Using modal derivatives to determine the behaviour of mode shapes and natural frequencies during large deflections

Mieke van den Belt

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EXAMINATION COMMITTEE
Prof.dr.ir. A. de Boer
Prof.dr.ir. D.M. Brouwer PDEng
Dr.ir. M.H.M. Ellenbroek
Ir. J.P. Schilder

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PREFACE

This report is written as a result of my master Mechanical Engineering and my graduation assignment in Applied Mechanics at the University of Twente. My graduation assignment, and therefore this report, is about modal derivatives, and how this method can be used to determine the behaviour of mode shapes and natural frequencies during large deflections. The report consists of two parts. The first and main part is my research paper. The second part contains appendices, in which derivations are presented that are used for the establishment of the research paper.

Six and a half years ago, I started my bachelor in Industrial Design. At that moment, I had never expected to be where I am now. However, during my bachelor I soon found out that my main interest lied in the mechanics courses, and it was my first dynamics course, by Bert Geijselaers, that made me think of actually switching to Mechanical Engineering. During my pre-master and master, I took several courses in the field of Applied Mechanics and found out that this was the field within Mechanical Engineering that I enjoyed most. I finally decided to do my graduation project in this field as well.

Before you now lies a report of which I never thought I was capable of writing a couple of years ago. I have learned a lot and there were many people that allowed me to achieve this, and I would like to thank some people in particular.

First of all thanks to my family for giving me the opportunity to do this and for always supporting me, no matter which decisions I made, even though you probably did not understand a thing of what I was doing most of the time. Especially thanks to my dad, from whom I probably inherited my love for Mechanical Engineering. I also would like to thank my boyfriend, for being there whenever I was stressed out and for believing in me even when I did not. I know I am going to hear a lot of *'I told you so'* after all of this, but thank you for making me believe that I might actually be able to do this.

Furthermore thanks to my roommates in N242. The coffee breaks, puzzles, football talks and pool competion at our room were the best distraction possible. Thanks to you, I now know that I have a very rare talent for counting the number of letters in words, but definitely not for playing pool. Thank you guys, for the great time I had with all of you.

The last person I would like to thank is my supervisor, Jurnan Schilder. This report probably would not have been here if it was not for you. You were not only a great supervisor during my graduation, but without your dynamics courses and your enthusiasm for dynamics, I might have never chosen to graduate in this subject. Thanks a lot for all your help and inspiration.

I am looking forward to start my PhD project at the University of Twente and I am sure that I will have a great time here for four more years.

Enschede, January 2017
Mieke van den Belt

CONTENTS

NOMENCLATURE

List of symbols

A	Cross sectional area	[m^2]
b	Width	[m]
E	Young's modulus	[N/m^2]
G	Shear modulus	[N/m^2]
h	Heigth	[m]
I	Second moment of area	[m^4]
I_{yy}	Second moment of area around y-axis	[m^4]
I_{zz}	Second moment of area around z-axis	[m^4]
$\mathbf{K}$	Stiffness matrix	[N/m]
L	Length	[m]
$\mathbf{M}$	Mass matrix	[kg]
$\mathbf{q}$	Generalized coordinates	[-]
$\mathbf{Q}$	Elastic forces	[N]
$\mathbf{S}$	Shape function matrix	[-]
$\mathbf{u}$	Displacement	[m]
U	Strain energy	[N·m]
V	Volume	[m^3]
$\boldsymbol{\gamma}$	Frequency derivatives	[-]
$\boldsymbol{\varepsilon}$	Strain matrix	[-]
$\hat{\boldsymbol{\varepsilon}}$	Strain vector with components of $\boldsymbol{\varepsilon}$	[-]
$\boldsymbol{\eta}$	Modal coordinates	[-]
$\boldsymbol{\theta}$	Modal derivatives	[-]
ρ	Density	[kg/m^3]
$\boldsymbol{\sigma}$	Stress matrix	[N/m^2]
$\hat{\boldsymbol{\sigma}}$	Stress vector with components of $\boldsymbol{\sigma}$	[N/m^2]
φ	Torsion angle	[rad]
$\boldsymbol{\phi}_i$	i^{th} mode shape	[-]
$\boldsymbol{\Phi}$	Modal matrix	[-]
ω_i	i^{th} natural frequency	[rad/s]

List of abbreviations

CFR	Corotational frame of reference
EVP	Eigenvalue problem
FDs	Frequency derivatives
FFR	Floating frame of reference
MDs	Modal derivatives
MOR	Model order reduction
NLFFR	Nonlinear floating frame of reference
VMs	Vibration modes/mode shapes

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Abstract. In this paper, a method will be presented with which it is possible to determine how mode shapes and natural frequencies change during large elastic deflections. This is done by applying a model order reduction technique in the floating frame of reference. This model order reduction technique makes use of modal derivatives, which are corrections on the mode shapes. Modal derivatives take higher order terms of the strain energy into account and therefore the method can account for geometric nonlinearities and can thus be applied on large elastic deflections. Using modal derivatives, mode shapes and natural frequencies in any given configuration can be determined. It is also possible to determine mode shapes and natural frequencies by solving the eigenvalue problem of a beam in a certain configuration, but this method requires significantly more calculation time. By comparing the results obtained using modal derivatives to the results obtained by using the eigenvalue problem, the method is validated. This validation shows that modal derivatives can indeed be used to correctly describe the behaviour of the mode shapes and natural frequencies for large deflections. The method is applied on a three-dimensional beam element as well, showing coupling between bending and torsion modes.

Keywords. Floating frame of reference, model order reduction, geometric nonlinearities, modal derivatives, frequency derivatives, precision engineering.

1 INTRODUCTION

In the floating frame of reference (FFR) formulation, the global position of a point P on a flexible body is written as a combination of the translation orientation of the body coordinates, the undeformed position of P with respect to these body coordinates and elastic coordinates of the body. This means that, especially for large systems, the number of coordinates increases significantly, therefore also increasing calculation time. A large advantage of the FFR formulation is its ability to apply model order reduction (MOR) [1,2]. The local generalized coordinates $\mathbf{q}$ are written as a linear combination of a small number of mode shapes: $\mathbf{q} = \mathbf{\Phi}\boldsymbol{\eta}$. The modal coordinates $\boldsymbol{\eta}$ describe how the mode shapes behave in time. Since only a limited amount of modal coordinates is required to accurately describe a systems motion, this reduces the amount of coordinates and therefore decreases calculation time. The mode shapes $\boldsymbol{\phi}_i$ are determined by solving the eigenvalue problem for free vibrations of an element: $(\mathbf{K} - \omega_i^2\mathbf{M})\boldsymbol{\phi}_i = \mathbf{0}$. However, these coordinates are still only valid for small deflections around $\mathbf{q}$ and thus do not take geometric nonlinearities into account. This means that for large deflections, the eigenvalue problem can no longer be solved for the equilibrium configuration, but needs to be solved for the deflected configuration. To this end, the stiffness matrix $\mathbf{K}$ has to updated for every step, after which the eigenvalue problem can be solved. Especially for large deflections and large systems, this is rather tedious job and therefore, in [2] and [3], Wu and Tiso present another MOR technique, based on modal derivatives. In this technique, static corrections on the mode shapes are taken into account. In earlier literature, the nonlinear floating frame of reference (NLFFR) was compared with traditional linear [4] and nonlinear [5] floating frame. Wu and Tiso [3] validated their code against numerical examples in [6], in which the theory of the corotational frame of reference (CFR) was used. These examples showed that a combination of mode shapes (VMs) and modal derivatives (MDs) yields better results than using only VMs and is just as accurate as CFR or NLFFR.

Modal derivatives are obtained by deriving the eigenvalue problem with respect to a set of coordinates. This can be done using either generalized coordinates $\mathbf{q}$ or modal coordinates $\boldsymbol{\eta}$. As stated before, using modal coordinates reduces the size of the systems and therefore the calculation time. In [2], Wu and Tiso state that the method 'does not pose additional conceptual difficulties for the extension into three dimensonal cases'. However, no numerical results are presented. In this paper, the method of using MDs is expanded to 3D beam elements and it is shown that indeed no difficulties occur. Also a technique is proposed to derive frequency derivatives (FDs) from the MDs, making it possible to define the decrease of natural frequencies for increasing deflections.

In for example precision engineering, the behaviour of the mode shapes and natural frequencies is very relevant. In this field, flexures are used that should be very compliant in one direction while constraining motions in another direction. Therefore it is highly important to know how the natural frequencies of this flexure behave in both directions during deflections. When optimizing system parameters in mechanisms such as flexure hinges, using MDs can significantly reduce calculation time compared to using the eigenvalue problem for large deflections.

In chapter 2, the method is presented for general cases. The derivativation of the modal derivatives and frequency derivatives is presented. Chapter 3 and 4 show the application of the method in 2D and 3D, respectively. Because of its applicability on leaf springs in precision engineering, the method is applied on beam elements for both the 2D and 3D cases. Chapter 5 shows the results for two numerical examples, a 2D simply supported beam and a 3D clamped-free beam. Finally the conclusions and recommendatios are presented in chapter 6.

2 METHOD: MODAL DERIVATIVES AND FREQUENCY DERIVATIVES

2.1 Stiffness

In order to derive the stiffness, first the displacement field is defined. For small deflections, linear theory can be applied. This in shown in Figure 1 (left). However, for large deflections the assumption of linear theory is not valid anymore (Figure 1 (right)).

Figure 1. Displacement using linear theory (left) and nonlinear theory (right)

The Green-Lagrange strain expression [7] is given as

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_j}\frac{\partial u_k}{\partial x_i}\right), \qquad (1)$$

where the subscripts i, j and k are replaced with 1 and 2, or 1, 2 and 3 for respectively 2D and 3D. For small deflections, the assumption of linear theory can be applied and thus this expression can be simplified. However, since we consider large deflections, we will not make this assumption. The displacements in (1) are defined as

$$\mathbf{u} = \mathbf{S}\mathbf{q}, \qquad (2)$$

in which $\mathbf{S}$ is the shape function matrix and $\mathbf{q}$ are the generalized coordinates. The strain energy [7] is expressed as

$$U = \frac{1}{2}\int_V \hat{\boldsymbol{\varepsilon}}^{\mathrm{T}}\hat{\boldsymbol{\sigma}}\, dV. \qquad (3)$$

where $\hat{\boldsymbol{\sigma}}$ and $\hat{\boldsymbol{\varepsilon}}$ are the stress and strain vector, respectively, including all terms of the stress and strain matrix.

The elastic forces can be directly determined from the differentiation of strain energy U as

$$Q_i(\mathbf{q}) = \frac{\partial U}{\partial q_i}. \tag{4}$$

The stiffness matrix and its derivatives are then given by

$$K_{ij}(\mathbf{q}) = \frac{\partial^2 U}{\partial q_i \partial q_j}, \tag{5}$$

$$\frac{\partial K_{ij}}{\partial q_k} = \frac{\partial^3 U}{\partial q_i \partial q_j \partial q_k}. \tag{6}$$

2.2 Eigenvalue problem

The eigenvalue problem for free vibrations around a certain configuration $\mathbf{q}$ is written as

$$\left(\mathbf{K} - \omega_i^2 \mathbf{M}\right)\boldsymbol{\phi}_i = \mathbf{0}. \tag{7}$$

The mass matrix [8] is defined as

$$\mathbf{M} = \int_V \rho \mathbf{S}^T \mathbf{S}\, dV, \tag{8}$$

in which ρ is the density and $\mathbf{S}$ is again the shape function matrix. Solving the eigenvalue problem yields mode shapes and natural frequencies that are valid for small deflections around $\mathbf{q}$. This means that for any large deflection $\mathbf{q}$, the system cannot be linearized around the equilibrium configuration $\overline{\mathbf{q}}$ anymore but needs to be linearized around the deflected configuration. Large deflections also change the stiffness matrix. A general expression for the stiffness matrix is given by

$$\mathbf{K} = \mathbf{K}(\mathbf{q}) = \overline{\mathbf{K}} + \frac{\partial \mathbf{K}}{\partial \mathbf{q}}(\mathbf{q} - \overline{\mathbf{q}}), \tag{9}$$

in which a bar $(\bar{\cdot})$ indicates the equilibrium configuration. During large deflections, the stiffness matrix thus needs to be updated for every step. After this, the eigenvalue problem can be solved and this yields mode shapes and natural frequencies for that current configuration. This procedure will from now on be referred to as the eigenvalue problem (EVP) method. This method requires a lot of calculation time, especially for large systems with many coordinates.

2.3 Modal derivatives

Since for large systems, the calculation time increases significantly, model order reduction is applied. This is performed using the method of modal derivatives, as presented by Wu and Tiso in [3].

MDs are derived by differentiating the eigenvalue problem with respect to the modal coordinates:

$$\left(\frac{\partial \mathbf{K}}{\partial \eta_j} - \frac{\partial \omega_i^2}{\partial \eta_j}\mathbf{M}\right)\overline{\boldsymbol{\phi}}_i + \left(\overline{\mathbf{K}} - \overline{\omega}_i^2 \mathbf{M}\right)\overline{\boldsymbol{\theta}}_{ij} = \mathbf{0}, \tag{10}$$

in which $\overline{\boldsymbol{\theta}}_{ij}$ represents the modal derivatives as

$$\overline{\boldsymbol{\theta}}_{ij} = \frac{\partial \overline{\boldsymbol{\phi}}_i}{\partial \eta_j}. \tag{11}$$

According to literature [9,10,11] all inertia terms in (10) can be neglected. This means that the modal derivatives are static corrections on the mode shapes. Rewriting gives an expression for the modal derivatives:

$$\bar{\boldsymbol{\theta}}_{ij} = -\bar{\mathbf{K}}^{-1}\frac{\partial \mathbf{K}}{\partial \eta_j}\bar{\boldsymbol{\phi}}_i. \tag{12}$$

The mode shapes in a deflected configuration $\boldsymbol{\eta}$ are then defined as a combination of the mode shapes in equilibrium configuration and the modal derivatives multiplied with the modal coordinates:

$$\boldsymbol{\phi}_i = \bar{\boldsymbol{\phi}}_i + \bar{\boldsymbol{\theta}}_{ij}\eta_j. \tag{13}$$

2.4 Frequency derivatives

When the MDs are known from (12), these can be backsubstituted in (10), now using the new mode shapes $\boldsymbol{\phi}_i$ instead of $\bar{\boldsymbol{\phi}}_i$. Then, the derivates of the natural frequencies (frequency derivatives, FDs) can be calculated. In order to obtain a unique solution, the expression is premultiplied with the transposed of the mode shapes ($\boldsymbol{\phi}_i^T$):

$$\boldsymbol{\phi}_i^T\left(\frac{\partial \mathbf{K}}{\partial \eta_j} - \frac{\partial \omega_i^2}{\partial \eta_j}\mathbf{M}\right)\boldsymbol{\phi}_i + \boldsymbol{\phi}_i^T\left(\bar{\mathbf{K}} - \bar{\omega}_i^2\mathbf{M}\right)\bar{\boldsymbol{\theta}}_{ij} = 0. \tag{14}$$

The derivatives are then defined as

$$\bar{\boldsymbol{\gamma}}_{ij} = \frac{\boldsymbol{\phi}_i^T\frac{\partial \mathbf{K}}{\partial \eta_j}\boldsymbol{\phi}_i + \boldsymbol{\phi}_i^T\left(\bar{\mathbf{K}} - \bar{\omega}_i^2\mathbf{M}\right)\bar{\boldsymbol{\theta}}_{ij}}{\boldsymbol{\phi}_i^T\mathbf{M}\boldsymbol{\phi}_i}, \tag{15}$$

in which $\bar{\boldsymbol{\gamma}}_{ij}$ represents the frequency derivatives as

$$\bar{\boldsymbol{\gamma}}_{ij} = \frac{\partial \omega_i^2}{\partial \eta_j}. \tag{16}$$

Note that the formulation for $\bar{\boldsymbol{\gamma}}_{ij}$ is quite similar to the definition of the Rayleigh quotient, which can be used to calculate natural frequencies from known mode shapes:

$$R = \frac{\boldsymbol{\phi}_i^T\mathbf{K}\boldsymbol{\phi}_i}{\boldsymbol{\phi}_i^T\mathbf{M}\boldsymbol{\phi}_i}. \tag{17}$$

The natural frequencies in a deflected state are defined in a similar way as the mode shapes in deflected state (13):

$$(\omega_i)^2 = \bar{\omega}_i^2 + \bar{\boldsymbol{\gamma}}_{ij}\eta_j. \tag{18}$$

3 METHOD: 2D BEAM FORMULATION

To illustrate the concept, this paper will first present a two-dimensional Euler-Bernoulli beam element. We take into account the full Green-Lagrange strain expression [12],

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right], \tag{19}$$

where u and v are the axial and transverse displacement at any point of the cross-section. The displacement field [12] is defined as

$$\begin{aligned} u &= u_0 - y\frac{\partial v_0}{\partial x}, \\ v &= v_0, \end{aligned} \tag{20}$$

in which the subscript $(\cdot)_0$ indicates a point on the neutral line in case of bending, and y is the transverse coordinate (Figure 2).

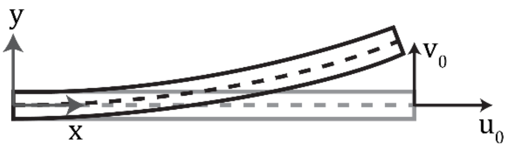

Figure 2. Axial and transverse displacement

Therefore, the strain expression equals

$$\varepsilon_{xx} = \frac{\partial u_0}{\partial x} - y\frac{\partial^2 v_0}{\partial x^2} + \frac{1}{2}\left(\frac{\partial u_0}{\partial x}\right)^2 - y\frac{\partial u_0}{\partial x}\frac{\partial^2 v_0}{\partial x^2} + \frac{1}{2}y^2\left(\frac{\partial^2 v_0}{\partial x^2}\right)^2 + \frac{1}{2}\left(\frac{\partial v_0}{\partial x}\right)^2. \tag{21}$$

The strain energy [13] is expressed as

$$\begin{aligned} U = \frac{1}{2}\int_0^L\left[b\cdot\int_{-\frac{h}{2}}^{\frac{h}{2}} E\varepsilon_{xx}^2 dy\right]dx = \frac{1}{2}\int_0^L \frac{1}{4}Ebh^5\left(\frac{\partial^2 v_0}{\partial x^2}\right)^4 dx + \frac{1}{2}\int_0^L \frac{3}{2}EI\left(\frac{\partial u_0}{\partial x}\right)^2\left(\frac{\partial^2 v_0}{\partial x^2}\right)^2 dx \\ + \frac{1}{2}\int_0^L 3EI\frac{\partial u_0}{\partial x}\left(\frac{\partial^2 v_0}{\partial x^2}\right)^2 dx + \frac{1}{2}\int_0^L \frac{1}{2}EI\left(\frac{\partial v_0}{\partial x}\right)^2\left(\frac{\partial^2 v_0}{\partial x^2}\right)^2 dx + \frac{1}{2}\int_0^L EI\left(\frac{\partial^2 v_0}{\partial x^2}\right)^2 dx \\ + \frac{1}{2}\int_0^L \frac{1}{4}EA\left(\frac{\partial u_0}{\partial x}\right)^4 dx + \frac{1}{2}\int_0^L EA\left(\frac{\partial u_0}{\partial x}\right)^3 dx + \frac{1}{2}\int_0^L \frac{1}{2}EA\left(\frac{\partial u_0}{\partial x}\right)^2\left(\frac{\partial v_0}{\partial x}\right)^2 dx \\ + \frac{1}{2}\int_0^L EA\left(\frac{\partial u_0}{\partial x}\right)^2 dx + \frac{1}{2}\int_0^L EA\frac{\partial u_0}{\partial x}\left(\frac{\partial v_0}{\partial x}\right)^2 dx + \frac{1}{2}\int_0^L \frac{1}{4}EA\left(\frac{\partial v_0}{\partial x}\right)^4 dx, \end{aligned} \tag{22}$$

in which E is Young's modulus, A is the cross sectional area of the beam, L is the length of the beam and I is the second moment of area. The axial and transverse displacements u_0 and v_0 of a planar beam are expressed as functions of the generalized coordinates using the shape function matrix **S** [5], consisting of Hermite cubics, which are standard interpolation functions for beams:

$$\begin{Bmatrix} u_0 \\ v_0 \end{Bmatrix} = \mathbf{Sq} = \begin{bmatrix} 1-\frac{x}{L} & 0 & 0 & \frac{x}{L} & 0 & 0 \\ 0 & 1-3\left(\frac{x}{L}\right)^2+2\left(\frac{x}{L}\right)^3 & L\left(\frac{x}{L}-2\left(\frac{x}{L}\right)^2+\left(\frac{x}{L}\right)^3\right) & 0 & 3\left(\frac{x}{L}\right)^2-2\left(\frac{x}{L}\right)^3 & L\left(\left(\frac{x}{L}\right)^3-\left(\frac{x}{L}\right)^2\right) \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}, \tag{23}$$

where the generalized coordinates **q** are defined as in Figure 3.

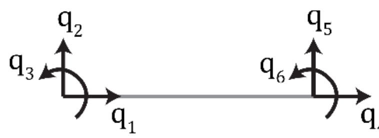

Figure 3. Generalized coordinates 2D beam element

For the two-dimensional Euler-Bernoulli beam element, the linear stiffness matrix can be determined from (4) as:

$$\bar{\mathbf{K}} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}. \tag{24}$$

4 METHOD: 3D BEAM FORMULATION

The model for the 2D beam can be expanded to a 3D beam. For a three-dimensional beam element, the displacement field is defined [1] as

$$\begin{aligned} u &= u_0 - y\frac{\partial v_0}{\partial x} - z\frac{\partial w_0}{\partial x}, \\ v &= v_0 - z\varphi_0, \\ w &= w_0 + y\varphi_0, \end{aligned} \tag{25}$$

in which φ_0 is the torsion angle. The displacements can be calculated by

$$\begin{Bmatrix} u_0 \\ v_0 \\ w_0 \\ \varphi_0 \end{Bmatrix} = \mathbf{Sq}, \tag{26}$$

in which the generalized coordinates **q** are defined as in Figure 4.

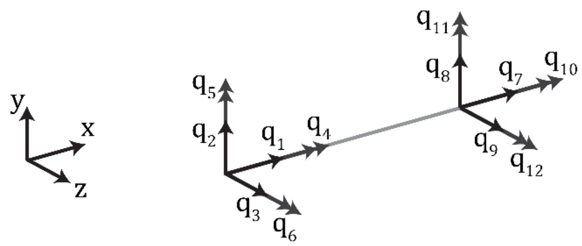

Figure 4. Generalized coordinates 3D beam element

and **S** is the shape function matrix given in (23), but expanded for 3D:

$$\mathbf{S}^{\mathrm{T}} = \begin{bmatrix} 1-\xi & 0 & 0 & 0 \\ 0 & 1-3\xi^2+2\xi^3 & 0 & 0 \\ 0 & 0 & 1-3\xi^2+2\xi^3 & 0 \\ 0 & 0 & 0 & \xi \\ 0 & 0 & (-\xi+2\xi^2-\xi^3)L & 0 \\ 0 & (\xi-2\xi^2+\xi^3)L & 0 & 0 \\ \xi & 0 & 0 & 0 \\ 0 & 3\xi^2-2\xi^3 & 0 & 0 \\ 0 & 0 & 3\xi^2-2\xi^3 & 0 \\ 0 & 0 & 0 & 1-\xi \\ 0 & 0 & (\xi^2-\xi^3)L & 0 \\ 0 & (-\xi^2+\xi^3)L & 0 & 0 \end{bmatrix}, \qquad (27)$$

in which $\xi = \frac{x}{L}$ and L is the length of the element.

