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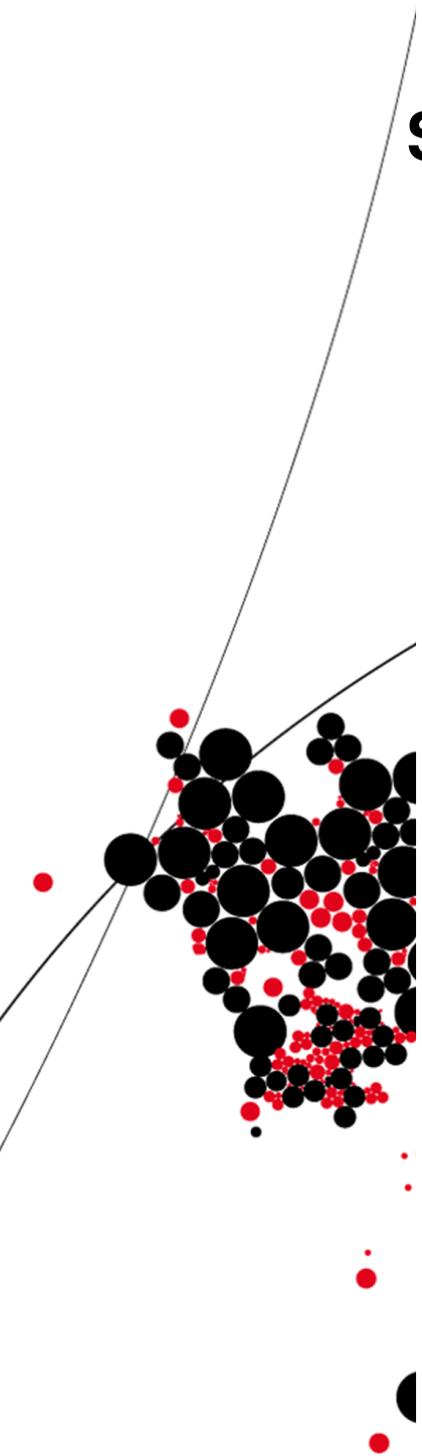
Structural dynamics of roller coaster structures: transient analysis

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Preface

During my study in mechanical engineering, I progressively developed more interest in dynamics and finite element applications. After I did an internship at Van Velzen Extern Engineering B.V. (now Intamin Holland B.V.), a company specialized in design and analysis of amusement rides, I was quite sure to do my master thesis about an amusement ride. Actually I am not that big of a fan of amusement rides, but the application of knowledge about subjects I really enjoy can wonderfully be used to design and especially analyse amusement rides.

Unfortunately due to personal circumstances I have taken a little too much time for my personal liking to perform this research, but I am very pleased with the final result. Hopefully this research does contribute in making it easier and more predictable to design a roller coaster in the future.

I would like to thank Jurnan Schilder for his supervision, without his help and knowledge this research was not possible. We always had very efficient meetings, helping me to get along. Furthermore I would like to thank my family, who stood by my side during tough times. At last I would like to thank Helmer van den Hoorn, who helped me by creating the multibody dynamic model used in the analysis.

Rob Mekers
Enschede, 3 January 2020

Contents

Structural dynamics of roller coaster structures: transient analysis

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ABSTRACT

In this paper a method for the analysis of vibrations in roller coaster structures is described. An initial design is required, from which the dimensions, material properties and the time histories of contact forces between vehicles and rails have to be retrieved as an input for the Finite Element model. The transient loads applied on the structure are obtained from a multibody dynamic model. The guiding principle of the Finite Element model is the equation of motion, which is solved iteratively to obtain the displacements of the structure for each load case. The proposed method is applied to an existing roller coaster to clarify the essential steps in the model and to test its capabilities. In the end, a second roller coaster is treated where undesired vibrations showed up, which could potentially have been identified before building it using the method proposed in this paper. The method is developed such that existing Finite Element models and multibody dynamic models of roller coasters can be reused to perform a transient analysis of the structure.

1. Introduction

Roller coasters are designed to be fun and safe at the same time. International norms dictate, therefore there are restrictions on the amount of G-forces on passengers and safety of the train, track and supporting structure. These restrictions directly influence the maximum allowable stresses, often by defining safety factors to multiply the calculated stresses with. These safety factors cover the uncertainties present in the stresses. Dynamic effects are considerably responsible for this, as these effects are not modelled precisely in industry practice. Hence if these dynamic effects can be modelled (more) accurately, safety factors can be reduced and roller coasters could be more fun without sacrificing structural integrity.

The current state of the art and international norms are tailored to the quasi-static analysis of the structure of roller coasters. Earlier papers like [1, 2] are assuming the structure to be "sufficiently stiff". The first paper [1] considers a procedure to properly and quickly size the rails, whereas the second paper [2] presents a tool for the design of supporting structure characteristics. With the help of these two papers roller coaster structures can be modelled properly on a global scale, however problems occur when dynamic effects are underestimated (Lost Gravity [3]) or when a structure is designed more lightweight (Karacho). In both roller coasters tuned mass dampers are installed after opening to minimize the (unexpected) structural vibrations. Therefore, in this paper a method is proposed to predict these vibrations before building the actual roller coasters, so the general design of roller coasters can be improved.

In this paper a Finite Element Method (FEM) (now also readily available) is proposed as a fundamental concept to solve the equation of motion. Performing a transient analysis gives the displacements of the structure while a train is running.

The representative loads on the track are obtained from a multibody dynamic (MBD) model; a complete roller coaster ride is simulated during one lap. The loads and positions of the wheels are retrieved for each time step, where each time step represents a slightly different position of the train on the track, so each time step can also be seen as a new load case. The obtained loads retrieved from the MBD-model are used as external loads in FEM, after which the stiffness and mass matrix can be determined.

The FEM-model is based on beam elements. The appropriate local properties are retrieved from CAD-models, from which the stiffness and mass matrix of each element can be obtained. These local matrices are rotated to the global coordinate system using rotation matrices, after which the stiffness and mass matrices are collected in two uncoupled global matrices. The constraint stiffness and mass matrix are obtained after coupling the degrees of freedom by means of a boolean matrix. The resulting equation of motion is solved iteratively using the Newmark-Beta integration method.

Throughout this paper the reference roller coaster is considered based on Zierer's Donderstenen, however it must be noted the method proposed is not limited to this track. A schematic overview of the track is shown in Fig. 1. The displacements obtained from the transient analysis are compared with the static solution for each load case to show the benefits of this method.

In the end, a second analysis is performed on a roller coaster where undesired transverse vibrations showed up. This model is inspired by MACK's Lost Gravity roller coaster and analysed as a test case for the proposed FEM-model to test its capabilities.

The proposed method in this paper is developed such that existing FEM- and MBD-models can be reused as an input to the developed transient analysis method. Therefore this method can be thought of as an additional analysis to already existing methods (e.g. static calculations), with the intention to make the method as efficient as possible to limit computational time.

