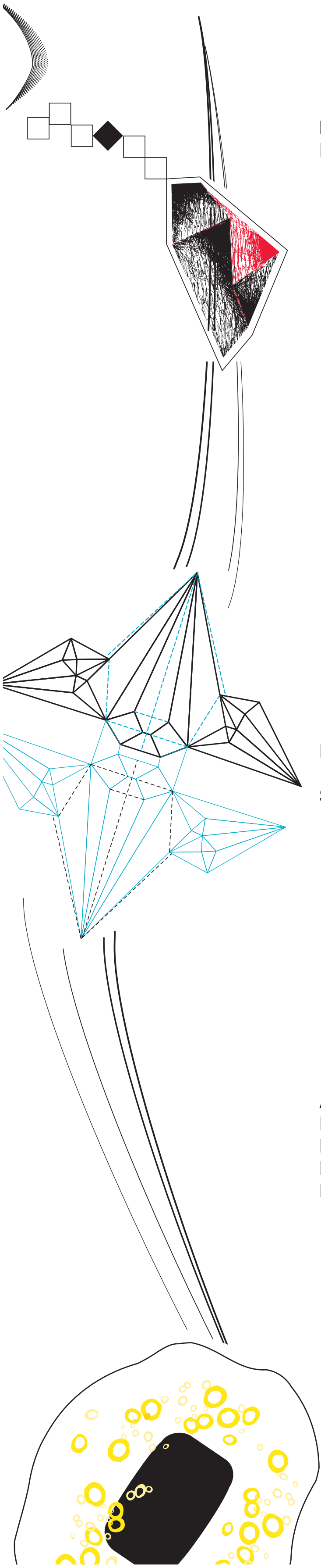

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Dynamic sharing of service capacity in a network

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Preface

This report is the result of my graduation project for the master Applied Mathematics at the University of Twente. I thank Judith Timmer and Werner Scheinhardt for providing me this interesting assignment and for their supervision. I would like to thank my family for their support.

Edwin Booltink

Abstract

In this report we study networks of queues in which the operators of the queues are allowed to share service capacity dynamically. We focus on the question whether it is beneficial for the individual operators to cooperate. One small extension of a Jackson network is considered where capacity can only be shared if there is a single customer in the entire network. Another extension is investigated where capacity is shared proportionally to the numbers of customers at each queue. For these networks corresponding cost games are formulated with the operators of the queues as players. The expected queue length and the server utilization are used to measure the performance of the network and to define the cost functions. For the first extension we focus on tandem networks where the service capacities of all queues are equal. We see that the value of this capacity determines whether it is beneficial to cooperate. For the second extension we focus on tandem networks where coalitions can only be formed by consecutive queues starting backwards from the queue at the end of the tandem. We find out that it is beneficial for all queues of the tandem to participate in the grand coalition. Although the service capacities of the queues might differ, the costs of the grand coalition can be distributed equally over the operators.

Contents

1 Introduction

1.1 Problem description

In this report we consider networks of queues in which each queue is operated by a different and independent operator. We study whether it is beneficial for the operators to share service capacity dynamically. The networks we focus on have infinite waiting rooms, a single server and a FIFO service discipline for all queues. The external arrival processes are Poisson processes. The service requirement is taken from an exponential distribution. The state of the networks is defined as the number of customers at each queue. A rule for sharing service capacity in a dynamic network specifies for each state in which way the operators are allowed to share their service capacity.

Having a network and a rule we must be able to compute the expected queue length and server utilization. This is because they are used to measure the performance of the network and to define the cost functions. As a result we have to be able to derive the equilibrium distribution. This means that our study is divided into two steps. First we investigate for which networks and rules we can find the equilibrium distribution. Second we want to find out whether it is beneficial for the individual operators to cooperate. We study the second step by formulating corresponding cost games with the operators of the queues as players. Our research questions reflect these steps and are as follows.

Can we find networks with state dependent rules for sharing service capacity such that

1. we can compute the performance measures?

2. a fair cost allocation is possible?

In this report we use two different ways to find such rule. The first method we use is to search for a rule by extending the Jackson network such that partial balance is maintained. This results into the rule of simple sharing. The second method we use is to find rules within the formulation of a Kelly Whittle network since for these networks an equilibrium distribution is known. This results into the rule of proportional sharing.

1.2 Literature review

Computing performance measures for a queueing network where the operators are allowed to share service capacity dynamically and studying the core of a corresponding cost game are the topics of our report. In this section we give a short review of state dependent networks and cost games in literature.

In literature many networks of queues are studied where cooperation is used to optimize performance measures. In these studies the individual operators are often replaced by a single operator. In [7] Timmer and Scheinhardt study Jackson networks where the operators remain independent. The operators of the queues are viewed as decision makers in a cooperative cost game. In the paper it is proven that it is beneficial for all individual operators of the network to share service capacity. The operators of a set of queues cooperate by redistributing the total service capacity of the coalition over all members of the coalition. In this setting each operator gets a new capacity which remains fixed and the total costs of the queues of the coalition is minimized. An explicit cost allocation is given that is proven to be in the core. In [3] Peters describes cooperative games. Basic concepts in theory of transferable utility games are covered such as the core, the Shapley value and the nucleolus. In our report we compute for several networks and their corresponding cost games the Shapley value and we investigate if the core is nonempty.

In [2] Van Dijk gives a practical approach for station balance. For a tandem of two queues with state independent service capacities station balance is used to derive a product form for the joint equilibrium distribution. Resing et al. [4] consider a tandem queue with coupled processors. When both stations are nonempty each queue has a fixed service capacity. If one of the station is empty, the service rate of the other queue changes. A functional equation for the generating function of the equilibrium distribution is derived and solved. In our report we use station balance to derive the equilibrium distribution of a small extension of a Jackson network. We investigate if service capacity can be reallocated when there is only one customer in the system.

R.F. Serfozo [5] studies networks with dependent nodes. The equilibrium distribution of a basic network process is given. The Jackson, BCMP and Kelly-Whittle processes are special cases of it. Virtamo [8] proves that the equilibrium distribution and performance measures are insensitive for certain networks. This means that they depend on the traffic characteristics only through the loads of different nodes. The author proves that insensitivity holds for networks where the nodes have balanced global state dependent service rates and where the service discipline within each node is symmetric. Dijk et al. [1] study parallel and tandem networks that consist of two queues. The servers of these queues share a common resource. It is proven that a product form for the equilibrium distribution exists if a constructed adjoint Markov chain is reversible. Tandems with proportional and α-unproportional sharing of service capacity are formulated as Kelly Whittle networks. In our report we compute performance measures for a network that shares service capacity proportionally. We formulate it as a Kelly Whittle network in which the service rates are balanced and for which the service discipline of the queues is symmetric.

1.3 Structure of the report

In section 2 we give a short overview of cost games, Jackson networks and Kelly Whittle networks. In section 3 we describe networks with the rule of simple sharing. For a tandem network we investigate the corresponding symmetric cost games. In the section 4 we study networks with the rule of proportional sharing. We investigate corresponding cost games for a tandem network. Conclusions and discussion can be found in section 5.

2 Preliminaries

In this report we define cost games for networks. We investigate if it is beneficial for the individual operators of the queues to cooperate. We consider networks that are extensions of Jackson networks and we formulate several networks as Kelly Whittle networks. In section 2.1 we give a short description of cost games. We give a short overview of Jackson and Kelly Whittle networks in sections 2.2 and 2.3 respectively.

2.1 Cost games

In a cost game denoted by (N, c), a set of players N can decide if they want to form coalitions. A cost function $c(S)$ assigns to each coalition S a cost. A cost game is monotone increasing if $S \subset T$ implies $c(S) \leq c(T)$ and monotone decreasing if $S \subset T$ implies $c(S) \geq c(T)$. A cost game is subadditive if $c(S \cup T \leq c(S) + c(T)$ is satisfied for any two disjoint coalitions S and T.

The core $C(N, c)$ of a cost game is defined as

$$C(N, c) = \{x \in \mathbb{R}^N\} \text{ such that} \tag{2.1.1}$$

$$\sum_{i \in N} x_i = c(N) \text{ and} \tag{2.1.2}$$

$$\sum_{i \in S} x_i \leq c(S) \text{ for all } S \subset N \tag{2.1.3}$$

If we can find a vector allocation x of the costs that is in the core, then it is beneficial for each coalition S to participate in the grand coalition N. This is called a fair cost allocation. We let the operators of the queues be players in a cost game. As costs for the cost function we choose the expected queue lengths or the server utilizations of the queues.

We denote the worth of coalition S with $v(S)$ and we define it as the difference between not cooperating and cooperating. We have

$$v(S) = \sum_{j \in S} c(\{j\}) - c(S) \tag{2.1.4}$$

For the cost games we compute the Shapley values. These are the average of the marginal contributions of each player. The Shapley value for player j is defined in [3] as

$$\Phi_j(N, v) = \sum_{S \subseteq N : J \notin S} \frac{|S|!\big(n - |S| - 1\big)!}{n!} \big[v(S \cup j) - v(S)\big] \tag{2.1.5}$$

2.2 Jackson networks

We describe an open Jackson network consisting of J queues. These J queues are single-class $m/m/s$-queues and have a FIFO server discipline. Arrivals from outside the system into queue i form a Poisson process with arrival rate γ_i. Each queue has s_i servers and their service times are exponentially distributed with parameter μ_i. An irreducible routing matrix P describes the transitions probabilities P_{ij}. After a customer has visited queue i, he will be directed to queue j with probability P_{ij} or he will leave the system with probability P_{i0} which gives

$$P_{i0} = 1 - \sum_{j=1}^{J} P_{ij} \tag{2.2.1}$$

The state of this Jackson network is described as $n = (n_1, n_2, ..., n_J)$. The state space is

$$\{(n_1, n_2, ..., n_J)\} \qquad \text{where } n_1, n_2, ..., n_J \in N. \tag{2.2.2}$$

Let the unit vector $(0, .., 0, 1, 0, ...)$ at position j be denoted by e_j. The transition rates are

$$q(n, n - e_j + e_k) = \phi_j(n_j)\mu_j P_{jk} \qquad \forall j, k \in \{1, 2, ..., J\} \tag{2.2.3}$$
$$q(n, n - e_j) = \phi_j(n_j)\mu_j P_{j0} \qquad \forall j \in \{1, 2, ..., J\} \tag{2.2.4}$$
$$q(n, n + e_k) = \gamma_k \qquad \forall k \in \{1, 2, ..., J\} \tag{2.2.5}$$

Jackson networks satisfy the traffic equations, which are

$$\lambda_i = \gamma_i + \sum_{j=1}^{J} \lambda_j P_{ji} \qquad \forall i \in \{1, ..., J\} \tag{2.2.6}$$

In case each queue has a single server we have the functions $\phi(n_j) = 1$ for $j \in \{1, 2, .., J\}$. The equilibrium distribution of the Jackson network is then described as

$$\pi(n) = \prod_{j=1}^{J} (1 - \rho_j)\rho_j^{n_j}, \qquad \text{where } \rho_j = \frac{\lambda_j}{\mu_j} \tag{2.2.7}$$

For the system to be stable we require $\rho_j < 1$. In case queue j has multiple servers s_j we have the functions $\phi(n_j) = \min\{n_j, s_j\}$. For the system to be stable we require $\rho_j < s_j$. The equilibrium distribution of the Jackson network is then described in [9] as

$$\pi(n) = \prod_{j=1}^{J} \frac{\rho_j^{n_j}}{m(n_j)} P_{0j}, \qquad \text{where } \rho_j = \frac{\lambda_j}{\mu_j} \tag{2.2.8}$$

$$m(n_j) = \begin{cases} n_j! & \text{for } 0 \le n_j \le s_j \\ s_j^{n_j - s_j} s_j! & \text{for } n_j \ge s_j \end{cases} \tag{2.2.9}$$

The normalization constant is denoted by P_{0j} and is given by

$$P_{0j} = \left(\sum_{n_j=0}^{s_j - 1} \frac{\rho_j^{n_j}}{n_j!} + \frac{\rho_j^{s_j}}{s_j!} \frac{s_j}{s_j - \rho_j} \right)^{-1} \tag{2.2.10}$$

In this report we assume that all networks consist of single server queues. We give here the case for multiple servers to show that in that case the function $\phi(n_j)$ only depends on the number of customers at queue j. In the next section we generalize this function $\phi(n_j)$ when we describe Kelly Whittle networks. In these networks the function $\phi(n_j)$ depends on the number of customers at all queues of the network.