The strain energy expression is now given by

$$U = \frac{1}{2}\int_V [E\varepsilon_{xx}^2 + 4G(\varepsilon_{xy}^2 + \varepsilon_{xz}^2)]dV. \qquad (28)$$

The linear stiffness matrix is then determined from (4) as

$$\bar{\mathbf{K}} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{12EI_{zz}}{L^3} & 0 & 0 & 0 & \frac{6EI_{zz}}{L^2} & 0 & -\frac{12EI_{zz}}{L^3} & 0 & 0 & 0 & \frac{6EI_{zz}}{L^2} \\ 0 & 0 & \frac{12EI_{yy}}{L^3} & 0 & -\frac{6EI_{yy}}{L^2} & 0 & 0 & 0 & -\frac{12EI_{yy}}{L^3} & 0 & -\frac{6EI_{yy}}{L^2} & 0 \\ 0 & 0 & 0 & \frac{G(I_{yy}+I_{zz})}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{G(I_{yy}+I_{zz})}{L} & 0 & 0 \\ 0 & 0 & -\frac{6EI_{yy}}{L^2} & 0 & \frac{4EI_{yy}}{L} & 0 & 0 & 0 & \frac{6EI_{yy}}{L^2} & 0 & \frac{2EI_{yy}}{L} & 0 \\ 0 & \frac{6EI_{zz}}{L^2} & 0 & 0 & 0 & \frac{4EI_{zz}}{L} & 0 & -\frac{6EI_{zz}}{L^2} & 0 & 0 & 0 & \frac{2EI_{zz}}{L} \\ -\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 & \frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{12EI_{zz}}{L^3} & 0 & 0 & 0 & -\frac{6EI_{zz}}{L^2} & 0 & \frac{12EI_{zz}}{L^3} & 0 & 0 & 0 & -\frac{6EI_{zz}}{L^2} \\ 0 & 0 & -\frac{12EI_{yy}}{L^3} & 0 & \frac{6EI_{yy}}{L^2} & 0 & 0 & 0 & \frac{12EI_{yy}}{L^3} & 0 & \frac{6EI_{yy}}{L^2} & 0 \\ 0 & 0 & 0 & -\frac{G(I_{yy}+I_{zz})}{L} & 0 & 0 & 0 & 0 & 0 & \frac{G(I_{yy}+I_{zz})}{L} & 0 & 0 \\ 0 & 0 & -\frac{6EI_{y}}{L^2} & 0 & \frac{2EI_{yy}}{L} & 0 & 0 & 0 & \frac{6EI_{yy}}{L^2} & 0 & \frac{4EI_{yy}}{L} & 0 \\ 0 & \frac{6EI_{zz}}{L^2} & 0 & 0 & 0 & \frac{2EI_{zz}}{L} & 0 & -\frac{6EI_{zz}}{L^2} & 0 & 0 & 0 & \frac{4EI_{zz}}{L} \end{bmatrix} \qquad (29)$$

5 RESULTS

5.1 2D simply supported beam

First of all the method is validated against the results of Wu and Tiso [3]. Therefore, a simply supported beam of 20 elements is modelled and analysed. Its dimensions are not relevant as the presented results are dimensionless. In Figure 5, the first two mode shapes (VMs) and corresponding MDs are presented as well as the results from Wu and Tiso [3]. The results are represented by respectively yellow and black lines.

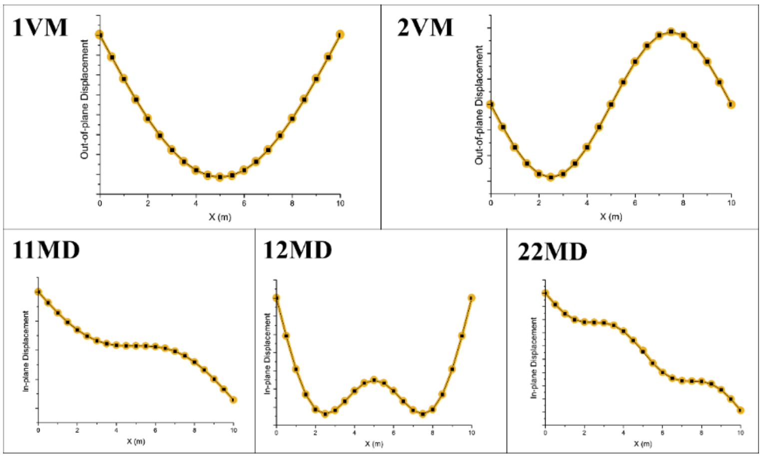

Figure 5. Mode shapes and modal derivatives (adapted from Wu and Tiso [3])

As stated before, it is possible to determine mode shapes in a deflected state in two ways. The first is to derive the eigenvalue problem for the coordinates $\mathbf{q}$ using $\mathbf{K} = \mathbf{K}(\mathbf{q})$, while the second method uses MDs:

$$\boldsymbol{\phi}_i = \overline{\boldsymbol{\phi}}_i + \overline{\boldsymbol{\theta}}_{ij}\eta_j. \tag{30}$$

These two methods are compared. The beam is deflected in the first mode shape, so $\eta_1 \neq 0$, while all other modal coordinates equal zero. Figure 6 shows the first three mode shapes in the deflected state, determined by both the eigenvalue problem (EVP) and modal derivatives (MDs).

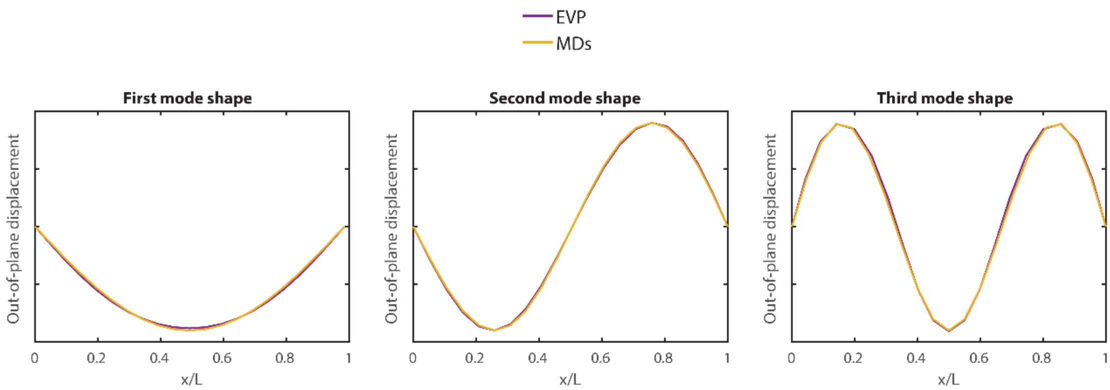

Figure 6. First three mode shapes for a deflection in the first mode

For an increasing deflection in the first mode, the natural frequencies decrease. For this example, the beam is deflected in $\boldsymbol{\phi}_1$. The decrease of a selection of the natural frequencies is shown in Figure 7, determined by both EVP and MDs. The decrease is plotted against the vertical displacement of the middle node of the beam. Since this is a 2D problem, all these natural frequencies correspond to in-plane bending mode shapes. As Figure 7 shows, the results of both methods match very well. Only the parts just before the natural frequencies actually reach zero show some small differences.

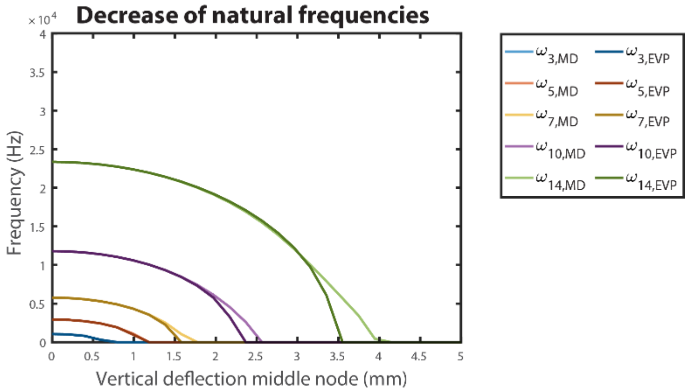

Figure 7. Decrease of natural frequencies for 2D simply supported beam

5.2 3D clamped-free beam

A 3D example is presented as well. We consider a clamped-free beam. Due to the applicability of the method on precision engineering and on for examples leaf springs, we consider typical dimensions of a leaf spring. The beam has a length of 100 mm, width of 44 mm, thickness of 0.5 mm, and is clamped along the z-axis on its short side. The first natural frequency corresponds to a bending mode shape around the z-axis. For an increasing deflection in this mode shape, the natural frequencies decrease. The beam is deflected in the first mode shape, until θ_z reaches 90 degrees (Figure 8).

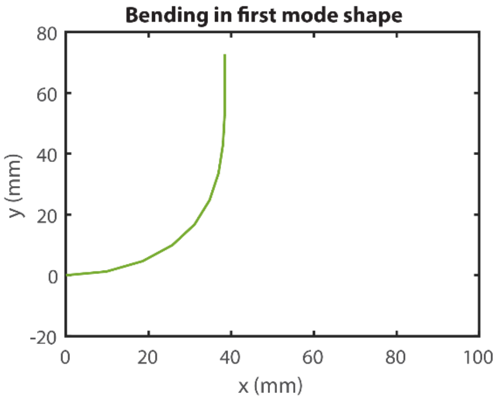

Figure 8. Bending in first mode shape

Figure 9 shows the decrease of a selection of the natural frequencies. In contradiction to the 2D problem, not all frequencies decrease with the same speed. This is due to the fact that not all shown frequencies correspond to bending mode shapes around the same axis. Blue lines correspond to bending mode shapes around the z-axis, red lines to bending mode shapes around the y-axis, and yellow lines to torsion mode shapes. As this figure shows, there is a clear difference between the types of mode shape the natural frequencies correspond to.

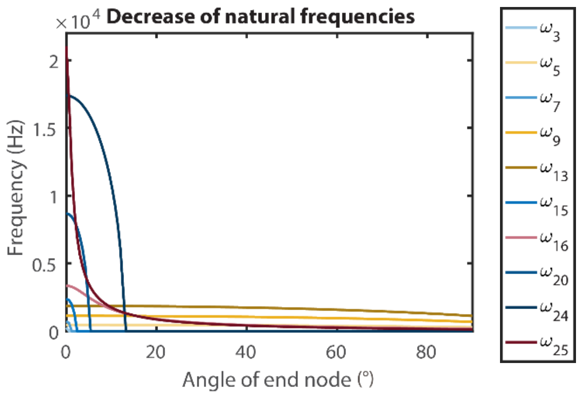

Figure 9. Decrease of natural frequencies for 3D clamped-free beam

For clarity, the natural frequencies per type of mode shape are presented seperately in Figure 10, note that the axes for each figure are different. A notable difference between the three types of mode shapes is that the torsional natural frequencies decrease much slower. Furthermore, the natural frequencies corresponding to bending around the z-axis go to zero, while those corresponding to bending around the y-axis do approach zero but they do not reach it.

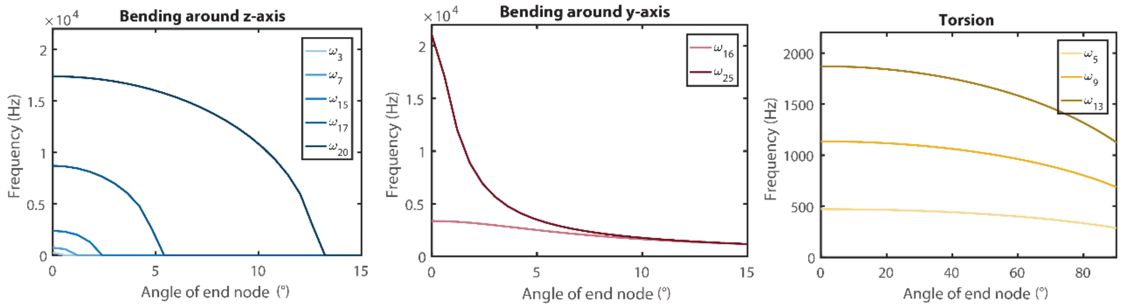

Figure 10. Decrease of natural frequencies for 3D clamped-free beam

6 CONCLUSIONS AND RECOMMENDATIONS

6.1 Conclusions

In this paper, a model order reduction technique using modal derivatives is treated. The method makes use of the full Green-Lagrange strain tensor, making it possible to take into account geometric nonlinearities. By applying this model order reduction technique, the number of coordinates and therefore the calculation time decrease significantly.

Modal derivatives are used to determine mode shapes in a deflected state. Using a 2D example, the results are verified with a method in which the mode shapes are determined by calculating the eigenvalue problem for the known coordinates. These results match. The second example in this paper shows that the model order reduction technique of using modal derivatives can indeed be applied on 3D beam elements as well. It is shown that using the method of modal derivatives, also frequency derivatives can be determined. These frequency derivatives can reliably be used to determine the way natural frequencies of a beam change during large deflections. As was to be expected, the method shows coupling between bending and torsion mode shapes as well. This can be concluded from for example the fact that for a deflection in a bending mode shape, also the torsional natural frequencies are affected. This property of the method makes it indeed very applicable in precision engineering, in which it is very relevant to determine the behaviour of natural frequencies in more directions than only the direction of the deflection.

6.2 Recommendations

In both examples, the effect on axial mode shapes and axial natural frequencies is left out, since this did not yield accurate results. This was also concluded by Wu and Tiso in [2], but no reason has yet been found. This should be further investigated, so that the method can be applied on axial mode shapes and axial natural frequencies as well. Furthermore, the 2D example showed some small differences in the close-to-zero region of the natural frequencies. A follow-up study into these specific parts should be performed to explain these differences.

In this paper, only examples of beam elements are presented, for reasons mentioned before. However, no major difficulties are expected when expanding the method to other types of elements. In follow-up studies, the method could be applied on for example plates, or any random other shape, to verify this expectation.

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APPENDICES

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CONTENTS

APPENDIX A
Derivation for 2D beam elements

A1 INTRODUCTION

The modal derivatives can be determined by deriving the eigenvalue problem with respect to the modal coordinates. The expression for the modal derivatives is given in (12) as:

$$\bar{\boldsymbol{\theta}}_{ij} = -\bar{\mathbf{K}}^{-1}\frac{\partial \mathbf{K}}{\partial \eta_j}\bar{\boldsymbol{\phi}}_i \,. \tag{31}$$

As this expression shows, we need the stiffness matrix, derivatives of the stiffness with respect to the modal coordinates, and the mode shapes. All will be derived in the following chapters. Note that determining the modal derivatives is only possible for constrained cases, since otherwise the stiffness matrix is singular and its inverse does not exist. However, all derivations are first performed for an unconstrained case. Boundary conditions can then be applied later on.

The stiffness is expressed as the second derivative of the strain energy with respect to the generalized coordinates (5). Therefore the strain energy is determined first. The stiffness will be expressed in terms of the generalized coordinates. Therefore, the stiffness cannot be derived with respect to the modal coordinates immediately. Therefore, the derivative of the stiffness with respect to the modal coordinates is expressed as following:

$$\frac{\partial \mathbf{K}}{\partial \eta_j} = \sum_{i=1}^{6}\frac{\partial \mathbf{K}}{\partial q_i}\frac{\partial q_i}{\partial \eta_j}. \tag{32}$$

The stiffness will thus first be derived with respect to the generalized coordinates and then multiplied with the derivatives of the generalized coordinates with respect to the modal coordinates.

In this report, we consider a planar Euler-Bernoulli beam element, of which the generalized coordinates are defined as following:

$$\mathbf{q} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix} = \begin{Bmatrix} x_1 \\ y_1 \\ \theta_1 \\ x_2 \\ y_2 \\ \theta_2 \end{Bmatrix}. \tag{33}$$

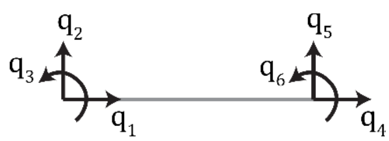


Figure 11. Generalized coordinates 2D beam

A2 STRAIN ENERGY

A2.1 Strain

In order to determine the strain energy, we first need an expression for the strain. The Green-Lagrange strain definition for a planar Euler-Bernoulli beam (19) equals:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]. \tag{34}$$

The deformation field (20) is defined as:

$$\begin{aligned} u &= u_0 - y\frac{\partial v_0}{\partial x}, \\ v &= v_0. \end{aligned} \tag{35}$$

Substituting these expressions yields the definition for the axial strain:

$$\begin{aligned} \varepsilon_{xx} &= \frac{\partial u_0}{\partial x} - y\frac{\partial^2 v_0}{\partial x^2} + \frac{1}{2}\left(\frac{\partial u_0}{\partial x} - y\frac{\partial^2 v_0}{\partial x^2}\right)^2 + \frac{1}{2}\left(\frac{\partial v_0}{\partial x}\right)^2 \\ &= \frac{\partial u_0}{\partial x} - y\frac{\partial^2 v_0}{\partial x^2} + \frac{1}{2}\left(\frac{\partial u_0}{\partial x}\right)^2 - y\frac{\partial u_0}{\partial x}\frac{\partial^2 v_0}{\partial x^2} + \frac{1}{2}y^2\left(\frac{\partial^2 v_0}{\partial x^2}\right)^2 + \frac{1}{2}\left(\frac{\partial v_0}{\partial x}\right)^2. \end{aligned} \tag{36}$$

A2.2 Strain energy

Now that we found an expression for the strain, we can find an expression for the strain energy. The strain energy is defined as:

$$U == \frac{1}{2}\int_V \hat{\boldsymbol{\varepsilon}}^{\mathrm{T}}\hat{\boldsymbol{\sigma}}\, dV = \frac{1}{2}\iiint \hat{\boldsymbol{\varepsilon}}^{\mathrm{T}}\hat{\boldsymbol{\sigma}}\, dx\, dy\, dz = \frac{1}{2}\int_0^L \left[b \cdot \int_{-\frac{h}{2}}^{\frac{h}{2}} E\varepsilon_{xx}^2 dy\right] dx. \tag{37}$$

To be substituted in the expression, ε_{xx}, has to be quadrated.

$$\begin{aligned} \varepsilon_{xx}^2 &= \left[\frac{\partial u_0}{\partial x} - y\frac{\partial^2 v_0}{\partial x^2} + \frac{1}{2}\left(\frac{\partial u_0}{\partial x}\right)^2 - y\frac{\partial u_0}{\partial x}\frac{\partial^2 v_0}{\partial x^2} + \frac{1}{2}y^2\left(\frac{\partial^2 v_0}{\partial x^2}\right)^2 + \frac{1}{2}\left(\frac{\partial v_0}{\partial x}\right)^2\right]^2 \\ &= \frac{1}{4}y^4\left(\frac{\partial^2 v_0}{\partial x^2}\right)^4 - y^3\frac{\partial u}{\partial x}\left(\frac{\partial^2 v_0}{\partial x^2}\right)^3 - y^3\left(\frac{\partial^2 v_0}{\partial x^2}\right)^3 + \frac{3}{2}y^2\left(\frac{\partial u_0}{\partial x}\right)^2\left(\frac{\partial^2 v_0}{\partial x^2}\right)^2 \\ &\qquad + 3y^2\frac{\partial u_0}{\partial x}\left(\frac{\partial^2 v_0}{\partial x^2}\right)^2 + \frac{1}{2}y^2\left(\frac{\partial v_0}{\partial x}\right)^2\left(\frac{\partial^2 v_0}{\partial x^2}\right)^2 + y^2\left(\frac{\partial^2 v_0}{\partial x^2}\right)^2 - y\left(\frac{\partial u_0}{\partial x}\right)^3\frac{\partial^2 v_0}{\partial x^2} \\ &\qquad - 3y\left(\frac{\partial u_0}{\partial x}\right)^2\frac{\partial^2 v_0}{\partial x^2} - y\frac{\partial u_0}{\partial x}\left(\frac{\partial v_0}{\partial x}\right)^2\frac{\partial^2 v_0}{\partial x^2} - 2y\frac{\partial u_0}{\partial x}\frac{\partial^2 v_0}{\partial x^2} - y\left(\frac{\partial v_0}{\partial x}\right)^2\frac{\partial^2 v_0}{\partial x} \\ &\qquad + \frac{1}{4}\left(\frac{\partial u_0}{\partial x}\right)^4 + \left(\frac{\partial u_0}{\partial x}\right)^3 + \frac{1}{2}\left(\frac{\partial u_0}{\partial x}\right)^2\left(\frac{\partial v_0}{\partial x}\right)^2 + \left(\frac{\partial u_0}{\partial x}\right)^2 + \frac{\partial u_0}{\partial x}\left(\frac{\partial v_0}{\partial x}\right)^2 \\ &\qquad + \frac{1}{4}\left(\frac{\partial v_0}{\partial x}\right)^4. \end{aligned} \tag{38}$$

When substituting each term in the expression for the strain energy, terms including y and y^3 will result in a zero-term, since:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} y\,dy = [y^2]_{-\frac{h}{2}}^{\frac{h}{2}} = \left[\frac{h^2}{4} - \frac{h^2}{4}\right] = 0, \tag{39}$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} y^3\,dy = \left[\frac{1}{4}y^4\right]_{-\frac{h}{2}}^{\frac{h}{2}} = \left[\left(\frac{1}{4}\cdot\frac{h^4}{16}\right) - \left(\frac{1}{4}\cdot\frac{h^4}{16}\right)\right] = 0. \tag{40}$$

All terms including y and y^3 will therefore already be left out, leaving us with the following expression for ε_{xx}^2:

$$\begin{aligned}\varepsilon_{xx}^2 = {} & \frac{1}{4}y^4\left(\frac{\partial^2 v_0}{\partial x^2}\right)^4 + \frac{3}{2}y^2\left(\frac{\partial u_0}{\partial x}\right)^2\left(\frac{\partial^2 v_0}{\partial x^2}\right)^2 + 3y^2\frac{\partial u_0}{\partial x}\left(\frac{\partial^2 v_0}{\partial x^2}\right)^2 + \frac{1}{2}y^2\left(\frac{\partial v_0}{\partial x}\right)^2\left(\frac{\partial^2 v_0}{\partial x^2}\right)^2 \\ & + y^2\left(\frac{\partial^2 v_0}{\partial x^2}\right)^2 + \frac{1}{4}\left(\frac{\partial u_0}{\partial x}\right)^4 + \left(\frac{\partial u_0}{\partial x}\right)^3 + \frac{1}{2}\left(\frac{\partial u_0}{\partial x}\right)^2\left(\frac{\partial v_0}{\partial x}\right)^2 + \left(\frac{\partial u_0}{\partial x}\right)^2 \\ & + \frac{\partial u_0}{\partial x}\left(\frac{\partial v_0}{\partial x}\right)^2 + \frac{1}{4}\left(\frac{\partial v_0}{\partial x}\right)^4.\end{aligned} \tag{41}$$

Furthermore, fourth order terms can be left out too. Fourth order terms of ε_{xx}^2 result in fourth order terms in U. In the required stiffness matrices, only the linear terms of the first and second derivatives of U will be used. In the first derivative of U, the originally fourth order terms will become third order, and in the second derivative, they will become second order. Since only linear terms are used, these terms will never be used and are for clarity left out from now on. This leaves us with the following expression for ε_{xx}^2:

$$\varepsilon_{xx}^2 = \left(\frac{\partial u_0}{\partial x}\right)^2 + \left(\frac{\partial u_0}{\partial x}\right)^3 + 3y^2\frac{\partial u_0}{\partial x}\left(\frac{\partial^2 v_0}{\partial x^2}\right)^2 + \frac{\partial u_0}{\partial x}\left(\frac{\partial v_0}{\partial x}\right)^2 + y^2\left(\frac{\partial^2 v_0}{\partial x^2}\right)^2. \tag{42}$$

Five terms remain in the expression for ε_{xx}^2. These terms can now be substituted in the strain energy expression.

