cubic spline interpolation on log discounts for the one month curve, we plotted these curves with their forwards in Figure 5.6. We used *Actual/Actual ISDA* as day count convention to determine the year fractions used for discounting. The piecewise-constant spread function $\eta(\cdot)$ is plotted in Figure 5.7.

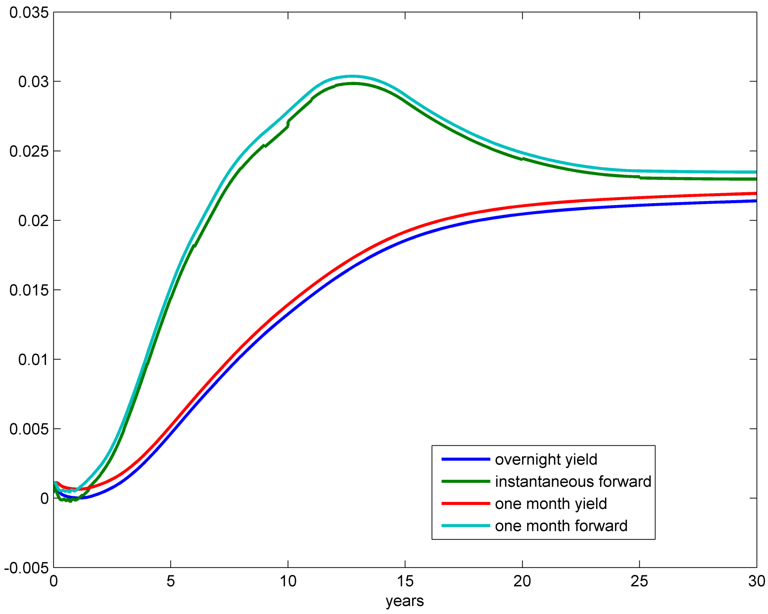

Figure 5.6: Overnight and One Month Yield and Forward Curves using Natural Cubic Spline Interpolation for the One Month Yield Curve. Swap rates in Table 5.3, 5.4 and 5.5.

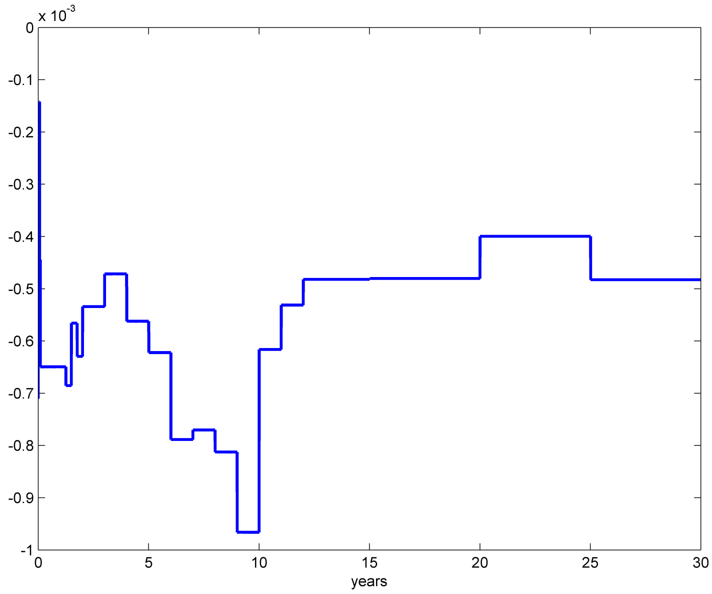

Figure 5.7: Piecewise Flat Spread Function between Instantaneous Forwards of Overnight Yield Curve and One Month Yield Curve. Swap rates in Table 5.3, 5.4 and 5.5.

In Figure 5.6, we have smooth curves to construct the one month forward rates. The instantaneous forward curve isn't smooth, but that is not a big problem as we mainly need the discount curve to discount cash flows. In the next subsection we will use Eurodollar futures to bootstrap the three month forward curve; we will use the same method we discussed in this subsection.

5.2.3 Simultaneous Bootstrapping using Eurodollar Futures

Eurodollar futures are traded on exchange markets for several maturities. This means that these products are very liquid. In Section 2.1.2, we explained that futures rates are not equal to forward rates, as there is a convexity adjustment. In Section 2.8 we explained ED futures contracts; we showed that they are quoted by

$$\text{Quote} = 1 - F(t; S, T),$$

where $F(t; S, T)$ is the futures rate on time t of an underlying index fixing at time S and maturing at time T. Using the convexity adjustment as explained in Section 2.1.2 and the notation used for forward rates of Section 4.4, we obtain the following equality:[5]

$$\text{Quote} = 1 - F(t; S, T) = 1 - CA(t; S, T) - \mathbb{E}_t^T\left(L(S, T)\right),$$

which implies

$$\mathbb{E}_t^T\left(L(S, T)\right) = 1 - \text{Quote} - CA(t; S, T).$$

And thus

$$\frac{1}{\widetilde{\alpha}}\left(\frac{P^{3\text{M}}(t, S)}{P^{3\text{M}}(t, T)} - 1\right) = 1 - \text{Quote} - CA(t; S, T), \tag{5.19}$$

where $\widetilde{\alpha}$ is the year fraction between S and T.

We will bootstrap the curves using Eonia swaps, three month fixed-floating IRS, three month-OIS basis swaps and Eurodollar futures. We used the products of [11], unfortunately the basis swaps are synthetics, but we chose to use them because we want to demonstrate the bootstrap principle using ED futures. The products we used are shown in Table 5.3, 5.6, 5.7 and 5.8. We computed the convexity adjustments for the ED futures by the method discussed in Section 6.1.2.

We will need to solve a system of equations similar to (5.10), with the addition of twenty equations like (5.19) at $t = 0$. Again, we used a piecewise-flat spread function between the instantaneous forward curves like we did in the previous subsection. We solved this system using OpenGamma, where we used natural cubic spline interpolation on the log discounts. The curves are shown in Figure 5.8 and the spread function is shown in Figure 5.9.

Maturity	Swap rate	Start Date	End Date
6 Years	0.8170%	13-12-2012	13-12-2018
7 Years	1.0000%	13-12-2012	13-12-2019
8 Years	1.1710%	13-12-2012	14-12-2020
9 Years	1.3240%	13-12-2012	13-12-2021
10 Years	1.4590%	13-12-2012	13-12-2022
12 Years	1.6920%	13-12-2012	13-12-2024
15 Years	1.9330%	13-12-2012	13-12-2027
20 Years	2.0990%	13-12-2012	13-12-2032
25 Years	2.1560%	13-12-2012	14-12-2037
30 Years	2.1860%	13-12-2012	15-12-2042

Table 5.6: 3M Euribor Swaps used for bootstrapping 3M yield curve.[6] Trade date is 11-12-2013.

Maturity	Basis Spread	Start Date	End Date
11 Years	0.178%	13-12-2012	13-12-2023
12 Years	0.176%	13-12-2012	13-12-2024
15 Years	0.169%	13-12-2012	13-12-2027
20 Years	0.160%	13-12-2012	13-12-2032
25 Years	0.153%	13-12-2012	14-12-2037
30 Years	0.148%	13-12-2012	15-12-2042

Table 5.7: Basis Swaps used for Bootstrapping Overnight and Three Month Yield Curve. Trade date is 11-12-2013.

[5]For ED futures, the fixing time is the same as the start time of the accrual period and the maturity is the same as the end time of the accrual period.

[6]Day count convention is *Actual/360* for both legs and the payment frequency is three month for both legs.

Quote	Convexity adjustment[7]	Start Date	End Date
99.8225 %	0.0000 %	19-12-2012	19-3-2013
99.8725 %	0.0000 %	20-3-2013	20-6-2013
99.8775 %	0.0000 %	19-6-2013	19-9-2013
99.8725 %	0.0000 %	18-9-2013	18-12-2013
99.8425 %	0.0000 %	18-12-2013	18-3-2014
99.8025 %	0.0001 %	19-3-2014	19-6-2014
99.7425 %	0.0001 %	18-6-2014	18-9-2014
99.6875 %	0.0002 %	17-9-2014	17-12-2014
99.6175 %	0.0004 %	17-12-2014	17-3-2015
99.5575 %	0.0007 %	18-3-2015	18-6-2015
99.4775 %	0.0011 %	17-6-2015	17-9-2015
99.3825 %	0.0017 %	16-9-2015	16-12-2015
99.2775 %	0.0026 %	16-12-2015	16-3-2016
99.1675 %	0.0038 %	16-3-2016	16-6-2016
99.0375 %	0.0055 %	15-6-2016	15-9-2016
98.9100 %	0.0077 %	21-9-2016	21-12-2016
98.7775 %	0.0104 %	21-12-2016	21-3-2017
98.6575 %	0.0136 %	15-3-2017	15-6-2017
98.5325 %	0.0173 %	21-6-2017	21-9-2017
98.4125 %	0.0216 %	20-9-2017	20-12-2017

Table 5.8: Eurodollar futures used for Bootstrapping Three Month Yield Curve. Trade date is 11-12-2013.

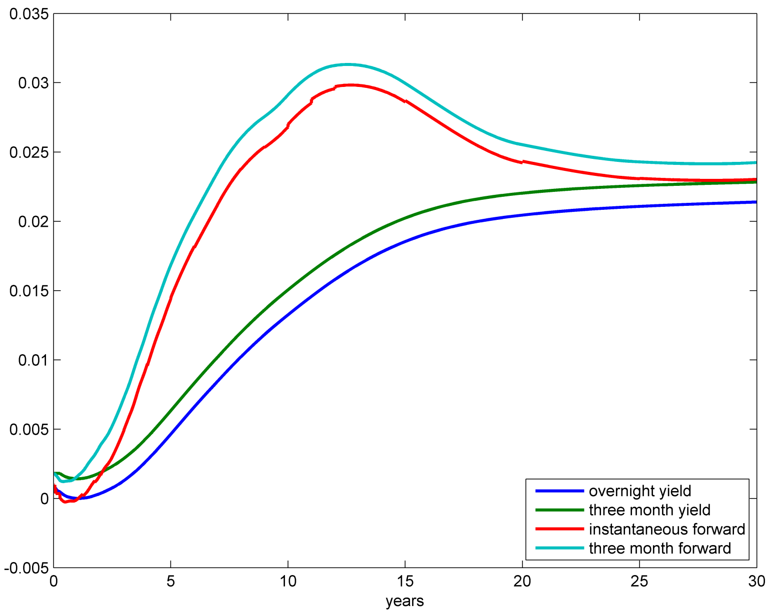

Figure 5.8: Overnight and Three Month Yield and Forward Curves using Natural Cubic Spline Interpolation for the Overnight Yield Curve. Swap rates in Table 5.3, 5.6, 5.7 and 5.8.