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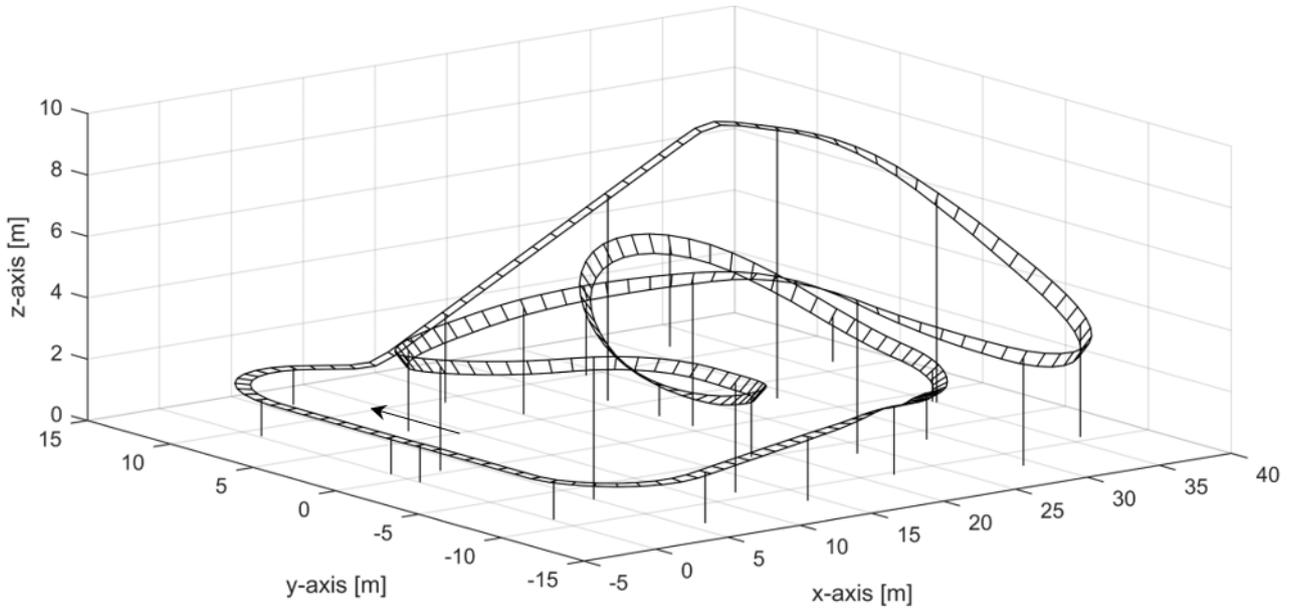
Fig. 1: Reference roller coaster track layout, the driving direction is indicated by the arrow.

2. Moving load application

The equation of motion to be solved is defined as:

$$\mathbf{M}_c\ddot{\mathbf{q}} + \mathbf{C}_c\dot{\mathbf{q}} + \mathbf{K}_c\mathbf{q} = \mathbf{f} \tag{1}$$

With $\mathbf{M}_c$ the constraint mass matrix, $\mathbf{C}_c$ the constraint damping matrix, $\mathbf{K}_c$ the constraint stiffness matrix, $\mathbf{q}$ the displacement vector and $\mathbf{f}$ the force vector.

Throughout this paper it is assumed that the damping matrix $\mathbf{C}_c$ is equal to zero, since it is not required to show the advantages of the proposed method and due to the fact that knowledge about the damping parameters of the considered roller coaster is lacking. The first step in solving the equation of motion is to define the loads applied on the track.

To obtain an as close to realistic model as possible, the loads applied on the track are retrieved from a MBD-model, which is created in Simscape [4]. In this model, a full circuit of the reference roller coaster is simulated. Parts of the train as well as all structural parts are modelled using CAD-software and loaded into the MBD-scheme.

A total of ten carts make up for the train, where each cart has four guide wheels and four running wheels, except the front cart, which has eight guide wheels and eight running wheels. Hence in total 88 wheels make contact with the track (which is rigid) at each time step resulting in 88 loads, consisting of components in orthogonal (y and z) direction as well as some tangent (x) components (friction force) according to Fig. 2. The wheels are connected to the track using point on curve constraints, from which the position (parametrized relative to the track) and loads can be retrieved for each time step.

The initial position of the train in the MBD-model is shown in Fig. 3. After positioning the train in the station, it is transported to the top of the lift with a prescribed velocity, whereafter the train completes a lap under the action of

Fig. 2: 12 degrees of freedom beam element used in the Finite Element Model.
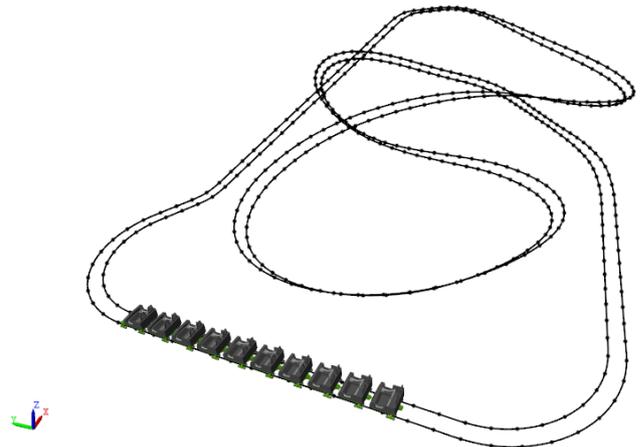
Fig. 3: Initial position of the train in the MBD-simulation, columns and cross ties are not shown.
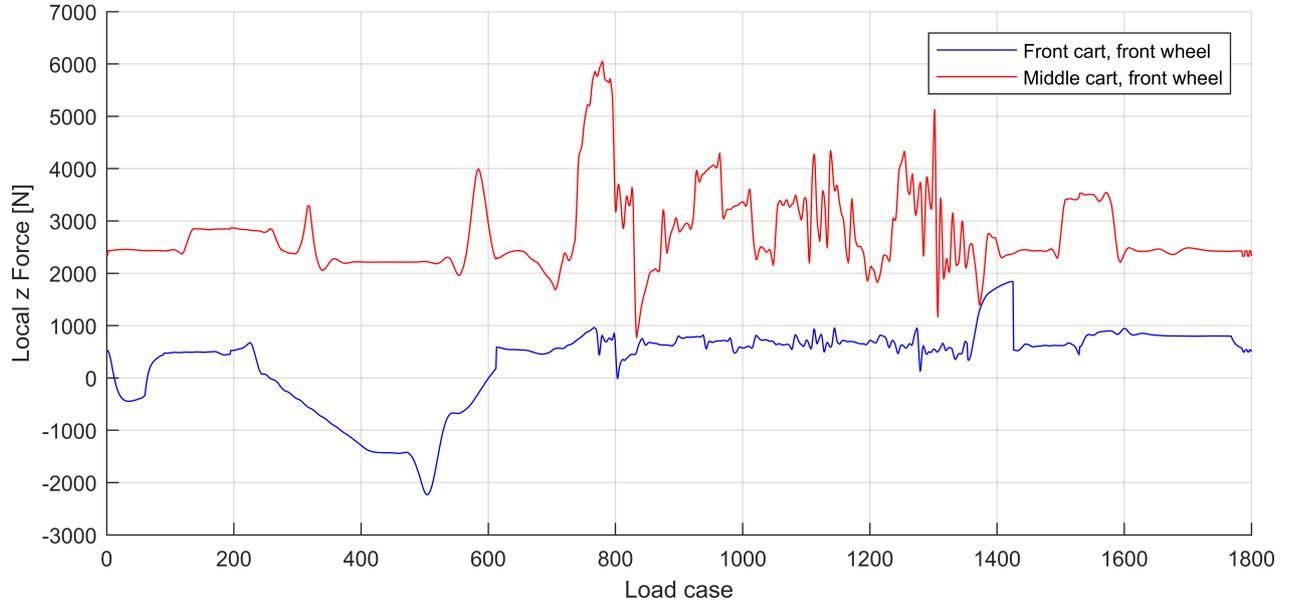
Fig. 4: Local reaction z-forces on the left track when following the train along the track of the reference roller coaster. Positive forces imply the train is pushing down on the track.

gravity until the brakes. A full circuit takes 45 seconds to complete; using a sample time of 0.0025 seconds, a total of $45/0.0025 = 18{,}000$ time steps are simulated. The sample time has to be sufficiently small in the MBD-simulation for stability reasons.

After the simulation has completed, the loads and positions of the wheels at each load case are exported to the FEM-model. To reduce computational time, the number of load cases solved in the MBD-simulation is reduced by a factor 10, so a total of 1,800 load cases remain. This directly affects the accuracy of the final solution, so it is important to choose the reduction factor wisely (or take this factor equal to one if the solution should be as accurate as possible). Hence a more accurate final solution can be obtained if the time step in the MBD-simulation is decreased, however this comes at the cost of increased computational time.