For the first term of ε_{xx}^2:

$$U_{(1)} = \frac{1}{2}\int_0^L\left[b\cdot\int_{-\frac{h}{2}}^{\frac{h}{2}} E\left(\frac{\partial u_0}{\partial x}\right)^2 dy\right]dx. \tag{43}$$

The part between the square brackets is solved first:

$$
\begin{aligned}
b \cdot \int_{-\frac{h}{2}}^{\frac{h}{2}} E \left(\frac{\partial u_0}{\partial x}\right)^2 dy &= Eb \left(\frac{\partial u_0}{\partial x}\right)^2 \cdot \int_{-\frac{h}{2}}^{\frac{h}{2}} dy \\
&= Eb \left(\frac{\partial u_0}{\partial x}\right)^2 \cdot [y]_{-\frac{h}{2}}^{\frac{h}{2}} \\
&= Eb \left(\frac{\partial u_0}{\partial x}\right)^2 \cdot \left[\frac{h}{2} + \frac{h}{2}\right] \\
&= Eb \left(\frac{\partial u_0}{\partial x}\right)^2 \cdot h \\
&= EA \left(\frac{\partial u_0}{\partial x}\right)^2.
\end{aligned} \tag{44}
$$

So for the strain energy:

$$U_{(1)} = \frac{1}{2} \int_0^L EA \left(\frac{\partial u_0}{\partial x}\right)^2 dx. \tag{45}$$

A similar procedure is followed for the other terms of ε_{xx}^2. This yields:

$$U_{(2)} = \frac{1}{2} \int_0^L EA \left(\frac{\partial u_0}{\partial x}\right)^3 dx, \tag{46}$$

$$U_{(3)} = \frac{1}{2} \int_0^L 3EI \frac{\partial u_0}{\partial x} \left(\frac{\partial^2 v_0}{\partial x^2}\right)^2 dx, \tag{47}$$

$$U_{(4)} = \frac{1}{2} \int_0^L EA \frac{\partial u_0}{\partial x} \left(\frac{\partial v_0}{\partial x}\right)^2 dx, \tag{48}$$

$$U_{(5)} = \frac{1}{2} \int_0^L EI \left(\frac{\partial^2 v_0}{\partial x^2}\right)^2 dx. \tag{49}$$

The total strain energy expression becomes:

$$
\begin{aligned}
U &= U_{(1)} + U_{(2)} + U_{(3)} + U_{(4)} + U_{(5)} \\
&= \frac{1}{2} \int_0^L EA \left(\frac{\partial u_0}{\partial x}\right)^2 dx + \frac{1}{2} \int_0^L EA \left(\frac{\partial u_0}{\partial x}\right)^3 dx + \frac{1}{2} \int_0^L 3EI \frac{\partial u_0}{\partial x} \left(\frac{\partial^2 v_0}{\partial x^2}\right)^2 dx \\
&\quad + \frac{1}{2} \int_0^L EA \frac{\partial u_0}{\partial x} \left(\frac{\partial v_0}{\partial x}\right)^2 dx + \frac{1}{2} \int_0^L EI \left(\frac{\partial^2 v_0}{\partial x^2}\right)^2 dx.
\end{aligned} \tag{50}
$$

A2.3 Coordinate transformation

In order to determine the stiffness matrix, we need the derivative of the strain energy with respect to the generalized coordinates. However, in the current expression, U is defined as a function of the displacements $\mathbf{u}$, while U has to be derived with respect to $\mathbf{q}$. Therefore, $\mathbf{u}$ has to be expressed as a function of $\mathbf{q}$. This can be achieved using coordinate transformation:

$$\mathbf{u} = \mathbf{Sq}. \tag{51}$$

In this equation, **S** is the shape function matrix, containing six Craig-Bampton mode shapes. These mode shapes are defined as:

$$\begin{aligned} S_1 &= 1 - \frac{x}{L}, \\ S_2 &= 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3, \\ S_3 &= \left(\frac{x}{L} - 2\left(\frac{x}{L}\right)^2 + \left(\frac{x}{L}\right)^3\right)L, \\ S_4 &= \frac{x}{L}, \\ S_5 &= 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3, \\ S_6 &= \left(-\left(\frac{x}{L}\right)^2 + \left(\frac{x}{L}\right)^3\right)L. \end{aligned} \tag{52}$$

The shape function matrix then equals:

$$\begin{aligned} \mathbf{S} &= \begin{bmatrix} S_1 & 0 & 0 & S_4 & 0 & 0 \\ 0 & S_2 & S_3 & 0 & S_5 & S_6 \end{bmatrix} \\ &= \begin{bmatrix} 1 - \frac{x}{L} & 0 & 0 & \frac{x}{L} & 0 & 0 \\ 0 & 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3 & \left(\frac{x}{L} - 2\left(\frac{x}{L}\right)^2 + \left(\frac{x}{L}\right)^3\right)L & 0 & 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3 & \left(-\left(\frac{x}{L}\right)^2 + \left(\frac{x}{L}\right)^3\right)L \end{bmatrix}. \end{aligned} \tag{53}$$

Substituting **S** and **q** in the coordinate transformation yields:

$$\begin{Bmatrix} u_0 \\ v_0 \end{Bmatrix} = \begin{bmatrix} S_1 & 0 & 0 & S_4 & 0 & 0 \\ 0 & S_2 & S_3 & 0 & S_5 & S_6 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}. \tag{54}$$

So for the horizontal displacement u_0 and vertical displacement v_0 along the element:

$$\begin{aligned} u_0 &= S_1 q_1 + S_4 q_4, \\ v_0 &= S_2 q_2 + S_3 q_3 + S_5 q_5 + S_6 q_6. \end{aligned} \tag{55}$$

Since the terms including a subscript 1 or 4 refer to the horizontal displacement u_0, and terms including a subscript 2, 3, 5 or 6 refer to the vertical displacement v_0, the following expressions are defined:

$$\begin{aligned} \mathbf{S}_u &= [S_1 \quad S_4], \\ \mathbf{S}_v &= [S_2 \quad S_3 \quad S_5 \quad S_6], \\ \mathbf{q}_u &= \{q_1 \quad q_4\}^T, \\ \mathbf{q}_v &= \{q_2 \quad q_3 \quad q_5 \quad q_6\}^T. \end{aligned} \tag{56}$$

Since the strain energy expression contains first and second derivatives of $\mathbf{q}$ with respect to x, an expression for these derivatives has to be found. A derivative with respect to x will be referred to with an apostrophe here.

The derivatives equal:

$$\frac{\partial \mathbf{u}}{\partial x} = \frac{\partial \mathbf{S}}{\partial x}\mathbf{q} = \mathbf{S}'\mathbf{q}, \tag{57}$$

$$\frac{\partial^2 \mathbf{u}}{\partial x^2} = \frac{\partial^2 \mathbf{S}}{\partial x^2}\mathbf{q} = \mathbf{S}''\mathbf{q}. \tag{58}$$

The first and second derivative of the shape function matrix **S** are:

$$\begin{aligned}\frac{\partial \mathbf{S}}{\partial x} = \mathbf{S}' &= \begin{bmatrix} \frac{\partial S_1}{\partial x} & 0 & 0 & \frac{\partial S_4}{\partial x} & 0 & 0 \\ 0 & \frac{\partial S_2}{\partial x} & \frac{\partial S_3}{\partial x} & 0 & \frac{\partial S_5}{\partial x} & \frac{\partial S_6}{\partial x} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{L} & 0 & 0 & \frac{1}{L} & 0 & 0 \\ 0 & \frac{6x^2}{L^3} - \frac{6x}{L^2} & L\left(\frac{1}{L} - \frac{4x}{L^2} + \frac{3x^2}{L^3}\right) & 0 & \frac{6x}{L^2} - \frac{6x^2}{L^3} & L\left(\frac{3x^2}{L^3} - \frac{2x}{L^2}\right) \end{bmatrix},\end{aligned} \tag{59}$$

$$\begin{aligned}\frac{\partial^2 \mathbf{S}}{\partial x^2} = \mathbf{S}'' &= \begin{bmatrix} \frac{\partial^2 S_1}{\partial x^2} & 0 & 0 & \frac{\partial^2 S_4}{\partial x^2} & 0 & 0 \\ 0 & \frac{\partial^2 S_2}{\partial x^2} & \frac{\partial^2 S_3}{\partial x^2} & 0 & \frac{\partial^2 S_5}{\partial x^2} & \frac{\partial^2 S_6}{\partial x^2} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{12x}{L^3} - \frac{6}{L^2} & L\left(\frac{6x}{L^3} - \frac{4}{L^2}\right) & 0 & \frac{6}{L^2} - \frac{12x}{L^3} & L\left(\frac{6x}{L^3} - \frac{2}{L^2}\right) \end{bmatrix}.\end{aligned} \tag{60}$$

So for the first and second derivative of u_0 and v_0:

$$\begin{aligned}\begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial x} \end{Bmatrix} &= \begin{bmatrix} \frac{\partial S_1}{\partial x} & 0 & 0 & \frac{\partial S_4}{\partial x} & 0 & 0 \\ 0 & \frac{\partial S_2}{\partial x} & \frac{\partial S_3}{\partial x} & 0 & \frac{\partial S_5}{\partial x} & \frac{\partial S_6}{\partial x} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix} \\ &= \begin{bmatrix} -\frac{1}{L} & 0 & 0 & \frac{1}{L} & 0 & 0 \\ 0 & \frac{6x^2}{L^3} - \frac{6x}{L^2} & L\left(\frac{1}{L} - \frac{4x}{L^2} + \frac{3x^2}{L^3}\right) & 0 & \frac{6x}{L^2} - \frac{6x^2}{L^3} & L\left(\frac{3x^2}{L^3} - \frac{2x}{L^2}\right) \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix},\end{aligned} \tag{61}$$

$$\begin{aligned}\begin{Bmatrix} \frac{\partial^2 u_0}{\partial x^2} \\ \frac{\partial^2 v_0}{\partial x^2} \end{Bmatrix} &= \begin{bmatrix} \frac{\partial^2 S_1}{\partial x^2} & 0 & 0 & \frac{\partial^2 S_4}{\partial x^2} & 0 & 0 \\ 0 & \frac{\partial^2 S_2}{\partial x^2} & \frac{\partial^2 S_3}{\partial x^2} & 0 & \frac{\partial^2 S_5}{\partial x^2} & \frac{\partial^2 S_6}{\partial x^2} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{12x}{L^3} - \frac{6}{L^2} & L\left(\frac{6x}{L^3} - \frac{4}{L^2}\right) & 0 & \frac{6}{L^2} - \frac{12x}{L^3} & L\left(\frac{6x}{L^3} - \frac{2}{L^2}\right) \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}.\end{aligned} \tag{62}$$

A2.4 Substitution in strain energy

We now can determine all required terms that have to be substituted in the strain energy expression.

For the five required terms:

$$\begin{aligned}\left(\frac{\partial u_0}{\partial x}\right)^2 &= \mathbf{q}_\mathrm{u}^\mathrm{T}\mathbf{S}_\mathrm{u}'^\mathrm{T}\mathbf{S}_\mathrm{u}'\mathbf{q}_\mathrm{u} \\ &= \{q_1 \quad q_4\}\begin{bmatrix}S_1' \\ S_4'\end{bmatrix}[S_1' \quad S_4']\begin{Bmatrix}q_1 \\ q_4\end{Bmatrix} \\ &= \frac{1}{L^2}[q_1 - 2q_1q_4 + q_4],\end{aligned} \tag{63}$$

$$\begin{aligned}\left(\frac{\partial u_0}{\partial x}\right)^3 &= \mathbf{S}_\mathrm{u}'\mathbf{q}_\mathrm{u}\mathbf{q}_\mathrm{u}^\mathrm{T}\mathbf{S}_\mathrm{u}'^\mathrm{T}\mathbf{S}_\mathrm{u}'\mathbf{q}_\mathrm{u} \\ &= [S_1' \quad S_4']\begin{Bmatrix}q_1 \\ q_4\end{Bmatrix}\{q_1 \quad q_4\}\begin{bmatrix}S_1' \\ S_4'\end{bmatrix}[S_1' \quad S_4']\begin{Bmatrix}q_1 \\ q_4\end{Bmatrix} \\ &= \frac{1}{L^3}[-q_1^3 + 3q_1^2q_4 - 3q_1q_4^2 + q_4^3],\end{aligned} \tag{64}$$

$$\begin{aligned}\frac{\partial u_0}{\partial x}\left(\frac{\partial^2 v_0}{\partial x^2}\right)^2 &= \mathbf{S}_\mathrm{u}'\mathbf{q}_\mathrm{u}\mathbf{q}_\mathrm{v}^\mathrm{T}\mathbf{S}_\mathrm{v}''^\mathrm{T}\mathbf{S}_\mathrm{v}''\mathbf{q}_\mathrm{v} \\ &= [S_1' \quad S_4']\begin{Bmatrix}q_1 \\ q_4\end{Bmatrix}\{q_2 \quad q_3 \quad q_5 \quad q_6\}\begin{bmatrix}S_2'' \\ S_3'' \\ S_5'' \\ S_6''\end{bmatrix}[S_2'' \quad S_3'' \quad S_5'' \quad S_6'']\begin{Bmatrix}q_2 \\ q_3 \\ q_5 \\ q_6\end{Bmatrix} \\ &= -\frac{4}{L^7}[q_1 \\ &\quad - q_4][3L\,q_2 - 3L\,q_5 - 6x\,q_2 + 6x\,q_5 + 2L^2\,q_3 + L^2\,q_6 - 3Lx\,q_3 - 3Lx\,q_6]^2,\end{aligned} \tag{65}$$

$$\begin{aligned}\frac{\partial u_0}{\partial x}\left(\frac{\partial v_0}{\partial x}\right)^2 &= \mathbf{S}_\mathrm{u}'\mathbf{q}_\mathrm{u}\mathbf{q}_\mathrm{v}^\mathrm{T}\mathbf{S}_\mathrm{v}'^\mathrm{T}\mathbf{S}_\mathrm{v}'\mathbf{q}_\mathrm{v} \\ &= [S_1' \quad S_4']\begin{Bmatrix}q_1 \\ q_4\end{Bmatrix}\{q_2 \quad q_3 \quad q_5 \quad q_6\}\begin{bmatrix}S_2' \\ S_3' \\ S_5' \\ S_6'\end{bmatrix}[S_2' \quad S_3' \quad S_5' \quad S_6']\begin{Bmatrix}q_2 \\ q_3 \\ q_5 \\ q_6\end{Bmatrix} \\ &= -\frac{1}{L^7}[q_1 \\ &\quad - q_4][L^3\,q_3 + 6x^2\,q_2 + 3Lx^2\,q_3 - 6x^2\,q_5 + 3Lx^2\,q_6 - 6Lx\,q_2 - 4L^2x\,q_3 \\ &\quad + 6Lx\,q_5 - 2L^2x\,q_6]^2,\end{aligned} \tag{66}$$

$$\begin{aligned}\left(\frac{\partial^2 v_0}{\partial x^2}\right)^2 &= \mathbf{q}_\mathrm{v}^\mathrm{T}\mathbf{S}_\mathrm{v}''^\mathrm{T}\mathbf{S}_\mathrm{v}''\mathbf{q}_\mathrm{v} \\ &= \{q_2 \quad q_3 \quad q_5 \quad q_6\}\begin{bmatrix}S_2'' \\ S_3'' \\ S_5'' \\ S_6''\end{bmatrix}[S_2'' \quad S_3'' \quad S_5'' \quad S_6'']\begin{Bmatrix}q_2 \\ q_3 \\ q_5 \\ q_6\end{Bmatrix} \\ &= \frac{1}{L^6}[12L\,q_2 - 12L\,q_5 - 24x\,q_2 + 24x\,q_5 + 8L^2\,q_3 + 4L^2\,q_6 - 12Lx\,q_3 - 12Lx\,q_6]^2.\end{aligned} \tag{67}$$

Now each term is substituted in the strain energy expression.

$$\frac{1}{2}\int_0^L EA\left(\frac{\partial u_0}{\partial x}\right)^2 dx = \frac{EA}{2L}(q_1 - 2q_1q_4 + q_4), \tag{68}$$

$$\frac{1}{2}\int_0^L EA\left(\frac{\partial u_0}{\partial x}\right)^3 dx = \frac{EA}{2L}(-q_1^3 + 3q_1^2q_4 - 3q_1q_4^2 + q_4^3), \tag{69}$$

$$\begin{aligned}\frac{1}{2}\int_0^L 3EI\frac{\partial u_0}{\partial x}\left(\frac{\partial^2 v_0}{\partial x^2}\right)^2 dx \\ &= \frac{3EI}{L^4}(2L^2q_3^2q_4 - 6q_1q_2^2 - 2L^2q_1q_6^2 - 2L^2q_1q_3^2 - 6q_1q_5^2 + 6q_2^2q_4 + 2L^2q_4q_6^2 \\ &+ 6q_4q_5^2 - 6Lq_1q_2q_3 - 2L^2q_1q_3q_6 - 6Lq_1q_2q_6 + 6Lq_1q_3q_5 + 6Lq_2q_3q_4 \\ &+ 12q_1q_2q_5 + 2L^2q_3q_4q_6 + 6Lq_1q_5q_6 + 6Lq_2q_4q_6 - 6Lq_3q_4q_5 - 12q_2q_4q_5 \\ &- 6Lq_4q_5q_6),\end{aligned} \tag{70}$$

$$\begin{aligned}\frac{1}{2}\int_0^L EA\frac{\partial u_0}{\partial x}\left(\frac{\partial v_0}{\partial x}\right)^2 dx \\ &= \frac{EA}{30L^2}\big(2L^2q_3^2q_4 - 2L^2q_1q_6^2 - 2L^2q_1q_3^2 + 2L^2q_4q_6^2 - 18q_1q_2^2 \\ &- 18q_1q_5^2 + 18q_2^2q_4 + 18q_4q_5^2 + L^2q_1q_3q_6 - L^2q_3q_4q_6 - 3Lq_1q_2q_3 \\ &- 3Lq_1q_2q_6 + 3Lq_1q_3q_5 + 3Lq_2q_3q_4 + 36q_1q_2q_5 + 3Lq_1q_5q_6 \\ &+ 3Lq_2q_4q_6 - 3Lq_3q_4q_5 - 36q_2q_4q_5 - 3Lq_4q_5q_6\big),\end{aligned} \tag{71}$$

$$\begin{aligned}\frac{1}{2}\int_0^L EI\left(\frac{\partial^2 v_0}{\partial x^2}\right)^2 dx \\ &= \frac{EI}{L^3}(2L^2q_3^2 + 6q_2^2 + 2L^2q_6^2 + 6q_5^2 + 6Lq_2q_3 + 2L^2q_3q_6 + 6Lq_2q_6 - 6Lq_3q_5 \\ &- 12q_2q_5 - 6Lq_5q_6).\end{aligned} \tag{72}$$

All these terms combined give us the strain energy:

$$\begin{aligned}U = \frac{EA}{2L}(q_1 - 2q_1q_4 + q_4) + \frac{EA}{2L}(-q_1^3 + 3q_1^2q_4 - 3q_1q_4^2 + q_4^3) \\ &+ \frac{3EI}{L^4}(2L^2q_3^2q_4 - 6q_1q_2^2 - 2L^2q_1q_6^2 - 2L^2q_1q_3^2 - 6q_1q_5^2 + 6q_2^2q_4 + 2L^2q_4q_6^2 \\ &+ 6q_4q_5^2 - 6Lq_1q_2q_3 - 2L^2q_1q_3q_6 - 6Lq_1q_2q_6 + 6Lq_1q_3q_5 + 6Lq_2q_3q_4 \\ &+ 12q_1q_2q_5 + 2L^2q_3q_4q_6 + 6Lq_1q_5q_6 + 6Lq_2q_4q_6 - 6Lq_3q_4q_5 - 12q_2q_4q_5 \\ &- 6Lq_4q_5q_6) \\ &+ \frac{EA}{30L^2}(2L^2q_3^2q_4 - 2L^2q_1q_6^2 - 2L^2q_1q_3^2 + 2L^2q_4q_6^2 - 18q_1q_2^2 - 18q_1q_5^2 \\ &+ 18q_2^2q_4 + 18q_4q_5^2 + L^2q_1q_3q_6 - L^2q_3q_4q_6 - 3Lq_1q_2q_3 - 3Lq_1q_2q_6 \\ &+ 3Lq_1q_3q_5 + 3Lq_2q_3q_4 + 36q_1q_2q_5 + 3Lq_1q_5q_6 + 3Lq_2q_4q_6 - 3Lq_3q_4q_5 \\ &- 36q_2q_4q_5 - 3Lq_4q_5q_6) \\ &+ \frac{EI}{L^3}(2L^2q_3^2 + 6q_2^2 + 2L^2q_6^2 + 6q_5^2 + 6Lq_2q_3 + 2L^2q_3q_6 + 6Lq_2q_6 - 6Lq_3q_5 \\ &- 12q_2q_5 - 6Lq_5q_6).\end{aligned} \tag{73}$$

A3 STIFFNESS

A3.1 Linear stiffness matrix

Now that we found the strain energy, we can determine the elastic forces. This is defined as:

$$\mathbf{Q} = \frac{\partial U}{\partial \mathbf{q}} = \bar{\mathbf{K}}\mathbf{q} + \mathbf{Q}_f \tag{74}$$

In this equation, $\bar{\mathbf{K}}$ represents the linear stiffness matrix and $\mathbf{Q}_f$ represents the higher order nonlinear forces. All derivatives of U with respect to $\mathbf{q}$ are determined and are presented here:

$$\begin{aligned}\frac{\partial U}{\partial q_1} = -\frac{E}{30L^4}(&2AL^4q_3^2 - AL^4q_3q_6 + 2AL^4q_6^2 - 30AL^3q_1 + 3AL^3q_2q_3 + 3AL^3q_2q_6 - 3AL^3q_3q_5 \\ &+ 30AL^3q_4 - 3AL^3q_5q_6 + 45AL^2q_1^2 - 90AL^2q_1q_4 + 18AL^2q_2^2 - 36AL^2q_2q_5 \\ &+ 180IL^2q_3^2 + 180IL^2q_3q_6 + 45AL^2q_4^2 + 18AL^2q_5^2 + 180IL^2q_6^2 + 540ILq_2q_3 \\ &+ 540ILq_2q_6 - 540ILq_3q_5 - 540ILq_5q_6 + 540Iq_2^2 - 1080Iq_2q_5 + 540Iq_5^2),\end{aligned} \tag{75}$$

$$\begin{aligned}\frac{\partial U}{\partial q_2} = \frac{E}{10L^4}(&120ILq_2 - 120ILq_5 - 360Iq_1q_2 + 360Iq_1q_5 + 360Iq_2q_4 - 360Iq_4q_5 + 60IL^2q_3 \\ &+ 60IL^2q_6 - 12AL^2q_1q_2 - AL^3q_1q_3 + 12AL^2q_1q_5 + 12AL^2q_2q_4 - AL^3q_1q_6 \\ &+ AL^3q_3q_4 - 12AL^2q_4q_5 + AL^3q_4q_6 - 180ILq_1q_3 - 180ILq_1q_6 + 180ILq_3q_4 \\ &+ 180ILq_4q_6),\end{aligned} \tag{76}$$

$$\begin{aligned}\frac{\partial U}{\partial q_3} = \frac{E}{30L^3}(&120IL^2q_3 + 180ILq_2 + 60IL^2q_6 - 180ILq_5 - 4AL^3q_1q_3 + AL^3q_1q_6 + 4AL^3q_3q_4 \\ &- AL^3q_4q_6 - 3AL^2q_1q_2 + 3AL^2q_1q_5 + 3AL^2q_2q_4 - 3AL^2q_4q_5 - 360ILq_1q_3 \\ &- 540Iq_1q_2 - 180ILq_1q_6 + 360ILq_3q_4 + 540Iq_1q_5 + 540Iq_2q_4 \\ &+ 180ILq_4q_6 - 540ILq_4q_5),\end{aligned} \tag{77}$$

$$\begin{aligned}\frac{\partial U}{\partial q_4} = \frac{E}{30L^4}(&2AL^4q_3^2 - AL^4q_3q_6 + 2AL^4q_6^2 - 30AL^3q_1 + 3AL^3q_2q_3 + 3AL^3q_2q_6 - 3AL^3q_3q_5 \\ &+ 30AL^3q_4 - 3AL^3q_5q_6 + 45AL^2q_1^2 - 90AL^2q_1q_4 + 18AL^2q_2^2 - 36AL^2q_2q_5 \\ &+ 180IL^2q_3^2 + 180IL^2q_3q_6 + 45AL^2q_4^2 + 18AL^2q_5^2 + 180IL^2q_6^2 + 540ILq_2q_3 \\ &+ 540ILq_2q_6 - 540ILq_3q_5 - 540ILq_5q_6 + 540Iq_2^2 - 1080Iq_2q_5 + 540Iq_5^2),\end{aligned} \tag{78}$$

$$\begin{aligned}\frac{\partial U}{\partial q_5} = -\frac{E}{10L^4}(&120ILq_2 - 120ILq_5 - 360Iq_1q_2 + 360Iq_1q_5 + 360Iq_2q_4 - 360Iq_4q_5 \\ &+ 60IL^2q_3 + 60IL^2q_6 - 12AL^2q_1q_2 - AL^3q_1q_3 + 12AL^2q_1q_5 + 12AL^2q_2q_4 \\ &- AL^3q_1q_6 + AL^3q_3q_4 - 12AL^2q_4q_5 + AL^3q_4q_6 - 180ILq_1q_3 - 180ILq_1q_6 \\ &+ 180ILq_3q_4 + 180ILq_4q_6),\end{aligned} \tag{79}$$

$$\begin{aligned}\frac{\partial U}{\partial q_6} = \frac{E}{30L^3}(&60IL^2q_3 + 180ILq_2 + 120IL^2q_6 - 180ILq_5 + AL^3q_1q_3 - 4AL^3q_1q_6 - AL^3q_3q_4 \\ &+ 4AL^3q_4q_6 - 3AL^2q_1q_2 + 3AL^2q_1q_5 + 3AL^2q_2q_4 - 3AL^2q_4q_5 - 180ILq_1q_3 \\ &- 540Iq_1q_2 - 540ILq_1q_6 + 180ILq_3q_4 + 540Iq_1q_5 + 540Iq_2q_4 \\ &+ 360ILq_4q_6 - 540Iq_4q_5).\end{aligned} \tag{80}$$