2.3 Kelly Whittle networks

In this section we give description of a general Kelly Whittle network. These networks are extensions of Jackson networks. Both types have Poisson arrival processes and exponentially distributed service requirements. In Jackson networks the service rate at a queue only depends on the number of customers at that queue. In Kelly Whittle networks the service rate at a queue depends on the number of customers at all queues of the network.

Let the network consist of J queues. The state of the network is described as $n = (n_1, n_2, ..., n_J)$. The state space is

$$\{(n_1, n_2, ..., n_J)\} \qquad \text{where } n_1, n_2, ..., n_J \in N \tag{2.3.1}$$

The unit vector $(0, .., 0, 1, 0, ...)$ at position j is denoted by e_j. The transition rates of an open Kelly Whittle network are

$$q(n, n - e_j + e_k) = \frac{\Psi(n - e_j)}{\Phi(n)} \theta_j P_{jk} \qquad \forall j, k \in \{1, 2, ..., J\} \tag{2.3.2}$$

$$q(n, n - e_j) = \frac{\Psi(n - e_j)}{\Phi(n)} \theta_j P_{j0} \qquad \forall j \in \{1, 2, ..., J\} \tag{2.3.3}$$

$$q(n, n + e_k) = \frac{\Psi(n)}{\Phi(n)} \gamma_k \qquad \forall k \in \{1, 2, ..., J\} \tag{2.3.4}$$

The external arrival rate at queue k is given by a Poisson process with rate γ_k and the variables $\Psi(n)$ and $\Phi(n)$. When a customer leaves queue j he continues to queue k with probability $P(j, k)$. The variables $\Psi(n), \Phi(n), \theta_j$ and $P(j, k)$ form the service rate at queue j. The variables $\Psi(n)$ and $\Phi(n)$ are functions of the state n. This reflects the fact that the service rates depend on the entire state of the network. The variable θ_j depends only on queue j.

The service requirement for each customer is taken from an exponential distribution with unit mean. Arriving customers from outside to the network, customers moving from one queue to another and departing customers from the network are possible movements for customers. All these movements do not change the expected service requirements of the customers in the tandem network. This is because of the memoryless property of the exponential distribution, which states that the expected residual service requirement at any point in time equals the expected service requirement. The service time is the service requirement divided by the service rate. Since the service rates change when the state of the network changes, the expected service times also change when the state of the network changes.

Kelly Whittle network satisfy the traffic equations, which are

$$\lambda_i = \gamma_i + \sum_{j=1}^{n} \lambda_j P_{ji} \qquad \forall i \in \{1, ..., n\} \tag{2.3.5}$$

The equilibrium distribution of this network is

$$\pi(n)^{KellyWhittle} = c\, \Phi(n) \prod_{j=1}^{J} \rho_j^{n_j} \qquad \text{with} \quad \rho_j = \frac{\lambda_j}{\theta_j} \tag{2.3.6}$$

The normalizing constant is denoted by c. The equilibrium distribution in 2.3.6 is independent of the function $\Psi(n)$. However other performance measures like the throughput, do depend on the function $\Psi(n)$.

If $\Psi(n) = \Phi(n)$ we get the functions $\phi_j(n)$

$$\phi_j(n) = \frac{\Phi(n - e_j)}{\Phi(n)} \tag{2.3.7}$$

The transition rates of this open Kelly Whittle network are

$$\begin{aligned}
q(n, n - e_j + e_k) &= \phi_j(n)\theta_j P_{jk} \qquad \forall j, k \in \{1, 2, ..., J\} && (2.3.8)\\
q(n, n - e_j) &= \phi_j(n)\theta_j P_{j0} \qquad \forall j \in \{1, 2, ..., J\} && (2.3.9)\\
q(n, n + e_k) &= \gamma_k \qquad \forall k \in \{1, 2, ..., J\} && (2.3.10)
\end{aligned}$$

The functions $\phi_j(n)$ satisfy the balance property

$$\frac{\phi_k(n - e_j)}{\phi_k(n)} = \frac{\phi_j(n - e_k)}{\phi_j(n)} \qquad \forall n \in \{(n_1, n_2, ..., n_J)\} \tag{2.3.11}$$

The left side of equation 2.3.11 can be viewed as the relative change in service rate for queue k when a customer of queue j leaves the network from state n. The right side of equation 2.3.11 is then the relative change in service rate for queue j when a customer of queue k leaves the network from state n. In this report we will assume expression (2.3.7).

3 Simple sharing

In this section we study a small extension of the Jackson network where the operators of the queues are only allowed to share service capacity when there is one customer in the entire network. In this report we denote it as a network with simple sharing. In subsection 3.1 we describe a general network with simple sharing and we study corresponding cost games. In subsection 3.2 we investigate a tandem network as a special case.

3.1 General network

We consider a general network with simple sharing. In subsection 3.1.1 a rule for sharing service capacity is formulated. Further we use partial balance to derive the equilibrium distribution from the equilibrium distribution of a Jackson network. In subsections 3.1.2 and 3.1.3 we formulate corresponding cost games where the expected queue length and the server utilization respectively are taken as cost functions.

3.1.1 Description

The network consists of J queues and the set of queues is indicated by $N = \{1, 2, ..., J\}$. We assume infinite waiting rooms, a single server and a FIFO service discipline for all queues. For queue j the external arrival process is a Poisson process with rate γ_j and the service distribution is exponential with rate μ_j. The state of the network is described as $n = (n_1, n_2, ..., n_J)$ with n_j as the number of customers at queue j. The state space is $\{(n_1, n_2, ..., n_J)\}$ where $n_1, n_2, ..., n_J \in \mathbb{N}$. The total arrival rate at queue j is λ_j and satisfies the traffic equations (2.2.6). The zero state is the state n with $n_j = 0$ for all $j \in N$. The set of states having one customer in the entire network is assigned the name $State_{one}$ and can be written as

$$State_{one} = \Big\{(n_1, n_2, ..., n_J)\Big|\ n_j = 1 \ \text{ and } \ n_i = 0 \ \ \forall i \in \{1, 2, ..., J\} \backslash j \ \Big\} \tag{3.1.1}$$

The set $S \subseteq N$ is the set of cooperating queues. We need the following definition.

$$Z_S = \frac{\sum\limits_{S} \mu_i}{\max\limits_{N}\{\mu_i\}} \tag{3.1.2}$$

A rule for sharing service capacity in a dynamic network specifies for each state in which way the operators are allowed to share their service capacity. In definition 1 we formulate the rule for sharing service capacity in a network with simple sharing.

Definition 1 *In a network with simple sharing the operators of the queues are only allowed to share their service capacities when the network is in a state of the set $State_{one}$. For these states the service capacity for queue j is defined as $Z_S\mu_j$. For the other states the service capacity of queue j is defined as μ_j.*

With the exterior of the network denoted as queue j with $j = 0$, we have the following service rates for the network for all $j \in \{1, ..., J\}$ and for all $k \in \{0, 1, ..., J\}$

$$q(n, n - e_j + e_k) = \begin{cases} Z_S \mu_j P_{jk} & \text{if } n \in State_{one} \\ \\ \mu_j P_{jk} & \text{otherwise} \end{cases} \tag{3.1.3}$$

In proposition 2 we give the equilibrium distribution and the normalization constant of this network.

Proposition 2 *The equilibrium distribution of the network of definition 1 depends on the coalition S.*

The probability of the zero state denoted by $\pi_{0,S}^{simple}$ is

$$\pi_{0,S}^{simple} \quad = \quad \frac{Z_S}{G_S^{simple}} \tag{3.1.4}$$

and the probabilities of the other states denoted by $\pi_S^{simple}(n_1, n_2, ..., n_J)$ are

$$\pi_S^{simple}(n_1, n_2, ..., n_J) \quad = \quad \frac{1}{G_S^{simple}} \Big(\frac{\lambda_1}{\mu_1}\Big)^{n_1} \Big(\frac{\lambda_2}{\mu_2}\Big)^{n_2} \; ... \; \Big(\frac{\lambda_J}{\mu_J}\Big)^{n_J} \tag{3.1.5}$$

The normalization constant denoted by G_S^{simple} is

$$G_S^{simple} = Z_S - 1 + \prod_{i \in N} \frac{\mu_i}{\mu_i - \lambda_i} \tag{3.1.6}$$

Proof.
The network with simple sharing is similar to the Jackson network except for the zero state and the states of set $State_{one}$. Therefore we construct a network with simple sharing from a Jackson network.

Each state in a Jackson networks satisfies partial balance. This means that the rate out of a state due to a departure from queue j equals the rate into that state due to an arrival at queue j. The partial balance equations for the queues of the network are

$$\sum_{k=0} \pi^{Jackson}(n) \; q(n, n - e_j + e_k) = \sum_{k=0} \pi^{Jackson}(n - e_j + e_k) \; q(n - e_j + e_k, n) \quad \forall j \in N \tag{3.1.7}$$

We can see arrivals from outside the network into queue j as departures from the exterior. We can regard departures from queue j out of the network as arrivals into the exterior. With this view partial balance is satisfied for this external queue as well. Therefore we have

$$\sum_{j \in N} \pi^{Jackson}(n) \; q(n, n + e_j) = \sum_{j \in N} \pi^{Jackson}(n + e_j) \; q(n + e_j, n) \tag{3.1.8}$$

Substituting the normalization constant $(G^{Jackson})^{-1}$ for $\pi_0^{Jackson}$ we can write the partial balance equations for the states $State_{one}$ as

$$\mu_j \; \pi^{Jackson}(0 + e_j) =$$

$$\lambda_j \; (G^{Jackson})^{-1} + \sum_{i=1} \mu_i P_{ij} \; \pi^{Jackson}(0 + e_i) \quad \forall j \in N \tag{3.1.9}$$

We can write the partial balance equations for the zero state as

$$\sum_{j=1} \lambda_j \; (G^{Jackson})^{-1} = \sum_{i=1} \mu_i P_{i0} \; \pi^{Jackson}(0 + e_i) \tag{3.1.10}$$

The equations (3.1.9) and (3.1.10) form a system of $|N| + 1$ linear equations. The service capacities μ_j for all queues $j \in N$ and the normalization constant $(G^{Jackson})^{-1}$ can be seen as the unique solution for this system. We assume the probabilities $\pi^{Jackson}(0 + e_i)$, the arrival rates λ_j and the routing probabilities P_{ij} to be constant. We multiply each equation with the same constant and we only change the service capacities μ_j for all queues $j \in N$. The system maintains a unique solution and therefore partial balance for the network is maintained as well.