The loads acting on the track directly underneath two wheels of the train are considered as a function of time. The
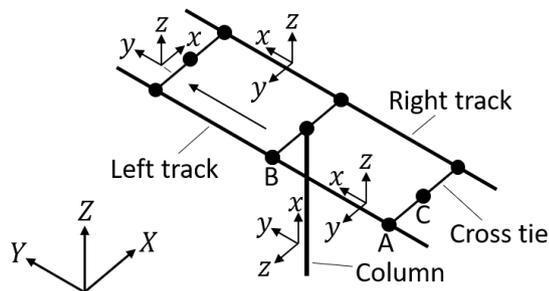
figure is shown in Fig. 4, where the front wheels of the front cart and the fifth cart (middle cart) are taken as the positions where the local z-loads according to Fig. 5 are plotted.

2.1. Equivalent nodal loads/moments

Before the loads can be used in the equation of motion, an additional check is performed to determine between which nodes (or on which node) the loads are applied. The loads are known on a [mm] level, but the elements of the FEM-model are created on a [m] level (section 3). This implies interpolation is required if loads are not applied directly on a node of an element (Appendix A). The tangent components of the loads (friction forces) are distributed linearly between the two nodes of an element, whereas the orthogonal components (normal forces) require some more attention; the equivalent nodal forces and moments are calculated with the help of beam deflection formulas. The situation is shown in Fig. 6, a cantilever beam is used to calculate equivalent nodal loads and moments. This method provides the statically and kinematically equivalent solution in the nodes, but only approximates the exact solution along the element.

To find the equivalent nodal forces and moments, standard formulas to calculate the deflection and slope of the beam are used [5]. The method is based on the fact that the deflection and slope of the beam caused by an arbitrary load should be equal to the deflection and slope caused by nodal forces and moments (kinematically equivalent).

The deflection and slope at the free end of the beam caused by an arbitrary load can be found using the following equations respectively:

Fig. 5: Structural elements of the reference roller coaster, the driving direction is indicated by the arrow. The local coordinate frames are shown for each element type.

$$v = -\frac{Fx^2}{6EI}(3L - x) \qquad (2)$$
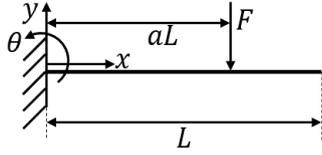
(a) Load retrieved from MBD-simulation applied between two nodes.
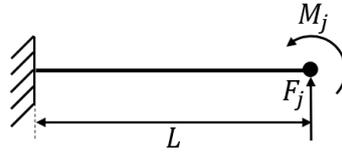
(b) Equivalent nodal force and moment for FEM-model.
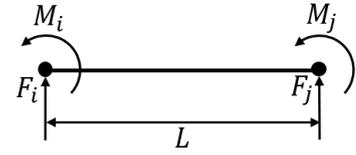
(c) Final FEM loads and moments, after replacing the clamping.

Fig. 6: If a load is applied between nodes, it has to be translated to equivalent nodal forces and moments.

$$\theta = -\frac{Fx^2}{2EI} \tag{3}$$

Where E is the Young's modulus, I the area moment of inertia and x the distance between the clamping and the load application point. Hence in this situation (Fig. 6a) $x = aL$ where $0 < a < 1$.

The deflection and slope caused by the applied load should be equal to the deflection and slope caused by the load and moment at node j (Fig. 6b):

$$v = -\frac{Fa^2L^2}{6EI}(3L - aL) = \frac{F_jL^3}{3EI} + \frac{M_jL^2}{2EI} \tag{4}$$

$$\theta = -\frac{Fa^2L^2}{2EI} = \frac{F_jL^2}{2EI} + \frac{M_jL}{EI} \tag{5}$$

From these two equations the equivalent force and moment at node j can be determined. Thereafter, static equilibrium is considered to determine the force and moment at node i, resulting in the following expressions:

$$F_j = \left(2a^3 - 3a^2\right)F \tag{6}$$

$$M_j = \left(a^2 - a^3\right)FL \tag{7}$$

$$F_i = \left(1 - 2a^3 + 3a^2\right)F \tag{8}$$

$$M_i = \left(a + 2a^2 - a^3\right)FL \tag{9}$$

After performing this procedure for all loads from the MBD-simulation, they are rotated using the corresponding element rotation matrices (10). All loads and moments are collected in a matrix with dimensions number of degrees of freedom by number of load cases (4,164×1,800) (section 3). For each load case the corresponding column of the matrix is selected to insert in the equation of motion (1).

3. The Finite Element Model

The reference roller coaster to be analysed is composed of several types of structural elements; track elements, cross ties and columns as shown in Fig. 5. Since the columns are connected at the center between the left and right track, all cross ties are split up in two beam elements to create nodes to connect the columns to. For programming reasons it is convenient to do this for all cross ties instead for those only where the columns are attached; the computational time does not increase disproportionately doing so; it takes approximately one minute to complete the calculations of the reference roller coaster including split up cross ties.

All of the aforementioned elements are modelled using beam elements with 12 degrees of freedom as shown in Fig. 2. The Craig-Bampton method [6] can be used to reduce the finite element model of e.g. a cross-tie to an equivalent beam. Although the degrees of freedom of all beams are equivalent, inertial properties differ for each element type and could be obtained from CAD-models that contain detailed information about geometry.

The complete model consists of 916 elements (222 left track elements, 222 right track elements, 444 cross ties and 28 columns) and 694 independent nodes (222 left track nodes, 222 right track nodes, 222 nodes between the left and right track and 28 columns). This leads to a total of $694 \times 6 = 4{,}164$ independent degrees of freedom.

After identifying each element type and assigning the correct inertial properties, the stiffness and mass matrix can be created. Both matrices are based on the convention in Fig. 2 when considering the local coordinate frame.

3.1. Stiffness matrix

The local element stiffness matrix used for all elements in the roller coaster model is stated in [7]. As the columns are connected to the fixed world, boundary conditions should be applied. There are two methods to accomplish this, namely by assuming a rigid connection (infinite stiffness) or estimating the foundation stiffnesses and including them in the stiffness matrices of the columns. Since the latter is a more accurate representation of reality, it is chosen to add additional stiffness at the bottom nodes of all columns as shown in Fig. 7.

The stiffness of each (massless) spring is added at the corresponding translational degree of freedom in the element stiffness matrix, with the assumption $k_x = k_y = k_z$.

The local matrices have to be rotated such that the local
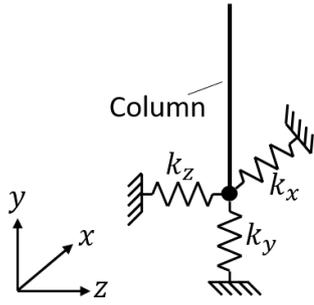
Fig. 7: Additional springs added to the lower node of a column to represent connectivity to the fixed world.

(a) Uncoupled situation.

(b) Coupled situation.

Fig. 8: Coupling of 2-D beam elements.

coordinate frame of each element is aligned with the global reference frame of the entire model. Each element therefore requires an unique rotation matrix, which is defined as:

$$
\mathbf{R} = \begin{bmatrix} \mathbf{R}_{3\times3} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{3\times3} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{R}_{3\times3} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{R}_{3\times3} \end{bmatrix} \tag{10}
$$

With $\mathbf{0}$ a 3×3 zeros matrix. $\mathbf{R}_{3\times3}$ is a direction cosine matrix, defined as follows:

$$
\mathbf{R}_{3\times3} = \begin{bmatrix} \cos(xX) & \cos(xY) & \cos(xZ) \\ \cos(yX) & \cos(yY) & \cos(yZ) \\ \cos(zX) & \cos(zY) & \cos(zZ) \end{bmatrix} \tag{11}
$$

The local coordinate system is indicated with the lower case letters whereas the global coordinate system is represented by the capital letters. To show the approach taken to define a rotation matrix, element AB (left track element) from Fig. 5 has been taken as an example (Appendix B).