[7] We used the method assuming stochastic volatility, which will be discussed in Section 6.1.2, to compute the convexity adjustments. We assumed that the start and end date of all futures were at the 18'th day of the month.

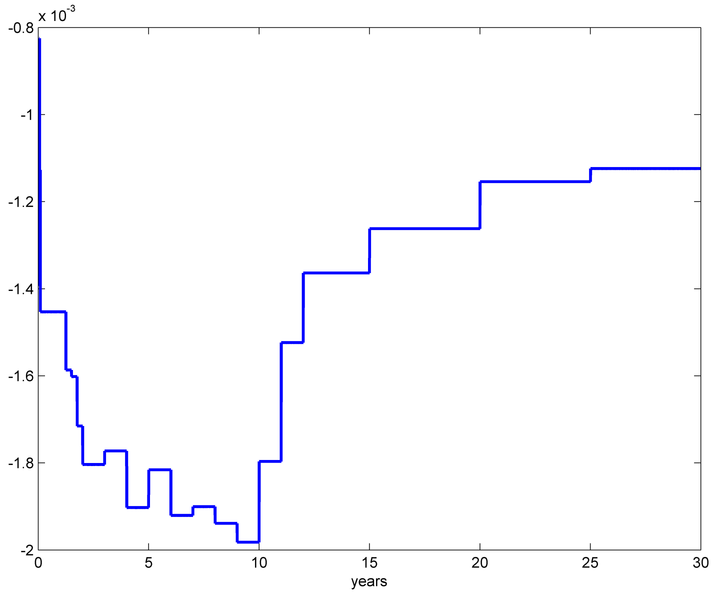

Figure 5.9: Piecewise Flat Spread Function between Instantaneous Forwards of Overnight Yield Curve and Three Month Yield Curve. Swap rates in Table 5.3, 5.6, 5.7 and 5.8.

Chapter 6

Eurodollar Futures

We explained a Eurodollar futures contract in Section 2.8. A big advantage of ED futures compared to interest rate swaps is the fact that ED futures are traded on exchange markets and therefore are very liquid. ED futures have a tenor of three months and prove to be very useful to bootstrap three month discount curves. We did bootstrap a three month discount curve in Section 5.2.3. In this chapter we will explain the existence of convexity adjustment and we will explain some formulas to approximate the convexity adjustment.

The implied futures rate of an ED futures contract is an expectation of a LIBOR rate at maturity. Forward LIBOR rates are expectations of LIBOR rates at maturity as well. However, both expectations are taken under a different measure and therefore we need to apply a small convexity adjustment in order to use market quotes of ED futures in our bootstrap algorithm.

In this chapter we will start by showing that the implied futures rate is the expected value of a LIBOR rate under the risk-neutral measure. We will continue by explaining some models to approximate the convexity adjustment. Finally, we will perform a test to compute the convexity adjustments of some futures rates. We applied the same test to the future quotes in Table 5.8. This chapter discusses the following topics:

- Futures Convexity Adjustment.
  - Hull-White Convexity Adjustment.
  - Stochastic Volatility Convexity Adjustment.
- Computing Convexity Adjustments using Stochastic Volatility.

6.1 Futures Convexity Adjustment

The following theorem will prove that the implied futures rate is an expectation of a LIBOR rate under the risk-neutral measure.
Recall that forward rates in the old resp. new framework were defined by $\mathbb{E}_t^{\widetilde{T}}\left(L(S,T)\right)$ resp. $\mathbb{E}_t^{\widetilde{T}^c}\left(L(S,T)\right)$. We will show that futures rates are expectations of LIBOR rates as well, under a different measure. Futures rates have no difference between the fixing date and the start of the accrual period as well as between the payment date and the end of the accrual period, therefore we have $S = \widetilde{S}$ and $\widetilde{T} = T$.

Theorem 6.1. *The implied futures rate of an ED futures contract is given by*

$$F(t; S, T) = \mathbb{E}_t^{Q}\left(L(S,T)\right). \tag{6.1}$$

Proof. The proof is based on [1, Ch4]. To clarify the proof, we used the theory of dividend paying assets of [9, Ch16].

We will use the theory based on the continuous risk-free rate bank account, although it may not be observable. Eurodollar futures are resettled every day, meaning the holder of a futures contract will gain the increased value of the futures rate. This gained money is put into a bank account.

We will proof the theorem by assuming continuous resettlement and neglect the probability of default. The gain process of the futures contract can be seen as the gain process of a dividend paying asset. We will consider the gain process corresponding to a futures contract maturing at T with tenor τ. Suppose we buy this futures contract at time t and sell it at time $\mathcal{T}$. The gain over the period $[t, \mathcal{T}]$ can be seen will be

$$G(t, \mathcal{T}) = V_{\text{fut}}(\mathcal{T}) + \int_t^{\mathcal{T}} e^{\int_s^{\mathcal{T}} r(s)ds} dF(s; T, T + \tau). \tag{6.2}$$

where $V_{\text{fut}}(t)$ denotes the value of the futures contract at time t. This value process is self-financing and therefore absence of arbitrage gives us

$$\begin{aligned} V_{\text{fut}}(t) &= B_t \mathbb{E}_t^Q \left(\frac{G(t, \mathcal{T})}{B_{\mathcal{T}}} \right) && (6.3) \\ &= \mathbb{E}_t^Q \left(e^{-\int_t^{\mathcal{T}} r(s)ds} V_{\text{fut}}(\mathcal{T}) + \int_t^{\mathcal{T}} e^{-\int_t^s r(s)ds} dF(s; T, T + \tau) \right). && (6.4) \end{aligned}$$

As it is always costless to enter into a futures contract, $V_{\text{fut}}(t) = V_{\text{fut}}(\mathcal{T}) = 0$ by definition, so we must have

$$\mathbb{E}_t^Q \left(\int_t^{\mathcal{T}} e^{-\int_t^s r(s)ds} dF(s; T, T + \tau) \right) = 0. \tag{6.5}$$

As $e^{-\int_t^s r(s)ds} > 0$, it follows that

$$\mathbb{E}_t^Q \left(dF(s, T, T + \tau) \right) = 0, \quad t \leq s \leq T, \tag{6.6}$$

which demonstrates that F is a Q-martingale. As $F(T; T, T + \tau) = L(T, T + \tau)$, we have (6.1). $\square$

For the old framework as well as the new framework the futures rates and forward rates are defined under a different expectation and therefore

$$\begin{aligned} F(t; S, T) &= \mathbb{E}_t^Q \left(L(S, T) \right) \neq \mathbb{E}_t^T \left(L(S, T) \right), && (6.7) \\ F(t; S, T) &= \mathbb{E}_t^Q \left(L(S, T) \right) \neq \mathbb{E}_t^{T^c} \left(L(S, T) \right). && (6.8) \end{aligned}$$

The theory in this chapter will hold for the old framework as well as the new framework. We will introduce some notations which we will use throughout this chapter:

Note: *In the remaining of this chapter we will denote the forward rate by*

$$L(t; S, T)$$

and the (pseudo) discount curve by

$$P(t, T),$$

i.e.

Old framework:

$$L(t; S, T) = \mathbb{E}_t^T \left(L(S, T) \right) = \frac{1}{\tau} \left(\frac{P(t, S)}{P(t, T)} - 1 \right). \tag{6.9}$$

New framework:

$$\begin{aligned} P(t, T) &:= P^\tau(t, T) && (6.10) \\ L(t; S, T) &:= \mathbb{E}_t^{T^c} \left(L(S, T) \right) = \frac{1}{\tau} \left(\frac{P^\tau(t, S)}{P^\tau(t, T)} - 1 \right) := \frac{1}{\tau} \left(\frac{P(t, S)}{P(t, T)} - 1 \right) = \mathbb{E}_t^T \left(L(S, T) \right) \end{aligned}$$

The difference between the futures and forward rate is called the convexity adjustment ($CA(t; S, T)$):

$$F(t; S, T) - L(t; S, T) = CA(t; S, T). \tag{6.11}$$

6.1.1 Hull-White Convexity Adjustment

We will start with computing the convexity adjustment using a deterministic volatility model, namely the Hull-White one-factor model. We use [12] to compute the convexity adjustment for this model. The Hull-White interest rate model is given by

$$dr(t) = (\Theta(t) - ar(t))\,dt + \sigma dW(t), \tag{6.12}$$

where $\Theta(t)$ is a deterministic function of time, a is the mean reversion factor and σ is the deterministic volatility. The corresponding HJM[1] formulation of the instantaneous forward rate is given by

$$df(t,T) = \alpha(t,T)dt + \sigma e^{-a(T-t)}dW(t), \tag{6.13}$$

where $\alpha(t)$ is a deterministic function.

We express P by

$$P(t,T) = \exp\left\{-\int_t^T f(t,s)ds\right\} \tag{6.14}$$

and use [12, Thm1] to express $F(t;S,T)$ in terms of $P(\cdot,\cdot)$, a and σ:

$$F(t;S,T) = \frac{1}{\widetilde{\alpha}}\left(\frac{P(t,S)}{P(t,T)}e^{Z(t)} - 1\right), \tag{6.15}$$

where

$$Z(t) = \frac{\sigma^2}{2a^3}\left(e^{-aS} - e^{-aT}\right)\left(e^{aS} - e^{at}\right)\left(2 - e^{-a(T-S)} - e^{-a(T-t)}\right). \tag{6.16}$$

6.1.2 Stochastic Volatility Convexity Adjustment

To give a better approximation of the convexity adjustment, we should be aware of the existence of stochastic volatility. We will use several assumptions in our model. Dropping these assumptions and use other methods, like Monte Carlo, will probably give better approximations. However, using Monte Carlo will decrease the speed of our model and therefore we chose to make some assumptions to improve the speed of our model.