Partial balance means that there is a circulation of flows between states. By multiplying with a constant we change the flows between the states of set $State_{one}$ and the zero state but these flows continue to form a circulation. We only change the service capacities μ_j of the flows of this circulation. Except for the equilibrium probability of the zero state we keep the equilibrium probabilities for all states the same. This means that the global balance equations for all states remain satisfied. Although the equilibrium probability of the zero state does change, the global balance equation for this state remains satisfied as well. After multiplying with a constant we need to substitute the normalization constant $(G_S^{simple})^{-1}$ for $(G^{Jackson})^{-1}$.

The total amount of capacity that can be shared is $\sum_S \mu_i$. When this sum is divided by the largest service capacity of the network $\max_N\{\mu_i\}$, we get the upper bound for the constant. In expression (3.1.2) this upper bound is assigned by Z_S.

We can write the partial balance equations for the states of set $State_{one}$ for a network with simple sharing as

$$Z_S \; \mu_j \; \pi_S^{simple}(0 + e_j) =$$

$$Z_S \; \lambda_j \; (G_S^{simple})^{-1} + \sum_{i=1} Z_S \; \mu_i P_{ij} \; \pi_S^{simple}(0 + e_i) \quad \forall j \in N \tag{3.1.11}$$

We can write the partial balance equations for the zero state as

$$Z_S \sum_{j=1} \lambda_j \; (G_S^{simple})^{-1} = \sum_{i=1} Z_S \; \mu_i P_{i0} \; \pi_S^{simple}(0 + e_i) \tag{3.1.12}$$

For the other states the partial balance equations of a network with simple sharing equal those of a Jackson network. Comparing all partial balance equations of both networks we can conclude that the equilibrium distributions for all states are the same except for the zero state and for the normalization constant. The probability of the zero state can be read from equation 3.1.12 by expressing the LHS into the factors $\sum_{j=1} \lambda_j$ and $Z_S(G_S^{simple})^{-1}$. The latter factor equals the probability of the zero state. Expressions 3.1.4 and 3.1.5 are therefore shown.

The normalization constant G_S^{simple} of expression (3.1.6) is computed by setting the summation over all probabilities equal to one.

$$\begin{aligned} 1 &= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} ... \sum_{n_J=0}^{\infty} \pi_S^{simple}(n_1, n_2, ..., n_J) \\ &= \pi_{0,S}^{simple} - \frac{1}{G_S^{simple}} + \frac{1}{G_S^{simple}} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} ... \sum_{n_J=0}^{\infty} \left(\frac{\lambda_1}{\mu_1}\right)^{n_1} \left(\frac{\lambda_2}{\mu_2}\right)^{n_2} ... \left(\frac{\lambda_J}{\mu_J}\right)^{n_J} \\ &= \frac{Z_S}{G_S^{simple}} - \frac{1}{G_S^{simple}} + \frac{1}{G_S^{simple}} \prod_{i \in N} \frac{\mu_i}{\mu_i - \lambda_i} \end{aligned}$$

□

Proposition 3 *The expectation of the queue length of queue j of the network with simple sharing denoted by $E(L_{j,S}^{simple})$ is*

$$E(L_{j,S}^{simple}) = \left(\frac{\lambda_j}{\mu_j - \lambda_j}\right) \frac{\prod_i \mu_i}{(Z_S - 1) \prod_i (\mu_i - \lambda_i) + \prod_i \mu_i} \tag{3.1.13}$$

Proof.
We start with the computation of the expected queue length of queue 1

$$\begin{aligned} E(L_{1,S}^{simple}) &= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} ... \sum_{n_J=0}^{\infty} n_1 \, \pi_S^{simple}(n_1, n_2, ..., n_J) \\ &= \frac{1}{G_S^{simple}} \sum_{n_1=0}^{\infty} n_1 \left(\frac{\lambda_1}{\mu_1}\right)^{n_1} \sum_{n_2=0}^{\infty} \left(\frac{\lambda_2}{\mu_2}\right)^{n_2} ... \sum_{n_J=0}^{\infty} \left(\frac{\lambda_J}{\mu_J}\right)^{n_J} \\ &= \frac{1}{G_S^{simple}} \left(\frac{\lambda_j}{\mu_j - \lambda_j}\right) \prod_{i \in N} \frac{\mu_i}{\mu_i - \lambda_i} \end{aligned}$$

Substituting the normalization constant of expression (3.1.6) into the last equation gives the result for the expected queue length of queue 1. Substituting queue j for queue 1 gives the same result for the expected queue length of queue j. □

We define h_S to be

$$h_S = \frac{\prod_i \mu_i}{(Z_S - 1) \prod_i (\mu_i - \lambda_i) + \prod_i \mu_i} \tag{3.1.14}$$

We can write the expected queue length of queue j as

$$E(L_{j,S}^{simple}) = E(L_j^{Jackson}) \, h_s \tag{3.1.15}$$

Proposition 4 *The server utilization of queue j of the simple tandem denoted by $\rho_{j,S}^{simple}$ is*

$$\rho_{j,S}^{simple} = \frac{\prod_i \mu_i - (\mu_j - \lambda_j)\prod_{i \neq j} \mu_i}{(Z_S - 1)\prod_i(\mu_i - \lambda_i) + \prod_i \mu_i} \qquad (3.1.16)$$

Proof.
We start with the computation of the server utilization of queue 1. The computation is similar to the computation in a Jackson tandem except for the zero state.

$$\begin{aligned}
\rho_{1,S}^{simple} &= 1 - \sum_{n_2=0}^{\infty} \dots \sum_{n_J=0}^{\infty} \pi_S^{simple}(0, n_2, \dots, n_J) \\
&= 1 - \frac{1}{G_S^{simple}} \sum_{n_2=0}^{\infty} \Big(\frac{\lambda_2}{\mu_2}\Big)^{n_2} \dots \sum_{n_J=0}^{\infty} \Big(\frac{\lambda_J}{\mu_2}\Big)^{n_J} - \frac{Z_S}{G_S^{simple}} + \frac{1}{G_S^{simple}} \\
&= 1 - \frac{1}{G_S^{simple}} \prod_{i \neq 1} \frac{\mu_i}{\mu_i - \lambda_i} - \frac{Z_S}{G_S^{simple}} + \frac{1}{G_S^{simple}}
\end{aligned}$$

Substituting the normalization constant of expression (3.1.6) into the last equation gives the result for server utilization of queue 1. Substituting queue j for queue 1 gives the same result for the server utilization of queue j. □

When we divide each term in expression (3.1.16) by $\prod_i(\mu_i)$ and make use of the expression $\frac{\mu_i - \lambda_i}{\mu_i} = 1 - \rho_i^{Jackson}$ we get

$$\rho_{j,S}^{simple} = \frac{\rho_j^{Jackson}}{(Z_S - 1)\prod_i\big(1 - \rho_i^{Jackson}\big) + 1}$$

We define r_S to be

$$r_S = \frac{1}{(Z_S - 1)\prod_i\big(1 - \rho_i^{Jackson}\big) + 1} \qquad (3.1.17)$$

The expression (3.1.17) for r_S is the same as the expression for h_S (3.1.14). We can write the server utilization of queue j as

$$\rho_{j,S}^{simple} = \rho_j^{Jackson}\, r_s \qquad (3.1.18)$$

3.1.2 Cost function: expected queue length

In this subsection we define a corresponding cost game for the network with simple sharing when the expected queue length is taken as cost function. We investigate if the core is nonempty. When the coalition consists of one queue, the cost for this coalition is the expected queue length from the Jackson network. When the coalition consists of two or more queues, the cost for the coalition is defined as follows. We first take the sum of the expected queue lengths of all queues of the network with simple sharing. Then we reduce this sum with the expected queue length from the Jackson network for every queue not participating in the coalition. The cost function $c(S)$ is

$$\begin{aligned}
c(j) &= E\big(L_j^{Jackson}\big) && \text{if} \quad j \in S \quad \text{and} \quad |S| = 1 \\
c(S) &= h_S \sum_{j \in N} E\big(L_j^{Jackson}\big) - \sum_{j \in N-S} E\big(L_j^{Jackson}\big) && \text{if } |S| > 1
\end{aligned} \qquad (3.1.19)$$

Definition 5 *We define a cost game for a network with simple sharing with the operators of the queues as players. The cost function for coalition S is given by the expressions (3.1.19).*

Proposition 6 *Let $S \subseteq T \subseteq N$. The cost game of definition (5) is monotone increasing if we have*

$$h_S - h_T \le \frac{\sum_{j \in T-S} E\big(L_j^{Jackson}\big)}{\sum_{j \in N} E\big(L_j^{Jackson}\big)} \quad \text{for all } S \subseteq N \tag{3.1.20}$$

Proof.
The network is monotone increasing if $c(S) \le c(T)$.

$$\begin{aligned} h_S \sum_{j \in N} E\big(L_j^{Jackson}\big) - \sum_{j \in N-S} E\big(L_j^{Jackson}\big) &\le h_T \sum_{j \in N} E\big(L_j^{Jackson}\big) - \sum_{j \in N-T} E\big(L_j^{Jackson}\big) \\ \sum_{j \in N} E\big(L_j^{Jackson}\big)\big(h_S - h_T\big) &\le \sum_{j \in T-S} E\big(L_j^{Jackson}\big) \end{aligned}$$

□

Proposition 7 *It is beneficial for the operators of the cost game of definition (5) to cooperate.*

Proof.
We show that the following inequality is satisfied

$$c(N) \le \sum_{j \in N} c(j)$$

Substituting the cost function of 3.1.19 gives

$$h_N \sum_{j \in N} E\big(L_j^{Jackson}\big) \le \sum_{j \in N} E\big(L_j^{Jackson}\big)$$

Since we always have $Z_N > 1$ the denominator of expression (3.1.14) is larger than one. Therefore we get $h_N < 1$. □

Proposition 8 *Let the vector x be a cost allocation vector of the cost game of definition (5). Let each operator be assigned an equal part of the total costs $c(N)$. The vector x is then in the core if*

$$h_S \ge \frac{|S|}{|N|} h_N - \frac{\sum_{j \in S} E\big(L_j^{Jackson}\big)}{\sum_{j \in N} E\big(L_j^{Jackson}\big)} + 1 \quad \text{for all } S \subseteq N \tag{3.1.21}$$

Proof.
The core is nonempty if

$$\sum_{i \in S} x_i \le c(S) \text{ for all } S \subseteq N \tag{3.1.22}$$

We have

$$x_i = \frac{c(N)}{|N|} \text{ for all } i \in N$$

Inequality (3.1.22) can then be written as

$$\frac{|S|}{|N|}c(N) \leq c(S) \text{ for all } S \subseteq N$$

$$\frac{|S|}{|N|}h_N \sum_{j\in N} E\big(L_j^{Jackson}\big) \leq h_S \sum_{j\in N} E\big(L_j^{Jackson}\big) - \sum_{j\in N-S} E\big(L_j^{Jackson}\big)$$