It is assumed the local x-axis lays in the direction of the element (from A to B). Accordingly, the first three direction cosines (the first row in (11)) can be determined. As the cross-ties are all orthogonal to the left track elements they can be used to create a second axis, giving the second three direction cosines. This y-axis is defined positive from C to A, so the remaining z-axis is automatically pointing upwards (cross product) because a Cartesian coordinate system is used. Similar procedures can be performed to obtain the rotation matrices for the remaining elements in Fig. 5.

After rotating all element stiffness matrices to the global coordinate system, they can be collected in an assembled (uncoupled) global stiffness matrix $\mathbf{K}_{ass}$ ($10,992 \times 10,992$):

$$
\mathbf{K}_{ass} = \begin{bmatrix} \mathbf{K}_{el,1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{el,2} & & \\ \vdots & & \ddots & \\ \mathbf{0} & & & \mathbf{K}_{el,n} \end{bmatrix} \tag{12}
$$

With $\mathbf{K}_{el,1}$ until $\mathbf{K}_{el,n}$ the n global element stiffness matrices and $\mathbf{0}$ a 12x12 zeros matrix. The order of the types of elements from top left to bottom right is as follows: left track - right track - cross ties (left track to center) - cross ties

(center to right track) - columns. In general this order can be arbitrary as long as it is consistent throughout the entire model.

The last step to obtain the global stiffness matrix is coupling the degrees of freedom. As an example two 2-D beams will be coupled like shown in Fig. 8.

Each beam has six degrees of freedom in total, where the degrees of freedom of the first beam ($e1$) will be numbered one to six, and the degrees of freedom of the second beam ($e2$) will be numbered four to nine. Hence they share three degrees of freedom, as the beams will be coupled at one node. The coupling can be captured using a boolean matrix $\mathbf{B}$ containing zeros and ones only:

$$\mathbf{B} =$$

	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9
$S_{1,e1}$	1								
$S_{2,e1}$		1							
$S_{3,e1}$			1						
$S_{4,e1}$				1					
$S_{5,e1}$					1				
$S_{6,e1}$						1			
$S_{4,e2}$				1					
$S_{5,e2}$					1				
$S_{6,e2}$						1			
$S_{7,e2}$							1		
$S_{8,e2}$								1	
$S_{9,e2}$									1

$$\tag{13}$$

The zeros are not shown for readability purposes. The columns represent the independent (global) degrees of freedom and the rows represent the (local) degrees of freedom of both elements. This principle can be extended to 3-D situations and applied to the reference roller coaster to find the constraint stiffness matrix:

$$
\mathbf{K}_c = \mathbf{B}^T \mathbf{K}_{ass} \mathbf{B} \tag{14}
$$

(a) Load case 1

(b) Load case 600

(c) Load case 750

(d) Load case 1,200
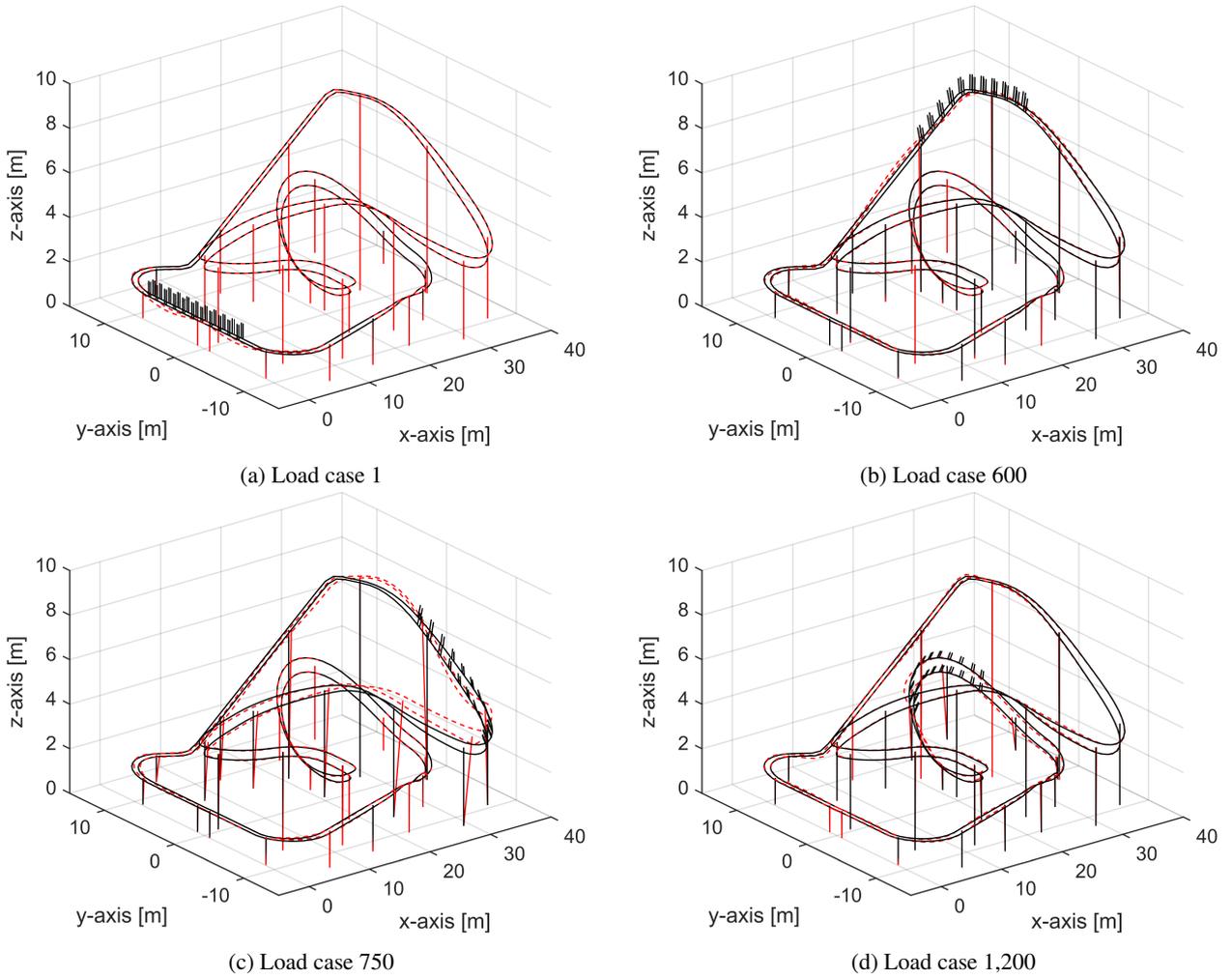
Fig. 9: Four load cases (out of 1,800) of the reference roller coaster. The original track is shown with the continuous black line whereas the deformed track is shown with the dashed red line (deformations are multiplied by a factor 800), cross ties are not shown for clarity. The positions of all (44) wheels are indicated by the black lines perpendicular to the track.

This constraint stiffness matrix is square with dimensions equal to the number of degrees of freedom of the entire roller coaster (4,164 × 4,164). It can be filled in the equation of motion (1).

3.2. Mass matrix

The local element mass matrix used for all elements in the roller coaster model is stated in [7]. The procedure to obtain the coupled global mass matrix is the same as for the stiffness matrix. Hence the same boolean matrix as before can be used, so the constraint mass matrix can be found by:

$$\mathbf{M}_c = \mathbf{B}^T \mathbf{M}_{ass} \mathbf{B} \qquad (15)$$

Where $\mathbf{M}_{ass}$ (the assembled (uncoupled) global mass matrix) is a sparse matrix composed of all element mass matrices, constructed in a similar fashion as $\mathbf{K}_{ass}$ (12).

The constraint mass matrix (4164 × 4164) can be filled in the equation of motion (1).