The method we will use to approximate the convexity adjustment is mainly based on [1, Sec16.8] and [2]. However, their method doesn't take the possibility into account that the trade date and the start date of an ED futures contract will be on a different date. They also do not explain how to compute the forward LIBOR rate at time zero. We will deal with these shortcomings and we will explain all our steps in detail.

In the first subsection we will use a Taylor approximation to give an equation for the forward rate in terms of futures rates. In the next subsection we will use a stochastic volatility model to obtain the final formula we will use for convexity adjustments. We will assume that the maturity time and the fixing time of two successive ED futures will be the same, $\{T_n\}_{n=0}^{N-1}$ is the collection of fixing times, also used as maturity times.

In the next subsections we will use stochastic volatility models to approximate the variance of forward rates and the correlation between forward rates. We will end with our final theorem.

Taylor Expansion around the Futures Value

We will start by denoting a new, discretely compounded, money market account $\widetilde{B}_t$:

$$\widetilde{B}(t) = \begin{cases} \dfrac{1}{P(0,t)} & \text{for } t \le T_0 \\ \dfrac{P(t,T_{n+1})}{P(0,T_0)} \cdot \prod_{i=0}^{n}\left(1 + \tau_i L(T_i,T_{i+1})\right), \quad T_n < t \le T_{n+1} & \text{for } t > T_0 \end{cases}, \tag{6.17}$$

We will show that the futures rate is an expectation of a LIBOR rate, using this money market account as numeraire:

[1] The Heath-Jarrow-Morton framework assumes that the instantaneous forward rate has a stochastic differential, see e.g. [9, Ch25].

Lemma 6.2. *The implied futures rate of an ED futures contract is given by*

$$F(t; S, T) = \mathbb{E}_t^{\widetilde{B}}\left(L(S, T)\right). \tag{6.18}$$

Proof. Again, we will assume no arbitrage and use the self-financing property of V_{fut}, with gain function (6.2). This time we use a different numeraire to price V_{fut}(t):

$$\begin{aligned} V_{\text{fut}}(t) &= \widetilde{B}_t \mathbb{E}_t^{\widetilde{B}}\left(\frac{G(t,\mathcal{T})}{\widetilde{B}_{\mathcal{T}}}\right) & (6.19)\\ &= \mathbb{E}_t^{\widetilde{B}}\left(\frac{\widetilde{B}_t}{\widetilde{B}_{\mathcal{T}}} V_{\text{fut}}(\mathcal{T}) + \int_t^{\mathcal{T}} \frac{e^{\int_s^{\mathcal{T}} r(s)ds}}{\widetilde{B}_{\mathcal{T}}} dF(s; T, T+\tau)\right). & (6.20)\end{aligned}$$

Again, we use $V_{\text{fut}}(t) = V_{\text{fut}}(\mathcal{T}) = 0$. This gives us

$$\mathbb{E}_t^{\widetilde{B}}\left(\int_t^{\mathcal{T}} \frac{e^{\int_s^{\mathcal{T}} r(s)ds}}{\widetilde{B}_{\mathcal{T}}} dF(s; T, T+\tau)\right) = 0. \tag{6.21}$$

As $e^{\int_t^{\mathcal{T}} r(s)ds} > 0$ and $\widetilde{B}_{\mathcal{T}} > 0$ we have

$$\mathbb{E}_t^{\widetilde{B}}\left(dF(s; T, T+\tau)\right) = 0, \qquad \text{for } t \le s \le T, \tag{6.22}$$

which demonstrates that F is a $\widetilde{B}$-martingale. As $F(T; T, T+\tau) = L(T, T+\tau)$, we have (6.18). □

We will change the numeraire using the Radon-Nikodym derivative to express the forward LIBOR rate in the measure associated with the numeraire $\widetilde{B}$:

Lemma 6.3. *For each n, $n = 0, \ldots N-1$,*

$$L(0, T_n, T_{n+1}) = \mathbb{E}_0^{\widetilde{B}}\left(\left[\prod_{i=0}^{n} \frac{1+\tau_i L(0, T_i, T_{i+1})}{1+\tau_i L(T_i, T_{i+1})}\right] L(T_n, T_{n+1})\right). \tag{6.23}$$

Proof. We have

$$L(0, T_n, T_{n+1}) = \mathbb{E}_0^{T_{n+1}}\left(L(T_n, T_{n+1})\right).$$

We use the change of numeraire, our Radon-Nikodym derivative is

$$\varsigma(t) = \frac{P(t, T_{n+1})/P(0, T_{n+1})}{\widetilde{B}(t)/\widetilde{B}(0)}.$$

This gives

$$\begin{aligned} \mathbb{E}_0^{T_{n+1}}\left(L(T_n, T_{n+1})\right) &= \mathbb{E}_0^{\widetilde{B}}\left(L(T_n, T_{n+1}) \frac{P(T_n, T_{n+1})/P(0, T_{n+1})}{\widetilde{B}(T_n)/\widetilde{B}(0)}\right) \\ &= \mathbb{E}_0^{\widetilde{B}}\left(L(T_n, T_{n+1}) \cdot \frac{1}{\prod_{i=0}^{n}\left(1+\tau_i L(T_i, T_{i+1})\right)} \cdot \frac{P(0, T_0)}{P(0, T_{n+1})}\right) \\ &= \mathbb{E}_0^{\widetilde{B}}\left(\left[\prod_{i=0}^{n} \frac{1+\tau_i L(0, T_i, T_{i+1})}{1+\tau_i L(T_i, T_{i+1})}\right] L(T_n, T_{n+1})\right). \end{aligned}$$

□

We define $L^\varepsilon(\cdot; \cdot, \cdot)$ and $V(\varepsilon)$ in the next two definitions.

Definition 6.4. *We fix $\varepsilon > 0$ and define $L^\varepsilon(t; T_n, T_{n+1})$ by*

$$L^\varepsilon(t; T_n, T_{n+1}) = \varepsilon(L(t; T_n, T_{n+1}) - F(0; T_n, T_{n+1})) + F(0; T_n, T_{n+}).$$

Note that

$$\begin{aligned} L^1(t; T_n, T_{n+1}) &= L(t; T_n, T_{n+1}) & (6.24)\\ L^0(t; T_n, T_{n+1}) &= F(0; T_n, T_{n+1}) & (6.25)\\ \frac{\partial L^\varepsilon(t; T_n, T_{n+1})}{\partial \varepsilon} &= L(t; T_n, T_{n+1}) - F(0; T_n, T_{n+1}). & (6.26)\end{aligned}$$

Definition 6.5. *We define*

$$V(\varepsilon) = \left[\prod_{i=0}^{n} \frac{1+\tau_i L(0;T_i,T_{i+1})}{1+\tau_i L^{\varepsilon}(T_i,T_{i+1}))}\right] L^{\varepsilon}(T_n,T_{n+1}).$$

We expand $V(\varepsilon)$ into a Taylor series:

$$V(\varepsilon) = V(0) + \frac{dV}{d\varepsilon}(0)\cdot\varepsilon + \frac{1}{2}\frac{d^2V}{d\varepsilon^2}(0)\cdot\varepsilon^2 + \mathcal{O}(\varepsilon^3), \tag{6.27}$$

which implies

$$V(1) \approx V(0) + \frac{dV}{d\varepsilon}(0) + \frac{1}{2}\frac{d^2V}{d\varepsilon^2}(0). \tag{6.28}$$

Using Lemma 6.3 we have,

$$L(0;T_n,T_{n+1}) = \mathbb{E}_0^{\widetilde{B}}\left(V(1)\right). \tag{6.29}$$

And thus we have an approximation for the forward LIBOR rate at time zero:

$$L(0;T_n,T_{n+1}) \approx V(0) + \mathbb{E}_0^{\widetilde{B}}\left(\frac{dV}{d\varepsilon}(0)\right) + \frac{1}{2}\mathbb{E}_0^{\widetilde{B}}\left(\frac{d^2V}{d\varepsilon^2}(0)\right). \tag{6.30}$$

We compute $\dfrac{dV}{d\varepsilon}$:

$$\begin{aligned}
\frac{dV(\varepsilon)}{d\varepsilon} &= \left(\sum_{j=0}^{n} \frac{\partial V}{\partial L^{\varepsilon}(T_j,T_{j+1})}\cdot\frac{\partial L^{\varepsilon}(T_j,T_{j+1})}{\partial\varepsilon}\right) && (6.31)\\
&= \left(\sum_{j=0}^{n} \frac{\partial V}{\partial L^{\varepsilon}(T_j,T_{j+1})}\cdot\left(L(T_j,T_{j+1}) - F(0;T_j,T_{j+1})\right)\right) && (6.32)\\
&= \left(\prod_{i=0}^{n}(1+\tau_i L(0;T_i,T_{i+1}))\right) && (6.33)\\
&\quad\times\left(\sum_{j=0}^{n} \frac{\partial\left(\left[\prod_{i=0}^{n}\frac{1}{1+\tau_i L^{\varepsilon}(T_i,T_{i+1})}\right]L^{\varepsilon}(T_n,T_{n+1})\right)}{\partial L^{\varepsilon}(T_j,T_{j+1})}\cdot\left(L(T_j,T_{j+1}) - F(0;T_j,T_{j+1})\right)\right).
\end{aligned}$$

We compute $\dfrac{d^2V}{d\varepsilon^2}$:

$$\begin{aligned}
\frac{d^2V(\varepsilon)}{d\varepsilon^2} &= \left(\prod_{i=0}^{n}(1+\tau_i L(0;T_i,T_{i+1}))\right) && (6.34)\\
&\quad\times\left(\sum_{j,m=0}^{n} \frac{\partial^2\left(\left[\prod_{i=0}^{n}\frac{1}{1+\tau_i L^{\varepsilon}(T_i,T_{i+1})}\right]L^{\varepsilon}(T_n,T_{n+1})\right)}{\partial L^{\varepsilon}(T_j,T_{j+1})\partial L^{\varepsilon}(T_m,T_{m+1})}\cdot\frac{\partial L^{\varepsilon}(T_m,T_{m+1})}{\partial\varepsilon}\cdot\left(L(T_j,T_{j+1}) - F(0;T_j,T_{j+1})\right)\right)\\
&= \left(\prod_{i=0}^{n}(1+\tau_i L(0;T_i,T_{i+1}))\right)\left(\sum_{j,m=0}^{n} \frac{\partial^2\left(\left[\prod_{i=0}^{n}\frac{1}{1+\tau_i L^{\varepsilon}(T_i,T_{i+1})}\right]L^{\varepsilon}(T_n,T_{n+1})\right)}{\partial L^{\varepsilon}(T_j,T_{j+1})\partial L^{\varepsilon}(T_m,T_{m+1})}\right. && (6.35)\\
&\quad\left.\times\left(L(T_j,T_{j+1}) - F(0;T_j,T_{j+1})\right)\left(L(T_m,T_{m+1}) - F(0;T_m,T_{m+1})\right)\right).
\end{aligned}$$