In order to determine the linear stiffness matrix, the linear terms are extracted from the derivatives:

$$\left(\frac{\partial U}{\partial q_1}\right)_{lin} = \frac{EA}{L}q_1 - \frac{EA}{L}q_4, \tag{81}$$

$$\left(\frac{\partial U}{\partial q_2}\right)_{lin} = \frac{12EI}{L^3}q_2 + \frac{6EI}{L^2}q_3 - \frac{12EI}{L^3}q_5 + \frac{6EI}{L^2}q_6, \tag{82}$$

$$\left(\frac{\partial U}{\partial q_3}\right)_{lin} = \frac{6EI}{L^2}q_2 + \frac{4EI}{L}q_3 - \frac{6EI}{L^2} + \frac{2EI}{L}q_6, \tag{83}$$

$$\left(\frac{\partial U}{\partial q_4}\right)_{lin} = -\frac{EA}{L}q_1 + \frac{EA}{L}q_4, \tag{84}$$

$$\left(\frac{\partial U}{\partial q_5}\right)_{lin} = -\frac{12EI}{L^3}q_2 - \frac{6EI}{L^2}q_3 + \frac{12EI}{L^3}q_5 - \frac{6EI}{L^2}q_6, \tag{85}$$

$$\left(\frac{\partial U}{\partial q_6}\right)_{lin} = \frac{6EI}{L^2}q_2 + \frac{2EI}{L}q_3 - \frac{6EI}{L^2}q_5 + \frac{4EI}{L}q_6. \tag{86}$$

The linear stiffness matrix can be determined and yields:

$$\bar{\mathbf{K}} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}. \tag{87}$$

A3.2 Derivatives of stiffness

We now have determined the linear stiffness matrix $\overline{\mathbf{K}}$ and the higher order nonlinear terms $\mathbf{Q}_f$. As stated in before, we also need the derivatives of $\mathbf{K}$ with respect to $\mathbf{q}$. Therefore, the derivatives of U with respect to $\mathbf{q}$ can be derived with respect to $\mathbf{q}$ again:

$$\frac{\partial \mathbf{K}}{\partial \mathbf{q}} = \frac{\partial U}{\partial \mathbf{q}\partial \mathbf{q}\partial \mathbf{q}}. \qquad (88)$$

First of all, all previously determined derivatives of U with respect to $\mathbf{q}$ are derived with respect to q_1, then with respect to q_2, etc. As was to be expected, no higher order terms occur. This is due to the fact that all fourth order terms of U were already left out before.

The derivatives are given by:

$$\frac{\partial}{\partial q_1}\left(\frac{\partial U}{\partial q_1}\right) = \frac{EA}{L^2}(L - 3q_1 + 3q_3), \qquad (89)$$

$$\frac{\partial}{\partial q_1}\left(\frac{\partial U}{\partial q_2}\right) = \frac{E}{10L^4}(-360Iq_2 - 180ILq_3 + 360Iq_5 - 180ILq_6 - 12AL^2q_2 - AL^3q_3 + 12AL^2q_5 - AL^3q_6), \qquad (90)$$

$$\frac{\partial}{\partial q_1}\left(\frac{\partial U}{\partial q_3}\right) = \frac{E}{30L^3}(-540Iq_2 - 360ILq_3 + 540Iq_5 - 180ILq_6 - 3AL^2q_2 - 4AL^3q_3 + 3AL^2q_5 + AL^3q_6), \qquad (91)$$

$$\frac{\partial}{\partial q_1}\left(\frac{\partial U}{\partial q_4}\right) = \frac{EA}{L^2}(L + 3q_1 - 3q_4), \qquad (92)$$

$$\frac{\partial}{\partial q_1}\left(\frac{\partial U}{\partial q_5}\right) = \frac{E}{10L^4}(360Iq_2 + 180ILq_3 - 360Iq_5 + 180ILq_6 + 12AL^2q_2 + AL^3q_3 - 12AL^2q_5 + AL^3q_6), \qquad (93)$$

$$\frac{\partial}{\partial q_1}\left(\frac{\partial U}{\partial q_6}\right) = \frac{E}{30L^3}(-540Iq_2 - 180ILq_3 + 540Iq_5 - 360ILq_6 - 3AL^2q_2 + AL^3q_3 + 3AL^2q_5 - 4AL^3q_6), \qquad (94)$$

$$\frac{\partial}{\partial q_2}\left(\frac{\partial U}{\partial q_1}\right) = \frac{E}{10L^4}(-360Iq_2 - 180ILq_3 + 360Iq_5 - 180ILq_6 - 12AL^2q_2 - AL^3q_3 + 12AL^2q_5 - AL^3q_6), \quad (95)$$

$$\frac{\partial}{\partial q_2}\left(\frac{\partial U}{\partial q_2}\right) = \frac{6E}{5L^4}(30Iq_4 - 30Iq_1 + 10IL - AL^2q_1 + AL^2q_4), \quad (96)$$

$$\frac{\partial}{\partial q_2}\left(\frac{\partial U}{\partial q_3}\right) = \frac{E}{10L^3}(180Iq_4 - 180Iq_1 + 60IL - AL^2q_1 + AL^2q_4), \quad (97)$$

$$\frac{\partial}{\partial q_2}\left(\frac{\partial U}{\partial q_4}\right) = \frac{E}{10L^4}(360Iq_2 - 360Iq_5 + 180ILq_3 + 180ILq_6 + 12AL^2q_2 + AL^3q_3 - 12AL^2q_5 + AL^3q_6), \quad (98)$$

$$\frac{\partial}{\partial q_2}\left(\frac{\partial U}{\partial q_5}\right) = \frac{6E}{5L^4}(-30Iq_4 + 30Iq_1 - 10IL + AL^2q_1 - AL^2q_4)\,\frac{\partial}{\partial q_1}\left(\frac{\partial U}{\partial q_5}\right) = \frac{E}{10L^4}(360Iq_2 + 180ILq_3 - 360Iq_5 + 180ILq_6 + 12AL^2q_2 + AL^3q_3 - 12AL^2q_5 + AL^3q_6), \quad (99)$$

$$\frac{\partial}{\partial q_2}\left(\frac{\partial U}{\partial q_6}\right) = \frac{E}{10L^3}(180Iq_4 - 180Iq_1 + 60IL - AL^2q_1 + AL^2q_4), \quad (100)$$

$$\frac{\partial}{\partial q_3}\left(\frac{\partial U}{\partial q_1}\right) = \frac{E}{30L^3}(-540Iq_2 - 360ILq_3 + 540Iq_5 - 180ILq_6 - 3AL^2q_2 - 4AL^3q_3 + 3AL^2q_5 + AL^3q_6), \quad (101)$$

$$\frac{\partial}{\partial q_3}\left(\frac{\partial U}{\partial q_2}\right) = \frac{E}{10L^3}(180Iq_4 - 180Iq_1 + 60IL - AL^2q_1 + AL^2q_4), \quad (102)$$

$$\frac{\partial}{\partial q_3}\left(\frac{\partial U}{\partial q_3}\right) = \frac{2E}{15L^2}(180Iq_4 - 180Iq_1 + 30IL - AL^2q_1 + AL^2q_4), \quad (103)$$

$$\frac{\partial}{\partial q_3}\left(\frac{\partial U}{\partial q_4}\right) = \frac{E}{30L^3}(540Iq_2 + 360ILq_3 - 540ILq_5 + 180ILq_6 + 3AL^2q_2 + 4AL^3q_3 - 3AL^2q_5 - AL^3q_6), \quad (104)$$

$$\frac{\partial}{\partial q_3}\left(\frac{\partial U}{\partial q_5}\right) = \frac{E}{10L^3}(-180Iq_4 + 180Iq_1 - 60IL + AL^2q_1 - AL^2q_4), \quad (105)$$

$$\frac{\partial}{\partial q_3}\left(\frac{\partial U}{\partial q_6}\right) = \frac{E}{30L^2}(180Iq_4 - 180Iq_1 + 60IL + AL^2q_1 - AL^2q_4), \quad (106)$$

$$\frac{\partial}{\partial q_4}\left(\frac{\partial U}{\partial q_1}\right) = \frac{EA}{L^2}(-L + 3q_1 - 3q_4), \tag{107}$$

$$\frac{\partial}{\partial q_4}\left(\frac{\partial U}{\partial q_2}\right) = \frac{E}{10L^4}(360Iq_2 - 360Iq_5 + 180ILq_3 + 180ILq_6 + 12AL^2q_2 + AL^3q_3 - 12AL^2q_5 + AL^3q_6), \tag{108}$$

$$\frac{\partial}{\partial q_4}\left(\frac{\partial U}{\partial q_3}\right) = \frac{E}{30L^3}(4AL^3q_3 - AL^3q_6 + 360ILq_3 + 540Iq_2 + 180ILq_6 - 540Iq_5 + 3AL^2q_2 - 3AL^2q_5), \tag{109}$$

$$\frac{\partial}{\partial q_4}\left(\frac{\partial U}{\partial q_4}\right) = \frac{EA}{L^2}(L - 3q_1 + 3q_4), \tag{110}$$

$$\frac{\partial}{\partial q_4}\left(\frac{\partial U}{\partial q_5}\right) = \frac{E}{10L^4}(-360Iq_2 + 360Iq_5 - 180ILq_3 - 180ILq_6 - 12AL^2q_2 - AL^3q_3 + 12AL^2q_5 - AL^3q_6), \tag{111}$$

$$\frac{\partial}{\partial q_4}\left(\frac{\partial U}{\partial q_6}\right) = \frac{E}{30L^3}(4AL^3q_6 - AL^3q_3 + 180ILq_3 + 540Iq_2 + 360ILq_6 - 540Iq_5 + 3AL^2q_2 - 3AL^2q_5), \tag{112}$$

$$\frac{\partial}{\partial q_5}\left(\frac{\partial U}{\partial q_1}\right) = \frac{E}{10L^4}(360Iq_2 - 360Iq_5 + 180ILq_3 + 180ILq_6 + 12AL^2q_2 + AL^3q_3 - 12AL^2q_5 + AL^3q_6), \tag{113}$$

$$\frac{\partial}{\partial q_5}\left(\frac{\partial U}{\partial q_2}\right) = \frac{6E}{5L^4}(-30Iq_4 + 30Iq_1 - 10IL + AL^2q_1 - AL^2q_4), \tag{114}$$

$$\frac{\partial}{\partial q_5}\left(\frac{\partial U}{\partial q_3}\right) = \frac{E}{10L^3}(-180Iq_4 + 180Iq_1 - 60IL + AL^2q_1 - AL^2q_4), \tag{115}$$

$$\frac{\partial}{\partial q_5}\left(\frac{\partial U}{\partial q_4}\right) = \frac{E}{10L^4}(-360Iq_2 + 360Iq_5 - 180ILq_3 - 180ILq_6 - 12AL^2q_2 - AL^3q_3 + 12AL^2q_5 - AL^3q_6), \tag{116}$$

$$\frac{\partial}{\partial q_5}\left(\frac{\partial U}{\partial q_5}\right) = \frac{6E}{5L^4}(30Iq_4 - 30Iq_1 + 10IL - AL^2q_1 + AL^2q_4), \tag{117}$$

$$\frac{\partial}{\partial q_5}\left(\frac{\partial U}{\partial q_6}\right) = \frac{E}{10L^3}(-180Iq_4 + 180Iq_1 - 60IL + AL^2q_1 - AL^2q_4), \tag{118}$$

$$\frac{\partial}{\partial q_6}\left(\frac{\partial U}{\partial q_1}\right) = \frac{E}{30L^3}(-540Iq_2 - 180ILq_3 + 540Iq_5 - 360ILq_6 - 3AL^2q_2 + AL^3q_3 + 3AL^2q_5 - 4AL^3q_6), \tag{119}$$

$$\frac{\partial}{\partial q_6}\left(\frac{\partial U}{\partial q_2}\right) = \frac{E}{10L^3}(180Iq_4 - 180Iq_1 + 60IL - AL^2q_1 + AL^2q_4), \tag{120}$$

$$\frac{\partial}{\partial q_6}\left(\frac{\partial U}{\partial q_3}\right) = \frac{E}{30L^2}(180Iq_4 - 180Iq_1 + 60IL - AL^2q_4 + AL^2q_1), \tag{121}$$

$$\frac{\partial}{\partial q_6}\left(\frac{\partial U}{\partial q_4}\right) = \frac{E}{30L^3}(540Iq_2 + 180ILq_3 - 540Iq_5 + 360ILq_6 + 3AL^2q_2 - AL^3q_3 - 3AL^2q_5 + 4AL^3q_6), \tag{122}$$

$$\frac{\partial}{\partial q_6}\left(\frac{\partial U}{\partial q_5}\right) = \frac{E}{10L^3}(180Iq_1 - 180Iq_4 - 60IL + AL^2q_1 - AL^2q_4), \tag{123}$$

$$\frac{\partial}{\partial q_6}\left(\frac{\partial U}{\partial q_6}\right) = \frac{E}{15L^2}(180Iq_4 - 180Iq_1 + 60IL - 2AL^2q_1 + 2AL^2q_4). \tag{124}$$

The six matrices that result from this are:

$$\frac{\partial \mathbf{K}}{\partial q_1}=\begin{bmatrix} -\frac{3EA}{L^2} & 0 & 0 & \frac{3EA}{L^2} & 0 & 0 \\ 0 & \frac{-180EI-6EAL^2}{5L^4} & \frac{-180EI-EAL^2}{10L^3} & 0 & \frac{180EI+6EAL^2}{5L^4} & \frac{-180EI-EAL^2}{10L^3} \\ 0 & \frac{-180EI-EAL^2}{10L^3} & \frac{-180EI-2EAL^2}{15L^2} & 0 & \frac{180EI+EAL^2}{10L^3} & \frac{-180EI+EAL^2}{30L^2} \\ \frac{3EA}{L^2} & 0 & 0 & -\frac{3EA}{L^2} & 0 & 0 \\ 0 & \frac{180EI+6EAL^2}{5L^4} & \frac{180EI+EAL^2}{10L^3} & 0 & \frac{-180EI-6EAL^2}{5L^4} & \frac{180EI+EAL^2}{10L^3} \\ 0 & \frac{-180EI-EAL^2}{10L^3} & \frac{-180EI+EAL^2}{30L^2} & 0 & \frac{180EI+EAL^2}{10L^3} & \frac{-180EI-2EAL^2}{15L^2} \end{bmatrix} \tag{125}$$

$$\frac{\partial \mathbf{K}}{\partial q_2}=\begin{bmatrix} 0 & \frac{-180EI-6EAL^2}{5L^4} & \frac{-180EI-EAL^2}{10L^3} & 0 & \frac{180EI+6EAL^2}{5L^4} & \frac{-180EI-EAL^2}{10L^3} \\ \frac{-180EI-6EAL^2}{5L^4} & 0 & 0 & \frac{180EI+6EAL^2}{5L^4} & 0 & 0 \\ \frac{-180EI-EAL^2}{10L^3} & 0 & 0 & \frac{180EI+EAL^2}{10L^3} & 0 & 0 \\ 0 & \frac{180EI+6EAL^2}{5L^4} & \frac{180EI+EAL^2}{10L^3} & 0 & \frac{-180EI-6EAL^2}{5L^4} & \frac{180EI+EAL^2}{10L^3} \\ \frac{180EI+6EAL^2}{5L^4} & 0 & 0 & \frac{-180EI-6EAL^2}{5L^4} & 0 & 0 \\ \frac{-180EI-EAL^2}{10L^3} & 0 & 0 & \frac{180EI+EAL^2}{10L^3} & 0 & 0 \end{bmatrix} \tag{126}$$

$$\frac{\partial \mathbf{K}}{\partial q_3}=\begin{bmatrix} 0 & \frac{-180EI-EAL^2}{10L^3} & \frac{-180EI-2EAL^2}{15L^2} & 0 & \frac{180EI+EAL^2}{10L^3} & \frac{-180EI+EAL^2}{30L^2} \\ \frac{-180EI-EAL^2}{10L^3} & 0 & 0 & \frac{180EI+EAL^2}{10L^3} & 0 & 0 \\ \frac{-180EI-2EAL^2}{15L^2} & 0 & 0 & \frac{180EI+2EAL^2}{15L^2} & 0 & 0 \\ 0 & \frac{180EI+EAL^2}{10L^3} & \frac{180EI+2EAL^2}{15L^2} & 0 & \frac{-180EI-EAL^2}{10L^3} & \frac{180EI-EAL^2}{30L^2} \\ \frac{180EI+EAL^2}{10L^3} & 0 & 0 & \frac{-180EI-EAL^2}{10L^3} & 0 & 0 \\ \frac{-180EI+EAL^2}{30L^2} & 0 & 0 & \frac{180EI-EAL^2}{30L^2} & 0 & 0 \end{bmatrix} \tag{127}$$

$$\frac{\partial \mathbf{K}}{\partial q_4} = \begin{bmatrix} \frac{3EA}{L^2} & 0 & 0 & -\frac{3EA}{L^2} & 0 & 0 \\ 0 & \frac{180EI+6EAL^2}{5L^4} & \frac{180EI+EAL^2}{10L^3} & 0 & \frac{-180EI-6EAL^2}{5L^4} & \frac{180EI+EAL^2}{10L^3} \\ 0 & \frac{180EI+EAL^2}{10L^3} & \frac{180EI+2EAL^2}{15L^2} & 0 & \frac{-180EI-EAL^2}{10L^3} & \frac{180EI-EAL^2}{30L^2} \\ -\frac{3EA}{L^2} & 0 & 0 & \frac{3EA}{L^2} & 0 & 0 \\ 0 & \frac{-180EI-6EAL^2}{5L^4} & \frac{-180EI-EAL^2}{10L^3} & 0 & \frac{180EI+6EAL^2}{5L^4} & \frac{-180EI-EAL^2}{10L^3} \\ 0 & \frac{180EI+EAL^2}{10L^3} & \frac{180EI-EAL^2}{30L^2} & 0 & \frac{-180EI-EAL^2}{10L^3} & \frac{180EI+2EAL^2}{15L^2} \end{bmatrix} \quad (128)$$

$$\frac{\partial \mathbf{K}}{\partial q_5} = \begin{bmatrix} 0 & \frac{180EI+6EAL^2}{5L^4} & \frac{180EI+EAL^2}{10L^3} & 0 & \frac{-180EI-6EAL^2}{5L^4} & \frac{180EI+EAL^2}{10L^3} \\ \frac{180EI+6EAL^2}{5L^4} & 0 & 0 & \frac{-180EI-6EAL^2}{5L^4} & 0 & 0 \\ \frac{180EI+EAL^2}{10L^3} & 0 & 0 & \frac{-180EI-EAL^2}{10L^3} & 0 & 0 \\ 0 & \frac{-180EI-6EAL^2}{5L^4} & \frac{-180EI-EAL^2}{10L^3} & 0 & \frac{180EI+6EAL^2}{5L^4} & \frac{-180EI-EAL^2}{10L^3} \\ \frac{-180EI-6EAL^2}{5L^4} & 0 & 0 & \frac{180EI+6EAL^2}{5L^4} & 0 & 0 \\ \frac{180EI+EAL^2}{10L^3} & 0 & 0 & \frac{-180EI-EAL^2}{10L^3} & 0 & 0 \end{bmatrix} \quad (129)$$

$$\frac{\partial \mathbf{K}}{\partial q_6} = \begin{bmatrix} 0 & \frac{-180EI-EAL^2}{10L^3} & \frac{-180EI+EAL^2}{30L^2} & 0 & \frac{180EI+EAL^2}{10L^3} & \frac{-180EI-2EAL^2}{15L^2} \\ \frac{-180EI-EAL^2}{10L^3} & 0 & 0 & \frac{180EI+EAL^2}{10L^3} & 0 & 0 \\ \frac{-180EI+EAL^2}{30L^2} & 0 & 0 & \frac{180EI-EAL^2}{30L^2} & 0 & 0 \\ 0 & \frac{180EI+EAL^2}{10L^3} & \frac{180EI-EAL^2}{30L^2} & 0 & \frac{-180EI-EAL^2}{10L^3} & \frac{180EI+2EAL^2}{15L^2} \\ \frac{180EI+EAL^2}{10L^3} & 0 & 0 & \frac{-180EI-EAL^2}{10L^3} & 0 & 0 \\ \frac{-180EI-2EAL^2}{15L^2} & 0 & 0 & \frac{180EI+2EAL^2}{15L^2} & 0 & 0 \end{bmatrix} \quad (130)$$

A4 MASS MATRIX

In order to determine the mass matrix, the earlier defined shape functions are needed. The expression for the mass matrix equals:

$$\mathbf{M} = \int_0^L \rho\,[\mathbf{S}^{\mathrm{T}}\mathbf{S}]\,A\;dx = \rho A \int_0^L [\mathbf{S}^{\mathrm{T}}\mathbf{S}]\,dx$$
$$= \frac{\rho A L}{420}\begin{bmatrix} 140 & 0 & 0 & 70 & 0 & 0 \\ 0 & 156 & 22L & 0 & 54 & -13L \\ 0 & 22L & 4L^2 & 0 & 13L & -3L^2 \\ 70 & 0 & 0 & 140 & 0 & 0 \\ 0 & 54 & 13L & 0 & 156 & -22L \\ 0 & -13L & -3L^2 & 0 & -22L & 4L^2 \end{bmatrix} \quad (131)$$