4. Solving the equation of motion

The equation of motion (1) is solved for the displacements $\mathbf{q}$ using an iterative Newmark-Beta integration scheme [8]. The scheme is based on two update rules:

$$\dot{\mathbf{q}}_{t+\Delta t} = \dot{\mathbf{q}}_t + \Delta t \left[(1 - \gamma)\ddot{\mathbf{q}}_t + \gamma \ddot{\mathbf{q}}_{t+\Delta t} \right] \qquad (16)$$

$$\mathbf{q}_{t+\Delta t} = \mathbf{q}_t + \Delta t \dot{\mathbf{q}}_t + \Delta t^2 \left[\left(\frac{1}{2} - \beta \right) \ddot{\mathbf{q}}_t + \beta \ddot{\mathbf{q}}_{t+\Delta t} \right] \qquad (17)$$

Where Δt is the time step, β the Newmark-Beta parameter ($0 \le \beta \le 1/4$) and γ the Newmark numerical damping parameter ($\gamma = 1/2$). The second equation can be rewritten into an expression for $\ddot{\mathbf{q}}_{t+\Delta t}$, which in turn can be filled into the equation of motion (when analysed at time $t + \Delta t$). Rewriting the equation of motion (without damping) finally gives an expression for $\mathbf{q}_{t+\Delta t}$ as a function of variables cal-

(a) Load case 1

(b) Load case 600

(c) Load case 750

(d) Load case 1,200
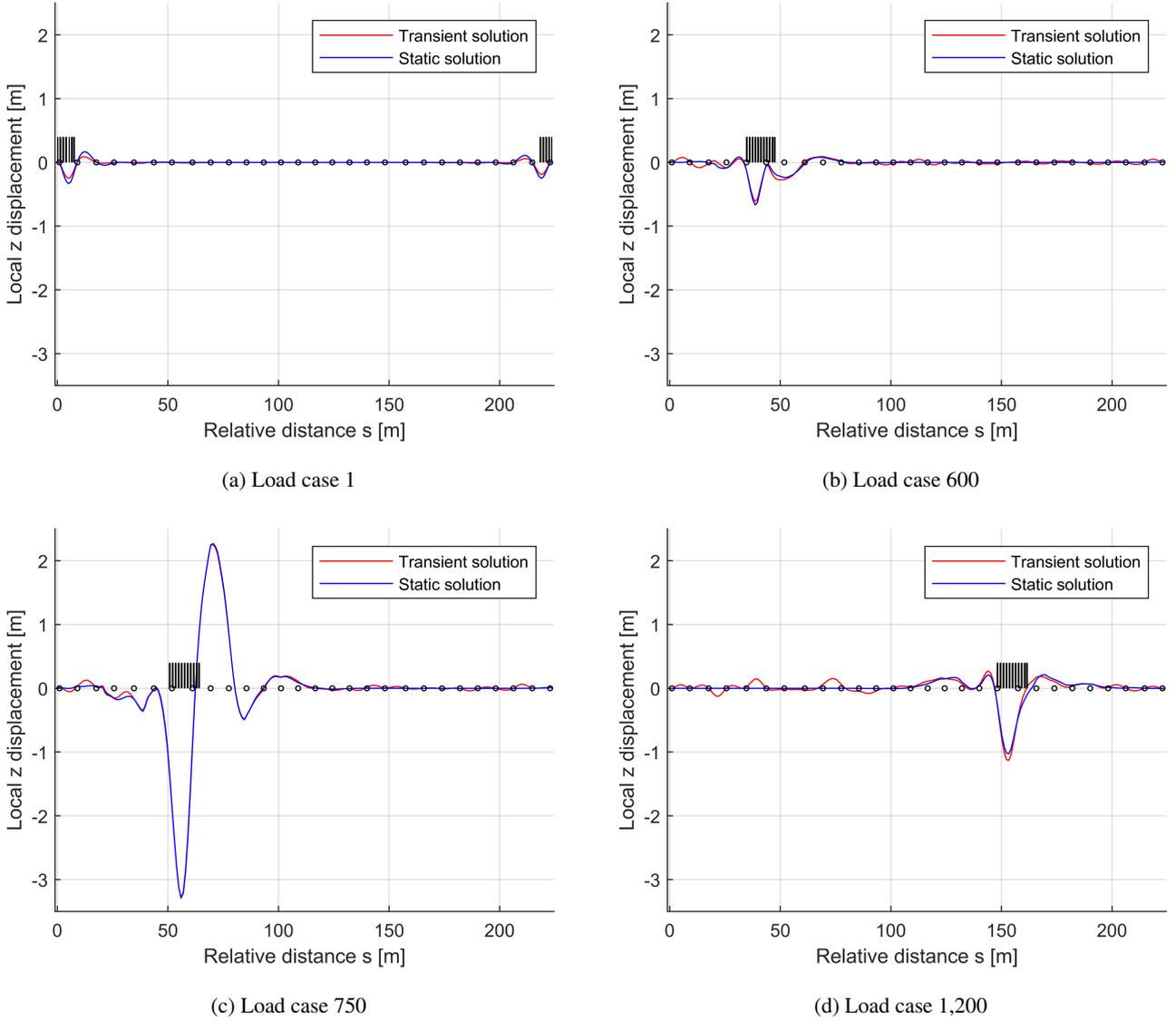
Fig. 10: Four load cases (out of 1,800) of the reference roller coaster, deformations are multiplied by a factor 800. The locations of the columns are shown with the markers.

culated at the previous time step t only:

$$\mathbf{q}_{t+\Delta t} = \left[\frac{1}{\beta \Delta t^2} \mathbf{M}_c + \mathbf{K}_c \right]^{-1} \left\{ \mathbf{f}_{t+\Delta t} \right.$$
$$\left. + \mathbf{M}_c \left[\frac{1}{\beta \Delta t^2} \mathbf{q}_t + \frac{1}{\beta \Delta t} \dot{\mathbf{q}}_t + \left(\frac{1}{2\beta} - 1 \right) \ddot{\mathbf{q}}_t \right] \right\} \quad (18)$$

It should be noted that the force vector $\mathbf{f}$ is required at the current time step, nevertheless this does not impose any problems as it is known for all time steps from the MBD-simulation.

Choosing γ equal to $1/2$ results in no numerical damping, whereas for stability reasons [9] β has been chosen equal to $1/4$, resulting in the average acceleration scheme which is implicit and unconditionally stable. Since the scheme is implicit, initial conditions have to be chosen: $\mathbf{q}_0 = \dot{\mathbf{q}}_0 = \ddot{\mathbf{q}}_0 = \mathbf{0}$.

After incorporating the constraint stiffness matrix $\mathbf{K}_c$ (14) and the constraint mass matrix $\mathbf{M}_c$ (15) the equations can be solved. Solving the equations for every time step Δt give the final displacements of every node for each load case. A few results of the 1,800 load cases of the reference roller coaster are shown in Fig. 9.

To quantify the displacements, it is convenient to show them in a 2D-plot as a function of the length of the track. This has been done for the left track elements (Fig. 5) in Fig. 10, where also the displacements obtained from the static calculation $\mathbf{q} = \mathbf{K}^{-1}\mathbf{f}$ are shown to compare with.