As $\mathbb{E}_0^{\widetilde{B}}\left(L(t;T_i,T_{i+1}) - F(0;T_i,T_{i+1})\right) = 0$, (6.33) gives us

$$\mathbb{E}_0^{\widetilde{B}}\left(\frac{dV}{d\varepsilon}(0)\right) = 0. \tag{6.36}$$

By (6.35) we have

$$\mathbb{E}_0^{\widetilde{B}}\left(\frac{d^2V}{d\varepsilon^2}(0)\right) = \left[\prod_{i=0}^{n}\frac{1+\tau_i L(0;T_i,T_{i+1})}{1+\tau_i F(0;T_i,T_{i+1})}\right] F(0;T_n,T_{n+1}) \tag{6.37}$$

$$\times \sum_{j,m=0}^{n} D_{j,m}\mathbb{E}_0^{\widetilde{B}}\left[(L(T_j,T_{j+1}) - F(0;T_j,T_{j+1}))(L(T_m,T_{m+1}) - F(0;T_m,T_{m+1})\right]$$

$$= V(0)\sum_{j,m=1}^{n} D_{j,m}\mathbb{E}_0^{\widetilde{B}}\left[(L(T_j,T_{j+1}) - \mathbb{E}_0^{\widetilde{B}}(L(T_j,T_{j+1})))(L(T_m,T_{m+1}) - \mathbb{E}_0^{\widetilde{B}}(L(T_m,T_{m+1})))\right]$$

$$= V(0)\sum_{j,m=1}^{n} D_{j,m}\mathrm{Cov}^{\widetilde{B}}\left(L(T_j,T_{j+1}), L(T_m,T_{m+1})\right), \tag{6.38}$$

where

$$D_{j,m} = \left(-\frac{\tau_j}{1+\tau_j F(0;T_j,T_{j+1})} + \frac{\mathbb{1}_{\{j=n\}}}{F(0;T_n,T_{n+1})}\right)\left(-\frac{\tau_m}{1+\tau_m F(0;T_m,T_{m+1})} + \frac{\mathbb{1}_{\{m=n\}}}{F(0;T_n,T_{n+1})}\right)$$
$$+ \mathbb{1}_{\{j=m\}}\left(\frac{\tau_j^2}{(1+\tau_j F(0;T_j,T_{j+1}))^2} - \frac{\mathbb{1}_{\{j=n\}}}{F(0;T_n,T_{n+1})^2}\right). \tag{6.39}$$

This will gives us the following theorem:

Theorem 6.6. *For any $n \in \{0,\ldots,N-1\}$, an approximation to the forward rate $L(0;T_n,T_{n+1})$ is given by*

$$L(0;T_n,T_{n+1}) = V(0) + \frac{1}{2}V(0)\sum_{j,m=0}^{n} D_{j,m}\mathrm{Cov}^{\widetilde{B}}\left(L(T_j,T_{j+1}), L(T_m,T_{m+1})\right), \tag{6.40}$$

where $V(0)$ and $D_{j,m}$ are given in Definition 6.5 and (6.39)*.*

Note: *For $T_0 = 0$ we have*

$$L(0;T_0,T_1) = L(0;0,T_1) = V(0). \tag{6.41}$$

Proof.

$$\begin{aligned}
L(0;0,T_1) &= V(0) + \frac{1}{2}V(0)D_{0,0}\mathrm{Cov}^{\widetilde{B}}\left(L(0,T_1), L(0,T_1)\right) \\
&= V(0) + \frac{1}{2}V(0)D_{0,0}\mathrm{Var}^{\widetilde{B}}\left(L(0,T_1)\right) \\
&= V(0) + \frac{1}{2}V(0)D_{0,0}\mathbb{E}^{\widetilde{B}}\left(\left(L(0,T_1) - \mathbb{E}^{\widetilde{B}}\left(L(0,T_1)\right)\right)^2\right) \\
&= V(0).
\end{aligned}$$

□

Forward Rate Variances

The formula (6.40) depends on covariances between several forward rates. By definition of the covariance we have

$$\mathrm{Cov}^{\widetilde{B}}(L(T_j,T_{j+1}), L(T_m,T_{m+1})) = \sqrt{\mathrm{Var}^{\widetilde{B}}\left(L(T_j,T_{j+1})\right)\mathrm{Var}^{\widetilde{B}}\left(L(T_m,T_{m+1})\right)}$$
$$\times\, \mathrm{Corr}^{\widetilde{B}}\left(L(T_j,T_{j+1}), L(T_m,T_{m+1})\right). \tag{6.42}$$

The variance in (6.42) is given in the $\widetilde{B}$-measure. As an approximation, we will compute the variance in the forward measure as the forward rate is a martingale in this measure,

$$\mathrm{Var}^{\widetilde{B}}\left(L(T_n,T_{n+1})\right) \approx \mathrm{Var}^{T_{n+1}}\left(L(T_n,T_{n+1})\right). \tag{6.43}$$

In [1, Ch16] they show that if you want to compute the variance of a forward rate using a model independent approach, we will need to integrate Eurodollar futures options prices. This may not be easy to compute in practice as only a discrete set of strikes is typically traded, and these are not very liquid. Therefore we need to adopt a model. We use a standard stochastic volatility model in which the forward rate is a martingale:

$$\begin{aligned} dL_n(t) &= \sigma_n(b_n L_n(t) + (1-b_n)L_n(0))\sqrt{z(t)}\,dW^{T_{n+1}}(t) \\ dz(t) &= \theta(z_0 - z(t))dt + \eta_n\sqrt{z(t)}\,dZ(t), \end{aligned} \tag{6.44}$$

with correlation $\langle dW^{T_{n+1}}(t), dZ(t)\rangle = \rho$,

where

$L_n(t)$:	Shorthand notation for $L(t;T_n,T_{n+1})$.
σ_n:	Volatility level.
b_n:	Slope of the volatility.
$z(t)$:	Stochastic variance process.
z_0:	Long run variance.
θ:	Mean reversion speed of the variance.
η_n:	Volatility of variance.

For simplicity assumptions, we will use $\rho = 0$ and $z_0 = 1$.

The next proposition shows that the SDE (6.44) can be integrated explicitly.

Proposition 6.7. *In* (6.44) *we have*

$$L_n(t) = \frac{1}{b_n}\left\{L_n(0)X(t) - (1-b_n)L_n(0)\right\}, \tag{6.45}$$

where

$$\frac{dX(t)}{X(t)} = \sigma_n b_n\sqrt{z(t)}\,dW^{T_{n+1}}(t), \qquad X(0) = 1, \tag{6.46}$$

i.e.

$$\ln X(t) = \sigma_n b_n \int_0^t \sqrt{z(s)}\,dW^{T_{n+1}}(s) - \frac{1}{2}\sigma_n^2 b_n^2 \int_0^t z(s)ds. \tag{6.47}$$

Proof. We define $X(t)$ and $Y(t)$ by

$$X(t) = \frac{1}{L_n(0)}\left\{b_n L_n(t) + (1-b_n)L_n(0)\right\} \tag{6.48}$$

$$Y(t) = \ln\left\{b_n L_n(t) + (1-b_n)L_n(0)\right\} \tag{6.49}$$

$$= \ln\{L_n(0)\} + \ln\{X(t)\}. \tag{6.50}$$

This gives

$$dY(t) = \frac{1}{X(t)}dX(t) - \frac{1}{2}\frac{1}{X(t)^2}dX(t)dX(t). \tag{6.51}$$

By (6.49) we have

$$L_n(t) = \frac{1}{b_n}\left\{e^{Y(t)} - (1-b_n)L_n(0)\right\}. \tag{6.52}$$

This implies

$$dL_n(t) = \frac{1}{b_n}e^{Y(t)}dY(t) + \frac{1}{2}\frac{1}{b_n}e^{Y(t)}dY(t)dY(t) \tag{6.53}$$

$$= \frac{1}{b_n}\left\{b_n L_n(t) + (1-b_n)L_n(0)\right\}dY(t) + \mathcal{O}(dY(t)dY(t)) \tag{6.54}$$

$$= \frac{1}{b_n}\left\{b_n L_n(t) + (1-b_n)L_n(0)\right\}\frac{dX(t)}{X(t)} + \mathcal{O}(dX(t)dX(t)). \tag{6.55}$$

This means (6.55) equals (6.44) iff

$$\frac{dX(t)}{X(t)} = \sigma_n b_n\sqrt{z(t)}\,dW^{T_{n+1}}(t). \tag{6.56}$$

By (6.48) we have $X(0) = 1$ and thus by (6.56) we have

$$X(t) = \exp\left\{\sigma_n b_n \int_0^t \sqrt{z(s)}\, dW^{T_{n+1}}(s) - \frac{1}{2}\sigma_n^2 b_n^2 \int_0^t z(s) ds\right\}, \tag{6.57}$$

converting (6.48) gives (6.45) and we have completed the proof. □

Proposition 6.8. *We have*

$$\mathrm{Var}^{T_{n+1}}(L(T_n, T_{n+1})) = \frac{L(0; T_n, T_{n+1})}{b_n^2} \left\{\mathbb{E}_0^{T_{n+1}}\left(e^{\sigma_n^2 b_n^2 \int_0^t z(s)ds}\right) - 1\right\} \tag{6.58}$$

Proof.