Dividing the last inequality with $\sum_{j\in N} E\big(L_j^{Jackson}\big)$ gives inequality (3.1.21). □

Proposition 9 *The Shapley value for queue j of the cost game of definition (5) is*

$$\Phi_j = \sum_{j\in N} E\big(L_j^{Jackson}\big) \sum_{S\subseteq N:\, j\notin S} \frac{|S|!\big(|N|-|S|-1\big)!}{N!}\big(h_S - h_{S\cup i}\big) \tag{3.1.23}$$

Proof.
We denote the worth of coalition S with $v(S)$ and we define it as the difference between not cooperating and cooperating. We have

$$v(S) = \sum_{j\in S} E\big(L_j^{Jackson}\big) - c(S) \tag{3.1.24}$$

Substituting expression (3.1.19) for $c(S)$ we can write this as

$$v(S) = \sum_{j\in N} E\big(L_j^{Jackson}\big)\big(1 - h_S\big) \tag{3.1.25}$$

We continue with

$$v(S\cup i) - v(S) = \sum_{j\in N} E\big(L_j^{Jackson}\big)\big(h_S - h_{S\cup i}\big) \tag{3.1.26}$$

Substituting the last expression into the formula for the Shapley value (2.1.5) gives proposition 9. □

3.1.3 Cost function: server utilization

In this subsection we define a corresponding cost game for the network with simple sharing when the server utilization is taken as cost function. We investigate if the core is nonempty. When the coalition consists of one queue, the cost for this coalition is the server utilization from the Jackson network. When the coalition consists of two or more queues, the cost for the coalition is defined as follows. We first take the sum of the server utilization of all queues of the network with simple sharing. Then we reduce this sum with the server utilization from the Jackson network for every queue not participating in the coalition. The cost function $c(S)$ is

$$\begin{aligned} c(j) &= \rho_j^{Jackson} && \text{if } j\in S \text{ and } |S|=1 \\ c(S) &= r_S \sum_{j\in N} \rho_j^{Jackson} - \sum_{j\in N-S} \rho_j^{Jackson} && \text{if } |S|>1 \end{aligned} \tag{3.1.27}$$

Definition 10 *We define a cost game for a network with simple sharing with the operators of the queues as players. The cost function for coalition S is given by the expressions (3.1.27).*

The following propositions are obtained in the same way as the propositions of the cost game of subsection (3.1.2). Therefore we only give here the results.

Proposition 11 *Let $S \subseteq T \subseteq N$. The cost game of definition (10) is monotone increasing if we have*

$$r_S - r_T \leq \frac{\sum\limits_{j \in T-S} \rho_j^{Jackson}}{\sum\limits_{j \in N} \rho_j^{Jackson}} \qquad \textit{for all } S \subseteq N \tag{3.1.28}$$

Proposition 12 *It is beneficial for the operators of the cost game of definition (10) to cooperate. Since we always have $Z_N > 1$ the denominator of expression (3.1.17) is larger than one. Therefore we get $r_N < 1$.*

Proposition 13 *Let the vector x be a cost allocation vector of the cost game of definition (10). Let each operator be assigned an equal part of the total costs $c(N)$. The vector x is then in the core if*

$$r_S \geq \frac{|S|}{|N|} r_N - \frac{\sum\limits_{j \in S} \rho_j^{Jackson}}{\sum\limits_{j \in N} \rho_j^{Jackson}} + 1 \qquad \textit{for all } S \subseteq N \tag{3.1.29}$$

Proposition 14 *The Shapley value for queue j of the cost game of definition (10) is*

$$\Phi_j = \sum_{j \in N} \rho_j^{Jackson} \sum_{S \subseteq N:\, j \notin S} \frac{|S|!\big(|N| - |S| - 1\big)!}{N!} \big(r_S - r_{S \cup i}\big) \tag{3.1.30}$$

3.2 Tandem network

In subsection 3.1.1 we derived the equilibrium distribution of a network with simple sharing by making use of partial balance. In this subsection we give another method of finding the equilibrium distribution of a network. We study a tandem network with a rule for sharing service capacity that differs from the rule of definition 1. We use the equilibrium distribution of a Jackson tandem to guess the equilibrium distribution of this tandem. We check this guessed solution by checking the global balance equations. In order to get an equilibrium distribution we will see that the rule of this section has to be restricted such that it is a special case of the rule of definition 1. In subsections 3.2.2 and 3.2.3 we formulate corresponding cost games where the expected queue length and the server utilization respectively are taken as cost functions.

3.2.1 Description

The tandem consists of J queues and the set of queues is indicated by $N = \{1, 2, ..., J\}$. We assume infinite waiting rooms, a single server and a FIFO service discipline for all queues. The external arrival process is a Poisson process with rate λ and the service distribution is exponential with rate μ_j for queue j. The state of the tandem is described as $n = (n_1, n_2, ..., n_J)$ with n_j as the number of customers at queue j. The state space is $\{(n_1, n_2, ..., n_J)\}$ where $n_1, n_2, ..., n_J \in \mathbb{N}$.

A rule for sharing service capacity in a dynamic network specifies for each state in which way the operators are allowed to share their service capacity. In definition 15 we formulate a rule for sharing service capacity that differs from the rule in definition 1.

Definition 15 *If in a tandem network there is one customer at queue j and all other queues are empty, then the service capacity of queue $j+1$ is added to the service capacity of queue j. If there is one customer at queue J and all other queues are empty, then the service capacity of queue 1 is added to the service capacity of queue J.*

The service rates for the tandem are for all $j \in \{1, 2, ..., J-1\}$

$$q(n, n - e_j + e_{j+1}) = \begin{cases} \mu_j + \mu_{j+1} & \text{if } n \in State_{one} \\ \mu_j & \text{otherwise} \end{cases} \tag{3.2.1}$$

The service rates for queue $j = J$ is

$$q(n, n - e_J) = \begin{cases} \mu_1 + \mu_J & \text{if } n_J = 1 \text{ and } n_i = 0 \ \forall i \in \{1, 2, ..., J-1\} \\ \mu_J & \text{otherwise} \end{cases} \tag{3.2.2}$$

If an equilibrium distribution of the tandem of definition 15 exists, it is unique. This equilibrium distribution can therefore be found by guessing a distribution and then checking the global balance equation for each state. If all balance equations are satisfied we have found the right distribution. Since the tandem of definition 15 is almost equivalent to the Jackson tandem we guess the following equilibrium distribution with $j \in \{1, 2, ..., J\}$.

$$\pi(n_1, n_2, ..., n_J) = \frac{1}{G^{simple}} \Big(\frac{\lambda}{\mu_1}\Big)^{n_1} \Big(\frac{\lambda}{\mu_2}\Big)^{n_2} ... \Big(\frac{\lambda}{\mu_J}\Big)^{n_J} \; \forall n_j \geq 0 | n = (0, 0, ..., 0) \tag{3.2.3}$$

The probability of the zero state is denoted as π_0^{simple}. In the following proposition we show that the guessed distribution satisfies the balance equation for each state only if the rule of definition 15 is restricted.

Proposition 16 *The tandem of definition 15 has an equilibrium distribution only if the service capacities of all queues are equal*

$$\mu_j = \mu \qquad \forall j \in \{1, 2, ..., J\} \tag{3.2.4}$$

In case equation 3.2.4 is satisfied, the equilibrium distribution of the tandem of definition 15 denoted by $\pi^{simple}(n)$ is

for the zero state

$$\pi_0^{simple} = \frac{2}{G^{simple}} \tag{3.2.5}$$

and for the other states

$$\pi^{simple}(n) = \frac{1}{G^{simple}} \Big(\frac{\lambda}{\mu}\Big)^{n_1} \Big(\frac{\lambda}{\mu}\Big)^{n_2} ... \Big(\frac{\lambda}{\mu}\Big)^{n_J} \tag{3.2.6}$$

The normalization constant G^{simple} is

$$G^{simple} = \Big(\frac{\mu}{\mu - \lambda}\Big)^J + 1 \tag{3.2.7}$$

In case equation 3.2.4 is satisfied, the rule for sharing service capacity of definition 15 is a special case of the rule of definition 1. The tandem of definition 15 then is a tandem with simple sharing having a value for $Z_S = 2$.

Proof.
The tandem of definition 15 has the same global balance equations as the Jackson tandem except for the zero state $(0, 0, ..., 0)$ and for the set of states from $State_{one}$. The normalization constant of both networks also differ. We first check the guessed equilibrium distribution (3.2.3) for the balance equations of the zero state and all states of expression (3.1.1). The balance equation for the zero state $n = (0, 0, ..., 0)$ is

$$\lambda \; \pi_0^{simple} = (\mu_1 + \mu_J)\pi^{simple}(0, 0, ..., 1)$$

$$\lambda \; \pi_0^{simple} = (\mu_1 + \mu_J)\frac{1}{G^{simple}} \frac{\lambda}{\mu_J}$$

$$\pi_0^{simple} = \frac{1}{G^{simple}} \Big(\frac{\mu_1}{\mu_J} + 1\Big) \tag{3.2.8}$$

The balance equation for the state $n = (1, 0, ..., 0)$ with $n_1 = 1$ and $n_j = 0$ for all $j \in \{2, ..., J\}$ is

$$
\begin{aligned}
(\lambda + \mu_1 + \mu_2)\ \pi^{simple}(1, 0, ..., 0) &= \lambda\ \pi_0^{simple} + \mu_J\ \pi^{simple}(1, 0, ..., 1) \\
(\lambda + \mu_1 + \mu_2)\ \frac{1}{G^{simple}} \frac{\lambda}{\mu_1} &= \lambda\ \pi_0^{simple} + \mu_J\ \frac{1}{G^{simple}}\ \frac{\lambda^2}{\mu_1 \mu_J} \\
\frac{1}{G^{simple}} \left(\frac{\lambda^2}{\mu_1} + \lambda + \frac{\lambda \mu_2}{\mu_1} \right) &= \lambda\ \pi_0^{simple} + \frac{1}{G^{simple}} \frac{\lambda^2}{\mu_1} \\
\pi_0^{simple} &= \frac{1}{G^{simple}} \left(\frac{\mu_2}{\mu_1} + 1 \right) \qquad (3.2.9)
\end{aligned}
$$

The balance equation for the state $(0, 0, ..., 0, 1)$ with $n_J = 1$ and $n_j = 0$ for all $j \in \{1, 2, ..., J-1\}$ is

$$
(\lambda + \mu_1 + \mu_J)\ \pi^{simple}(0, 0, ..., 0, 1) =
$$

$$
(\mu_{J-1} + \mu_J)\ \pi^{simple}(0, 0, ..., 1, 0) + \mu_J\ \pi^{simple}(0, 0, ..., 0, 2)
$$

$$
\begin{aligned}
(\lambda + \mu_1 + \mu_J)\ \frac{1}{G} \frac{\lambda}{\mu_J} &= (\mu_{J-1} + \mu_J) \frac{1}{G^{simple}} \frac{\lambda}{\mu_{J-1}} + \mu_J\ \frac{1}{G^{simple}} \frac{\lambda^2}{\mu_J^2} \\
\frac{1}{G^{simple}} \left(\frac{\lambda^2}{\mu_J} + \frac{\lambda \mu_1}{\mu_J} + \lambda \right) &= \frac{1}{G^{simple}} \left(\lambda + \frac{\lambda \mu_J}{\mu_{J-1}} + \frac{\lambda^2}{\mu_J} \right) \qquad (3.2.10)
\end{aligned}
$$