The differences between the transient and the static solutions are limited, especially at the location of the train they are negligible. However some discrepancy shows up when the train is further up along the track, as vibrations are not damped (no damping was used during these simulations). The absolute maximum local z-displacements (without mul-
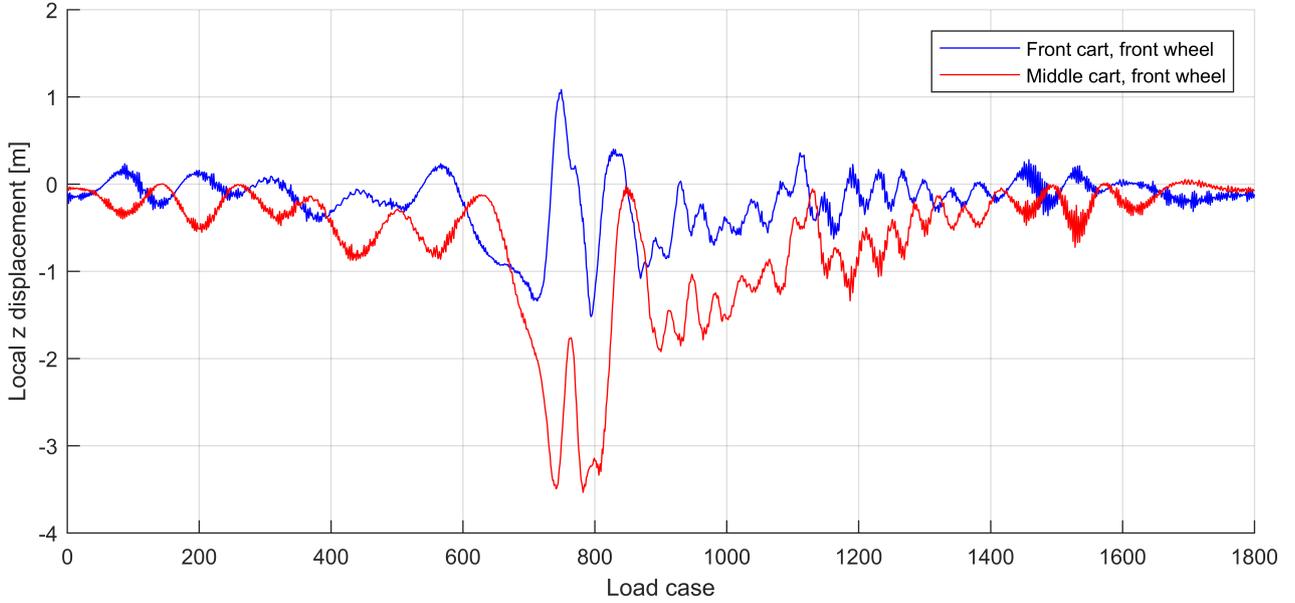
Fig. 11: Local z-displacements (multiplied by a factor 800) of the left track when following the train along the track of the reference roller coaster.

Table 1
Absolute maximum local z-displacements per load case (without multiplication factor).

	Maximum (static) displacement [mm]	Maximum (transient) displacement [mm]
LC 1	0.415	0.313
LC 600	1.180	1.319
LC 750	5.066	5.237
LC 1200	1.306	1.496
LC 783	6.593	7.089

tiplication factor) per load case considering all track elements are shown in Table 1. Load case 783 is added since it result in the largest displacements for the transient as well as the static analysis.

In general it can be concluded that the displacements calculated using the transient analysis are slightly higher than those of the static analysis. Considering the first few load cases this is the other way around, due to the fact that no vibrations caused by the train are present yet. Hence vibrations amplify displacements, so when another lap is completed after the first one, displacements will be even higher. Adding damping to the transient model can reduce this amplification significantly, hence estimating dynamic effects and thereby identifying the occurring vibrations is a very important step in designing a roller coaster.

To conclude, the displacements of the track directly underneath two wheels of the train are considered as a function of time. The figure is shown in Fig. 11, where the front wheels of the front cart and the fifth cart (middle cart) are taken as the positions where the local z-displacements (Fig. 5) are calculated. Since the deformations are only known at the nodes, interpolation is required to find the deformations for

each load case. This interpolation has been implemented using piecewise polynomials (spline method).

5. Test case

In this section a test case is analysed, where an attempt has been made to mimic (undesired) transverse vibrations in roller coasters. The model is inspired by MACK's Lost Gravity roller coaster; a schematic overview of the part of the structure where the vibrations showed up is shown in Fig. 12.

5.1. Moving load application

The (single) cart has four running wheels, so for each load case two wheels make contact with the left track and two wheels make contact with the right track. The loads applied on the structure are the same for each load case, where only normal and friction forces are taken into account (in total eight loads are simultaneously acting on the track). The magnitudes of the normal forces are chosen to be $5kN$ on the left track and $2kN$ on the right track respectively, representing an asymmetrical load case e.g. caused by passengers seating on one side of the cart only.

The friction forces can be calculated as a function of normal forces:

$$\mathbf{f}_f = -\mu \mathbf{f}_n \qquad (19)$$

With $\mathbf{f}_f$ the friction force vector, μ the friction coefficient (chosen to be 0.15) and $\mathbf{f}_n$ the normal force vector.

In this simulation 10 load cases per element are taken, where each load case represents a shift of 0.03 meters (hence the loads are applied equidistant instead of isochronal as is the case for the reference roller coaster), resulting in a total of 406 load cases. All loads are again converted to nodal
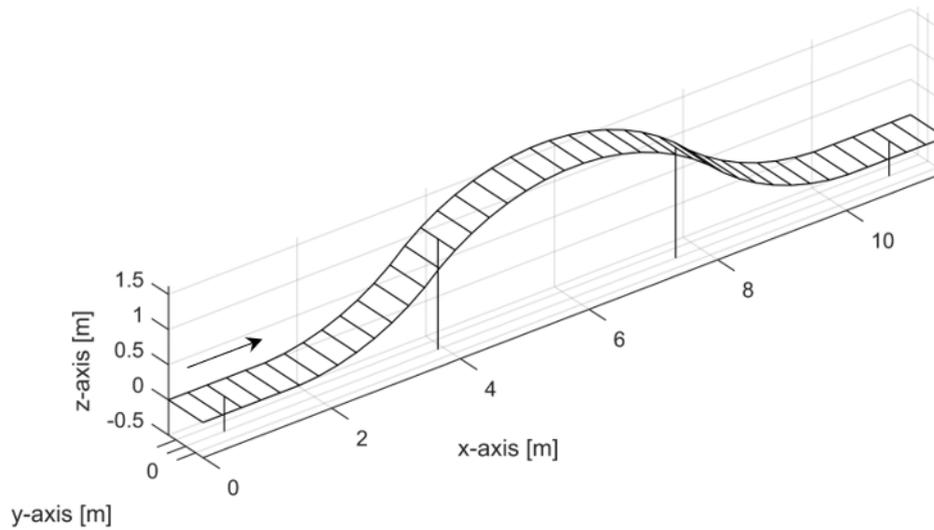
Fig. 12: Part of the test case track, the driving direction is indicated by the arrow.
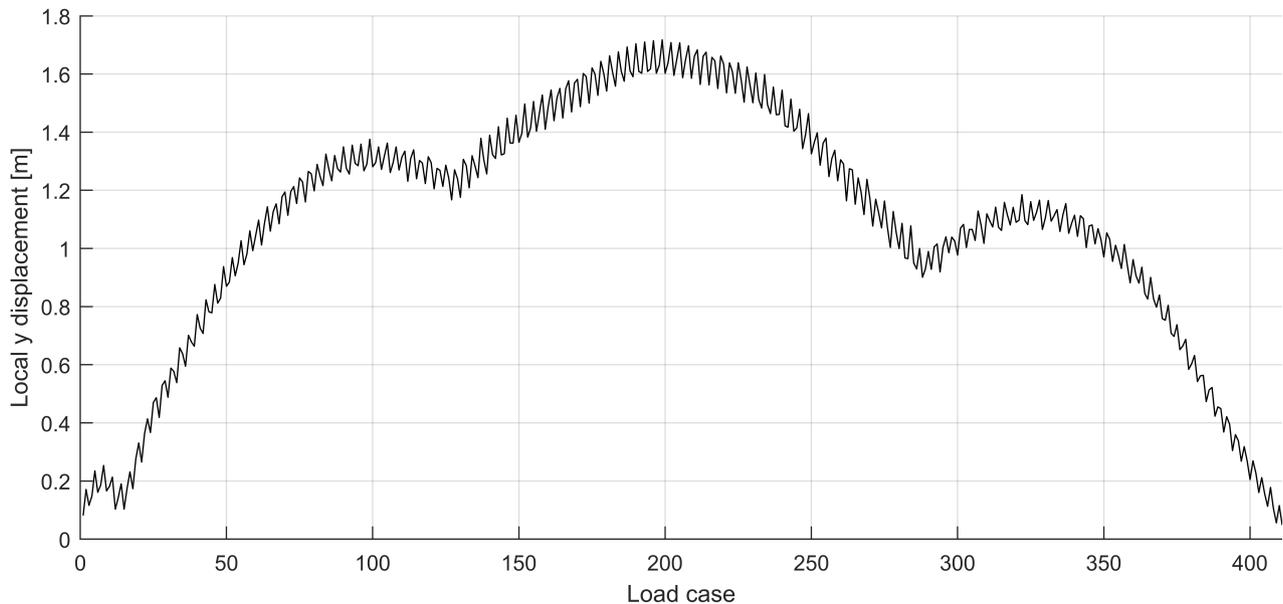
Fig. 13: Displacements (multiplied by a factor 800) of node 22 of the left track (top of the hill).

loads according the method explained in subsection 2.1. In the end, all loads and moments are collected in a matrix with dimensions of number of degrees of freedom by number of load cases (780 × 406) (subsection 5.2).