A5 EXAMPLE: SIMPLY SUPPORTED BEAM

A5.1 Stiffness

By applying boundary conditions, the linear stiffness matrix and the derivatives of the stiffness reduce to:

$$\bar{\mathbf{K}} = \begin{bmatrix} \frac{4EI}{L} & 0 & \frac{2EI}{L} \\ 0 & \frac{EA}{L} & 0 \\ \frac{2EI}{L} & 0 & \frac{4EI}{L} \end{bmatrix}, \quad (132)$$

$$\frac{\partial \mathbf{K}}{\partial q_3} = \begin{bmatrix} 0 & \frac{360EI + 4EAL^2}{30L^2} & 0 \\ \frac{360EI + 4EAL^2}{30L^2} & 0 & \frac{180EI - EAL^2}{30L^2} \\ 0 & \frac{180EI - EAL^2}{30L^2} & 0 \end{bmatrix}, \quad (133)$$

$$\frac{\partial \mathbf{K}}{\partial q_4} = \begin{bmatrix} \frac{360EI + 4EAL^2}{30L^2} & 0 & \frac{180EI - EAL^2}{30L^2} \\ 0 & \frac{3EA}{L^2} & 0 \\ \frac{180EI - EAL^2}{30L^2} & 0 & \frac{360EI + 4EAL^2}{30L^2} \end{bmatrix}, \quad (134)$$

$$\frac{\partial \mathbf{K}}{\partial q_6} = \begin{bmatrix} 0 & \frac{180EI - EAL^2}{30L^2} & 0 \\ \frac{180EI - EAL^2}{30L^2} & 0 & \frac{360EI + 4EAL^2}{30L^2} \\ 0 & \frac{360EI + 4EAL^2}{30L^2} & 0 \end{bmatrix}. \quad (135)$$

A5.2 Natural frequencies and mode shapes

The first three natural frequencies (for a simply supported beam) are:

$$\omega_1 = \sqrt{\frac{3E}{\rho L^2}}, \quad \omega_2 = \sqrt{\frac{120EI}{\rho A L^4}}, \quad \omega_3 = \sqrt{\frac{2520EI}{\rho A L^4}}. \tag{136}$$

Substituting these natural frequencies in the eigenvalue problem yields three mode shapes:

$$\boldsymbol{\Phi_1} = \begin{Bmatrix} 0 \\ \alpha_1 \\ 0 \end{Bmatrix}, \qquad \boldsymbol{\Phi_2} = \begin{Bmatrix} -\alpha_2 \\ 0 \\ \alpha_2 \end{Bmatrix}, \qquad \boldsymbol{\Phi_3} = \begin{Bmatrix} \alpha_3 \\ 0 \\ \alpha_3 \end{Bmatrix}. \tag{137}$$

Therefore, the modal matrix equals:

$$\boldsymbol{\Phi} = \begin{bmatrix} 0 & -\alpha_2 & \alpha_3 \\ \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & \alpha_3 \end{bmatrix}. \tag{138}$$

A5.3 Modal derivatives

We need to know the derivatives of $\mathbf{K}$ with respect to η_j. However, we already know the derivatives with respect to $\mathbf{q}$. Therefore, we need to apply coordinate transformation:

$$\mathbf{q} = \boldsymbol{\Phi\eta}, \tag{139}$$

$$\mathbf{q} = \begin{bmatrix} 0 & -\alpha_2 & \alpha_3 \\ \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & \alpha_3 \end{bmatrix} \begin{Bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{Bmatrix}, \tag{140}$$

$$q_3 = -\alpha_2\eta_2 + \alpha_3\eta_3, \quad q_4 = \alpha_1\eta_1, \quad q_6 = \alpha_2\eta_2 + \alpha_3\eta_3. \tag{141}$$

For q_3:

$$\frac{\partial q_3}{\partial \eta_1} = 0, \quad \frac{\partial q_3}{\partial \eta_2} = -\alpha_2, \quad \frac{\partial q_3}{\partial \eta_3} = \alpha_3. \tag{142}$$

For q_4:

$$\frac{\partial q_4}{\partial \eta_1} = \alpha_1, \quad \frac{\partial q_4}{\partial \eta_2} = 0, \quad \frac{\partial q_4}{\partial \eta_3} = 0. \tag{143}$$

For q_6:

$$\frac{\partial q_6}{\partial \eta_1} = 0, \quad \frac{\partial q_6}{\partial \eta_2} = \alpha_2, \quad \frac{\partial q_6}{\partial \eta_3} = \alpha_3. \tag{144}$$

We need to know expressions for the derivatives of **K** with respect to η_j. Since **K** depends on **q**, and **q** depends on **η**, this expression can be stated as following:

$$\frac{\partial \mathbf{K}}{\partial \eta_j} = \frac{\partial \mathbf{K}}{\partial q_i}\frac{\partial q_i}{\partial \eta_j}. \tag{145}$$

The derivatives of **K** with respect to q_i are already determined and now the derivatives of q_i with respect to η_j are determined too. Therefore, the derivative of **K** with respect to η_j can be determined:

$$\begin{aligned}
\frac{\partial \mathbf{K}}{\partial \eta_1} &= \sum_{i=1}^{6} \frac{\partial \mathbf{K}}{\partial q_i}\frac{\partial q_i}{\partial \eta_1} = \frac{\partial K}{\partial q_3}\frac{\partial q_3}{\partial \eta_1} + \frac{\partial K}{\partial q_4}\frac{\partial q_4}{\partial \eta_1} + \frac{\partial K}{\partial q_6}\frac{\partial q_6}{\partial \eta_1} \\
&= \begin{bmatrix} 0 & \frac{360EI+4EAL^2}{30L^2} & 0 \\ \frac{360EI+4EAL^2}{30L^2} & 0 & \frac{180EI-EAL^2}{30L^2} \\ 0 & \frac{180EI-EAL^2}{30L^2} & 0 \end{bmatrix} \cdot 0 \\
&+ \begin{bmatrix} \frac{360EI+4EAL^2}{30L^2} & 0 & \frac{180EI-EAL^2}{30L^2} \\ 0 & \frac{3EA}{L^2} & 0 \\ \frac{180EI-EAL^2}{30L^2} & 0 & \frac{360EI+4EAL^2}{30L^2} \end{bmatrix} \cdot \alpha_1 \\
&+ \begin{bmatrix} 0 & \frac{180EI-EAL^2}{30L^2} & 0 \\ \frac{180EI-EAL^2}{30L^2} & 0 & \frac{360EI+4EAL^2}{30L^2} \\ 0 & \frac{360EI+4EAL^2}{30L^2} & 0 \end{bmatrix} \cdot 0 \\
&= \begin{bmatrix} \frac{360EI+4EAL^2}{30L^2}\alpha_1 & 0 & \frac{180EI-EAL^2}{30L^2}\alpha_1 \\ 0 & \frac{3EA}{L^2}\alpha_1 & 0 \\ \frac{180EI-EAL^2}{30L^2}\alpha_1 & 0 & \frac{360EI+4EAL^2}{30L^2}\alpha_1 \end{bmatrix},
\end{aligned} \tag{146}$$

$$
\begin{aligned}
\frac{\partial \mathbf{K}}{\partial \eta_2} &= \sum_{i=1}^{6} \frac{\partial \mathbf{K}}{\partial q_i}\frac{\partial q_i}{\partial \eta_2} = \frac{\partial K}{\partial q_3}\frac{\partial q_3}{\partial \eta_2} + \frac{\partial K}{\partial q_4}\frac{\partial q_4}{\partial \eta_2} + \frac{\partial K}{\partial q_6}\frac{\partial q_6}{\partial \eta_2} \\
&= \begin{bmatrix} 0 & \frac{360EI+4EAL^2}{30L^2} & 0 \\ \frac{360EI+4EAL^2}{30L^2} & 0 & \frac{180EI-EAL^2}{30L^2} \\ 0 & \frac{180EI-EAL^2}{30L^2} & 0 \end{bmatrix} \cdot -\alpha_2 \\
&+ \begin{bmatrix} \frac{360EI+4EAL^2}{30L^2} & 0 & \frac{180EI-EAL^2}{30L^2} \\ 0 & \frac{3EA}{L^2} & 0 \\ \frac{180EI-EAL^2}{30L^2} & 0 & \frac{360EI+4EAL^2}{30L^2} \end{bmatrix} \cdot 0 \\
&+ \begin{bmatrix} 0 & \frac{180EI-EAL^2}{30L^2} & 0 \\ \frac{180EI-EAL^2}{30L^2} & 0 & \frac{360EI+4EAL^2}{30L^2} \\ 0 & \frac{360EI+4EAL^2}{30L^2} & 0 \end{bmatrix} \cdot \alpha_2 \\
&= \begin{bmatrix} 0 & \frac{-36EI-EAL^2}{6L^2}\alpha_2 & 0 \\ \frac{-36EI-EAL^2}{6L^2}\alpha_2 & 0 & \frac{36EI+EAL^2}{6L^2}\alpha_2 \\ 0 & \frac{36EI+EAL^2}{6L^2}\alpha_2 & 0 \end{bmatrix},
\end{aligned}
\tag{147}
$$

$$
\begin{aligned}
\frac{\partial \mathbf{K}}{\partial \eta_3} &= \sum_{i=1}^{6} \frac{\partial \mathbf{K}}{\partial q_i}\frac{\partial q_i}{\partial \eta_3} = \frac{\partial K}{\partial q_3}\frac{\partial q_3}{\partial \eta_3} + \frac{\partial K}{\partial q_4}\frac{\partial q_4}{\partial \eta_3} + \frac{\partial K}{\partial q_6}\frac{\partial q_6}{\partial \eta_3} \\
&= \begin{bmatrix} 0 & \frac{360EI+4EAL^2}{30L^2} & 0 \\ \frac{360EI+4EAL^2}{30L^2} & 0 & \frac{180EI-EAL^2}{30L^2} \\ 0 & \frac{180EI-EAL^2}{30L^2} & 0 \end{bmatrix} \cdot \alpha_3 \\
&+ \begin{bmatrix} \frac{360EI+4EAL^2}{30L^2} & 0 & \frac{180EI-EAL^2}{30L^2} \\ 0 & \frac{3EA}{L^2} & 0 \\ \frac{180EI-EAL^2}{30L^2} & 0 & \frac{360EI+4EAL^2}{30L^2} \end{bmatrix} \cdot 0 \\
&+ \begin{bmatrix} 0 & \frac{180EI-EAL^2}{30L^2} & 0 \\ \frac{180EI-EAL^2}{30L^2} & 0 & \frac{360EI+4EAL^2}{30L^2} \\ 0 & \frac{360EI+4EAL^2}{30L^2} & 0 \end{bmatrix} \cdot \alpha_3 \\
&= \begin{bmatrix} 0 & \frac{180EI+EAL^2}{10L^2}\alpha_3 & 0 \\ \frac{180EI+EAL^2}{10L^2}\alpha_3 & 0 & \frac{180EI+EAL^2}{10L^2}\alpha_3 \\ 0 & \frac{180EI+EAL^2}{10L^2}\alpha_3 & 0 \end{bmatrix}.
\end{aligned}
\tag{148}
$$

We now have the derivatives of **K** with respect to η_j. This means that we now have all necessary terms in order to compute the modal derivatives.

The modal derivatives can now be calculated:

$$\boldsymbol{\theta}_{ij} = -\bar{\mathbf{K}}^{-1}\frac{\partial \mathbf{K}}{\partial \eta_j}\boldsymbol{\phi}_i, \tag{149}$$

$$\begin{aligned}
\theta_{11} &= -\bar{\mathbf{K}}^{-1}\frac{\partial \mathbf{K}}{\partial \eta_1}\boldsymbol{\phi}_1 \\
&= -\begin{bmatrix} \frac{L}{3EI} & 0 & -\frac{L}{6EI} \\ 0 & \frac{L}{EA} & 0 \\ -\frac{L}{6EI} & 0 & \frac{L}{3EI} \end{bmatrix}\begin{bmatrix} \frac{360EI+4EAL^2}{30L^2}\alpha_1 & 0 & \frac{180EI-EAL^2}{30L^2}\alpha_1 \\ 0 & \frac{3EA}{L^2}\alpha_1 & 0 \\ \frac{180EI-EAL^2}{30L^2}\alpha_1 & 0 & \frac{360EI+4EAL^2}{30L^2}\alpha_1 \end{bmatrix}\begin{Bmatrix} 0 \\ \alpha_1 \\ 0 \end{Bmatrix} \\
&= -\begin{bmatrix} \frac{L}{3EI} & 0 & -\frac{L}{6EI} \\ 0 & \frac{L}{EA} & 0 \\ -\frac{L}{6EI} & 0 & \frac{L}{3EI} \end{bmatrix}\begin{Bmatrix} 0 \\ \frac{3EA}{L^2}\alpha_1^2 \\ 0 \end{Bmatrix} \\
&= \begin{Bmatrix} 0 \\ -\frac{3}{L}\alpha_1^2 \\ 0 \end{Bmatrix},
\end{aligned} \tag{150}$$

$$\begin{aligned}
\theta_{12} &= -\bar{\mathbf{K}}^{-1}\frac{\partial \mathbf{K}}{\partial \eta_2}\boldsymbol{\phi}_1 \\
&= -\begin{bmatrix} \frac{L}{3EI} & 0 & -\frac{L}{6EI} \\ 0 & \frac{L}{EA} & 0 \\ -\frac{L}{6EI} & 0 & \frac{L}{3EI} \end{bmatrix}\begin{bmatrix} 0 & \frac{-36EI-EAL^2}{6L^2}\alpha_2 & 0 \\ \frac{-36EI-EAL^2}{6L^2}\alpha_2 & 0 & \frac{36EI+EAL^2}{6L^2}\alpha_2 \\ 0 & \frac{36EI+EAL^2}{6L^2}\alpha_2 & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ \alpha_1 \\ 0 \end{Bmatrix} \\
&= -\begin{bmatrix} \frac{L}{3EI} & 0 & -\frac{L}{6EI} \\ 0 & \frac{L}{EA} & 0 \\ -\frac{L}{6EI} & 0 & \frac{L}{3EI} \end{bmatrix}\begin{Bmatrix} \frac{-36EI-EAL^2}{6L^2}\alpha_1\alpha_2 \\ 0 \\ \frac{36EI+EAL^2}{6L^2}\alpha_1\alpha_2 \end{Bmatrix} \\
&= \begin{Bmatrix} \frac{36I+AL^2}{12IL}\alpha_1\alpha_2 \\ 0 \\ \frac{-36I-AL^2}{12IL}\alpha_1\alpha_2 \end{Bmatrix},
\end{aligned} \tag{151}$$

$$\theta_{13} = -\bar{\mathbf{K}}^{-1}\frac{\partial \mathbf{K}}{\partial \eta_3}\boldsymbol{\phi}_1$$

$$= -\begin{bmatrix} \frac{L}{3EI} & 0 & -\frac{L}{6EI} \\ 0 & \frac{L}{EA} & 0 \\ -\frac{L}{6EI} & 0 & \frac{L}{3EI} \end{bmatrix}\begin{bmatrix} 0 & \frac{180EI + EAL^2}{10L^2}\alpha_3 & 0 \\ \frac{180EI + EAL^2}{10L^2}\alpha_3 & 0 & \frac{180EI + EAL^2}{10L^2}\alpha_3 \\ 0 & \frac{180EI + EAL^2}{10L^2}\alpha_3 & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ \alpha_1 \\ 0 \end{Bmatrix}$$

$$= -\begin{bmatrix} \frac{L}{3EI} & 0 & -\frac{L}{6EI} \\ 0 & \frac{L}{EA} & 0 \\ -\frac{L}{6EI} & 0 & \frac{L}{3EI} \end{bmatrix}\begin{Bmatrix} \frac{180EI + EAL^2}{10L^2}\alpha_1\alpha_3 \\ 0 \\ \frac{180EI + EAL^2}{10L^2}\alpha_1\alpha_3 \end{Bmatrix} \quad (152)$$

$$= \begin{Bmatrix} \frac{-180I - AL^2}{60IL}\alpha_1\alpha_3 \\ 0 \\ \frac{-180I - AL^2}{60IL}\alpha_1\alpha_3 \end{Bmatrix},$$

$$\theta_{21} = -\bar{\mathbf{K}}^{-1}\frac{\partial \mathbf{K}}{\partial \eta_1}\boldsymbol{\phi}_2$$

$$= -\begin{bmatrix} \frac{L}{3EI} & 0 & -\frac{L}{6EI} \\ 0 & \frac{L}{EA} & 0 \\ -\frac{L}{6EI} & 0 & \frac{L}{3EI} \end{bmatrix}\begin{bmatrix} \frac{360EI + 4EAL^2}{30L^2}\alpha_1 & 0 & \frac{180EI - EAL^2}{30L^2}\alpha_1 \\ 0 & \frac{3EA}{L^2}\alpha_1 & 0 \\ \frac{180EI - EAL^2}{30L^2}\alpha_1 & 0 & \frac{360EI + 4EAL^2}{30L^2}\alpha_1 \end{bmatrix}\begin{Bmatrix} -\alpha_2 \\ 0 \\ \alpha_2 \end{Bmatrix}$$

$$= -\begin{bmatrix} \frac{L}{3EI} & 0 & -\frac{L}{6EI} \\ 0 & \frac{L}{EA} & 0 \\ -\frac{L}{6EI} & 0 & \frac{L}{3EI} \end{bmatrix}\begin{Bmatrix} \frac{-36EI - EAL^2}{6L^2}\alpha_1\alpha_2 \\ 0 \\ \frac{36EI + EAL^2}{6L^2}\alpha_1\alpha_2 \end{Bmatrix} \quad (153)$$

$$= \begin{Bmatrix} \frac{36I + AL^2}{12IL}\alpha_1\alpha_2 \\ 0 \\ \frac{-36I - AL^2}{12IL}\alpha_1\alpha_2 \end{Bmatrix},$$

$$\begin{aligned}
\theta_{22} &= -\bar{\mathbf{K}}^{-1}\frac{\partial \mathbf{K}}{\partial \eta_2}\boldsymbol{\phi}_2 \\
&= -\begin{bmatrix} \frac{L}{3EI} & 0 & -\frac{L}{6EI} \\ 0 & \frac{L}{EA} & 0 \\ -\frac{L}{6EI} & 0 & \frac{L}{3EI} \end{bmatrix}\begin{bmatrix} 0 & \frac{-36EI-EAL^2}{6L^2}\alpha_2 & 0 \\ \frac{-36EI-EAL^2}{6L^2}\alpha_2 & 0 & \frac{36EI+EAL^2}{6L^2}\alpha_2 \\ 0 & \frac{36EI+EAL^2}{6L^2}\alpha_2 & 0 \end{bmatrix}\begin{Bmatrix} -\alpha_2 \\ 0 \\ \alpha_2 \end{Bmatrix} \\
&= -\begin{bmatrix} \frac{L}{3EI} & 0 & -\frac{L}{6EI} \\ 0 & \frac{L}{EA} & 0 \\ -\frac{L}{6EI} & 0 & \frac{L}{3EI} \end{bmatrix}\begin{Bmatrix} 0 \\ \frac{36EI+EAL^2}{3L^2}\alpha_2^2 \\ 0 \end{Bmatrix} \\
&= \begin{Bmatrix} 0 \\ \frac{-36I-AL^2}{3AL}\alpha_2^2 \\ 0 \end{Bmatrix},
\end{aligned} \tag{154}$$

$$\begin{aligned}
\theta_{23} &= -\bar{\mathbf{K}}^{-1}\frac{\partial \mathbf{K}}{\partial \eta_3}\boldsymbol{\phi}_2 \\
&= -\begin{bmatrix} \frac{L}{3EI} & 0 & -\frac{L}{6EI} \\ 0 & \frac{L}{EA} & 0 \\ -\frac{L}{6EI} & 0 & \frac{L}{3EI} \end{bmatrix}\begin{bmatrix} 0 & \frac{180EI+EAL^2}{10L^2}\alpha_3 & 0 \\ \frac{180EI+EAL^2}{10L^2}\alpha_3 & 0 & \frac{180EI+EAL^2}{10L^2}\alpha_3 \\ 0 & \frac{180EI+EAL^2}{10L^2}\alpha_3 & 0 \end{bmatrix}\begin{Bmatrix} -\alpha_2 \\ 0 \\ \alpha_2 \end{Bmatrix} \\
&= -\begin{bmatrix} \frac{L}{3EI} & 0 & -\frac{L}{6EI} \\ 0 & \frac{L}{EA} & 0 \\ -\frac{L}{6EI} & 0 & \frac{L}{3EI} \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \\
&= \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix},
\end{aligned} \tag{155}$$

$$\theta_{31} = -\bar{\mathbf{K}}^{-1}\frac{\partial \mathbf{K}}{\partial \eta_1}\boldsymbol{\phi}_3$$
$$= -\begin{bmatrix} \frac{L}{3EI} & 0 & -\frac{L}{6EI} \\ 0 & \frac{L}{EA} & 0 \\ -\frac{L}{6EI} & 0 & \frac{L}{3EI} \end{bmatrix}\begin{bmatrix} \frac{360EI+4EAL^2}{30L^2}\alpha_1 & 0 & \frac{180EI-EAL^2}{30L^2}\alpha_1 \\ 0 & \frac{3EA}{L^2}\alpha_1 & 0 \\ \frac{180EI-EAL^2}{30L^2}\alpha_1 & 0 & \frac{360EI+4EAL^2}{30L^2}\alpha_1 \end{bmatrix}\begin{Bmatrix} \alpha_3 \\ 0 \\ \alpha_3 \end{Bmatrix}$$
$$= -\begin{bmatrix} \frac{L}{3EI} & 0 & -\frac{L}{6EI} \\ 0 & \frac{L}{EA} & 0 \\ -\frac{L}{6EI} & 0 & \frac{L}{3EI} \end{bmatrix}\begin{Bmatrix} \frac{180EI+EAL^2}{10L^2}\alpha_1\alpha_3 \\ 0 \\ \frac{180EI+EAL^2}{10L^2}\alpha_1\alpha_3 \end{Bmatrix} \quad (156)$$
$$= \begin{Bmatrix} \frac{-180I-AL^2}{60IL}\alpha_1\alpha_3 \\ 0 \\ \frac{-180I-AL^2}{60IL}\alpha_1\alpha_3 \end{Bmatrix},$$

$$\theta_{32} = -\bar{\mathbf{K}}^{-1}\frac{\partial \mathbf{K}}{\partial \eta_2}\boldsymbol{\phi}_3$$
$$= -\begin{bmatrix} \frac{L}{3EI} & 0 & -\frac{L}{6EI} \\ 0 & \frac{L}{EA} & 0 \\ -\frac{L}{6EI} & 0 & \frac{L}{3EI} \end{bmatrix}\begin{bmatrix} 0 & \frac{-36EI-EAL^2}{6L^2}\alpha_2 & 0 \\ \frac{-36EI-EAL^2}{6L^2}\alpha_2 & 0 & \frac{36EI+EAL^2}{6L^2}\alpha_2 \\ 0 & \frac{36EI+EAL^2}{6L^2}\alpha_2 & 0 \end{bmatrix}\begin{Bmatrix} \alpha_3 \\ 0 \\ \alpha_3 \end{Bmatrix}$$
$$= -\begin{bmatrix} \frac{L}{3EI} & 0 & -\frac{L}{6EI} \\ 0 & \frac{L}{EA} & 0 \\ -\frac{L}{6EI} & 0 & \frac{L}{3EI} \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad (157)$$
$$= \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix},$$

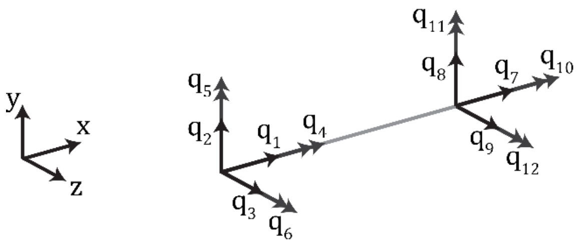
$$\theta_{33} = -\bar{\mathbf{K}}^{-1}\frac{\partial \mathbf{K}}{\partial \eta_3}\boldsymbol{\phi}_3$$
$$= -\begin{bmatrix} \frac{L}{3EI} & 0 & -\frac{L}{6EI} \\ 0 & \frac{L}{EA} & 0 \\ -\frac{L}{6EI} & 0 & \frac{L}{3EI} \end{bmatrix}\begin{bmatrix} 0 & \frac{180EI+EAL^2}{10L^2}\alpha_3 & 0 \\ \frac{180EI+EAL^2}{10L^2}\alpha_3 & 0 & \frac{180EI+EAL^2}{10L^2}\alpha_3 \\ 0 & \frac{180EI+EAL^2}{10L^2}\alpha_3 & 0 \end{bmatrix}\begin{Bmatrix} \alpha_3 \\ 0 \\ \alpha_3 \end{Bmatrix}$$
$$= -\begin{bmatrix} \frac{L}{3EI} & 0 & -\frac{L}{6EI} \\ 0 & \frac{L}{EA} & 0 \\ -\frac{L}{6EI} & 0 & \frac{L}{3EI} \end{bmatrix}\begin{Bmatrix} 0 \\ \frac{180EI+EAL^2}{5L^2}\alpha_3^2 \\ 0 \end{Bmatrix} \quad (158)$$
$$= \begin{Bmatrix} 0 \\ \frac{-180I-AL^2}{5AL}\alpha_3^2 \\ 0 \end{Bmatrix}.$$