The balance equations for the set of states (3.1.1) with $j \in \{2, ..., J-1\}$ are

$$
(\lambda + \mu_j + \mu_{j+1})\ \pi^{simple}(0, ..., 0, 1, 0, ..., 0) =
$$

$$
(\mu_{j-1} + \mu_j)\ \pi^{simple}(0, ..., 1, 0, 0, ..., 0) + \mu_J\ \pi^{simple}(0, ..., 0, 1, 0, ..., 1)
$$

$$
\begin{aligned}
(\lambda + \mu_j + \mu_{j+1})\ \frac{1}{G^{simple}} \frac{\lambda}{\mu_j} &= (\mu_{j-1} + \mu_j)\ \frac{1}{G^{simple}} \frac{\lambda}{\mu_{j-1}} + \mu_J \frac{1}{G^{simple}} \frac{\lambda^2}{\mu_j \mu_J} \\
\frac{1}{G^{simple}} \left(\frac{\lambda^2}{\mu_j} + \lambda + \frac{\lambda \mu_{j+1}}{\mu_j} \right) &= \frac{1}{G^{simple}} \left(\lambda + \frac{\lambda \mu_j}{\mu_{j-1}} + \frac{\lambda^2}{\mu_j} \right) \qquad (3.2.11)
\end{aligned}
$$

The balance equations 3.2.9, 3.2.10 and 3.2.11 are only satisfied if $\mu_j = \mu$ for all $j \in \{1, 2, ..., J\}$. Equations 3.2.8 and 3.2.9 immediately give the equilibrium probability for the zero state.

We can write the rest of the states from the state space as

$$n \in \Big\{(n_1, n_2, ..., n_J) \text{ where } n_1, n_2, ..., n_J \in \mathbb{N} \;\Big|\; \text{zero state and } n \in State_{one}\Big\} \tag{3.2.12}$$

and the global balance equations of these states are

$$\begin{aligned}\big(\lambda + \sum_{j=1}^{J} \mu_j | 1_{n_j>0}\big)\ \pi^{simple}(n) = \lambda 1_{n_1>0} \pi^{simple}(n - e_1) + \\ \sum_{j=1}^{J-1} \mu_j | 1_{n_{j+1}>0}\ \pi^{simple}(n + e_j - e_{j+1}) + \mu_J\ \pi^{simple}(n + e_J)\end{aligned} \tag{3.2.13}$$

When the guessed equilibrium distribution (3.2.6) is substituted for $\pi^{simple}(n)$, the equations (3.2.13) are satisfied. This is because these equations are similar to the equations of the Jackson tandem except for the normalization constant G^{simple}. Since this constant replaces the constant of the Jackson tandem for each term the global balance equations remain satisfied.

The normalization constant G^{simple} is computed by setting the summation over all probabilities equal to one.

$$\begin{aligned} 1 &= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} ... \sum_{n_J=0}^{\infty} \pi^{simple}(n_1, n_2, ..., n_J) \\ &= \pi_0^{simple} - \frac{1}{G^{simple}} + \frac{1}{G^{simple}} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} ... \sum_{n_J=0}^{\infty} \Big(\frac{\lambda}{\mu}\Big)^{n_1} \Big(\frac{\lambda}{\mu}\Big)^{n_2} ... \Big(\frac{\lambda}{\mu}\Big)^{n_J} \\ &= \frac{1}{G^{simple}} \Big(\frac{\mu}{\mu - \lambda}\Big)^J + \frac{1}{G^{simple}} \end{aligned}$$

□

We see in the last equation that the value for $Z_S = 2$.

3.2.2 Cost function: expected queue length

In this subsection we give a corresponding cost game for a tandem network with simple sharing from definition 1. We assume $\mu_i = \mu$ for all $i \in N$ and we set the arrival rate at $\lambda = 1$. Using definition (3.1.6) for the normalization constant G^{simple} we get

$$G_S^{simple} = Z_S - 1 + \prod_{i \in N} \frac{\mu}{\mu - 1} \tag{3.2.14}$$

We use proposition (3) to formulate the expected queue length.

$$E\big(L_{j,S}^{simple}\big) = \frac{\mu^{|N|}}{(Z_S - 1)(\mu - 1)^{|N|} + \mu^{|N|}} \Big(\frac{1}{\mu - 1}\Big) \tag{3.2.15}$$

The value for h_S becomes

$$h_S = \frac{\mu^{|N|}}{(Z_S - 1)(\mu - 1)^{|N|} + \mu^{|N|}} \tag{3.2.16}$$

We take the expected queue length as cost function which is defined according to (3.1.19) as.

$$\begin{aligned} c(j) &= \frac{1}{\mu - 1} && \text{if} \quad j \in S \quad \text{and} \quad |S| = 1 \\ c(S) &= h_S \frac{|N|}{\mu - 1} - \frac{|N| - |S|}{\mu - 1} && \text{if } |S| > 1 \end{aligned} \tag{3.2.17}$$

We apply several propositions of subsection 3.1.2 to this tandem. Because of proposition 6 the cost game is monotone increasing if we have

$$h_S - h_T \leq \frac{|T| - |S|}{|N|} \qquad \text{for all } \; S \subseteq N \tag{3.2.18}$$

Because of proposition 7 it is beneficial for the operators of the tandem to cooperate.
Let the vector x be a cost allocation vector of the cost game. Let each operator be assigned an equal part of the total costs $c(N)$. The vector x is then in the core according to proposition 8 if

$$\frac{h_S - 1}{h_N - 1} \leq \frac{|S|}{|N|} \qquad \text{for all } \; S \subseteq N \tag{3.2.19}$$

Proposition 17 *The Shapley value for queue j of the cost game of the tandem is*

$$\Phi_j = \frac{1}{|N|} c(N) \tag{3.2.20}$$

Proof.
Using expression (3.1.26) we get

$$v(S \cup i) - v(S) = \frac{|N|}{(\mu - 1)} \big(h_S - h_{S \cup i} \big) \tag{3.2.21}$$

The marginal vector can be computed using expression (3.2.21). Since the service capacities of all queues are equal, all marginal vectors contain the same entries. This means that each queue has the same total marginal contributions. The Shapley value therefore is the same for each queue. This gives (3.2.20). □

Proposition 18 *In case the cost game consists of two queues the core is always nonempty.*

Proof.
The core is nonempty if

$$c(1, 2) \leq c(1) + c(2) \tag{3.2.22}$$

This follows from proposition (7). □

If we set $S = \{1,2\}$ then the core can be described as

$$C(N,c) = \left\{ \left(x, \frac{2h_S}{(\mu-1)} - x \right) \middle| \frac{2h_S - 1}{(\mu-1)} \leq x \leq \frac{1}{\mu-1} \right\} \tag{3.2.23}$$

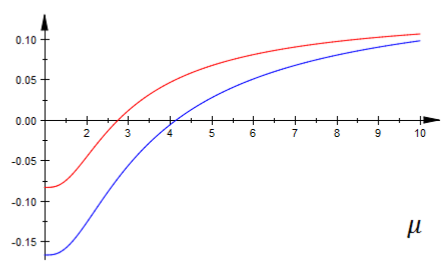
Core for tandems with three or more queues

The core of the tandem with three queues is nonempty if expression 3.2.19 is satisfied for $S = 2$. This expression is satisfied for values $1 < \mu < y$. When we solve expression $y^3 - 6*y^2 + 6*y - 2$ for y we get the upper bound for μ. This upper bound is $y \approx 4.85$.

We change expression 3.2.19 into a function $V(\mu)$ of μ

$$V(\mu) = \frac{h_S - 1}{h_N - 1} - \frac{|S|}{|N|} \tag{3.2.24}$$

In figure 3.1 this function is shown in case $N = 4$. The blue curve corresponds to a subset with $|S| = 2$ and the red curve corresponds to a subset with $|S| = 3$. It shows that the core of the tandem with four queues is nonempty if expression 3.2.19 is satisfied for $S = 3$. This expression is satisfied for values $1 < \mu < z$. When we solve expression $z^4 - (24*z^3)/5 + (36*z^2)/5 - (24*z)/5 + 6$ for z we get the upper bound for μ. This upper bound is $z \approx 2.77$.



Figuur 3.1: Function $V(\mu)$ with $N = 4$, $S = 2$ in blue and $S = 3$ in red

3.2.3 Cost function: server utilization

In this subsection we give another corresponding cost game for a tandem network with simple sharing from definition 1. As in subsection 3.2.2 we assume $\mu_i = \mu$ for all $i \in N$ and we set the arrival rate at $\lambda = 1$. Using definition (3.1.6) for the normalization constant G^{simple} we get

$$G_S^{simple} = Z_S - 1 + \prod_{i \in N} \frac{\mu}{\mu - 1} \tag{3.2.25}$$

We use proposition (4) to formulate the server utilization.

$$\rho_{j,S}^{simple} = \frac{\mu^{|N|} - (\mu-1)\mu^{|N|-1}}{(Z_S - 1)(\mu-1)^{|N|} + \mu^{|N|}} \tag{3.2.26}$$

The value for r_S becomes

$$r_S = \frac{1}{(Z_S - 1)\left(1 - \frac{1}{\mu}\right)^{|N|} + 1} \tag{3.2.27}$$

We take the server utilization as cost function which is defined according to (3.1.27) as

$$
\begin{aligned}
c(j) &= \frac{1}{\mu} && \text{if} \quad j \in S \quad \text{and} \quad |S| = 1 \\
c(S) &= r_S \, \frac{|N|}{\mu} - \frac{|N| - |S|}{\mu} && \text{if } |S| > 1
\end{aligned}
\tag{3.2.28}
$$

We apply several propositions of subsection (3.1.3) to the tandem.
Because of proposition 11 the cost game is monotone increasing if we have

$$
r_S - r_T \leq \frac{|T| - |S|}{|N|} \qquad \text{for all } \ S \subseteq N \tag{3.2.29}
$$

Because of proposition 12 it is beneficial for the operators of the tandem to cooperate.

Let the vector x be a cost allocation vector of the cost game. Let each operator be assigned an equal part of the total costs $c(N)$. The vector x is then in the core according to proposition 13 if

$$
\frac{r_S - 1}{r_N - 1} \leq \frac{|S|}{|N|} \qquad \text{for all } \ S \subseteq N \tag{3.2.30}
$$

Proposition 19 *The Shapley value for queue j of the cost game of the tandem is*

$$
\Phi_j = \frac{1}{|N|} c(N) \tag{3.2.31}
$$

Proof.
Using expression (3.1.26) we get

$$
v(S \cup i) - v(S) = \frac{|N|}{(\mu)} \big(r_S - r_{S \cup i} \big) \tag{3.2.32}
$$

The marginal vector can be computed using proposition 14. Since the service capacities of all queues are equal, all marginal vectors contain the same entries. This means that each queue has the same total marginal contributions. The Shapley value therefore is the same for each queue. This gives (3.2.31). □

Proposition 20 *In case the cost game consists of two queues the core is always nonempty.*

Proof.
The core is nonempty if

$$
c(1, 2) \leq c(1) + c(2) \tag{3.2.33}
$$

This follows from proposition (12). □

If we set $S = \{1, 2\}$ then the core can be described as

$$C(N, c) = \left\{ \left(x, \frac{2r_S}{\mu} - x \right) \,\middle|\, \frac{2r_S - 1}{\mu} \leq x \leq \frac{1}{\mu} \right\} \tag{3.2.34}$$

Core for tandems with three or more queues

We now have the same results as for the cost function with expected queue length. The core of the tandem with three queues and four queues are nonempty for the same values of μ as before.This result is not surprising since $h_S = r_S$.