5.2. The Finite Element Model

The structure is composed of the same elements as the reference coaster like shown in Fig. 5, where each beam has 12 degrees of freedom according Fig. 2. Hence the cross ties are split up in two elements again, resulting in a total of 170 elements (41 left track elements, 41 right track elements, 84 cross ties and 4 columns) and 130 nodes (42 left track nodes, 42 right track nodes, 42 nodes between the left and right track and 4 columns). This leads to a total of $130 \times 6 = 780$ independent degrees of freedom.

The columns are connected to the fixed world in a similar fashion as those of the reference roller coaster, namely making use of massless springs like shown in Fig. 7. Furthermore, the first and last node of the left and right track are also connected to massless springs to represent the stiffness of the remainder of the track. These stiffnesses are added to the corresponding translational degrees of freedom in the element stiffness matrices, where it is assumed every stiffness is equal in size.

The constraint stiffness and mass matrix are constructed according (14) and (15) respectively, both having dimensions equal to the number of degrees of freedom of the entire structure (780×780). Both matrices are filled in the equation of motion (1).
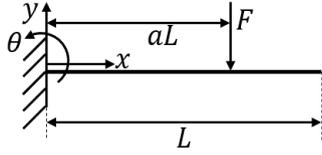
(a) Load retrieved from MBD-simulation applied between two nodes.
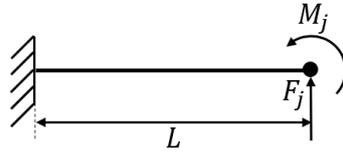
(b) Equivalent nodal force and moment for FEM-model.
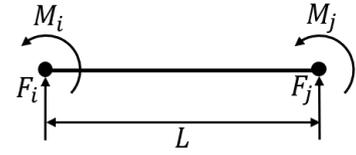
(c) Final FEM loads and moments, after replacing the clamping.

Fig. 14: If a load is applied between nodes, it has to be translated to equivalent nodal forces and moments.

5.3. Solving the equation of motion

The local y-displacements obtained from iteratively solving the equation of motion using the Newmark-Beta integration scheme (section 4) are shown for node 22 (on top of the hill) of the left track in Fig. 13. Again interpolation has been used to obtain a solution for each load case since the solution is only known at the nodes.

Unfortunately the model created is not able to capture the transverse vibrations occurring in roller coasters. Some small vibrations are visible, however these are not of the order that is expected beforehand (the displacements do not switch sign). Most likely another phenomenon is responsible for these vibrations, e.g. structural imperfections or wind loads.

6. Conclusions

In this paper a method to analyse vibrations in roller coaster structures is proposed. Existing FEM and MBD-models can be used to perform a transient analysis of roller coaster structures, making it possible to identify possible vibrations before building the actual roller coaster. This can potentially enhance the lifespan of roller coasters, as vibrations can cause the track to wear out faster. Furthermore, safety factors can be reduced significantly, striving for the complete removal of them. Hence this method can be incorporated with existing models efficiently, making it a practical tool to better design roller coasters in the future.

A. Nodal loads

The deflection and slope at the free end of the beam caused by an arbitrary load can be found using the following equations respectively:

$$v = -\frac{Fx^2}{6EI}(3L - x) \tag{20}$$

$$\theta = -\frac{Fx^2}{2EI} \tag{21}$$

Where E is the Young's modulus, I the area moment of inertia and x the distance between the clamping and the load application point. Hence in this situation (Fig. 14a) $x = aL$ where $0 < a < 1$.

The deflection and slope caused by the applied load should be equal to the deflection and slope caused by the load and moment at node j (Fig. 14b):

$$v = -\frac{Fa^2L^2}{6EI}(3L - aL) = \frac{F_jL^3}{3EI} + \frac{M_jL^2}{2EI} \tag{22}$$

$$\theta = -\frac{Fa^2L^2}{2EI} = \frac{F_jL^2}{2EI} + \frac{M_jL}{EI} \tag{23}$$

Which can be simplified to:

$$\frac{1}{3}Fa^3L - Fa^2L = \frac{2}{3}F_jL + M_j \tag{24}$$

$$-\frac{1}{2}Fa^2L = \frac{1}{2}F_jL + M_j \tag{25}$$

Equation (25) can be rewritten to find an expression for M_j, which in turn can be plugged in (24). This gives:

$$\frac{1}{3}Fa^3L - Fa^2L = \frac{2}{3}F_jL - \frac{1}{2}Fa^2L - \frac{1}{2}F_jL \tag{26}$$

Grouping the terms results in the expression for F_j:

$$F_j = \left(2a^3 - 3a^2\right)F \tag{27}$$

Equation (27) can be filled back in (25) to find the expression for M_j:

$$M_j = \left(a^2 - a^3\right)FL \tag{28}$$

F_i (Fig. 14c) can be found using force equilibrium in y-direction:

$$\Sigma F_y = F_i + F_j - F = 0 \tag{29}$$

Equation (27) can be filled in this equation to find the expression for F_i:

$$F_i = \left(1 - 2a^3 + 3a^2\right)F \tag{30}$$

M_i can be found using moment balance:

$$\Sigma M_i = M_i + M_j + F_jL - FaL = 0 \tag{31}$$

Equation (27) and (28) can be filled in this equation to find the expression for M_i:

$$M_i = \left(a + 2a^2 - a^3\right)FL \tag{32}$$
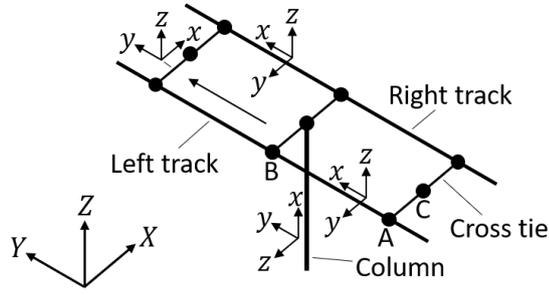
Fig. 15: Structural elements of the reference roller coaster, the driving direction is indicated by the arrow. The local coordinate frames are shown for each element type.

B. Rotation matrix

The local matrices have to be rotated such that the local coordinate frame of each element is aligned with the global reference frame of the entire model. Each element therefore requires an unique rotation matrix, which is defined as:

$$\mathbf{R} = \begin{bmatrix} \mathbf{R}_{3\times3} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{3\times3} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{R}_{3\times3} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{R}_{3\times3} \end{bmatrix} \quad (33)$$

With $\mathbf{0}$ a 3×3 zeros matrix. $\mathbf{R}_{3\times3}$ is a direction cosine matrix, defined as follows:

$$\mathbf{R}_{3\times3} = \begin{bmatrix} \cos(xX) & \cos(xY) & \cos(xZ) \\ \cos(yX) & \cos(yY) & \cos(yZ) \\ \cos(zX) & \cos(zY) & \cos(zZ) \end{bmatrix} \quad (34)$$

The local coordinate system is indicated with the lower case letters whereas the global coordinate system is represented by the capital letters. To show the approach taken to define a rotation matrix, element AB (left track element) from Fig. 15 has been taken as an example.