$$\begin{aligned}
\mathrm{Var}^{T_{n+1}}\left(L(T_n, T_{n+1})\right) &= \mathbb{E}_0^{T_{n+1}}\left(\left(L_n(T_n) - \mathbb{E}_0^{T_{n+1}}(L_n(T_n))\right)^2\right) && (6.59)\\
&= \mathbb{E}_0^{T_{n+1}}\left(\left(\frac{L_n(0)}{b_n}X(T_n) - \frac{L_n(0)}{b_n}\mathbb{E}(X(T_n))\right)^2\right) && (6.60)\\
&= \frac{L_n(0)^2}{b_n^2}\left\{\mathbb{E}_0^{T_{n+1}}\left(X(T_n) - 1\right)^2\right\} && (6.61)\\
&= \frac{L_n(0)^2}{b_n^2}\left\{\mathbb{E}_0^{T_{n+1}}\left(X(T_n)^2\right) - 1\right\} && (6.62)\\
&= \frac{L_n(0)^2}{b_n^2}\left\{\mathbb{E}_0^{T_{n+1}}\left(\exp\left\{2\sigma_n b_n \int_0^{T_n} \sqrt{z(s)}\, dW^{T_{n+1}}(s) - \sigma_n^2 b_n^2 \int_0^{T_n} z(s) ds\right\}\right) - 1\right\} &&\\
&= \frac{L_n(0)^2}{b_n^2}\left\{\mathbb{E}_0^{T_{n+1}}\left(\varsigma(T_n)\exp\left\{\sigma_n^2 b_n^2 \int_0^{T_n} z(s) ds\right\}\right) - 1\right\}, && (6.63)
\end{aligned}$$

where $\varsigma(t)$ is the exponential martingale

$$\begin{aligned}
\varsigma(t) &= \exp\left\{2\sigma_n b_n \int_0^t \sqrt{z(s)}\, dW^{T_{n+1}}(s) - \frac{1}{2}4\sigma_n^2 b_n^2 \int_0^t z(s) ds\right\} && (6.64)\\
&= \mathcal{E}\left(2\sigma_n b_n \int_0^t \sqrt{z(s)}\, dW^{T_{n+1}}(s)\right), && (6.65)
\end{aligned}$$

where $\mathcal{E}(\cdot)$ is the *Doleans exponential.* We use $\varsigma(t)$ as density process for a measure change. By Girsanov's theorem and Doleans exponential we have

$$d\widetilde{W}(t) = dW^{T_{n+1}}(t) - 2\sigma_n b_n \sqrt{z(t)}\, dt, \tag{6.66}$$

where $\widetilde{W}(t)$ is a $\widetilde{P}$-Brownian Motion. This gives us

$$\mathbb{E}_0^{T_{n+1}}\left(\varsigma(T_n)\exp\left\{\sigma_n^2 b_n^2 \int_0^{T_n} z(s) ds\right\}\right) = \mathbb{E}_0^{\widetilde{P}}\left(\exp\left\{\sigma_n^2 b_n^2 \int_0^{T_n} z(s) ds\right\}\right). \tag{6.67}$$

Under the old measure T_{n+1} we had

$$dZ(t) = \rho dW_{T_{n+1}}(t) + \sqrt{1-\rho^2}\, dB(t). \tag{6.68}$$

with $B(t)$ some Brownian Motion. The corresponding process under the new measure will be

$$d\widetilde{Z}(t) = \rho d\widetilde{W}(t) + \sqrt{1-\rho^2}\, dB(t), \tag{6.69}$$

with $\widetilde{Z}$ a $\widetilde{P}$-Brownian Motion. Combining (6.66), (6.68) and (6.69) gives us

$$dZ(t) = d\widetilde{Z}(t) + \rho 2\sigma_n b_n \sqrt{z(t)}\, dt. \tag{6.70}$$

This means that under the new probability measure $\widetilde{P}$ the process for $z(\cdot)$ is

$$dz(t) = \left(\theta(z_0 - z(t)) + \rho 2\eta_n \sigma_n b_n z(t)\right) dt + \eta_n \sqrt{z(t)}\, d\widetilde{Z}(t), \tag{6.71}$$

as we assumed $\rho = 0$ and $z_0 = 1$, this will be

$$dz(t) = \theta(1 - z(t))dt + \eta_n \sqrt{z(t)}\, d\widetilde{Z}(t). \tag{6.72}$$

As this equation is equal to the one in (6.44), we have $\widetilde{Z} = Z$. This means the change of measure does not affect $z(\cdot)$ and therefore

$$\mathbb{E}_0^{\widetilde{P}}\left(\exp\left\{\sigma_n^2 b_n^2 \int_0^{T_n} z(s)ds\right\}\right) = \mathbb{E}_0^{T_{n+1}}\left(\exp\left\{\sigma_n^2 b_n^2 \int_0^{T_n} z(s)ds\right\}\right). \tag{6.73}$$

Using (6.63), (6.67) and (6.73) gives us (6.58). $\square$

Proposition 6.9. *For*

$$\mathbb{E}_0^{T_{n+1}}\left(\exp\left\{\sigma_n b_n \int_0^{T_n} z(s)ds\right\}\right),$$

with

$$dz(t) = \theta(z_0 - z(t))dt + \eta_n \sqrt{z(t)}\, dZ(t), \qquad z(0) = z_0,$$

we have

$$\begin{aligned}
\mathbb{E}_0^{T_{n+1}}\left(\exp\left\{\sigma_n b_n \int_0^{T_n} z(s)ds\right\}\right) &= \exp\{A(0, T_n) + B(0, T_n)z_0\}, \\
A(0, T_n) &= \frac{\theta z_0}{\eta_n^2}\left[2\ln\left\{\frac{2\gamma}{\theta + \gamma - e^{-\gamma T_n}(\theta - \gamma)}\right\} + (\theta - \gamma)T_n\right], \\
B(0, T_n) &= \frac{2\sigma_n^2 b_n^2 (1 - e^{-\gamma T_n})}{(\theta + \gamma)(1 - e^{-\gamma T_n}) + 2\gamma e^{-\gamma T_n}}, \\
\gamma &= \sqrt{\theta^2 - 2\eta_n^2 \sigma_n^2 b_n^2}.
\end{aligned}$$

Proof. We denote a function $G(t, z)$ and process $z(t)$ by

$$G(t, z) = \mathbb{E}_t^{T_{n+1}}\left(\exp\left\{\int_t^{T_n} \sigma_n^2 b_n^2 z(s)ds\right\} \middle| \, z(t) = z\right) \tag{6.74}$$

$$dz(t) = \theta(z_0 - z(t))dt + \eta_n \sqrt{z(t)}\, dZ(t), \qquad z(0) = z_0. \tag{6.75}$$

We assume $G(t, z)$ has an *affine term structure*:

$$G(t, z) = \exp\{A(t, T_n) + B(t, T_n)z\}. \tag{6.76}$$

By (6.74) and (6.75) follows that $G(t, z)$ is a solution of the Feynman-Kac problem, which means that $G(t, z)$ is a solution of the following boundary value problem:

$$\frac{\partial G(t, z)}{\partial t} + \theta(1 - z)\frac{\partial G(t, z)}{\partial z} + \frac{1}{2}\eta_n^2 z \frac{\partial^2 G(t, z)}{\partial z^2} = 0 \tag{6.77}$$

$$G(T_n, z) = 1. \tag{6.78}$$

Filling in $G(t, z)$ in the form of (6.76) in this differential equation gives us

$$\frac{dA(t, T_n)}{dt}G(t, z) + \frac{dB(t, T_n)}{dt}zG(t, z) + \theta(z_0 - z)B(t, z)G(t, z) + \frac{1}{2}\eta_n^2 z B(t, z)^2 G(t, z) = 0, \tag{6.79}$$

we divide by $G(t, z)$, as $G(t, z) > 0$:

$$\frac{dA(t, T_n)}{dt} + \frac{dB(t, T_n)}{dt}z + \theta(z_0 - z)B(t, z) + \frac{1}{2}\eta_n^2 z B(t, z)^2 = 0. \tag{6.80}$$

Or written differently

$$\frac{dA(t, T_n)}{dt} + \theta z_0 B(t, z) + z\left(\frac{dB(t, T_n)}{dt} - \theta B(t, z) + \frac{1}{2}\eta_n^2 B(t, z)^2\right) = 0. \tag{6.81}$$

As z can be anything, we will have the following system of differential equations

$$\frac{dA(t,T_n)}{dt} + \theta z_0 B(t,z) = 0 \tag{6.82}$$

$$\frac{dB(t,T_n)}{dt} - \theta B(t,z) + \frac{1}{2}\eta_n^2 B(t,z)^2 = 0, \tag{6.83}$$

with terminal conditions (by $G(T_n, z) = 1$)

$$A(T_n, T_n) = 0, \quad B(T_n, T_n) = 0. \tag{6.84}$$

One can show that the following equations are direct solutions of the system of equations (6.82),(6.83).

$$A(t,T_n) = \frac{\theta z_0}{\eta_n^2}\left[2\ln\left\{\frac{2\gamma}{\theta+\gamma-e^{-\gamma(T_n-t)}(\theta-\gamma)}\right\} + (\theta-\gamma)(T_n-t)\right], \tag{6.85}$$

$$B(t,T_n) = \frac{2\sigma_n^2 b_n^2(1-e^{-\gamma(T_n-t)})}{(\theta+\gamma)(1-e^{-\gamma(T_n-t)}) + 2\gamma e^{-\gamma(T_n-t)}}, \tag{6.86}$$

$$\gamma = \sqrt{\theta^2 - 2\eta_n^2\sigma_n^2 b_n^2}. \tag{6.87}$$

As $G(0,z_0) = \mathbb{E}_0^{T_{n+1}}\left(\exp\left\{\sigma_n b_n \int_0^{T_n} z(s)ds\right\}\right)$, the result follows by (6.76). □

Forward Rate Correlations

To compute forward rate correlations, we will use a slightly different model, where we assume that the dynamics of LIBOR (Euribor) rates originate from a time-stationary Gaussian process of the mean-reverting type

$$dL_i(t) = \mathcal{O}(dt) + \sigma_0 e^{-\chi(T_i-t}dW_i(t), \tag{6.88}$$

where σ_0 is the volatility level at $t = T_i$ and χ is reflecting the effect of time on the change of the volatility. $W_j(t)$ and $W_m(t)$ are scalar $\widetilde{B}$-Brownian Motions with correlation independent of t:

$$\langle dW_j(t), dW_m(t)\rangle = \rho_{j,m}dt.$$

Proposition 6.10. *Under the forward rate model given by* (6.88)*, the correlation between two forward rates is given by*

$$\mathrm{Corr}^{\widetilde{B}}\left(L_j(T_j), L_m(T_m)\right) = \rho_{j,m}\sqrt{\frac{e^{2\chi\min\{T_j,T_m\}}-1}{e^{2\chi\max\{T_j,T_m\}}-1}}.$$