4 Proportional sharing

In this section we study another extension of a Jackson network for which the individual operators are allowed to cooperate in a dynamic way. The service capacities of all queues are added together and each queue receives that amount that is proportional to the number of customers at its queue. In this report we denote it as a network with proportional sharing. In section 4.1 we formulate a general network and in section 4.2 we consider a tandem network. It is difficult to find an equilibrium distribution for a general network if subsets of queues are sharing service capacity proportionally. Therefore we only study corresponding cost games for a tandem network in section 4.2.

4.1 General network: description

In this subsection we describe a general network with proportional sharing as a Kelly Whittle network. We start with a description of a network where the amount that an operator is allowed to use from his own service capacity depends on the number of customers at all queues of the network. For this network we give the equilibrium distribution, the expected queue length and the server utilization. Then we will apply these results to the case where the service capacities of all queues are added together before they are shared proportionally.

The set of queues of the network is indicated by $N = \{1, 2, ..., J\}$. We assume infinite waiting rooms, a single server and a FIFO service discipline for all queues. For queue j the external arrival process is a Poisson process with rate γ_j. The state of the network is the number of customers at each queue and is described as $n = (n_1, n_2, ..., n_J)$. The state space is $\{(n_1, n_2, ..., n_J)\}$ where $n_1, n_2, ..., n_J \in \mathbb{N}$. The total arrival rate at queue j is λ_j and satisfies the traffic equations (2.3.5).

The amount of service capacity the operator of queue j is allowed to use from his own service capacity θ_j is proportional to the number of customers at his queue with respect to the total number of customers at all queues of the network. A network with this type of sharing can be formulated as a Kelly Whittle network if the function $\Phi(n)$ of expression (2.3.6) is written as

$$\Phi(n) = \binom{n_1 + n_2 + ... + n_J}{n_1, n_2, ..., n_J} \tag{4.1.1}$$

The function $\phi_j(n)$ of expression (2.3.7) for queue j is

$$\phi_j(n) = \frac{n_1}{n_1 + n_2 + ... + n_J} \quad \text{with} \quad j \in \{1, 2, ..., J\} \tag{4.1.2}$$

The transition rates of this open Kelly Whittle network then become

$$q(n, n - e_j + e_k) = \frac{n_j}{n_1 + n_2 + ... + n_J}\theta_j P_{jk} \qquad \forall j, k \in \{1, 2, ..., J\} \tag{4.1.3}$$

$$q(n, n - e_j) = \frac{n_j}{n_1 + n_2 + ... + n_J}\theta_j P_{j0} \qquad \forall j \in \{1, 2, ..., J\} \tag{4.1.4}$$

$$q(n, n + e_k) = \gamma_k \qquad \forall k \in \{1, 2, ..., J\} \tag{4.1.5}$$

Proposition 21 *The equilibrium distribution of a Kelly Whittle network is denoted by $\pi^{KellyWhittle}(n_1, n_2, ..., n_J)$. If the function $\Phi(n)$ is defined by expression (4.1.1) the equilibrium distribution is*

$$\pi^{KellyWhittle}(n_1, n_2, ..., n_J) = c \binom{n_1 + n_2 + ... + n_J}{n_1, n_2, ..., n_J} (\frac{\lambda_1}{\theta_1})^{n_1} (\frac{\lambda_2}{\theta_2})^{n_2} ... (\frac{\lambda_J}{\theta_J})^{n_J} \tag{4.1.6}$$

The normalization constant c for this shared service network is

$$c = 1 - \frac{\lambda_1}{\theta_1} - \frac{\lambda_2}{\theta_2} - ... - \frac{\lambda_J}{\theta_J} \tag{4.1.7}$$

provided that $|\frac{\lambda_1}{\theta_1} + \frac{\lambda_2}{\theta_2} + ... + \frac{\lambda_J}{\theta_J}| < 1$.

Proof.
The equilibrium distribution follows from substituting function (4.1.1) into expression (2.3.6).

The normalization constant can be computed as follows. The sum of all probabilities is equal to 1.

$$\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} ... \sum_{n_J=0}^{\infty} \pi^{KellyWhittle}(n_1, n_2, ..., n_J) = 1$$

When we set $k = n_1 + n_2 + ... + n_J$ and substitute equilibrium distribution (4.1.6) into the last expression we get

$$c = \left(\sum_{k=0}^{\infty} \sum_{n_1=0}^{k} \sum_{n_2=0}^{k-n_1} ... \sum_{n_{J-1}=0}^{k-n_1-...-n_{J-2}} \binom{k}{n_1, n_2, ..., n_J} (\frac{\lambda_1}{\theta_1})^{n_1} (\frac{\lambda_2}{\theta_2})^{n_2} ... (\frac{\lambda_J}{\theta_J})^{n_J} \right)^{-1}$$

We make use of the multinomial theorem which states

$$\sum_{n_1+n_2+...+n_J=k, n_i \in \mathbb{N}} \binom{k}{n_1, n_2, ..., n_J} x_1^{n_1} x_2^{n_2} ... x_J^{n_J} = (x_1 + x_2 + ... + x_J)^k \tag{4.1.8}$$

We therefore get

$$c = \left(\sum_{k=0}^{\infty} \left(\frac{\lambda_1}{\theta_1} + \frac{\lambda_2}{\theta_2} + ... + \frac{\lambda_J}{\theta_J} \right)^k \right)^{-1}$$

This gives the result for the normalization constant. □

Proposition 22 *The expectation of the queue length of queue j of a Kelly Whittle network is denoted by $E\big(L_j^{KellyWhittle}\big)$. If the function $\Phi(n)$ is defined by expression (4.1.1) the expectation of the queue length of queue j is*

$$E\big(L_j^{KellyWhittle}\big) \quad = \quad \frac{\frac{\lambda_j}{\theta_j}}{(1-\frac{\lambda_1}{\theta_1}-\frac{\lambda_2}{\theta_2}-...-\frac{\lambda_J}{\theta_J})} \tag{4.1.9}$$

Proof.
We start with the computation of the expected queue length of queue 1.

$$\begin{aligned} E\big(L_1^{KellyWhittle}\big) &= \sum_{n_1=1}^{\infty}\sum_{n_2=0}^{\infty}...\sum_{n_J=0}^{\infty} n_1\ \pi^{KellyWhittle}(n_1,n_2,...,n_J) \\ &= c\sum_{k=1}^{\infty}\sum_{n_1=1}^{k}\sum_{n_2=0}^{k-n_1}...\sum_{n_{J-1}=0}^{k-n_1-...-n_{J-2}} n_1\binom{k}{n_1,n_2,...,n_J}(\frac{\lambda_1}{\theta_1})^{n_1}(\frac{\lambda_2}{\theta_2})^{n_2}...(\frac{\lambda_J}{\theta_J})^{n_J} \end{aligned}$$

We substitute s for n_1-1.

$$E\big(L_1^{KellyWhittle}\big) \quad = \quad c\sum_{k=1}^{\infty}\frac{k\lambda_1}{\theta_1}\sum_{s=0}^{k-1}\sum_{n_2=0}^{k-s-1}...\sum_{n_{J-1}=0}^{k-s-1-...-n_{J-2}}\frac{(k-1)!}{s!n_2!...n_J!}(\frac{\lambda_1}{\theta_1})^{s}(\frac{\lambda_2}{\theta_2})^{n_2}...(\frac{\lambda_J}{\theta_J})^{n_J}$$

Because of the multinomial theorem (4.1.8) we get

$$E\big(L_1^{KellyWhittle}\big) \quad = \quad c\sum_{k=1}^{\infty}\frac{k\lambda_1}{\theta_1}\Big(\frac{\lambda_1}{\theta_1}+\frac{\lambda_2}{\theta_2}+...+\frac{\lambda_J}{\theta_J}\Big)^{k-1} \tag{4.1.10}$$

Taking the derivative of the geometric series gives the following identity.

$$\sum_{k=1}^{\infty}k\ x^{k-1} \quad = \quad \frac{1}{(1-x)^2} \tag{4.1.11}$$

Substituting the normalization constant of expression (4.1.7) into equation (4.1.10) and making use of the derivative of the geometric series gives the result for the expected queue length of queue 1.

$$E\big(L_1^{KellyWhittle}\big) \quad = \quad \frac{\frac{\lambda_1}{\theta_1}}{(1-\frac{\lambda_1}{\theta_1}-\frac{\lambda_2}{\theta_2}-...-\frac{\lambda_J}{\theta_J})}$$

The expectations of the queue length for queue j, $E(L_j^{KellyWhittle})$ can be computed similarly.
□

Proposition 23 *The server utilization of queue j of a Kelly Whittle network is denoted by $\rho_j^{KellyWhittle}$. If the function $\Phi(n)$ is defined by expression (4.1.1) the server utilization of queue j is*

$$\rho_j^{KellyWhittle} = \frac{\frac{\lambda_j}{\theta_j}}{1-\frac{\lambda_1}{\theta_1}-...-\frac{\lambda_{j-1}}{\theta_{j-1}}-\frac{\lambda_{j+1}}{\theta_{j+1}}-...\frac{\lambda_J}{\theta_J}} \tag{4.1.12}$$

In expression (4.1.12) we find the term $\frac{\lambda_j}{\theta_j}$ in the numerator but not in the denominator.

Proof.
We first compute the service utilization of queue 1, $\rho_1^{KellyWhittle}$. This can be determined by a summation over all states where queue 1 is nonempty, or equivalently

$$\rho_1^{KellyWhittle} = 1-\sum_{n_2=0}^{\infty}...\sum_{n_J=0}^{\infty}\pi^{KellyWhittle}(0,n_2,...,n_J)$$

We set $k = n_2 + ... + n_J$ to get

$$\rho_1^{KellyWhittle} = 1-c\sum_{k=0}^{\infty}\sum_{n_2=0}^{k}\sum_{n_3=0}^{k-n_2}......\sum_{n_{J-1}=0}^{k-n_2-...-n_{J-2}}\binom{k}{n_2,...,n_J}(\frac{\lambda_2}{\theta_2})^{n_2}......(\frac{\lambda_J}{\theta_J})^{n_J}$$

Because of the multinomial theorem (4.1.8) we can write

$$\begin{aligned}\rho_1^{KellyWhittle} &= 1-c\sum_{k=0}^{\infty}(\frac{\lambda_2}{\theta_2}+...+\frac{\lambda_J}{\theta_J})^k \\ &= \frac{\frac{\lambda_1}{\theta_1}}{1-\frac{\lambda_2}{\theta_2}-...-\frac{\lambda_J}{\theta_J}}\end{aligned}$$

After substituting the normalization constant we find in the last expression the term $\frac{\lambda_1}{\theta_1}$ in the numerator but not in the denominator. The server utilization $\rho_j^{KellyWhittle}$ for queue j with $j \in \{2,...,J\}$ can be determined similarly. □

In definition 24 we apply the results of the equilibrium distribution, the expected queue length and the server utilization to the rule where the operators share their service capacity proportionally. The individual capacities are added together. The amount of service the operator of queue j receives from this total service capacity is proportional to the number of customers at queue j with respect to the total number of customers in the network.