APPENDIX B
Derivation for 3D beam elements

B1 INTRODUCTION

In this appendix, the derivations for a 3D beam element are presented. We consider a beam element, of which the generalized coordinates are defined as following:

$$\mathbf{q} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \\ q_9 \\ q_{10} \\ q_{11} \\ q_{12} \end{Bmatrix} = \begin{Bmatrix} x_1 \\ y_1 \\ z_1 \\ \theta_{x1} \\ \theta_{y1} \\ \theta_{z1} \\ x_2 \\ y_2 \\ z_2 \\ \theta_{x2} \\ \theta_{y2} \\ \theta_{z2} \end{Bmatrix}. \quad (159)$$



B2 STRAIN ENERGY

B2.1 Strain definition

In order to determine the strain energy, we first need an expression for the strain. The Green-Lagrange strain definition equals:

$$\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right], \\
\varepsilon_{xy} &= \frac{1}{2}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right], \\
\varepsilon_{xz} &= \frac{1}{2}\left[\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} + \frac{\partial u}{\partial z}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial v}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial w}{\partial x}\right].
\end{aligned} \quad (160)$$

The deformation field is defined as:

$$\begin{aligned}
u &= u_0 - y\frac{\partial v_0}{\partial x} - z\frac{\partial w_0}{\partial x}, \\
v &= v_0 - z\varphi_0, \\
w &= w_0 + y\varphi_0.
\end{aligned} \quad (161)$$

The derivatives of u, v and w with respect to x, y and z are given as:

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u_0}{\partial x} - y\frac{\partial^2 v_0}{\partial x^2} - z\frac{\partial^2 w_0}{\partial x^2},\\ \frac{\partial v}{\partial x} &= \frac{\partial v_0}{\partial x} - z\frac{\partial \varphi_0}{\partial x},\\ \frac{\partial w}{\partial x} &= \frac{\partial w_0}{\partial x} + y\frac{\partial \varphi_0}{\partial x}.\end{aligned} \tag{162}$$

$$\begin{aligned}\frac{\partial u}{\partial y} &= \frac{\partial u_0}{\partial y} - \frac{\partial v_0}{\partial x} - y\frac{\partial^2 v_0}{\partial x \partial y} - z\frac{\partial^2 w_0}{\partial x \partial y},\\ \frac{\partial v}{\partial y} &= \frac{\partial v_0}{\partial y} - z\frac{\partial \varphi_0}{\partial y},\\ \frac{\partial w}{\partial y} &= \frac{\partial w_0}{\partial y} + \varphi_0 + y\frac{\partial \varphi_0}{\partial y}.\end{aligned} \tag{163}$$

$$\begin{aligned}\frac{\partial u}{\partial z} &= \frac{\partial u_0}{\partial z} - y\frac{\partial^2 v_0}{\partial x \partial z} - \frac{\partial w_0}{\partial x} - z\frac{\partial^2 w_0}{\partial x \partial z},\\ \frac{\partial v}{\partial z} &= \frac{\partial v_0}{\partial z} - \varphi_0 - z\frac{\partial \varphi_0}{\partial z},\\ \frac{\partial w}{\partial z} &= \frac{\partial w_0}{\partial z} + y\frac{\partial \varphi_0}{\partial z}.\end{aligned} \tag{164}$$

B2.2 Coordinate transformation

We will now apply coordinate transformation, just as for 2D:

$$\mathbf{u} = \mathbf{S}\mathbf{q}. \tag{165}$$

The shape function matrix equals:

$$\mathbf{S}^{\mathrm{T}} = \begin{bmatrix} 1-\xi & 0 & 0 & 0 \\ 0 & 1-3\xi^2+2\xi^3 & 0 & 0 \\ 0 & 0 & 1-3\xi^2+2\xi^3 & 0 \\ 0 & 0 & 0 & \xi \\ 0 & 0 & (-\xi+2\xi^2-\xi^3)L & 0 \\ 0 & (\xi-2\xi^2+\xi^3)L & 0 & 0 \\ \xi & 0 & 0 & 0 \\ 0 & 3\xi^2-2\xi^3 & 0 & 0 \\ 0 & 0 & 3\xi^2-2\xi^3 & 0 \\ 0 & 0 & 0 & 1-\xi \\ 0 & 0 & (\xi^2-\xi^3)L & 0 \\ 0 & (-\xi^2+\xi^3)L & 0 & 0 \end{bmatrix}. \tag{166}$$

Substituting **S** and **q** in the coordinate transformation yields:

$$\begin{Bmatrix} u_0 \\ v_0 \\ w_0 \\ \varphi_0 \end{Bmatrix} = \begin{bmatrix} 1-\frac{x}{L} & 0 & 0 & 0 \\ 0 & 1-3\left(\frac{x}{L}\right)^2+2\left(\frac{x}{L}\right)^3 & 0 & 0 \\ 0 & 0 & 1-3\left(\frac{x}{L}\right)^2+2\left(\frac{x}{L}\right)^3 & 0 \\ 0 & 0 & 0 & \frac{x}{L} \\ 0 & 0 & \left(-\frac{x}{L}+2\left(\frac{x}{L}\right)^2-\left(\frac{x}{L}\right)^3\right)L & 0 \\ 0 & \left(\frac{x}{L}-2\left(\frac{x}{L}\right)^2+\left(\frac{x}{L}\right)^3\right)L & 0 & 0 \\ \frac{x}{L} & 0 & 0 & 0 \\ 0 & 3\left(\frac{x}{L}\right)^2-2\left(\frac{x}{L}\right)^3 & 0 & 0 \\ 0 & 0 & 3\left(\frac{x}{L}\right)^2-2\left(\frac{x}{L}\right)^3 & 0 \\ 0 & 0 & 0 & 1-\frac{x}{L} \\ 0 & 0 & \left(\left(\frac{x}{L}\right)^2-\left(\frac{x}{L}\right)^3\right)L & 0 \\ 0 & \left(-\left(\frac{x}{L}\right)^2+\left(\frac{x}{L}\right)^3\right)L & 0 & 0 \end{bmatrix}^T \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \\ q_9 \\ q_{10} \\ q_{11} \\ q_{12} \end{Bmatrix}. \tag{167}$$

So for the displacements of the neutral line:

$$u_0 = \left[1-\frac{x}{L}\right] q_1 + \left[\frac{x}{L}\right] q_7, \tag{168}$$

$$\begin{aligned} v_0 = {} & \left[1-3\left(\frac{x}{L}\right)^2+2\left(\frac{x}{L}\right)^3\right] q_2 + \left[\left(\frac{x}{L}-2\left(\frac{x}{L}\right)^2+\left(\frac{x}{L}\right)^3\right)L\right] q_6 + \left[3\left(\frac{x}{L}\right)^2-2\left(\frac{x}{L}\right)^3\right] q_8 \\ & + \left[\left(-\left(\frac{x}{L}\right)^2+\left(\frac{x}{L}\right)^3\right)L\right] q_{12}, \end{aligned} \tag{169}$$

$$\begin{aligned} w_0 = {} & \left[1-3\left(\frac{x}{L}\right)^2+2\left(\frac{x}{L}\right)^3\right] q_3 + \left[\left(-\frac{x}{L}+2\left(\frac{x}{L}\right)^2-\left(\frac{x}{L}\right)^3\right)L\right] q_5 + \left[3\left(\frac{x}{L}\right)^2-2\left(\frac{x}{L}\right)^3\right] q_9 \\ & + \left[\left(\left(\frac{x}{L}\right)^2-\left(\frac{x}{L}\right)^3\right)L\right] q_{11}, \end{aligned} \tag{170}$$

$$\varphi_0 = \left[\frac{x}{L}\right] q_4 + \left[1-\frac{x}{L}\right] q_{10}. \tag{171}$$

The first derivatives of the shape function matrix $\boldsymbol{S}$ are:

$$\left(\frac{\partial \boldsymbol{S}}{\partial x}\right)^T = \begin{bmatrix} -\frac{1}{L} & 0 & 0 & 0 \\ 0 & \frac{6x^2}{L^3}-\frac{6x}{L^2} & 0 & 0 \\ 0 & 0 & \frac{6x^2}{L^3}-\frac{6x}{L^2} & 0 \\ 0 & 0 & 0 & \frac{1}{L} \\ 0 & 0 & -L\left(\frac{1}{L}-\frac{4x}{L^2}+\frac{3x^2}{L^3}\right) & 0 \\ 0 & L\left(\frac{1}{L}-\frac{4x}{L^2}+\frac{3x^2}{L^3}\right) & 0 & 0 \\ \frac{1}{L} & 0 & 0 & 0 \\ 0 & \frac{6x}{L^2}-\frac{6x^2}{L^3} & 0 & 0 \\ 0 & 0 & \frac{6x}{L^2}-\frac{6x^2}{L^3} & 0 \\ 0 & 0 & 0 & -\frac{1}{L} \\ 0 & 0 & L\left(\frac{2x}{L^2}-\frac{3x^2}{L^3}\right) & 0 \\ 0 & -L\left(\frac{3x}{L^2}-\frac{2x^2}{L^3}\right) & 0 & 0 \end{bmatrix}, \qquad (172)$$

$$\frac{\partial u_0}{\partial x} = -\left[\frac{1}{L}\right] q_1 + \left[\frac{1}{L}\right] q_7, \qquad (173)$$

$$\frac{\partial v_0}{\partial x} = \left[\frac{6x^2}{L^3}-\frac{6x}{L^2}\right] q_2 + \left[L\left(\frac{1}{L}-\frac{4x}{L^2}+\frac{3x^2}{L^3}\right)\right] q_6 + \left[\frac{6x}{L^2}-\frac{6x^2}{L^3}\right] q_8 - \left[L\left(\frac{3x}{L^2}-\frac{2x^2}{L^3}\right)\right] q_{12}, \qquad (174)$$

$$\frac{\partial w_0}{\partial x} = \left[\frac{6x^2}{L^3}-\frac{6x}{L^2}\right] q_3 - \left[L\left(\frac{1}{L}-\frac{4x}{L^2}+\frac{3x^2}{L^3}\right)\right] q_6 + \left[\frac{6x}{L^2}-\frac{6x^2}{L^3}\right] q_9 + \left[L\left(\frac{2x}{L^2}-\frac{3x^2}{L^3}\right)\right] q_{11}, \qquad (175)$$

$$\frac{\partial \varphi_0}{\partial x} = \left[\frac{1}{L}\right] q_4 - \left[\frac{1}{L}\right] q_{10}. \qquad (176)$$

$$\left(\frac{\partial \boldsymbol{S}}{\partial y}\right)^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \left(\frac{\partial \boldsymbol{S}}{\partial z}\right)^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{177}$$

$$\frac{\partial u_0}{\partial y} = \frac{\partial v_0}{\partial y} = \frac{\partial w_0}{\partial y} = \frac{\partial \varphi_0}{\partial y} = \frac{\partial u_0}{\partial z} = \frac{\partial v_0}{\partial z} = \frac{\partial w_0}{\partial z} = \frac{\partial \varphi_0}{\partial z} = 0. \tag{178}$$

The second derivatives of the shape function matrix $\boldsymbol{S}$ are:

$$\left(\frac{\partial^2 \boldsymbol{S}}{\partial x^2}\right)^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{12x}{L^3} - \frac{6}{L^2} & 0 & 0 \\ 0 & 0 & \frac{12x}{L^3} - \frac{6}{L^2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -L\left(\frac{6x}{L^3} - \frac{4}{L^2}\right) & 0 \\ 0 & L\left(\frac{6x}{L^3} - \frac{4}{L^2}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{6}{L^2} - \frac{12x}{L^3} & 0 & 0 \\ 0 & 0 & \frac{6}{L^2} - \frac{12x}{L^3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -L\left(\frac{6x}{L^3} - \frac{2}{L^2}\right) & 0 \\ 0 & L\left(\frac{6x}{L^3} - \frac{2}{L^2}\right) & 0 & 0 \end{bmatrix}, \tag{179}$$

$$\frac{\partial^2 u_0}{\partial x^2} = 0, \tag{180}$$

$$\frac{\partial^2 v_0}{\partial x^2} = \left[\frac{12x}{L^3} - \frac{6}{L^2}\right] q_2 + \left[L\left(\frac{6x}{L^3} - \frac{4}{L^2}\right)\right] q_6 + \left[\frac{6}{L^2} - \frac{12x}{L^3}\right] q_8 + \left[L\left(\frac{6x}{L^3} - \frac{2}{L^2}\right)\right] q_{12}, \tag{181}$$

$$\frac{\partial^2 w_0}{\partial x^2} = \left[\frac{12x}{L^3} - \frac{6}{L^2}\right] q_3 - \left[L\left(\frac{6x}{L^3} - \frac{4}{L^2}\right)\right] q_5 + \left[\frac{6}{L^2} - \frac{12x}{L^3}\right] q_9 - \left[L\left(\frac{6x}{L^3} - \frac{2}{L^2}\right)\right] q_{11}, \tag{182}$$

$$\frac{\partial^2 \varphi_0}{\partial x^2} = 0. \tag{183}$$

$$\left(\frac{\partial^2 \boldsymbol{S}}{\partial x \partial y}\right)^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \left(\frac{\partial^2 \boldsymbol{S}}{\partial x \partial z}\right)^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \left(\frac{\partial^2 \boldsymbol{S}}{\partial y^2}\right)^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad (184)$$

$$\frac{\partial^2 u_0}{\partial x \partial y} = \frac{\partial^2 v_0}{\partial x \partial y} = \frac{\partial^2 w_0}{\partial x \partial y} = \frac{\partial^2 \varphi_0}{\partial x \partial y} = \frac{\partial^2 u_0}{\partial x \partial z} = \frac{\partial^2 v_0}{\partial x \partial z} = \frac{\partial^2 w_0}{\partial x \partial z} = \frac{\partial^2 \varphi_0}{\partial x \partial z} = \frac{\partial^2 u_0}{\partial y^2} = \frac{\partial^2 v_0}{\partial y^2} = \frac{\partial^2 w_0}{\partial y^2} = \frac{\partial^2 \varphi_0}{\partial y^2} = 0. \qquad (185)$$

$$\left(\frac{\partial^2 \boldsymbol{S}}{\partial y \partial z}\right)^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \left(\frac{\partial^2 \mathbf{S}}{\partial z^2}\right)^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad (186)$$

$$\frac{\partial^2 u_0}{\partial y \partial z} = \frac{\partial^2 v_0}{\partial y \partial z} = \frac{\partial^2 w_0}{\partial y \partial z} = \frac{\partial^2 \varphi_0}{\partial y \partial z} = \frac{\partial^2 u_0}{\partial z^2} = \frac{\partial^2 v_0}{\partial z^2} = \frac{\partial^2 w_0}{\partial z^2} = \frac{\partial^2 \varphi_0}{\partial z^2} = 0. \qquad (187)$$

Removing all zero-terms in the derivatives of u, v and w yields:

$$\frac{\partial u}{\partial x} = \frac{\partial u_0}{\partial x} - y\frac{\partial^2 v_0}{\partial x^2} - z\frac{\partial^2 w_0}{\partial x^2},$$
$$\frac{\partial v}{\partial x} = \frac{\partial v_0}{\partial x} - z\frac{\partial \varphi_0}{\partial x}, \quad (188)$$
$$\frac{\partial w}{\partial x} = \frac{\partial w_0}{\partial x} + y\frac{\partial \varphi_0}{\partial x}.$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v_0}{\partial x},$$
$$\frac{\partial v}{\partial y} = 0, \quad (189)$$
$$\frac{\partial w}{\partial y} = \varphi_0.$$

$$\frac{\partial u}{\partial z} = -\frac{\partial w_0}{\partial x},$$
$$\frac{\partial v}{\partial z} = -\varphi_0, \quad (190)$$
$$\frac{\partial w}{\partial z} = 0.$$

Substitution in the energy expression yields:

$$\varepsilon_{xx} = \frac{\partial u_0}{\partial x} - y\frac{\partial^2 v_0}{\partial x^2} - z\frac{\partial^2 w_0}{\partial x^2} + \frac{1}{2}\left[\left(\frac{\partial u_0}{\partial x} - y\frac{\partial^2 v_0}{\partial x^2} - z\frac{\partial^2 w_0}{\partial x^2}\right)^2 + \left(\frac{\partial v_0}{\partial x} - z\frac{\partial \varphi_0}{\partial x}\right)^2 + \left(\frac{\partial w_0}{\partial x} + y\frac{\partial \varphi_0}{\partial x}\right)^2\right],$$
$$\varepsilon_{xy} = \frac{1}{2}\left[-\frac{\partial v_0}{\partial x} + \frac{\partial v_0}{\partial x} - z\frac{\partial \varphi_0}{\partial x} + \left(\frac{\partial u_0}{\partial x} - y\frac{\partial^2 v_0}{\partial x^2} - z\frac{\partial^2 w_0}{\partial x^2}\right)\left(-\frac{\partial v_0}{\partial x}\right) + \left(\frac{\partial w_0}{\partial x} + y\frac{\partial \varphi_0}{\partial x}\right)\varphi_0\right], \quad (191)$$
$$\varepsilon_{xz} = \frac{1}{2}\left[\frac{\partial w_0}{\partial x} + y\frac{\partial \varphi_0}{\partial x} - \frac{\partial w_0}{\partial x} + \left(-\frac{\partial w_0}{\partial x}\right)\left(\frac{\partial u_0}{\partial x} - y\frac{\partial^2 v_0}{\partial x^2} - z\frac{\partial^2 w_0}{\partial x^2}\right) - \varphi_0\left(\frac{\partial v_0}{\partial x} - z\frac{\partial \varphi_0}{\partial x}\right)\right].$$

These expressions still have to be quadrated. When substituting each term in the expression for the strain energy, terms including y and y^3 will result in a zero-term, since:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} y\, dy = [y^2]_{-\frac{h}{2}}^{\frac{h}{2}} = \left[\frac{h^2}{4} - \frac{h^2}{4}\right] = 0, \quad (192)$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} y^3\, dy = \left[\frac{1}{4}y^4\right]_{-\frac{h}{2}}^{\frac{h}{2}} = \left[\left(\frac{1}{4}\cdot\frac{h^4}{16}\right) - \left(\frac{1}{4}\cdot\frac{h^4}{16}\right)\right] = 0. \quad (193)$$

The same holds for terms including z and z^3. All these terms are therefore already left out. Furthermore, fourth order terms can be left out too. Fourth order terms in the strain expression result in fourth order terms in U. In the required stiffness matrices, only the linear terms of the first and second derivatives of U will be used. In the first derivative of U, the originally fourth order terms will become third order, and in the second derivative, they will become second order. Since only linear terms are used, these terms will never be used and are thus left out as well.
This leaves us with the following expressions:

$$
\begin{aligned}
\varepsilon_{xx}^2 = &\left(\frac{\partial u_0}{\partial x}\right)^2 + y^2\left(\frac{\partial^2 v_0}{\partial x^2}\right)^2 + z^2\left(\frac{\partial^2 w_0}{\partial x^2}\right) + \left(\frac{\partial u_0}{\partial x}\right)^3 + \left(\frac{\partial u_0}{\partial x}\right)\left(\frac{\partial v_0}{\partial x}\right)^2 + \left(\frac{\partial u_0}{\partial x}\right)\left(\frac{\partial w_0}{\partial x}\right)^2 \\
&+ y^2\left(\frac{\partial u_0}{\partial x}\right)\left(\frac{\partial \varphi_0}{\partial x}\right)^2 + z^2\left(\frac{\partial u_0}{\partial x}\right)\left(\frac{\partial \varphi_0}{\partial x}\right)^2 + 3y^2\left(\frac{\partial u_0}{\partial x}\right)\left(\frac{\partial^2 v_0}{\partial x^2}\right)^2 \\
&- 2y^2\left(\frac{\partial w_0}{\partial x}\right)\left(\frac{\partial \varphi_0}{\partial x}\right)\left(\frac{\partial^2 v_0}{\partial x^2}\right) + 3z^2\left(\frac{\partial u_0}{\partial x}\right)\left(\frac{\partial^2 w_0}{\partial x^2}\right)^2 \\
&+ 2z^2\left(\frac{\partial v_0}{\partial x}\right)\left(\frac{\partial \varphi_0}{\partial x}\right)\left(\frac{\partial^2 w_0}{\partial x^2}\right), \qquad (194) \\
\varepsilon_{xy}^2 = &\frac{1}{4}z^2\left(\frac{\partial \varphi_0}{\partial x}\right)^2 - \frac{1}{2}z^2\left(\frac{\partial v_0}{\partial x}\right)\left(\frac{\partial \varphi_0}{\partial x}\right)\left(\frac{\partial^2 w_0}{\partial x^2}\right), \\
\varepsilon_{xz}^2 = &\frac{1}{4}y^2\left(\frac{\partial \varphi_0}{\partial x}\right)^2 + \frac{1}{2}y^2\left(\frac{\partial w_0}{\partial x}\right)\left(\frac{\partial \varphi_0}{\partial x}\right)\left(\frac{\partial v_0}{\partial x^2}\right).
\end{aligned}
$$

The expressions of the derivatives of u_0, v_0, w_0 and φ_0 can now be substituted, so the strain expression is given in terms of $\mathbf{q}$.