Proof. The forward rate using (6.88) can be integrated explicitly by

$$L_j(t) = L_j(0) + \int_0^t f(s)ds + \int_0^t \sigma_0 e^{-\chi(T_j-s)}dW_j(s). \tag{6.89}$$

The covariance between $L_j(t)$ and $L_m(T_m)$ is then given by

$$\begin{aligned}
\mathrm{Cov}^{\widetilde{B}}\left(L_j(T_j), L_m(T_m)\right) &= \mathbb{E}^{\widetilde{B}}\left(\left(L_j(T_j) - \mathbb{E}^{\widetilde{B}}(L_j(T_j))\right)\left(L_m(T_m) - \mathbb{E}^{\widetilde{B}}(L_m(T_m))\right)\right) && (6.90)\\
&= \mathbb{E}^{\widetilde{B}}\left(\int_0^{T_j}\sigma_0 e^{-\chi(T_j-s)}dW_j(s)\int_0^{T_m}\sigma_0 e^{-\chi(T_m-s)}dW_m(s)\right) && (6.91)\\
&= \sigma_0^2\int_0^{\min\{T_j,T_m\}} e^{-\chi(T_j-s)}e^{-\chi(T_m-s)}\rho_{j,m}ds && (6.92)\\
&= \frac{\sigma_0^2\rho_{j,m}}{2\chi}\left(e^{-\chi(\max\{T_j,T_m\}-\min\{T_j,T_m\})} - e^{-\chi(T_j+T_m)}\right). && (6.93)
\end{aligned}$$

And similarly the variance of $L_j(T_j)$ by

$$\begin{aligned}
\mathrm{Var}^{\widetilde{B}}\left(L_j(T_j)\right) &= \mathbb{E}^{\widetilde{B}}\left(\left(L_j(T_j) - \mathbb{E}^{\widetilde{B}}(L_j(T_j))\right)\left(L_j(T_j) - \mathbb{E}^{\widetilde{B}}(L_j(T_j))\right)\right) && (6.94)\\
&= \mathbb{E}^{\widetilde{B}}\left(\int_0^{T_j}\sigma_0 e^{-\chi(T_j-s)}dW_j(s)\int_0^{T_j}\sigma_0 e^{-\chi(T_j-s)}dW_j(s)\right) && (6.95)\\
&= \sigma_0^2\int_0^{T_j} e^{-2\chi(T_j-s)}ds && (6.96)\\
&= \frac{\sigma_0^2}{2\chi}\left(1 - e^{-2\chi T_j}\right). && (6.97)
\end{aligned}$$

Now we are able to give the correlation by the definition of the correlation

$$\mathrm{Corr}^{\widetilde{B}}\left(L_j(T_j), L_m(T_m)\right) = \frac{\mathrm{Cov}^{\widetilde{B}}(L_j(T_j), L_m(T_m))}{\sqrt{\mathrm{Var}^{\widetilde{B}}(L_j(T_j))\mathrm{Var}^{\widetilde{B}}(L_m(T_m))}} \tag{6.98}$$

$$= \rho_{j,m}\left(\frac{e^{-\chi(\max\{T_j,T_m\}-\min\{T_j,T_m\})} - e^{-\chi(T_j+T_m)}}{\sqrt{(1-e^{-2\chi T_j})(1-e^{-2\chi T_m})}}\right) \tag{6.99}$$

$$= \rho_{j,m}\sqrt{\frac{e^{2\chi\min\{T_j,T_m\}}-1}{e^{2\chi\max\{T_j,T_m\}}-1}}. \tag{6.100}$$

□

This brings us to our final theorem:

Theorem 6.11. *For each n, $n = 0, \ldots, N-1$, the forward rate $L(0;T_n,T_{n+1})$ is obtained from the futures rates $\{F(0;T_i,T_{i+1})\}_{i=0}^{n}$ and forward rates for previous periods $\{L(0;T_i,T_{i+1})\}_{i=0}^{n-1}$ by solving the following equation,*

$$L_n(0) = V(0)\left(1 + \frac{1}{2}\sum_{j,m=0}^{n} D_{j,m}\frac{L_j(0)L_m(0)}{b_j b_m} \sqrt{\mathbb{E}^{\widetilde{B}}\left(\exp\left\{\sigma_j b_j \int_0^{T_j} z(s)ds\right\}\right)\mathbb{E}^{\widetilde{B}}\left(\exp\left\{\sigma_m b_m \int_0^{T_m} z(s)ds\right\}\right)} c_{j,m}\right), \tag{6.101}$$

with $V(0)$ and $D_{j,m}$ are given in Definition 6.5 and (6.39). $\mathbb{E}^{\widetilde{B}}\left(\exp\left\{\sigma_j b_j \int_0^{T_j} z(s)ds\right\}\right)$ *is given in Proposition 6.9 and*

$$c_{j,m} = \mathrm{Corr}^{\widetilde{B}}\left(L_j(T_j), L_m(T_m)\right),$$

as given by Proposition 6.10.

Note: *For $T_0 = 0$ we have*

$$L_0(0) = V(0). \tag{6.102}$$

Proof. See (6.41). □

6.2 Computing Convexity Adjustments using Stochastic Volatility

We performed a test to compute convexity adjustments using stochastic volatility. We used Theorem 6.11 to obtain the convexity adjustments to the futures rates in Table 6.1. The variables we used are given in Table 6.2 and 6.3.

We computed the convexity adjustments of the futures rates with OpenGamma and Matlab, the results are given in Table 6.4. We used the same method to compute the convexity adjustments of Table 5.8, used in Section 5.2.3 to bootstrap multiple discount curves.

n	T_n	$F(0; T_n, T_{n+1})$
1	0.25	0.012
2	0.50	0.014
3	0.75	0.016
4	1.00	0.018
5	1.25	0.020
6	1.50	0.022
7	1.75	0.024
8	2.00	0.026
9	2.25	0.028
10	2.50	0.030
11	2.75	0.032
12	3.00	0.034
13	3.25	0.036
14	3.50	0.038
15	3.75	0.040
16	4.00	0.042
17	4.25	0.044
18	4.50	0.046
19	4.75	0.048
20	5.00	0.050

Table 6.1: Forward rates used.[2]

$$
\begin{gathered}
\tau_n = 0.25 \ \forall \ n \\
b_n = 0.5 \ \forall \ n \\
\eta_n = 0.5 \ \forall \ n \\
z_0 = 1 \\
\rho_{j,m} = 1 \ \forall \ j, m \\
\chi = 0.05
\end{gathered}
$$

Table 6.2: Static variables used for Test 6.

[2] We used $T_{21} = 5.25$ for the last forward rate.

n	σ_n
1	0.49125
2	0.1825
3	0.47375
4	0.465
5	0.45625
6	0.4475
7	0.43875
8	0.43
9	0.42125
10	0.4125
11	0.40375
12	0.395
13	0.38625
14	0.3775
15	0.36875
16	0.36
17	0.35125
18	0.3425
19	0.33375
20	0.325

Table 6.3: Volatility levels used.

n	OpenGamma	Matlab	Absolute difference[3]
1	2.18e-06	2.18e-06	5.21e-18
2	8.26e-06	8.26e-06	3.47e-18
3	2.02e-05	2.02e-05	3.50e-18
4	4.01e-05	4.01e-05	3.39e-18
5	7.01e-05	7.01e-05	<1e-18
6	1.13e-04	1.13e-04	6.99e-18
7	1.70e-04	1.70e-04	<1e-18
8	2.43e-04	2.43e-04	3.01e-18
9	3.35e-04	3.35e-04	<1e-18
10	4.47e-04	4.47e-04	4.01e-18
11	5.80e-04	5.80e-04	7.05e-18
12	7.34e-04	7.34e-04	<1e-18
13	9.12e-04	9.12e-04	2.09e-17
14	1.11e-03	1.11e-03	<1e-18
15	1.33e-03	1.33e-03	1.02e-17
16	1.58e-03	1.58e-03	<1e-18
17	1.84e-03	1.84e-03	9.97e-18
18	2.13e-03	2.13e-03	9.97e-18
19	2.43e-03	2.43e-03	<1e-18
20	2.75e-03	2.75e-03	9.97e-18

Table 6.4: Results Test: Convexity adjustments computed using OpenGamma and Matlab.

[3]The difference is caused by the precision of Java and Matlab.

Chapter 7

Conclusion and Further Research

Our main goal was to use OpenGamma for bootstrapping multiple discount curves. We used different bootstrapping techniques and different interest rate products to bootstrap multiple curves. We started with single curve bootstrapping, which was done before the start of the financial crisis. We explained that the pricing methods used before the start of the crisis don't hold anymore. As the counterparty risk has increased by the crisis, most contracts nowadays are collateralized. We explained the pricing methods for collateralized interest rate products after the start of the financial crisis.

We used OpenGamma and the market quotes of collateralized interest rate products to bootstrap multiple discount curves. We also used Eurodollar futures to bootstrap three month forward curves. Actually, there will be a difference between the futures rate and the forward rate, namely the convexity adjustment. We used a deterministic volatility model and a stochastic volatility model to compute the convexity adjustment of Eurodollar futures, using parameters.

We succeeded to implement the multiple curve bootstrapping algorithms in OpenGamma and we also succeeded to write a program in OpenGamma, which computes the convexity adjustments of futures rates. Finally, we will give some advice for further research in the next section.

Everything we did in this thesis was in a single currency. I would suggest performing bootstrapping for multiple currencies. One could use cross currency swaps to bootstrap multiple discount curves.

To compute the convexity adjustment of Eurodollar futures, we used some parameters in different models. I would suggest writing a program, which approximates the parameters in order to give a better approximation of the convexity adjustment.

Appendix A

Zero-Coupon Bonds

A bond consists of a holder and an issuer. The issuer agrees to pay the holder money at some pre-determined dates in the future. In exchange for these payments, the holder will pay the issuer an amount of money at the beginning. The type of bond we will use is the so-called *zero-coupon bond*. For the zero-coupon bond there will be no coupons, i.e. there is only one date at which the issuer will have to pay the holder. In our case, we will assume this payment to be equal to one unit.