Definition 24 *Let μ_j be the service capacity for queue j. If the individual capacities of the network of proposition (21) are added together we refer to it as a network with proportional sharing. We define the variables θ_j to be equal to the sum of the service capacities. We have*

$$\theta_j = \sum_{i\in N}\mu_i \quad for\ j\in\{1,2,...,J\} \tag{4.1.13}$$

Proposition 25 *The equilibrium distribution, the expected queue length and the server utilization of the network of definition 24 are denoted respectively by*

$\pi^{proportional}(n_1, n_2, ..., n_J)$, $E\big(L_j^{proportional}\big)$ *and* $\rho_j^{proportional}$.

We get

$$\pi^{proportional}(n_1, n_2, ..., n_J) = c \binom{n_1 + n_2 + ... + n_J}{n_1, n_2, ..., n_J} \frac{\lambda_1^{n_1} \lambda_2^{n_2} ... \lambda_J^{n_J}}{(\sum_N \mu_i)^{(n_1+n_2+...+n_J)}} \tag{4.1.14}$$

with normalization constant c

$$c = 1 - \frac{\sum_{i=1}^{J} \lambda_i}{\sum_{i=1}^{J} \mu_i} \tag{4.1.15}$$

and we have

$$E\big(L_j^{proportional}\big) = \frac{\lambda_j}{\sum_{i=1}^{J} (\mu_i - \lambda_i)} \tag{4.1.16}$$

$$\rho_j^{proportional} = \frac{\lambda_j}{\sum_{i=1}^{J} \mu_i - \sum_{i \neq j} \lambda_i} \tag{4.1.17}$$

Proof.
Substituting expression (4.1.13) into the results of a network with proportional sharing gives these result. Since we assume $\mu_j > \lambda_j$ for queue j, we find

$$\sum_{i=1}^{J} \mu_i > \sum_{i=1}^{J} \lambda_i \tag{4.1.18}$$

Because of inequality 4.1.18 the condition $|\frac{\lambda_1}{\theta_1} + \frac{\lambda_2}{\theta_2} + ... + \frac{\lambda_J}{\theta_J}| < 1$ for the network of proposition (21) is satisfied as well. □

4.2 Tandem network

In this section we study cost games for a tandem network with proportional sharing of service capacities. In section 4.2.1 we give a description of a tandem network. In sections 4.2.2 and 4.2.3 we study cost games where the expected queue length and server utilization respectively are taken as cost functions.

4.2.1 Description

The set of queues of the tandem is indicated by $N = \{1, 2, ..., J\}$. We assume infinite waiting rooms, a single server and a FIFO service discipline for all queues. The external arrival process is a Poisson process with rate λ. The service capacity of queue j is exponential with rate μ_j. The service capacities of all queues are added together. Each queue receives that amount of this total capacity that is proportional to the number of customers at his queue. The state of the tandem is described as $n = (n_1, n_2, ..., n_J)$ with n_j as the number of customers at queue j. The state space is $\{(n_1, n_2, ..., n_J)\}$ where $n_1, n_2, ..., n_J \in \mathbb{N}$.

The transition rates of this tandem network then become

$$q(n, n - e_j + e_{j+1}) = \frac{n_j}{n_1 + n_2 + ... + n_J} \sum_{i=1}^{J} \mu_i \qquad \forall j \in \{1, 2, ..., J-1\} \tag{4.2.1}$$

$$q(n, n - e_J) = \frac{n_J}{n_1 + n_2 + ... + n_J} \sum_{i=1}^{J} \mu_i \tag{4.2.2}$$

$$q(n, n + e_1) = \lambda \tag{4.2.3}$$

In this section we assume the external arrival rate to be $\lambda = 1$. When we apply the results of proposition(25) to this tandem we get

$$\pi^{proportional}(n_1, n_2, ..., n_J) = c \binom{n_1 + n_2 + ... + n_J}{n_1, n_2, ..., n_J} \frac{1}{(\sum_N \mu_i)^{(n_1+n_2+...+n_J)}} \tag{4.2.4}$$

with normalization constant c

$$c = 1 - \frac{|N|}{\sum_{i=1}^{J} \mu_i} \tag{4.2.5}$$

When we apply the results of propositions 22 and 23 we get

$$E(L_j^{proportional}) = \frac{1}{\sum_{i=1}^{J} \mu_i - |N|} \tag{4.2.6}$$

$$\rho_j^{proportional} = \frac{1}{\sum_{i=1}^{J} \mu_i - |N| + 1} \tag{4.2.7}$$

4.2.2 Cost function: expected queue length

In this section we define a cost game for the tandem network with proportional sharing of service capacity. We take the expected queue length as cost function. The cost function is

$$
\begin{aligned}
c(j) &= \frac{1}{\mu_j - 1} && \text{if } j \in S \text{ and } |S| = 1 \\
c(S) &= \frac{|S|}{\sum\limits_{i=J-|S|+1}^{J} \mu_i - |S|} && \text{if } S = \{J-|S|+1, J-|S|+2, ..., J-1, J\}
\end{aligned}
\tag{4.2.8}
$$

In the next definition we formulate a cost game for this tandem.

Definition 26 *We define a cost game for a tandem network with proportional sharing of service capacity with the operators of the queues as players. Coalitions can only be formed by consecutive queues starting backwards from the queue at the end of the tandem. The cost function for coalition S is given by the expressions (4.2.8).*

In the following proposition we show that we can find an equilibrium distribution for the tandem network of definition 26.

Proposition 27 *Let in a tandem network with proportional sharing of service capacity the operators cooperate according to definition 26. The equilibrium distribution of the cooperating queues is independent of the equilibrium distribution of the queues not taking part in the coalition.*

Proof.
Let a tandem network consist of the queues $\{1, 2, ..., J\}$ and let S be the set of cooperating queues. According to definition 26 the set S is the set of queues $\{J-|S|+1, J-|S|+2, ..., J-1, J\}$. Since we assume the external arrival process to be a Poisson process, the departure processes of the queues not taking part in the coalition are Poisson processes as well. The number of customers at queue $J-|S|$ is therefore independent of the departure process of this queue. Because this departure process is the arrival process of the set cooperating queues, the equilibrium distributions of the coalition is independent of the equilibrium distribution of queue $J-|S|$. □

We need the following expression for a set S.

$$
\bar{\mu_S} = \frac{\sum\limits_{i=J-|S|+1}^{J} \mu_i}{|S|} \tag{4.2.9}
$$

Proposition 28 *Let $S \subseteq T \subseteq N$. The cost game of definition (26) is monotone increasing if we have*

$$
\bar{\mu_S} \geq \bar{\mu_T} \tag{4.2.10}
$$

Proof.
The network is monotone increasing if $c(S) \leq c(T)$. If we express $c(S)$ and $c(T)$ according to the cost function 4.2.8 and divide them by $|S|$ and $|T|$ respectively we get

$$\frac{1}{\bar{\mu_S} - 1} \leq \frac{1}{\bar{\mu_T} - 1} \tag{4.2.11}$$

This is true if inequality (4.2.10) is satisfied. □

Proposition 29 *It is beneficial for the operators of the cost game of definition (26) to cooperate.*

Proof.
We need to check if cooperation of all J queues leads to smaller costs than when working separately. Therefore we need to check

$$c(1, 2, ..., J) < \sum_{i=1}^{J} c(i) \tag{4.2.12}$$

We continue with

$$c(1, 2, ..., J) = \frac{|N|}{\sum_{i=1}^{J} \mu_i - |N|} = \frac{1}{\bar{\mu} - 1}$$

The function $f(\mu) = \frac{1}{\mu - 1}$ is a convex function which means

$$f(\bar{\mu}) \leq \frac{1}{|N|} \sum_{i=1}^{J} f(\mu_i)$$

Therefore we can write

$$\frac{1}{\bar{\mu} - 1} \leq \frac{1}{|N|} \left(\sum_{i=1}^{J} \frac{1}{\mu_i - 1} \right) \leq \sum_{i=1}^{J} \frac{1}{\mu_i - 1} = \sum_{i=1}^{J} c(i)$$

and we can conclude that inequality (4.2.12) is satisfied. □

Proposition 30 *Let the vector x be a cost allocation vector of the cost game of definition (26). Let each operator be assigned an equal part of the costs of the grand coalition $c(N)$. The vector x is then in the core.*

Proof.
The core is nonempty if

$$\sum_{i \in S} x_i \quad \leq \quad c(S) \text{ for all } S \subseteq N \tag{4.2.13}$$

We have

$$x_i = \frac{c(N)}{|N|} \text{ for all } i \in N$$

Inequality (4.2.13) can then be written as

$$\frac{|S|}{|N|}c(N) \leq c(S) \text{ for all } S \subseteq N$$

Substituting the cost function (4.2.8) into the last inequality gives

$$\frac{|S|}{\sum_{i=1}^{J} \mu_i - |N|} \leq \frac{|S|}{\sum_{i=J-|S|+1}^{J} \mu_i - |S|} \text{ for all } S \subseteq N$$

Since $u_i - 1 > 0$ for all queues i, we have

$$\sum_{i=1}^{J} \mu_i - |N| \geq \sum_{i=J-|S|+1}^{J} \mu_i - |S| \tag{4.2.14}$$

Therefore inequality (4.2.13) is satisfied. □

Proposition 30 shows that if each operator is assigned an equal part of the costs of the grand coalition, it is still beneficial for all operators to cooperate. This result is independent of the values for the service capacities of the individual queues μ_i. This is remarkable since there is no need for different contributions to the total costs when the service capacities of the individual queues differ. It can be explained by the fact that queues with relatively low service capacities are allowed to extend their service capacity for many states n of the tandem network. The costs for these low capacity queues strongly decrease when sharing service capacities proportionally. This effect is that large that the costs of the grand coalition can be divided into equal parts. This equal part is even smaller than the individual costs for the queues with high capacity. We give an example.

Example
Let $|N| = 3$ with $\{\mu_1, \mu_2, \mu_3\} = \{2, 3, 100\}$. Queue 3 is the last queue of the tandem. The possible coalitions and costs are $c(\{3, 2, 1\}) = 3/102$, $c(\{3, 2\}) = 2/101$ and $c(\{3\}) = 1/100$. We see that the operator of queue 3 also benefits from cooperation if the costs are equally divided. It is even possible to find a cost allocation in the core where the operators of queues 1 and 2 pay an amount to the operator of queue 3. For example the cost allocation $\{1/4, 1/4, -8/17\}$ is also in the core.