B2.3 Strain energy

Now that we found an expression for the strain, we can find an expression for the strain energy. The strain energy is defined as:

$$U = \frac{1}{2}\int_V \left[E\varepsilon_{xx}^2 + 4G\left(\varepsilon_{xy}^2 + \varepsilon_{xz}^2\right)\right] dV = \frac{1}{2}\int_V E\varepsilon_{xx}^2\, dV + \frac{1}{2}\int_V 4G\varepsilon_{xy}^2\, dV + \frac{1}{2}\int_V 4G\varepsilon_{xz}^2\, dV. \qquad (195)$$

For clarity, this expression is split up in three parts: $U_{\varepsilon_{xx}}$, $U_{\varepsilon_{xy}}$ and $U_{\varepsilon_{xz}}$.

$$\begin{aligned}
U_{\varepsilon_{xx}} &= \frac{1}{2}\int_V E\varepsilon_{xx}^2\, dV \\
&= \frac{1}{2}\int_V E\Bigg[\left(\frac{\partial u_0}{\partial x}\right)^2 + y^2\left(\frac{\partial^2 v_0}{\partial x^2}\right)^2 + z^2\left(\frac{\partial^2 w_0}{\partial x^2}\right) + \left(\frac{\partial u_0}{\partial x}\right)^3 + \left(\frac{\partial u_0}{\partial x}\right)\left(\frac{\partial v_0}{\partial x}\right)^2 \\
&\qquad + \left(\frac{\partial u_0}{\partial x}\right)\left(\frac{\partial w_0}{\partial x}\right)^2 + y^2\left(\frac{\partial u_0}{\partial x}\right)\left(\frac{\partial \varphi_0}{\partial x}\right)^2 + z^2\left(\frac{\partial u_0}{\partial x}\right)\left(\frac{\partial \varphi_0}{\partial x}\right)^2 \\
&\qquad + 3y^2\left(\frac{\partial u_0}{\partial x}\right)\left(\frac{\partial^2 v_0}{\partial x^2}\right)^2 - 2y^2\left(\frac{\partial w_0}{\partial x}\right)\left(\frac{\partial \varphi_0}{\partial x}\right)\left(\frac{\partial^2 v_0}{\partial x^2}\right) + 3z^2\left(\frac{\partial u_0}{\partial x}\right)\left(\frac{\partial^2 w_0}{\partial x^2}\right)^2 \\
&\qquad + 2z^2\left(\frac{\partial v_0}{\partial x}\right)\left(\frac{\partial \varphi_0}{\partial x}\right)\left(\frac{\partial^2 w_0}{\partial x^2}\right)\Bigg] dV \\
&= \frac{1}{2}\int_V E\left[\left(\frac{\partial u_0}{\partial x}\right)^2\right] dV + \frac{1}{2}\int_V E\left[y^2\left(\frac{\partial^2 v_0}{\partial x^2}\right)^2\right] dV + \frac{1}{2}\int_V E\left[z^2\left(\frac{\partial^2 w_0}{\partial x^2}\right)\right] dV \\
&\qquad + \frac{1}{2}\int_V E\left[\left(\frac{\partial u_0}{\partial x}\right)^3\right] dV + \frac{1}{2}\int_V E\left[\left(\frac{\partial u_0}{\partial x}\right)\left(\frac{\partial v_0}{\partial x}\right)^2\right] dV \\
&\qquad + \frac{1}{2}\int_V E\left[\left(\frac{\partial u_0}{\partial x}\right)\left(\frac{\partial w_0}{\partial x}\right)^2\right] dV + \frac{1}{2}\int_V E\left[y^2\left(\frac{\partial u_0}{\partial x}\right)\left(\frac{\partial \varphi_0}{\partial x}\right)^2\right] dV \\
&\qquad + \frac{1}{2}\int_V E\left[z^2\left(\frac{\partial u_0}{\partial x}\right)\left(\frac{\partial \varphi_0}{\partial x}\right)^2\right] dV + \frac{1}{2}\int_V E\left[3y^2\left(\frac{\partial u_0}{\partial x}\right)\left(\frac{\partial^2 v_0}{\partial x^2}\right)^2\right] dV \\
&\qquad + \frac{1}{2}\int_V E\left[-2y^2\left(\frac{\partial w_0}{\partial x}\right)\left(\frac{\partial \varphi_0}{\partial x}\right)\left(\frac{\partial^2 v_0}{\partial x^2}\right)\right] dV \\
&\qquad + \frac{1}{2}\int_V E\left[3z^2\left(\frac{\partial u_0}{\partial x}\right)\left(\frac{\partial^2 w_0}{\partial x^2}\right)^2\right] dV + \frac{1}{2}\int_V E\left[2z^2\left(\frac{\partial v_0}{\partial x}\right)\left(\frac{\partial \varphi_0}{\partial x}\right)\left(\frac{\partial^2 w_0}{\partial x^2}\right)\right] dV.
\end{aligned} \qquad (196)$$

Each term is treated separately:

$$U_{\varepsilon_{xx}1} = \frac{1}{2}\int_V E\left[\left(\frac{\partial u_0}{\partial x}\right)^2\right] dV = \frac{EA}{2L}[q_1 - q_7]^2, \tag{197}$$

$$\begin{aligned} U_{\varepsilon_{xx}2} &= \frac{1}{2}\int_V E\left[y^2\left(\frac{\partial^2 v_0}{\partial x^2}\right)^2\right] dV \\ &= \frac{EI_{zz}}{6L}[q_6^2 + q_6 q_{12} + q_{12}^2] + \frac{EI_{zz}}{6L^2}[3q_2q_6 + 3q_2q_{12} - 3q_6q_8 - 3q_8q_{12}] \\ &\quad + \frac{EI_{zz}}{6L^3}[3q_2^2 - 6q_2q_8 + 3q_8^2], \end{aligned} \tag{198}$$

$$\begin{aligned} U_{\varepsilon_{xx}3} &= \frac{1}{2}\int_V E\left[z^2\left(\frac{\partial^2 w_0}{\partial x^2}\right)\right] dV \\ &= \frac{EI_{yy}}{6L}[q_5^2 + q_5 q_{11} + q_{11}^2] - \frac{EI_{yy}}{6L^2}[3q_3q_5 + 3q_3q_{11} - 3q_5q_9 - 3q_9q_{11}] \\ &\quad + \frac{EI_{yy}}{6L^3}[3q_3^2 - 6q_3q_9 + 3q_9^2], \end{aligned} \tag{199}$$

$$U_{\varepsilon_{xx}4} = \frac{1}{2}\int_V E\left[\left(\frac{\partial u_0}{\partial x}\right)^3\right] dV = -\frac{EA}{2L^2}[q_1 - q_7]^3, \tag{200}$$

$$\begin{aligned} U_{\varepsilon_{xx}5} &= \frac{1}{2}\int_V E\left[\left(\frac{\partial u_0}{\partial x}\right)\left(\frac{\partial v_0}{\partial x}\right)^2\right] dV \\ &= -\frac{EA}{30L^2}[q_1 - q_7][18q_2^2 - 36q_2q_8 + 18q_8^2] \\ &\quad - \frac{EA}{30L}[q_1 - q_7][3q_2q_6 + 3q_2q_{12} - 3q_6q_8 - 3q_8q_{12}] \\ &\quad - \frac{EA}{30}[q_1 - q_7][2q_6^2 - q_6q_{12} + 2q_{12}^2], \end{aligned} \tag{201}$$

$$\begin{aligned} U_{\varepsilon_{xx}6} &= \frac{1}{2}\int_V E\left[\left(\frac{\partial u_0}{\partial x}\right)\left(\frac{\partial w_0}{\partial x}\right)^2\right] dV \\ &= -\frac{EA}{30L^2}[q_1 - q_7][18q_3^2 - 36q_3q_9 + 18q_9^2] \\ &\quad + \frac{EA}{30L}[q_1 - q_7][3q_3q_5 + 3q_3q_{11} - 3q_5q_9 - 3q_9q_{11}] \\ &\quad - \frac{EA}{30}[q_1 - q_7][2q_5^2 - q_5q_{11} + 2q_{11}^2], \end{aligned} \tag{202}$$

$$U_{\varepsilon_{xx}7} = \frac{1}{2}\int_V E\left[y^2\left(\frac{\partial u_0}{\partial x}\right)\left(\frac{\partial \varphi_0}{\partial x}\right)^2\right] dV = -\frac{EI_{zz}}{2L^2}[q_1 - q_7][q_4 - q_{10}]^2, \tag{203}$$

$$U_{\varepsilon_{xx}8} = \frac{1}{2}\int_V E\left[z^2\left(\frac{\partial u_0}{\partial x}\right)\left(\frac{\partial \varphi_0}{\partial x}\right)^2\right] dV = -\frac{EI_{yy}}{2L^2}[q_1 - q_7][q_4 - q_{10}]^2, \tag{204}$$

$$U_{\varepsilon_{xx}9} = \frac{1}{2}\int_V E\left[3y^2\left(\frac{\partial u_0}{\partial x}\right)\left(\frac{\partial^2 v_0}{\partial x^2}\right)^2\right] dV = -\frac{6EI_{zz}}{L^4}[q_1 - q_7][q_6^2L^2 + q_6q_{12}L^2 + q_{12}^2L^2 + 3q_2q_6L + 3q_2q_{12}L - 3q_6q_8L - 3q_8q_{12}L + 3q_2^2 - 6q_2q_8 + 3q_8^2], \tag{205}$$

$$U_{\varepsilon_{xx}10} = \frac{1}{2}\int_V E\left[-2y^2\left(\frac{\partial w_0}{\partial x}\right)\left(\frac{\partial \varphi_0}{\partial x}\right)\left(\frac{\partial^2 v_0}{\partial x^2}\right)\right] dV = -\frac{EI_{zz}}{4L^2}[q_4 - q_{10}][2q_2q_5 + 2q_3q_6 - 2q_2q_{11} - 2q_5q_8 - 2q_3q_{12} - 2q_6q_9 + 2q_8q_{11} + 2q_9q_{12}] - \frac{EI_{zz}}{2L}[q_4 - q_{10}][q_5q_6 + q_5q_{12} - q_6q_{11} - q_{11}q_{12}], \tag{206}$$

$$U_{\varepsilon_{xx}11} = \frac{1}{2}\int_V E\left[3z^2\left(\frac{\partial u_0}{\partial x}\right)\left(\frac{\partial^2 w_0}{\partial x^2}\right)^2\right] dV = -\frac{6EI_{yy}}{L^4}[q_1 - q_7][q_5^2L^2 + q_5q_{11}L^2 + q_{11}^2L^2 - 3q_3q_5L - 3q_3q_{11}L + 3q_5q_9L + 3q_9q_{11}L + 3q_3^2 - 6q_3q_9 + 3q_9^2], \tag{207}$$

$$U_{\varepsilon_{xx}12} = \frac{1}{2}\int_V E\left[2z^2\left(\frac{\partial v_0}{\partial x}\right)\left(\frac{\partial \varphi_0}{\partial x}\right)\left(\frac{\partial^2 w_0}{\partial x^2}\right)\right] dV = -\frac{EI_{yy}}{4L^2}[q_4 - q_{10}][2q_2q_5 + 2q_3q_6 - 2q_2q_{11} - 2q_5q_8 - 2q_3q_{12} - 2q_6q_9 + 2q_8q_{11} + 2q_9q_{12}] + \frac{EI_{yy}}{2L}[q_4 - q_{10}][q_5q_6 - q_5q_{12} + q_6q_{11} - q_{11}q_{12}]. \tag{208}$$

All terms together yield:

$$
\begin{aligned}
U_{\varepsilon_{xx}} = {} & \frac{EA}{2L}[q_1 - q_7]^2 + \frac{EI_{zz}}{6L}[q_6^2 + q_6 q_{12} + q_{12}^2] + \frac{EI_{zz}}{6L^2}[3q_2 q_6 + 3q_2 q_{12} - 3q_6 q_8 - 3q_8 q_{12}] \\
& + \frac{EI_{zz}}{6L^3}[3q_2^2 - 6q_2 q_8 + 3q_8^2] + \frac{EI_{yy}}{6L}[q_5^2 + q_5 q_{11} + q_{11}^2] \\
& - \frac{EI_{yy}}{6L^2}[3q_3 q_5 + 3q_3 q_{11} - 3q_5 q_9 - 3q_9 q_{11}] + \frac{EI_{yy}}{6L^3}[3q_3^2 - 6q_3 q_9 + 3q_9^2] \\
& - \frac{EA}{2L^2}[q_1 - q_7]^3 - \frac{EA}{30L^2}[q_1 - q_7][18q_2^2 - 36q_2 q_8 + 18q_8^2] \\
& - \frac{EA}{30L}[q_1 - q_7][3q_2 q_6 + 3q_2 q_{12} - 3q_6 q_8 - 3q_8 q_{12}] \\
& - \frac{EA}{30}[q_1 - q_7][2q_6^2 - q_6 q_{12} + 2q_{12}^2] - \frac{EA}{30L^2}[q_1 - q_7][18q_3^2 - 36q_3 q_9 + 18q_9^2] \\
& + \frac{EA}{30L}[q_1 - q_7][3q_3 q_5 + 3q_3 q_{11} - 3q_5 q_9 - 3q_9 q_{11}] \\
& - \frac{EA}{30}[q_1 - q_7][2q_5^2 - q_5 q_{11} + 2q_{11}^2] - \frac{EI_{zz}}{2L^2}[q_1 - q_7][q_4 - q_{10}]^2 \\
& - \frac{EI_{yy}}{2L^2}[q_1 - q_7][q_4 - q_{10}]^2 \\
& - \frac{6EI_{zz}}{L^4}[q_1 - q_7][q_6^2 L^2 + q_6 q_{12} L^2 + q_{12}^2 L^2 + 3q_2 q_6 L + 3q_2 q_{12} L - 3q_6 q_8 L \\
& - 3q_8 q_{12} L + 3q_2^2 - 6q_2 q_8 + 3q_8^2] \\
& - \frac{EI_{zz}}{4L^2}[q_4 - q_{10}][2q_2 q_5 + 2q_3 q_6 - 2q_2 q_{11} - 2q_5 q_8 - 2q_3 q_{12} - 2q_6 q_9 \\
& + 2q_8 q_{11} + 2q_9 q_{12}] - \frac{EI_{zz}}{2L}[q_4 - q_{10}][q_5 q_6 + q_5 q_{12} - q_6 q_{11} - q_{11} q_{12}] \\
& - \frac{6EI_{yy}}{L^4}[q_1 - q_7][q_5^2 L^2 + q_5 q_{11} L^2 + q_{11}^2 L^2 - 3q_3 q_5 L - 3q_3 q_{11} L + 3q_5 q_9 L \\
& + 3q_9 q_{11} L + 3q_3^2 - 6q_3 q_9 + 3q_9^2] \\
& - \frac{EI_{yy}}{4L^2}[q_4 - q_{10}][2q_2 q_5 + 2q_3 q_6 - 2q_2 q_{11} - 2q_5 q_8 - 2q_3 q_{12} - 2q_6 q_9 \\
& + 2q_8 q_{11} + 2q_9 q_{12}] + \frac{EI_{yy}}{2L}[q_4 - q_{10}][q_5 q_6 - q_5 q_{12} + q_6 q_{11} - q_{11} q_{12}].
\end{aligned}
\tag{209}
$$

$$U_{\varepsilon_{xy}} = \frac{1}{2}\int_V 4G\varepsilon_{xy}^2\, dV = \frac{1}{2}\int_V 4G\left[\frac{1}{4}z^2\left(\frac{\partial\varphi_0}{\partial x}\right)^2 - \frac{1}{2}z^2\left(\frac{\partial v_0}{\partial x}\right)\left(\frac{\partial\varphi_0}{\partial x}\right)\left(\frac{\partial^2 w_0}{\partial x^2}\right)\right] dV$$
$$= \frac{1}{2}\int_V 4G\left[\frac{1}{4}z^2\left(\frac{\partial\varphi_0}{\partial x}\right)^2\right] dV + \frac{1}{2}\int_V 4G\left[-\frac{1}{2}z^2\left(\frac{\partial v_0}{\partial x}\right)\left(\frac{\partial\varphi_0}{\partial x}\right)\left(\frac{\partial^2 w_0}{\partial x^2}\right)\right] dV. \quad (210)$$

For each term separately:

$$U_{\varepsilon_{xy}1} = \frac{1}{2}\int_V 4G\left[\frac{1}{4}z^2\left(\frac{\partial\varphi_0}{\partial x}\right)^2\right] dV$$
$$= \frac{GI_{yy}}{2L}[q_4 - q_{10}]^2, \quad (211)$$

$$U_{\varepsilon_{xy}2} = \frac{1}{2}\int_V 4G\left[-\frac{1}{2}z^2\left(\frac{\partial v_0}{\partial x}\right)\left(\frac{\partial\varphi_0}{\partial x}\right)\left(\frac{\partial^2 w_0}{\partial x^2}\right)\right] dV$$
$$= \frac{GI_{yy}}{2L^2}[q_4 - q_{10}][2q_2q_5 + 2q_3q_6 - 2q_2q_{11} - 2q_5q_8 - 2q_3q_{12} - 2q_6q_9 + 2q_8q_{11}$$
$$+ 2q_9q_{12}] - \frac{GI_{yy}}{2L}[q_4 - q_{10}][q_5q_6 - q_5q_{12} + q_6q_{11} - q_{11}q_{12}]. \quad (212)$$

All terms together yield:

$$U_{\varepsilon_{xy}} = \frac{GI_{yy}}{2L}[q_4 - q_{10}]^2$$
$$+ \frac{GI_{yy}}{2L^2}[q_4 - q_{10}][2q_2q_5 + 2q_3q_6 - 2q_2q_{11} - 2q_5q_8 - 2q_3q_{12} - 2q_6q_9$$
$$+ 2q_8q_{11} + 2q_9q_{12}] - \frac{GI_{yy}}{2L}[q_4 - q_{10}][q_5q_6 - q_5q_{12} + q_6q_{11} - q_{11}q_{12}]. \quad (213)$$

$$U_{\varepsilon_{xz}} = \frac{1}{2}\int_V 4G\varepsilon_{xz}^2\,dV = \frac{1}{2}\int_V 4G\left[\frac{1}{4}y^2\left(\frac{\partial\varphi_0}{\partial x}\right)^2 + \frac{1}{2}y^2\left(\frac{\partial w_0}{\partial x}\right)\left(\frac{\partial\varphi_0}{\partial x}\right)\left(\frac{\partial v_0}{\partial x^2}\right)\right] dV$$
$$= \frac{1}{2}\int_V 4G\left[\frac{1}{4}y^2\left(\frac{\partial\varphi_0}{\partial x}\right)^2\right] dV + \frac{1}{2}\int_V 4G\left[\frac{1}{2}y^2\left(\frac{\partial w_0}{\partial x}\right)\left(\frac{\partial\varphi_0}{\partial x}\right)\left(\frac{\partial v_0}{\partial x^2}\right)\right] dV. \quad (214)$$

For each term separately:

$$U_{\varepsilon_{xz}1} = \frac{1}{2}\int_V 4G\left[\frac{1}{4}y^2\left(\frac{\partial\varphi_0}{\partial x}\right)^2\right] dV$$
$$= \frac{GI_{zz}}{2L}[q_4 - q_{10}]^2, \quad (215)$$

$$U_{\varepsilon_{xz}2} = \frac{1}{2}\int_V 4G\left[\frac{1}{2}y^2\left(\frac{\partial w_0}{\partial x}\right)\left(\frac{\partial\varphi_0}{\partial x}\right)\left(\frac{\partial v_0}{\partial x^2}\right)\right] dV$$
$$= \frac{GI_{zz}}{2L^2}[q_4 - q_{10}][2q_2q_5 + 2q_3q_6 - 2q_2q_{11} - 2q_5q_8 - 2q_3q_{12} - 2q_6q_9 + 2q_8q_{11}$$
$$+ 2q_9q_{12}] - \frac{GI_{zz}}{2L}[q_4 - q_{10}][q_5q_6 - q_5q_{12} + q_6q_{11} - q_{11}q_{12}]. \quad (216)$$

All terms together:

$$U_{\varepsilon_{xz}} = \frac{GI_{zz}}{2L}[q_4 - q_{10}]^2$$
$$+ \frac{GI_{zz}}{2L^2}[q_4 - q_{10}][2q_2q_5 + 2q_3q_6 - 2q_2q_{11} - 2q_5q_8 - 2q_3q_{12} - 2q_6q_9$$
$$+ 2q_8q_{11} + 2q_9q_{12}] - \frac{GI_{zz}}{2L}[q_4 - q_{10}][q_5q_6 - q_5q_{12} + q_6q_{11} - q_{11}q_{12}]. \quad (217)$$

The complete strain expression thus equals:

$$
\begin{aligned}
U = {} & \frac{EA}{2L}[q_1 - q_7]^2 + \frac{EI_{zz}}{6L}[q_6^2 + q_6q_{12} + q_{12}^2] + \frac{EI_{zz}}{6L^2}[3q_2q_6 + 3q_2q_{12} - 3q_6q_8 - 3q_8q_{12}] \\
& + \frac{EI_{zz}}{6L^3}[3q_2^2 - 6q_2q_8 + 3q_8^2] + \frac{EI_{yy}}{6L}[q_5^2 + q_5q_{11} + q_{11}^2] - \frac{EI_{yy}}{6L^2}[3q_3q_5 + 3q_3q_{11} - 3q_5q_9 - 3q_9q_{11}] \\
& + \frac{EI_{yy}}{6L^3}[3q_3^2 - 6q_3q_9 + 3q_9^2] - \frac{EA}{2L^2}[q_1 - q_7]^3 - \frac{EA}{30L^2}[q_1 - q_7][18q_2^2 - 36q_2q_8 + 18q_8^2] \\
& - \frac{EA}{30L}[q_1 - q_7][3q_2q_6 + 3q_2q_{12} - 3q_6q_8 - 3q_8q_{12}] - \frac{EA}{30}[q_1 - q_7][2q_6^2 - q_6q_{12} + 2q_{12}^2] \\
& - \frac{EA}{30L^2}[q_1 - q_7][18q_3^2 - 36q_3q_9 + 18q_9^2] + \frac{EA}{30L}[q_1 - q_7][3q_3q_5 + 3q_3q_{11} - 3q_5q_9 - 3q_9q_{11}] \\
& - \frac{EA}{30}[q_1 - q_7][2q_5^2 - q_5q_{11} + 2q_{11}^2] - \frac{EI_{zz}}{2L^2}[q_1 - q_7][q_4 - q_{10}]^2 - \frac{EI_{yy}}{2L^2}[q_1 - q_7][q_4 - q_{10}]^2 \\
& - \frac{6EI_{zz}}{L^4}[q_1 - q_7][q_6^2L^2 + q_6q_{12}L^2 + q_{12}^2L^2 + 3q_2q_6L + 3q_2q_{12}L - 3q_6q_8L - 3q_8q_{12}L + 3q_2^2 - 6q_2q_8 + 3q_8^2] \\
& - \frac{EI_{zz}}{4L^2}[q_4 - q_{10}][2q_2q_5 + 2q_3q_6 - 2q_2q_{11} - 2q_5q_8 - 2q_3q_{12} - 2q_6q_9 + 2q_8q_{11} + 2q_9q_{12}] \\
& - \frac{EI_{zz}}{2L}[q_4 - q_{10}][q_5q_6 + q_5q_{12} - q_6q_{11} - q_{11}q_{12}] \\
& - \frac{6EI_{yy}}{L^4}[q_1 - q_7][q_5^2L^2 + q_5q_{11}L^2 + q_{11}^2L^2 - 3q_3q_5L - 3q_3q_{11}L + 3q_5q_9L + 3q_9q_{11}L + 3q_3^2 - 6q_3q_9 + 3q_9^2] \\
& - \frac{EI_{yy}}{4L^2}[q_4 - q_{10}][2q_2q_5 + 2q_3q_6 - 2q_2q_{11} - 2q_5q_8 - 2q_3q_{12} - 2q_6q_9 + 2q_8q_{11} + 2q_9q_{12}] \\
& + \frac{EI_{yy}}{2L}[q_4 - q_{10}][q_5q_6 - q_5q_{12} + q_6q_{11} - q_{11}q_{12}] + \frac{GI_{yy}}{2L}[q_4 - q_{10}]^2 \\
& + \frac{GI_{yy}}{2L^2}[q_4 - q_{10}][2q_2q_5 + 2q_3q_6 - 2q_2q_{11} - 2q_5q_8 - 2q_3q_{12} - 2q_6q_9 + 2q_8q_{11} + 2q_9q_{12}] \\
& - \frac{GI_{yy}}{2L}[q_4 - q_{10}][q_5q_6 - q_5q_{12} + q_6q_{11} - q_{11}q_{12}] + \frac{GI_{zz}}{2L}[q_4 - q_{10}]^2 \\
& + \frac{GI_{zz}}{2L^2}[q_4 - q_{10}][2q_2q_5 + 2q_3q_6 - 2q_2q_{11} - 2q_5q_8 - 2q_3q_{12} - 2q_6q_9 + 2q_8q_{11} + 2q_9q_{12}] \\
& - \frac{GI_{zz}}{2L}[q_4 - q_{10}][q_5q_6 - q_5q_{12} + q_6q_{11} - q_{11}q_{12}].
\end{aligned} \tag{218}
$$

B3 STIFFNESS

B3.1 Linear stiffness matrix

Now that we found the strain energy, we can determine the stiffness. This is defined as:

$$\mathbf{K} = \frac{\partial U}{\partial \mathbf{q} \partial \mathbf{q}} \tag{219}$$