In general the holder will receive one unit at maturity. Actually, there is a possibility the issuer will default. In this case the holder will receive nothing or a recovery value. A zero-coupon bond with no probability of default of the issuer is called a *risk-free* zero-coupon bond.

Definition A.1. *We denote by $P(t,T)$ the value of a risk-free zero-coupon bond at time t, paying one unit at time T.*

Using absence of arbitrage, the price of a risk-free zero-coupon bond can be used for discounting risk-free cash flows, as the payoff of a risk-free cash flow corresponds to the payoff of a risk-free zero-coupon bond.

Assumption A.2. *Using absence of arbitrage, we assume the following:*

- *The price of a risk-free zero-coupon bond $P(t,T)$ is equal to the discount factor used to compute the present value at time t of a cash flow of one unit at time T.*
- *The relation $P(t,t) = 1$ holds for al t.*
- *The relation $P(t,T) = P(t,S)P(S,T)$ holds for all $t \leq S \leq T$.*

Definition A.3. *We define the yield, instantaneous forward rate and short rate;*

- *The continuous compounded yield over a period $[S,T]$ is defined by:*
$$R(S,T) = -\frac{\log\{P(S,T)\}}{\tau(S,T)}, \tag{A.1}$$
where $\tau(S,T)$ is the year fraction between S and T.
- *The instantaneous forward rate with maturity T, contracted at time t is defined by:*
$$f(t,T) = -\frac{\partial \log\{P(t,T)\}}{\partial T}. \tag{A.2}$$
- *The instantaneous short rate at time t is defined by:*
$$r(t) = f(t,t). \tag{A.3}$$

As an immediate consequence of Definition A.3, we have:

Lemma A.4. *For $S \leq T$ we have*
$$P(S,T) = \exp\{-R(S,T)\tau(S,T)\},$$

Lemma A.5. *For $t \leq s \leq T$ we have*
$$P(t,T) = \exp\left\{-\int_t^T f(t,s)ds\right\}.$$

In Appendix B, we will show that the price of a risk-free zero-coupon can be used for computing interest rates under certain assumptions.

Appendix B

Pricing Risk-Free Cash Flows

We want to price future cash flows of securities. We will use [1] to obtain pricing formulas. Interest rates can be seen as an asset which is self-financing, as there are no exogenous infusion or withdrawal of money. We define the risk-free money market account:

Definition B.1. *The risk-free money market account is denoted by*

$$B(t) = e^{\int_0^t r(s)ds}.$$

Proof. See [1]. ☐

We use the definition of this money market account and the fact that interest rates can be seen as a self-financing portfolio to give the following theorem:

Theorem B.2. *In the absence of arbitrage the process of the value of an asset $h(t)$, divided by the numeraire $B(t)$, must be a martingale under the measure Q associated with the numeraire $B(t)$, i.e.*

$$h(t) = B(t)\mathbb{E}_t^Q\left(\frac{h(T)}{B(T)}\right).$$

Proof. See [1]. ☐

We introduce the numeraire $P(t,T)$, which is the price of a risk-free zero-coupon bond and therefore by Definition B.1 and Theorem B.2 is

$$P(t,T) = \mathbb{E}_t^Q\left(e^{-\int_t^T r(s)ds}\right).$$

By changing the measure we obtain the forward pricing formula:

Theorem B.3. *We consider the numeraires $B(t)$ and $P(t,T)$, considering equivalent martingale measures Q and Q^T, then we are allowed to change the numeraire such that*

$$h(t) = B(t)\mathbb{E}_t^Q\left(\frac{h(T)}{B(T)}\right) = P(t,T)\mathbb{E}_t^T\left(\frac{h(T)}{P(T,T)}\right) = P(t,T)\mathbb{E}_t^T\left(h(T)\right),$$

where $\mathbb{E}^T$ denotes the expectation taken under measure Q^T.

Appendix C

Conventions and Calendars

C.1 Day Count Conventions

Actual/360: The year fraction between two dates is computed by dividing the actual days between two dates by 360:

$$\text{Year Fraction} = \frac{\text{Actual days}}{360}. \tag{C.1}$$

Actual/365: The year fraction between two dates is computed by dividing the actual days between two dates by 365:

$$\text{Year Fraction} = \frac{\text{Actual days}}{365}. \tag{C.2}$$

Actual/Actual ISDA: The year fraction is computed by making difference between days in a leap year and days in a non-leap year:

$$\text{Year Fraction} = \frac{\text{Days not in leap year}}{365} + \frac{\text{Days in leap year}}{366}. \tag{C.3}$$

30/360: This method assumes months to have 30 days and years to have 360 days:

$$\text{Year Fraction} = \frac{360 \cdot (Y2 - Y1) + 30 \cdot (M2 - M1) + (D2 - D1)}{360}. \tag{C.4}$$

C.2 Calendars

No Holiday Calendar: This calendar assumes all days to be business days, ignoring weekends and holidays.

Monday to Friday Calendar: This calendar assumes all weekdays to be business days and weekends to be non-business days, ignoring holidays.

C.3 Date Roll Conventions

Following Business Day Convention: If a payment date falls on a non-business day, the payment date is set to the next business day.

Modified Following Business Day Convention: If a payment date falls on a non-business day, the payment date is set to the next business day. With the exception that if the next business day falls in a new month, in that case the payment date is set to the last business day.

C.4 End-Of-Month Convention

End-Of-Month convention: If the spot date falls on the last business day of the month, all regular dates will be set to the last business day of the month as well.

Appendix D

Tests

During this thesis, we executed several tests. All tests are done with our open source software OpenGamma. To verify our results, we used Microsoft Excel or Matlab.

Five tests will compute the present value of interest rate swaps and one test computed convexity adjustments of Eurodollar futures.

D.1 Test 1: Pricing a Fixed-Floating IRS using a Single Discount Curve

We computed the present value of a fixed-floating IRS, using the pricing method which has been used before the start of the crisis. This means that our fixed-floating IRS is uncollateralized. As discussed in Chapter 3, we will use one single discount curve to compute the forward rate and the discount factor.

The fixed-floating IRS we will price is not traded on the market, as fixed-floating IRS traded on the market will adjust the fixed rate such that the product has a zero present value. We will use the fixed rate defined in the contract and we won't adjust this rate. The contract of the fixed-floating IRS we will price is given in Table D.1. We will use a discount curve which we created ourselves.

We used the open source software OpenGamma to price the fixed-floating IRS and we used Microsoft Excel to verify the results.

To construct a discount curve, we started by constructing a yield curve. We constructed the yield curve by using linear interpolation on the nodal points of Table D.2 (see Section 3.2.1). The discount curve is then constructed by converting the yield curve:

$$P(t) = \exp\{-R(t) \cdot t\}.$$

We used (3.1) and Definition 2.5 to compute the present value at time $t = 0$, recall that $P(0,t) \equiv P(t)$ and $R(0,t) \equiv R(t)$:

$$PV(0) = \sum_{i=1}^{N} \frac{\alpha_i}{\widetilde{\alpha}_i} \left(\frac{P(S_i)}{P(T_i)} - 1 \right) P(\widetilde{T}_i) - k \sum_{j=1}^{M} \tau_j P(t_j),$$

where we used the notations of Section 3.1.1. We used *Actual/Actual ISDA* as day count fraction to determine the year fractions of the payment times and the start time and end time of the accrual periods.

The results of Test 1 are shown in Table D.3.

IRS Contract		
Trade date[1]	13-3-2013	
Effective date	13-3-2013	
Maturity date	13-3-2043	
Fixed rate	2%	
Forward index	Euribor6M	
Notional value	EUR 100	
	Fixed leg	Floating leg
Frequency	Semi-annual	Semi-annual
Spot Lag	0 days	0 days
Day Count Convention	Act/365	Act/365
Calendar	No Holiday	No Holiday
Date Roll Convention	n/a	n/a
End-Of-Month Convention	not used	not used

Table D.1: IRS contract used for pricing using a single curve (dates are dd-mm-yyyy)

i	T_i	$R(T_i)$
1	0	0.0025
2	4	0.0027
3	8	0.0028
4	16.2	0.0035
5	24.5	0.0042
6	32	0.005

Table D.2: Nodal yield points, used for constructing yield curve.

	OpenGamma	Own Input	Difference[2]
Test 1	EUR -43.27	EUR -43.27	<10e-13

Table D.3: Results Test 1.

D.2 Test 2: Pricing an OIS

In this section we will price an overnight index swap. As discussed in Section 3.1.2 and 4.5.1, pricing collateralized and uncollateralized OIS both use only one discount curve. The pricing formula of both swaps is the same, except for the fact that they use a different discount curve. In this test, we will use a single discount curve which denotes the discount curve $P(\cdot,\cdot)$ in Section 3.1.2 and $P^c(\cdot,\cdot)$ in Section 4.5.1, i.e.

$$P(t,T) := P^c(t,T).$$

The OIS we will price is not traded on the market, as OIS traded on the market will adjust the OIS rate such that the product has a zero present value. We will use the OIS rate defined in the contract and we won't adjust this rate. The contract of the OIS we will price is given in Table D.4. We will use a discount curve which we created ourselves.

We used the open source software OpenGamma to price the fixed-floating IRS and we used Microsoft Excel to verify the results.

To construct a discount curve, we started by constructing a yield curve. We constructed the yield curve by using linear interpolation on the nodal points of Table D.5 (see Section 3.2.1). The discount curve is then constructed by converting the yield curve:

$$P(t) = \exp\{-R(t) \cdot t\}.$$

[1]By trade date we mean the date at which the present value is calculated, not the date at which the swap rate is determined.
[2]The difference is caused by the precision of Microsoft Excel.

We used (3.7) (or (4.18)) to compute the present value at time $t = 0$, recall that $P(0, t) \equiv P(t)$ and $R(0, t) \equiv R(t)$:

$$PV(0) = P(T_0) - P(T_N) - k \sum_{i=1}^{N} \tau_i P(T_i),$$

where we used the notations of Section 2.6. We used *Actual/Actual ISDA* as day count fraction to determine the year fractions of the payment times and the start time.

The results of Test 2 are shown in Table D.6.