Proposition 31 *The Shapley value of the cost game of definition 26 for queue J is*

$$\Phi_J \quad = \quad 0 \tag{4.2.15}$$

and for queues $J-|S|$ *with* $1 \leq |S| \leq J-1$ *it is*

$$\Phi_{(J-|S|)} \quad = \quad \frac{1}{\mu_{(J-|S|)}-1} + \frac{|S|}{\sum\limits_{k=J-|S|+1}^{J} \mu_k - |S|} - \frac{|S|+1}{\sum\limits_{k=J-|S|}^{J} \mu_k - |S| - 1} \tag{4.2.16}$$

Proof.
We denote the worth of coalition S with $v(S)$ and we define it as the difference between not cooperating and cooperating. We have

$$v(S) = \sum_{j \in S} c(\{j\}) - c(S) \tag{4.2.17}$$

Substituting the cost function (4.2.8) for $c(S)$ we can write this as

$$v(S) = \sum_{k \in S} \frac{1}{\mu_k - 1} - \frac{|S|}{\sum\limits_{k=J-|S|+1}^{J} \mu_k - |S|}$$

We continue with the marginal contribution of queue $J-|S|$ to coalition $S = \{J-|S|+1, J-|S|+2, ..., J-1, J\}$.

$$v\Big(S \cup (J-|S|)\Big) - v(S) = \frac{1}{\mu_{(J-|S|)}-1} + \frac{|S|}{\sum\limits_{k=J-|S|+1}^{J} \mu_k - |S|} - \frac{|S|+1}{\sum\limits_{k=J-|S|}^{J} \mu_k - |S| - 1} \tag{4.2.18}$$

The Shapley value is also defined as a vector with the average of the marginal contributions for each player. The cost game of definition 26 does not allow for queues to enter a coalition in a different order. There is only one marginal vector and each entry is described by expression (4.2.18). □

The core for a tandem with two queues can be described as

$$C(N,c) = \left\{ \left(x, \frac{2}{\mu_1+\mu_2-2} - x\right) \;\middle|\; \frac{\mu_2-\mu_1}{(\mu_1+\mu_2-2)(\mu_2-1)} \leq x \leq \frac{1}{\mu_1 - 1} \right\} \tag{4.2.19}$$

4.2.3 Cost function: server utilization

In this section we define another cost game for the tandem with proportional sharing of service capacity. We take the server utilization as cost function.

The cost function is

$$
\begin{aligned}
c(j) &= \frac{1}{\mu_j} && \text{if } j \in S \text{ and } |S| = 1 \\
c(S) &= \frac{|S|}{\sum\limits_{i=J-|S|+1}^{J} \mu_i - |S| + 1} && \text{if } S = \{J-|S|+1, J-|S|+2, ..., J-1, J\}
\end{aligned}
\tag{4.2.20}
$$

Definition 32 *We define a cost game for a tandem network with proportional sharing of service capacity with the operators of the queues as players. Coalitions can only be formed by consecutive queues starting backwards from the queue at the end of the tandem. The cost function for coalition S is given by the expressions (4.2.20).*

The following propositions are obtained in the same way as propositions 28, 29, 30 and 31. Therefore we only give here the results.

Proposition 33 *Let $S \subseteq T \subseteq N$. The cost game of definition (32) is monotone increasing if we have*

$$\bar{\mu_S} \geq \bar{\mu_T} \tag{4.2.21}$$

Proposition 34 *It is beneficial for the operators of the cost game of definition (32) to cooperate.*

Proposition 35 *Let the vector x be a cost allocation vector of the cost game of definition (32). Let each operator be assigned an equal part of the costs of the grand coalition $c(N)$. The vector x is then in the core.*

Proposition 36 *The Shapley value of the cost game of definition 32 for queue J is*

$$\Phi_J = 0 \tag{4.2.22}$$

and for queues $J - |S|$ with $1 \leq |S| \leq J-1$ it is

$$\Phi_{(J-|S|)} = \frac{1}{\mu_{(J-|S|)}} + \frac{|S|}{\sum\limits_{k=J-|S|+1}^{J} \mu_k - |S| + 1} - \frac{|S|+1}{\sum\limits_{k=J-|S|}^{J} \mu_k - |S|} \tag{4.2.23}$$

The core for a tandem with two queues can be described as

$$C(N,c) = \left\{ \left(x, \frac{2}{\mu_1 + \mu_2 - 1} - x \right) \;\middle|\; \frac{\mu_2 - \mu_1 + 1}{\mu_2(\mu_1 + \mu_2 - 1)} \leq x \leq \frac{1}{\mu_1} \right\} \tag{4.2.24}$$

.

5 Conclusions and discussion

In this section we first give the conclusions of our study. Then we give suggestions for future work.

5.1 Conclusions

In this section we focus on the results of our study and to what extent these results have answered our research questions. We wanted to investigate whether it is beneficial in a network for individual operators of queues to cooperate dynamically. The research questions divided this goal into two steps. We first wanted to know for which networks and rules we could find an equilibrium distribution. Second we wanted to see if a fair cost allocation is possible. We have found two rules for sharing service capacity for which we can compute performance measures resulting in networks with simple sharing and networks with proportional sharing. We will now describe these results in more detail.

In section 3.1.1 we computed the equilibrium distribution and performance measures of a network with simple sharing. In sections 3.1.2 and 3.1.3 we formulated corresponding cost games for which we computed some properties. These properties were rather general descriptions. The rule for sharing service capacity depends on the coalition of the cost game and is interpreted in the following way. There can only be one coalition formed in the network at the same time. Each coalition has a specific value for Z_S. The service capacity of a queue is multiplied with this value if there is one customer in the entire system at that queue. The costs for a coalition are defined as the total costs of the network minus the costs of the queues not taking part in the coalition which are the usual costs for a queue in a Jackson network. In sections 3.2.2 and 3.2.3 we applied the results of section 3.1 to a tandem network where all queues have the same service capacities. The cost games then become symmetric. Their Shapley values are computed by dividing the costs of the grand coalition into equal parts. For tandem networks with two queues we have shown that the core is nonempty for all values of μ. For networks with three and four queues we have given the values of μ such that the core is nonempty.

In section 4.1 we described another extension of a Jackson network in which the operators shared service capacity dynamically. This network with proportional sharing can be formulated as a Kelly Whittle network because the global state dependent function can be expressed by the multinomial coefficient. When all operators share service capacity proportionally we have an equilibrium distribution. When a subset of operators is sharing their service capacities proportionally we in general do not have an equilibrium distribution for the network. By restricting coalitions to subsets that are formed by consecutive queues starting backwards from the queue at the end of the tandem, we have an equilibrium distribution for the network. The reason for this is that we could make use of the fact that the departure process of a Jackson tandem is a Poisson process. We showed that if each operator is assigned an equal part of the costs of the grand coalition, it is still beneficial for all operators to cooperate. This result is independent of the values for the service capacities of the individual queues μ_i. This is remarkable since there is no need for different contributions to the total costs when the service capacities of the individual queues differ. It can be explained by the fact that the costs for queues with relatively low service capacities are strongly decreasing. They benefit from the additional service capacities when their queues have relatively many customers with respect to the other queues.

Studying our results we can conclude that we have found networks and rules for which it is beneficial for the operators to cooperate dynamically. We can conclude that we have answered the research questions. But we have to make a comment. For the network with simple sharing we only have detailed results for tandem networks with equal service capacities. For the network with proportional sharing we had to restrict the network to a tandem network where coalitions can only be formed by consecutive queues starting backwards from the queue at the end of the tandem.

5.2 Discussion

Allowing independent operators of queues to share service capacity based on the state of the network creates possibilities to improve performances of a Jackson network. Our focus was mainly on formulating rules that resulted into networks for which we could compute performance measures. In this report we used two different ways to find such rule. One method was to extend the Jackson network such that partial balance is maintained. The other method was to find a rule that can be formulated as the global state dependent function of a Kelly Whittle network. In Dijk et al. [1] we found results for networks of two queues with global state dependent service capacities. Future work could define cost games for these networks.

For the definition of the cost games with simple sharing in this report we have made the assumption that operators not participating have similar costs as if they were in a Jackson network. There are other possible definitions of the cost functions that could be of interest for future work. Specific results were derived for a tandem network. Future work could also focus on other network topologies.
For the definition of the cost games with proportional sharing in this report we have restricted the possible coalitions in a tandem network. Future work could focus on other types of coalitions in the tandem network.

6 Bibliography

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7 Notation

Symbol	Description
$\mathbb{N}$	Set of natural numbers: $\{0, 1, 2, ...\}$
N	Set of queues of a network: $\{1, 2, ..., J\}$
S	$S \subseteq N$ is set of cooperating queues
$queue_0$	Exterior of a network
e_j	vector $(m_1, m_2, ..., m_J)$ such that $m_j = 1$ and $m_i = 0$ $\forall i \in \{1, 2, ..., J\} \backslash j$
n	State $(n_1, n_2, ..., n_J)$ of a network with n_j customers at queue j
$zero\ \ state$	State with $n_j = 0$ for all $j \in N$ in a network
$State_{one}$	Set of states n with $n_j = 1$ and $n_i = 0$ $\forall i \in \{1, 2, ..., J\} \backslash j$
γ_j	External arrival rate at queue j
λ_j	Total arrival rate queue j
μ_j	Rate of the service capacity at queue j
$c(S)$	Costs of coalition of set S
$C(N, c)$	Cost game with N players and cost function c
$v(S)$	Worth of coalition S. It is defined as $v(S) = \sum_{j \in S} c(\{j\}) - c(S)$

Tabel 7.1: Symbols: General

Symbol	Description
$\pi_0^{Jackson}$	Probability of the zero state
$\pi^{Jackson}(n)$	Equilibrium distribution
$E\big(L_j^{Jackson}\big)$	Expected queue length of queue j
$\rho_j^{Jackson}$	Server utilization of queue j

Tabel 7.2: Symbols: Jackson network

Symbol	Description
Z_S	Maximum factor to multiply service capacity with
G_S^{simple}	Normalization constant
$\pi_{0,S}^{simple}$	Probabilty of the zero state
$\pi_S^{simple}(n)$	Equilibrium distribution
$E\big(L_{j,S}^{simple}\big)$	Expected queue length of queue j
$\rho_{j,S}^{simple}$	Server utilization of queue j
h_S	$= \frac{\prod_i \mu_i}{\left(Z_S-1\right)\prod_i\left(\mu_i-\lambda_i\right)+\prod_i \mu_i}$
$V(\mu)$	$= \frac{h_S-1}{h_N-1} - \frac{\lvert S\rvert}{\lvert N\rvert}$
r_S	$= \frac{1}{\left(Z_S-1\right)\prod_i\left(1-\rho_i^{Jackson}\right)+1}$
$W(\mu)$	$= \frac{r_S-1}{r_N-1} - \frac{\lvert S\rvert}{\lvert N\rvert}$

Tabel 7.3: Symbols: Network with simple sharing depending on coalition S

Symbol	Description
$\pi(n)^{KellyWhittle}$	Equilibrium distribution
$E\big(L_j^{KellyWhittle}\big)$	Expected queue length of queue j
$\rho_j^{KellyWhittle}$	Server utilization of queue j
$\pi(n)^{proportional}$	Equilibrium distribution
$E\big(L_{j,S}^{proportional}\big)$	Expected queue length of queue j
$\rho_{j,S}^{proportional}$	Server utilization of queue j

Tabel 7.4: Symbols: Kelly Whittle network and network with proportional sharing