When extracting the linear terms from these derivatives, the linear stiffness matrix can be determined and yields:

$$\bar{\boldsymbol{K}} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{12EI_{zz}}{L^3} & 0 & 0 & 0 & \frac{6EI_{zz}}{L^2} & 0 & -\frac{12EI_{zz}}{L^3} & 0 & 0 & 0 & \frac{6EI_{zz}}{L^2} \\ 0 & 0 & \frac{12EI_{yy}}{L^3} & 0 & -\frac{6EI_{yy}}{L^2} & 0 & 0 & 0 & -\frac{12EI_{yy}}{L^3} & 0 & -\frac{6EI_{yy}}{L^2} & 0 \\ 0 & 0 & 0 & \frac{G(I_{yy}+I_{zz})}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{G(I_{yy}+I_{zz})}{L} & 0 & 0 \\ 0 & 0 & -\frac{6EI_{yy}}{L^2} & 0 & \frac{4EI_{yy}}{L} & 0 & 0 & 0 & \frac{6EI_{yy}}{L^2} & 0 & \frac{2EI_{yy}}{L} & 0 \\ 0 & \frac{6EI_{zz}}{L^2} & 0 & 0 & 0 & \frac{4EI_{zz}}{L} & 0 & -\frac{6EI_{zz}}{L^2} & 0 & 0 & 0 & \frac{2EI_{zz}}{L} \\ -\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 & \frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{12EI_{zz}}{L^3} & 0 & 0 & 0 & -\frac{6EI_{zz}}{L^2} & 0 & \frac{12EI_{zz}}{L^3} & 0 & 0 & 0 & -\frac{6EI_{zz}}{L^2} \\ 0 & 0 & -\frac{12EI_{yy}}{L^3} & 0 & \frac{6EI_{yy}}{L^2} & 0 & 0 & 0 & \frac{12EI_{yy}}{L^3} & 0 & \frac{6EI_{yy}}{L^2} & 0 \\ 0 & 0 & 0 & -\frac{G(I_{yy}+I_{zz})}{L} & 0 & 0 & 0 & 0 & 0 & \frac{G(I_{yy}+I_{zz})}{L} & 0 & 0 \\ 0 & 0 & -\frac{6EI_{y}}{L^2} & 0 & \frac{2EI_{yy}}{L} & 0 & 0 & 0 & \frac{6EI_{yy}}{L^2} & 0 & \frac{4EI_{yy}}{L} & 0 \\ 0 & \frac{6EI_{zz}}{L^2} & 0 & 0 & 0 & \frac{2EI_{zz}}{L} & 0 & -\frac{6EI_{zz}}{L^2} & 0 & 0 & 0 & \frac{4EI_{zz}}{L} \end{bmatrix} \tag{220}$$

B4 MASS MATRIX

In order to determine the mass matrix, the earlier defined shape functions are needed. The expression for the mass matrix equals:

$$\mathbf{M} = \int_V \rho\,[\mathbf{S}^{\mathrm{T}}\mathbf{S}]\;dV = \rho\int_V [\mathbf{S}^{\mathrm{T}}\mathbf{S}]\,dV$$
$$= \frac{\rho AL}{420}\begin{bmatrix} 140 & 0 & 0 & 0 & 0 & 0 & 70 & 0 & 0 & 0 & 0 & 0 \\ 0 & 156 & 0 & 0 & 0 & 22L & 0 & 54 & 0 & 0 & 0 & -13L \\ 0 & 0 & 156 & 0 & -22L & 0 & 0 & 0 & 54 & 0 & 13L & 0 \\ 0 & 0 & 0 & 140 & 0 & 0 & 0 & 0 & 0 & 70 & 0 & 0 \\ 0 & 0 & -22L & 0 & 4L^2 & 0 & 0 & 0 & -13L & 0 & -3L^2 & 0 \\ 0 & 22L & 0 & 0 & 0 & 4L^2 & 0 & 13L & 0 & 0 & 0 & -3L^2 \\ 70 & 0 & 0 & 0 & 0 & 0 & 140 & 0 & 0 & 0 & 0 & 0 \\ 0 & 54 & 0 & 0 & 0 & 13L & 0 & 156 & 0 & 0 & 0 & -22L \\ 0 & 0 & 54 & 0 & -13L & 0 & 0 & 0 & 156 & 0 & 22L & 0 \\ 0 & 0 & 0 & 70 & 0 & 0 & 0 & 0 & 0 & 140 & 0 & 0 \\ 0 & 0 & 13L & 0 & -3L^2 & 0 & 0 & 0 & 22L & 0 & 4L^2 & 0 \\ 0 & -13L & 0 & 0 & 0 & -3L^2 & 0 & -22L & 0 & 0 & 0 & 4L^2 \end{bmatrix} \qquad (221)$$

B5 EXAMPLE: CLAMPED-FREE BEAM

B5.1 Stiffness

By applying boundary conditions, the linear stiffness matrix and the derivatives of the stiffness yield:

$$\overline{\mathbf{K}} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{12EI_{zz}}{L^3} & 0 & 0 & 0 & -\frac{6EI_{zz}}{L^2} \\ 0 & 0 & \frac{12EI_{yy}}{L^3} & 0 & \frac{6EI_{yy}}{L^2} & 0 \\ 0 & 0 & 0 & \frac{G(I_{yy}+I_{zz})}{L} & 0 & 0 \\ 0 & 0 & \frac{6EI_{yy}}{L^2} & 0 & \frac{4EI_{yy}}{L} & 0 \\ 0 & -\frac{6EI_{zz}}{L^2} & 0 & 0 & 0 & \frac{4EI_{zz}}{L} \end{bmatrix}, \qquad (222)$$

$$\frac{\partial\mathbf{K}}{\partial q_7} = \begin{bmatrix} \frac{3EA}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{36EI_{zz}}{L^4}+\frac{6EA}{5L^2} & 0 & 0 & 0 & -\frac{36EI_{zz}}{2L^3}-\frac{EA}{10L} \\ 0 & 0 & \frac{36EI_{yy}}{L^4}+\frac{6EA}{5L^2} & 0 & \frac{36EI_{yy}}{2L^3}+\frac{EA}{10L} & 0 \\ 0 & 0 & 0 & \frac{EI_{zz}}{L^2} & 0 & 0 \\ 0 & 0 & \frac{36EI_{yy}}{2L^3}+\frac{EA}{10L} & 0 & \frac{2EA}{15}+\frac{12EI_{yy}}{L^2} & 0 \\ 0 & -\frac{36EI_{zz}}{2L^3}-\frac{EA}{10L} & 0 & 0 & 0 & \frac{2EA}{15}+\frac{12EI_{zz}}{L^2} \end{bmatrix}, \qquad (223)$$

$$\frac{\partial \mathbf{K}}{\partial q_8} = \begin{bmatrix} 0 & \frac{36EI_{zz}}{L^4} + \frac{6EA}{5L^2} & 0 & 0 & 0 & -\frac{36EI_{zz}}{2L^3} - \frac{EA}{10L} \\ \frac{36EI_{zz}}{L^4} + \frac{6EA}{5L^2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{E\left(I_{zz}+I_{yy}\right) - G(I_{zz}+I_{yy})}{L^2} & 0 \\ 0 & 0 & 0 & \frac{E\left(I_{zz}+I_{yy}\right) - G(I_{zz}+I_{yy})}{L^2} & 0 & 0 \\ -\frac{36EI_{zz}}{2L^3} - \frac{EA}{10L} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad (224)$$

$$\frac{\partial \mathbf{K}}{\partial q_9} = \begin{bmatrix} 0 & 0 & \frac{36EI_{yy}}{L^4} + \frac{6EA}{5L^2} & 0 & \frac{36EI_{yy}}{2L^3} + \frac{EA}{10L} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{36EI_{yy}}{L^4} + \frac{6EA}{5L^2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{E\left(I_{zz}+I_{yy}\right) - G(I_{zz}+I_{yy})}{L^2} \\ \frac{36EI_{yy}}{2L^3} + \frac{EA}{10L} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{E\left(I_{zz}+I_{yy}\right) - G(I_{zz}+I_{yy})}{L^2} & 0 & 0 \end{bmatrix}, \qquad (225)$$

$$\frac{\partial \mathbf{K}}{\partial q_{10}} = \begin{bmatrix} 0 & 0 & 0 & \frac{E(I_{zz}+I_{yy})}{L^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{E\left(I_{zz}+I_{yy}\right) - G(I_{zz}+I_{yy})}{L^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{E\left(I_{zz}+I_{yy}\right) - G(I_{zz}+I_{yy})}{L^2} \\ \frac{E(I_{zz}+I_{yy})}{L^2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{E\left(I_{zz}+I_{yy}\right) - G(I_{zz}+I_{yy})}{L^2} & 0 & 0 & 0 & \frac{E\left(I_{yy}-I_{zz}\right) + G(I_{zz}-I_{yy})}{2L} \\ 0 & 0 & \frac{E\left(I_{zz}+I_{yy}\right) - G(I_{zz}+I_{yy})}{L^2} & 0 & \frac{E\left(I_{yy}-I_{zz}\right) + G(I_{zz}-I_{yy})}{2L} & 0 \end{bmatrix}, \qquad (226)$$

$$\frac{\partial \mathbf{K}}{\partial q_{11}} = \begin{bmatrix} 0 & 0 & \frac{36EI_{yy}}{2L^3} + \frac{EA}{10L} & 0 & \frac{2EA}{15} + \frac{12EI_{yy}}{L^2} & 0 \\ 0 & 0 & 0 & \frac{E\left(I_{zz}+I_{yy}\right) - G(I_{zz}+I_{yy})}{L^2} & 0 & 0 \\ \frac{36EI_{yy}}{2L^3} + \frac{EA}{10L} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{E\left(I_{zz}+I_{yy}\right) - G(I_{zz}+I_{yy})}{L^2} & 0 & 0 & 0 & \frac{E\left(I_{yy}-I_{zz}\right) + G(I_{zz}-I_{yy})}{2L} \\ \frac{2EA}{15} + \frac{12EI_{yy}}{L^2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{E\left(I_{yy}-I_{zz}\right) + G(I_{zz}-I_{yy})}{2L} & 0 & 0 \end{bmatrix}, \qquad (227)$$

$$\frac{\partial \mathbf{K}}{\partial q_{12}} = \begin{bmatrix} 0 & -\frac{36EI_{zz}}{2L^3} - \frac{EA}{10L} & 0 & 0 & 0 & \frac{2EA}{15} + \frac{12EI_{zz}}{L^2} \\ -\frac{36EI_{zz}}{2L^3} - \frac{EA}{10L} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{E\left(I_{zz}+I_{yy}\right) - G(I_{zz}+I_{yy})}{L^2} & 0 & 0 \\ 0 & 0 & \frac{E\left(I_{zz}+I_{yy}\right) - G(I_{zz}+I_{yy})}{L^2} & 0 & \frac{E\left(I_{zz}+I_{yy}\right) - G(I_{zz}+I_{yy})}{L^2} & 0 \\ 0 & 0 & 0 & \frac{E\left(I_{zz}+I_{yy}\right) - G(I_{zz}+I_{yy})}{L^2} & 0 & 0 \\ \frac{2EA}{15} + \frac{12EI_{zz}}{L^2} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \qquad (228)$$

B5.2 Natural frequencies and mode shapes

The first six natural frequencies are:

$$
\begin{aligned}
\omega_1 &= \sqrt{\frac{Eb^2(8\sqrt{39}-51)}{\rho L^4}}, \\
\omega_2 &= \sqrt{\frac{Eb^2(8\sqrt{39}+51)}{\rho L^4}}, \\
\omega_3 &= \sqrt{\frac{Eh^2\left(8\sqrt{39}-51\right)}{\rho L^4}}, \\
\omega_4 &= \sqrt{\frac{Eh^2\left(8\sqrt{39}+51\right)}{\rho L^4}}, \\
\omega_5 &= \frac{\sqrt{3E}}{\rho L^2}, \\
\omega_6 &= \sqrt{\frac{G(b^2+h^2)}{4\rho L^2}}.
\end{aligned} \tag{229}
$$

Substituting these natural frequencies in the eigenvalue problem yields mode shapes:

$$
\boldsymbol{\phi}_1 = \begin{Bmatrix} 0 \\ 0 \\ -\frac{L\left(\sqrt{39}+9\right)}{21}\alpha_1 \\ 0 \\ \alpha_1 \\ 0 \end{Bmatrix}, \quad
\boldsymbol{\phi}_2 = \begin{Bmatrix} 0 \\ 0 \\ \frac{L\left(\sqrt{39}-9\right)}{21}\alpha_2 \\ 0 \\ \alpha_2 \\ 0 \end{Bmatrix}, \quad
\boldsymbol{\phi}_3 = \begin{Bmatrix} 0 \\ \frac{L\left(\sqrt{39}+9\right)}{21}\alpha_3 \\ 0 \\ 0 \\ 0 \\ \alpha_3 \end{Bmatrix}.
$$
$$
\boldsymbol{\phi}_4 = \begin{Bmatrix} 0 \\ -\frac{L\left(\sqrt{39}-9\right)}{21}\alpha_4 \\ 0 \\ 0 \\ 0 \\ \alpha_4 \end{Bmatrix}, \quad
\boldsymbol{\phi}_5 = \begin{Bmatrix} \alpha_5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \quad
\boldsymbol{\phi}_6 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \alpha_6 \\ 0 \\ 0 \end{Bmatrix}. \tag{230}
$$

Therefore, the modal matrix equals:

$$
\boldsymbol{\Phi} = \begin{bmatrix}
0 & 0 & 0 & 0 & \alpha_5 & 0 \\
0 & 0 & \frac{L\left(\sqrt{39}+9\right)}{21}\alpha_3 & -\frac{L\left(\sqrt{39}-9\right)}{21}\alpha_4 & 0 & 0 \\
-\frac{L\left(\sqrt{39}+9\right)}{21}\alpha_1 & \frac{L\left(\sqrt{39}-9\right)}{21}\alpha_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \alpha_6 \\
\alpha_1 & \alpha_2 & 0 & 0 & 0 & 0 \\
0 & 0 & \alpha_3 & \alpha_4 & 0 & 0
\end{bmatrix}. \tag{231}
$$

B5.3 Modal derivatives

Since we need to know derivatives of **K** with respect to η_j, we apply coordinate transformation:

$$\mathbf{q} = \mathbf{\Phi}\boldsymbol{\eta}, \tag{232}$$

$$\mathbf{q} = \begin{bmatrix} 0 & 0 & 0 & 0 & \alpha_5 & 0 \\ 0 & 0 & \frac{L(\sqrt{39}+9)}{21}\alpha_3 & -\frac{L(\sqrt{39}-9)}{21}\alpha_4 & 0 & 0 \\ -\frac{L(\sqrt{39}+9)}{21}\alpha_1 & \frac{L(\sqrt{39}-9)}{21}\alpha_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_6 \\ \alpha_1 & \alpha_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_3 & \alpha_4 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \\ \eta_6 \end{Bmatrix}, \tag{233}$$

$$\begin{aligned} q_7 &= \alpha_5\eta_5, \\ q_8 &= \frac{L(\sqrt{39}+9)}{21}\alpha_3\eta_3 - \frac{L(\sqrt{39}-9)}{21}\alpha_4\eta_4, \\ q_9 &= -\frac{L(\sqrt{39}+9)}{21}\alpha_1\eta_1 + \frac{L(\sqrt{39}-9)}{21}\alpha_2\eta_2, \\ q_{10} &= \alpha_6\eta_6, \\ q_{11} &= \alpha_1\eta_1 + \alpha_2\eta_2, \\ q_{12} &= \alpha_3\eta_3 + \alpha_4\eta_4 \end{aligned} \tag{234}$$

Knowing these expressions, the derivatives of q_i with respect to η_j can be determined. Therefore all derivatives of **K** with respect to η_j can be determined, and knowing these the modal derivatives can be determined:

$$\boldsymbol{\theta}_{ij} = -\bar{\mathbf{K}}^{-1}\frac{\partial \mathbf{K}}{\partial \eta_j}\boldsymbol{\phi}_i, \tag{235}$$

$$\theta_{11} = -\bar{\mathbf{K}}^{-1}\frac{\partial \mathbf{K}}{\partial \eta_1}\boldsymbol{\phi}_1 = \begin{Bmatrix} -\frac{(275+29\sqrt{39})L^2 + (390-15\sqrt{39})b^2}{735L} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \tag{236}$$

$$\theta_{12} = -\bar{\mathbf{K}}^{-1}\frac{\partial \mathbf{K}}{\partial \eta_2}\boldsymbol{\phi}_1 = \begin{Bmatrix} -\frac{17L}{105} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \tag{237}$$

$$\theta_{13} = -\bar{\mathbf{K}}^{-1}\frac{\partial \mathbf{K}}{\partial \eta_3}\boldsymbol{\phi}_1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -\dfrac{(b^2-h^2)(E-G)}{2G(b^2+h^2)} \\ 0 \\ 0 \end{Bmatrix}, \quad (238)$$

$$\theta_{14} = -\bar{\mathbf{K}}^{-1}\frac{\partial \mathbf{K}}{\partial \eta_4}\boldsymbol{\phi}_1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{(E-G)\left(\left(4\sqrt{39}-21\right)b^2+\left(4\sqrt{39}+21\right)h^2\right)}{42G(b^2+h^2)} \\ 0 \\ 0 \end{Bmatrix}, \quad (239)$$

$$\theta_{15} = -\bar{\mathbf{K}}^{-1}\frac{\partial \mathbf{K}}{\partial \eta_5}\boldsymbol{\phi}_1 = \begin{Bmatrix} 0 \\ 0 \\ \dfrac{\left(5\sqrt{39}+45\right)b^2+\left(7\sqrt{39}+77\right)L^2}{35b^2} \\ 0 \\ -\dfrac{\left(2\sqrt{39}+25\right)L^2+21b^2}{7b^2L} \\ 0 \end{Bmatrix}, \quad (240)$$

$$\theta_{16} = -\bar{\mathbf{K}}^{-1}\frac{\partial \mathbf{K}}{\partial \eta_6}\boldsymbol{\phi}_1 = \begin{Bmatrix} 0 \\ \dfrac{L(E-G)\left(\left(2\sqrt{39}-31\right)b^2+\left(2\sqrt{39}+9\right)h^2\right)}{84Eh^2} \\ 0 \\ 0 \\ 0 \\ \dfrac{(E-G)\left(\left(\sqrt{39}-12\right)b^2+\left(\sqrt{39}+9\right)h^2\right)}{21Eh^2} \end{Bmatrix}, \quad (241)$$

$$\theta_{22} = -\bar{\mathbf{K}}^{-1}\frac{\partial \mathbf{K}}{\partial \eta_2}\boldsymbol{\phi}_2 = \begin{Bmatrix} -\dfrac{\left(15\sqrt{39}+390\right)b^2+\left(275-29\sqrt{39}\right)L^2}{735L} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \quad (242)$$

$$\theta_{23} = -\bar{\mathbf{K}}^{-1}\frac{\partial \mathbf{K}}{\partial \eta_3}\boldsymbol{\phi}_2 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -\dfrac{(E-G)\left(\left(4\sqrt{39}+21\right)b^2+\left(4\sqrt{39}-21\right)h^2\right)}{42G(b^2+h^2)} \\ 0 \\ 0 \end{Bmatrix}, \quad (243)$$

$$\theta_{24} = -\bar{\mathbf{K}}^{-1}\frac{\partial \mathbf{K}}{\partial \eta_4}\boldsymbol{\phi}_2 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -\dfrac{(b^2-h^2)(E-G)}{2G(b^2+h^2)} \\ 0 \\ 0 \end{Bmatrix}, \quad (244)$$

$$\theta_{25} = -\bar{\mathbf{K}}^{-1}\frac{\partial \mathbf{K}}{\partial \eta_5}\boldsymbol{\phi}_2 = \begin{Bmatrix} 0 \\ 0 \\ -\frac{(5\sqrt{39}-45)b^2+(7\sqrt{39}-77)L^2}{35b^2} \\ 0 \\ -\frac{21b^2+(25-2\sqrt{39})L^2}{7b^2L} \\ 0 \end{Bmatrix}, \qquad (245)$$

$$\theta_{26} = -\bar{\mathbf{K}}^{-1}\frac{\partial \mathbf{K}}{\partial \eta_6}\boldsymbol{\phi}_2 = \begin{Bmatrix} -\frac{L(E-G)\left((2\sqrt{39}+31)b^2+(2\sqrt{39}-11)h^2\right)}{84Eh^2} \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{(E-G)\left((\sqrt{39}+12)b^2+(\sqrt{39}-9)h^2\right)}{21Eh^2} \end{Bmatrix}, \qquad (246)$$

$$\theta_{33} = -\bar{\mathbf{K}}^{-1}\frac{\partial \mathbf{K}}{\partial \eta_3}\boldsymbol{\phi}_3 = \begin{Bmatrix} -\frac{(29\sqrt{39}+275)L^2+(390-15\sqrt{39})h^2}{735L} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \qquad (247)$$

$$\theta_{34} = -\bar{\mathbf{K}}^{-1}\frac{\partial \mathbf{K}}{\partial \eta_4}\boldsymbol{\phi}_3 = \begin{Bmatrix} -\frac{17L}{105} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \qquad (248)$$

$$\theta_{35} = -\bar{\mathbf{K}}^{-1}\frac{\partial \mathbf{K}}{\partial \eta_5}\boldsymbol{\phi}_3 = \begin{Bmatrix} 0 \\ -\frac{(5\sqrt{39}+45)h^2+(7\sqrt{39}+77)L^2}{35h^2} \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \qquad (249)$$

$$\theta_{36} = -\bar{\mathbf{K}}^{-1}\frac{\partial \mathbf{K}}{\partial \eta_6}\boldsymbol{\phi}_3 = \begin{Bmatrix} 0 \\ 0 \\ \frac{L(E-G)\left((2\sqrt{39}+11)b^2+(2\sqrt{39}-31)h^2\right)}{82b^2E} \\ 0 \\ -\frac{(E-G)\left((\sqrt{39}+9)b^2+(\sqrt{39}-12)h^2\right)}{21b^2E} \\ 0 \end{Bmatrix}, \qquad (250)$$

$$\theta_{44} = -\bar{\mathbf{K}}^{-1}\frac{\partial \mathbf{K}}{\partial \eta_4}\boldsymbol{\phi}_4 = \begin{Bmatrix} -\frac{(15\sqrt{39}+390)h^2+(275-29\sqrt{39})L^2}{735L} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \qquad (251)$$

$$\theta_{45} = -\bar{\mathbf{K}}^{-1}\frac{\partial \mathbf{K}}{\partial \eta_5}\boldsymbol{\phi}_4 = \begin{Bmatrix} 0 \\ \frac{\left(5\sqrt{39}-45\right)h^2+\left(7\sqrt{39}-77\right)L^2}{35h^2} \\ 0 \\ 0 \\ 0 \\ -\frac{21h^2+\left(25-2\sqrt{39}\right)L^2}{7h^2L} \end{Bmatrix}, \quad (252)$$

$$\theta_{46} = -\bar{\mathbf{K}}^{-1}\frac{\partial \mathbf{K}}{\partial \eta_6}\boldsymbol{\phi}_4 = \begin{Bmatrix} 0 \\ 0 \\ -\frac{L(E-G)\left(\left(2\sqrt{39}-11\right)b^2+\left(2\sqrt{39}+31\right)h^2\right)}{84b^2E} \\ 0 \\ \frac{(E-G)\left(\left(\sqrt{39}-9\right)b^2+\left(\sqrt{39}+12\right)h^2\right)}{21b^2E} \\ 0 \end{Bmatrix}, \quad (253)$$

$$\theta_{55} = -\bar{\mathbf{K}}^{-1}\frac{\partial \mathbf{K}}{\partial \eta_5}\boldsymbol{\phi}_5 = \begin{Bmatrix} -\frac{3}{L} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \quad (254)$$

$$\theta_{56} = -\bar{\mathbf{K}}^{-1}\frac{\partial \mathbf{K}}{\partial \eta_6}\boldsymbol{\phi}_5 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -\frac{E}{GL} \\ 0 \\ 0 \end{Bmatrix}, \quad (255)$$

$$\theta_{66} = -\bar{\mathbf{K}}^{-1}\frac{\partial \mathbf{K}}{\partial \eta_6}\boldsymbol{\phi}_6 = \begin{Bmatrix} -\frac{b^2+h^2}{12L} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \quad (256)$$