OIS Contract	
Trade date[3]	3-4-2013
Effective date	5-4-2013
Maturity date swap	7-4-2014
Fixed rate	2%
Overnight index	Eonia
Settlement delay	0 days
Notional value	EUR 1
Frequency	Annual
Spot Lag	2 days
Day Count Convention	Act/360
Calendar	Monday-to-Friday
Date Roll Convention	Mod. Following
End-Of-Month Convention	not used

Table D.4: OIS contract used for Test 2 (dates are dd-mm-yyyy).

i	T_i	$R(T_i)$
1	0	0.0025
2	0.1	0.0027
3	0.5	0.0028
4	0.9	0.0035
5	1.3	0.0042
6	32	0.005

Table D.5: Nodal yield points, used for constructing yield curve.

	OpenGamma	Own Input	Difference[4]
Test 1	EUR -1.66	EUR -1.66	<1e-13

Table D.6: Results Test 2.

D.3 Test 3: Pricing a Basis Swap using a Single Discount Curve

We computed the present value of a basis swap, using the pricing method which has been used before the start of the crisis. This means that our basis swap is uncollateralized. As discussed in Chapter 3, we will use one single discount curve to compute the forward rate and the discount factor.

The basis swap we will price is not traded on the market, as basis swaps traded on the market will adjust the spread such that the product has a zero present value. We will use the spread defined in the contract and we won't adjust this spread. The contract of the basis swap we will price is given in Table D.7. We will

[3]By trade date we mean the date at which the present value is calculated, not the date at which the swap rate is determined.
[4]The difference is caused by the precision of Microsoft Excel.

use a discount curve which we created ourselves.

We used the open source software OpenGamma to price the basis swap and we used Microsoft Excel to verify the results.

To construct a discount curve, we started by constructing a yield curve. We constructed the yield curve by using linear interpolation on the nodal points of Table D.8 (see Section 3.2.1). The discount curve is then constructed by converting the yield curve:

$$P(t) = \exp\{-R(t) \cdot t\}.$$

We used (3.9) and Definition 2.5 to compute the present value at time $t = 0$, recall that $P(0,t) \equiv P(t)$ and $R(0,t) \equiv R(t)$:

$$PV(0) = \sum_{i=1}^{N} \frac{\alpha_i}{\widetilde{\alpha}_i} \left(\frac{P(S_i^A)}{P(T_i^A)} - 1 \right) P(\widetilde{T}_i^B) - \sum_{j=1}^{M} \tau_j \left(\frac{1}{\widetilde{\tau}_j} \left(\frac{P(S_i^B)}{P(T_i^B)} - 1 \right) + s \right) P(\widetilde{T}_j^A),$$

where we used the notations of Section 3.1.3. We used *Actual/Actual ISDA* as day count fraction to determine the year fractions of the payment times and the start time and end time of the accrual periods.

The results of Test 3 are shown in Table D.9.

Basis Swap Contract		
Trade date[5]	13-3-2013	
Effective date	15-3-2013	
Maturity date swap	15-3-2043	
Notional value	EUR 1	
Spread	2%	
	Leg A	Leg B
Index	Euribor1M	Euribor3M
Frequency	Monthly	Quarterly
Spot Lag	2 days	2 days
Day Count Convention	Act/365	Act/365
Calendar	No Holiday	No Holiday
Date Roll Convention	n/a	n/a
End-Of-Month Convention	not used	not used

Table D.7: Basis Swap Contract used for Test 3 (dates are dd-mm-yyyy)

i	T_i	$R(T_i)$
1	0	0.0025
2	0.1	0.0027
3	0.5	0.0028
4	0.9	0.0035
5	1.3	0.0042
6	32	0.005

Table D.8: Nodal yield points, used for constructing yield curve.

	OpenGamma	Own Input	Difference[6]
Test 3	EUR -0.57	EUR -0.57	<1e-13

Table D.9: Results Test 3.

[5]By trade date we mean the date at which the present value is calculated, not the date at which the swap rate is determined.
[6]The difference is caused by the precision of Microsoft Excel.

D.4 Test 4: Pricing a Collateralized Fixed-Floating IRS using Multiple Discount Curves

We computed the present value of a collateralized fixed-floating IRS, using the pricing method which has been used since the start of the crisis. As discussed in Chapter 4, we will have to use different discount curves. We will need one curve to denote the collateral payments and we need one curve to compute the forward rates.

The fixed-floating IRS we will price is not traded on the market, as fixed-floating IRS traded on the market will adjust the fixed rate such that the product has a zero present value. We will use the fixed rate defined in the contract and we won't adjust this rate. The contract of the fixed-floating IRS we will price is given in Table D.10. We will use discount curves which we created ourselves.

We used the open source software OpenGamma to price the collateralized fixed-floating IRS and we used Microsoft Excel to verify the results.

To construct a discount curve, we started by constructing a yield curve. We constructed the yield curve by using linear interpolation on the nodal points of Table D.11 (see Section 3.2.1). The discount curve is then constructed by converting the yield curve:

$$P(t) = \exp\{-R(t) \cdot t\}.$$

We used (4.21) and Definition 4.4 to compute the present value at time $t = 0$, recall that $P(0,t) \equiv P(t)$ and $R(0,t) \equiv R(t)$:

$$PV(0) = \sum_{i=1}^{N} \frac{\alpha_i}{\widetilde{\alpha}_i} \left(\frac{P^{\tau}(S_i)}{P^{\tau}(T_i)} - 1 \right) P^c(\widetilde{T}_i) - k \sum_{j=1}^{M} \tau_j P^c(t_j),$$

where we used the notations of Section 4.5.2. We used *Actual/Actual ISDA* as day count fraction to determine the year fractions of the payment times and the start time and end time of the accrual periods.

The results of Test 4 are shown in Table D.12.

IRS Contract		
Trade date[7]	13-3-2013	
Effective date	13-3-2013	
Maturity date	13-3-2043	
Fixed rate	2%	
Forward index	Euribor6M	
Notional value	EUR 100	
	Fixed leg	Floating leg
Frequency	Semi-annual	Semi-annual
Spot Lag	0 days	0 days
Day Count Convention	Act/365	Act/365
Calendar	No Holiday	No Holiday
Date Roll Convention	n/a	n/a
End-Of-Month Convention	not used	not used

Table D.10: IRS contract used for pricing using a single curve (dates are dd-mm-yyyy)

[7] By trade date we mean the date at which the present value is calculated, not the date at which the swap rate is determined.

i	T_i	$R^c(T_i)$	$R^{6\text{M}}(T_i)$
1	0	0.0025	0.0035
2	0.1	0.0027	0.0037
3	0.5	0.0028	0.0038
4	0.9	0.0035	0.0055
5	1.3	0.0042	0.0072
6	32	0.005	0.01

Table D.11: Nodal yield points, used for constructing yield curves.

	OpenGamma	Own Input	Difference[8]
Test 4	EUR -30.96	EUR -30.96	$<$1e-13

Table D.12: Results Test 4.

D.5 Test 5: Pricing a Collateralized Basis Swap using Multiple Discount Curves

We computed the present value of a collateralized basis swap, using the pricing method which has been used since the start of the crisis. As discussed in Chapter 4, we will have to use different discount curves. We will need one curve to denote the collateral payments and we need curves to compute the forward rates for each tenor.

The basis swap we will price is not traded on the market, as basis swaps traded on the market will adjust the spread such that the product has a zero present value. We will use the spread defined in the contract and we won't adjust this spread. The contract of the basis swap we will price is given in Table D.13. We will use discount curves which we created ourselves.

We used the open source software OpenGamma to price the collateralized fixed-floating IRS and we used Microsoft Excel to verify the results.

To construct a discount curve, we started by constructing a yield curve. We constructed the yield curve by using linear interpolation on the nodal points of Table D.14 (see Section 3.2.1). The discount curve is then constructed by converting the yield curve:

$$P(t) = \exp\{-R(t) \cdot t\}.$$

We used (4.24) and Definition 4.4 to compute the present value at time $t = 0$, recall that $P(0,t) \equiv P(t)$ and $R(0,t) \equiv R(t)$:

$$PV(0) = \sum_{i=1}^{N} \frac{\alpha_i}{\widetilde{\alpha}_i} \left(\frac{P^{\tau_B}(S_i^B)}{P^{\tau_B}(T_i^B)} - 1 \right) P^c(\widetilde{T}_i^B) - \sum_{j=1}^{M} \tau_j \left(\frac{1}{\widetilde{\tau}_j} \left(\frac{P^{\tau_A}(S_j^A)}{P^{\tau_A}(T_j^A)} - 1 \right) + s \right) P^c(\widetilde{T}_j^A),$$

where we used the notations of Section 4.5.3. We used *Actual/Actual ISDA* as day count fraction to determine the year fractions of the payment times and the start time and end time of the accrual periods.

The results of Test 5 are shown in Table D.15.

[8] The difference is caused by the precision of Microsoft Excel.

Basis Swap Contract		
Trade date[9]	13-3-2013	
Effective date	15-3-2013	
Maturity date swap	15-3-2043	
Notional value	EUR 1	
Spread	2%	
	Leg A	Leg B
Index	Euribor3M	Euribor12M
Frequency	Quarterly	Yearly
Spot Lag	2 days	2 days
Day Count Convention	Act/365	Act/365
Calendar	No Holiday	No Holiday
Date Roll Convention	n/a	n/a
End-Of-Month Convention	not used	not used

Table D.13: Basis Swap Contract used for Test 5 (dates are dd-mm-yyyy)

i	T_i	$R^c(T_i)$	$R^{3\text{M}}(T_i)$	$R^{12\text{M}}(T_i)$
1	0	0.0025	0.0035	0.0045
2	0.1	0.0027	0.0037	0.0047
3	0.5	0.0028	0.0038	0.0048
4	0.9	0.0035	0.0045	0.0055
5	1.3	0.0042	0.0052	0.0062
6	32	0.005	0.006	0.007

Table D.14: Nodal yield points, used for constructing yield curves.

	OpenGamma	Own Input	Difference[10]
Test 5	EUR -0.54	EUR -0.54	<1e-13

Table D.15: Results Test 5.

[9]By trade date we mean the date at which the present value is calculated, not the date at which the swap rate is determined.
[10]The difference is caused by the precision of Microsoft Excel.

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