It is assumed the local x-axis lays in the direction of the element (from A to B). Now the first three direction-cosines are calculated to be:

$$\cos xX = l_x = \frac{X_B - X_A}{L_{AB}} \quad (35)$$

$$\cos xY = m_x = \frac{Y_B - Y_A}{L_{AB}} \quad (36)$$

$$\cos xZ = n_x = \frac{Z_B - Z_A}{L_{AB}} \quad (37)$$

Hence the (normalised) unit vector along the local x-axis is given by:

$$\hat{\mathbf{x}} = l_x\mathbf{i} + m_x\mathbf{j} + n_x\mathbf{k} \quad (38)$$

Either the y-axis or the z-axis still has to be chosen. Since a Cartesian coordinate system is used, the remaining axis can

be found using the cross product. As all cross ties are always orthogonal to the left track (and the right track), the coordinates of them can be used to create a local y-axis (positive from C to A):

$$\cos yX = l_y = \frac{X_A - X_C}{L_{AC}} \quad (39)$$

$$\cos yY = m_y = \frac{Y_A - Y_C}{L_{AC}} \quad (40)$$

$$\cos yZ = n_y = \frac{Z_A - Z_C}{L_{AC}} \quad (41)$$

Hence the unit vector along the local y-axis is given by:

$$\hat{\mathbf{y}} = l_y\mathbf{i} + m_y\mathbf{j} + n_y\mathbf{k} \quad (42)$$

The last step in obtaining the rotation matrix is to determine the z-axis by using the cross product. This is done by calculating the determinant of the following matrix:

$$\hat{\mathbf{z}} = \det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ l_x & m_x & n_x \\ l_y & m_y & n_y \end{vmatrix} \quad (43)$$

Resulting in:

$$\cos zX = l_z = m_x n_y - m_y n_x \quad (44)$$

$$\cos zY = m_z = -l_x n_y + l_y n_x \quad (45)$$

$$\cos zZ = n_z = l_x m_y - l_y m_x \quad (46)$$

And accordingly the unit vector along the local z-direction becomes:

$$\hat{\mathbf{z}} = l_z\mathbf{i} + m_z\mathbf{j} + n_z\mathbf{k} \quad (47)$$

Now all rotations are defined, the results can be gathered so the rotation matrix $\mathbf{R}_{3\times3}$ for an arbitrary element along the left track becomes:

$$\mathbf{R}_{3\times3} = \begin{bmatrix} n_x & m_x & n_x \\ n_y & m_y & n_y \\ n_z & m_z & n_z \end{bmatrix} \quad (48)$$

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Progress discussion

This report is based on a paper written for publication at Elsevier. As only the final results are stated in this paper, the progress during developing the method is discussed in this section. The Finite Element method has been developed and implemented for increasingly complex structures, until it was working flawlessly and could be used to analyse the track of the reference coaster. The structures used during the development are shown in Fig. 16.

First a simple two-dimensional track with two columns was modelled (Fig. 16a). This model has been made using truss elements (2 degrees of freedom per element); instead of using a boolean matrix to assemble the global stiffness matrix this has been done manually for each element. Forces are only acting on the nodes, so no interpolation is required to translate forces to nodal forces. The mass matrix has not been used yet since a static calculation was performed to check if the displacements looked realistic (magnitude and direction). After this simulation was running successfully, the truss elements are replaced by beam elements (6 degrees of freedom) and the same procedure was performed again.

Hereafter, a boolean matrix has been used to assemble the global stiffness matrix. This saves a significant amount of coding, making the method more comprehensible and easier to adapt. Furthermore, a global mass matrix has been added whereafter the displacements are calculated using the Newmark-Beta integration method.

The next model (Fig. 16b) is even simpler than the first one, used to implement the method to translate forces to nodal forces only. Initially, only normal forces are used since they are the hardest to distribute along the nodes. Later on friction forces were added and linearly distributed along the nodes. After this worked for this model, it has been implemented in the first model (Fig. 16a)

The next step was to extend the model to a three-dimensional structure. This model is shown in Fig. 16c, where the track makes a curve of exactly 90 degrees to be able to check the rotation matrices. Hence only one curve has been plotted instead of both the left and right track, which was easier to start with.

At this point in time the model is working nicely, so the reference roller coaster based on the Donderstenen (Fig. 16d) could be analysed. First this has been done using one curve only, whereafter the left and right track and ties were added. The forces applied were initially fictive, as no multibody dynamic model was at hand yet. It took a lot of time to correctly apply the forces for each load case, since they required rotation to the global coordinate system, translation to nodal forces and placing them in the correct position in a matrix.

The final step in completing the model was to incorporate the multibody dynamic model. Helmer van den Hoorn created an initial model in Simscape, whereafter modifications to the scheme are applied. These modifications are mainly about retrieving the positions of every wheel for each load case. Furthermore the starting position of the train has to be related to the starting position of the track, which has been implemented in the code.
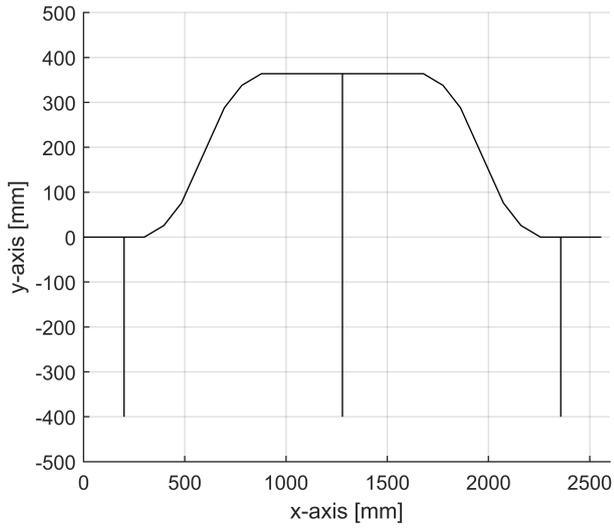
(a) First model
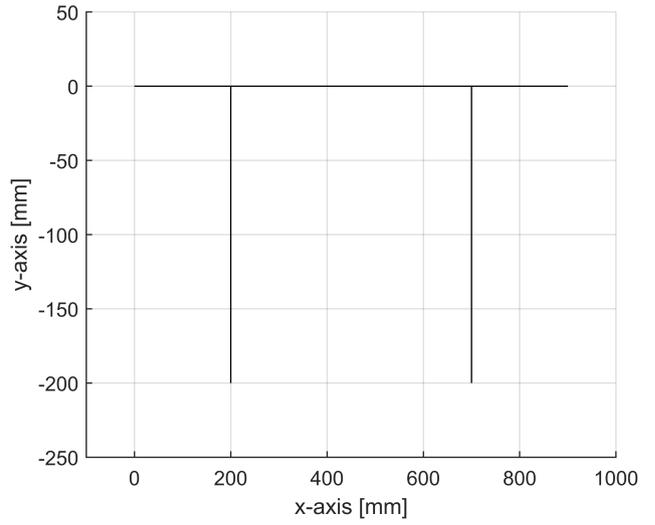
(b) Second model
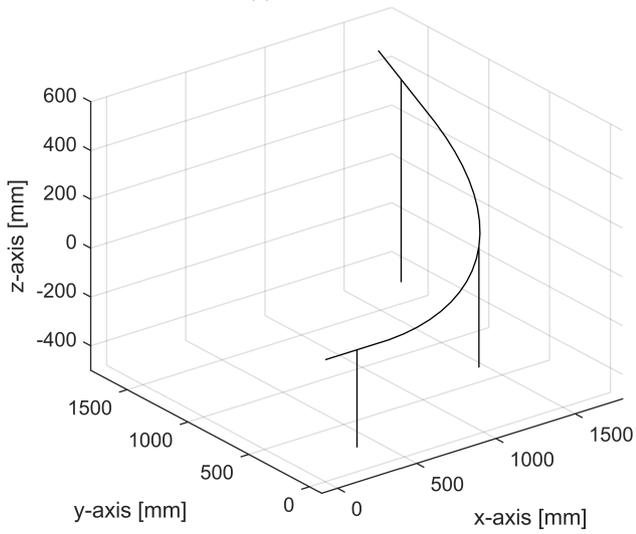
(c) Third model
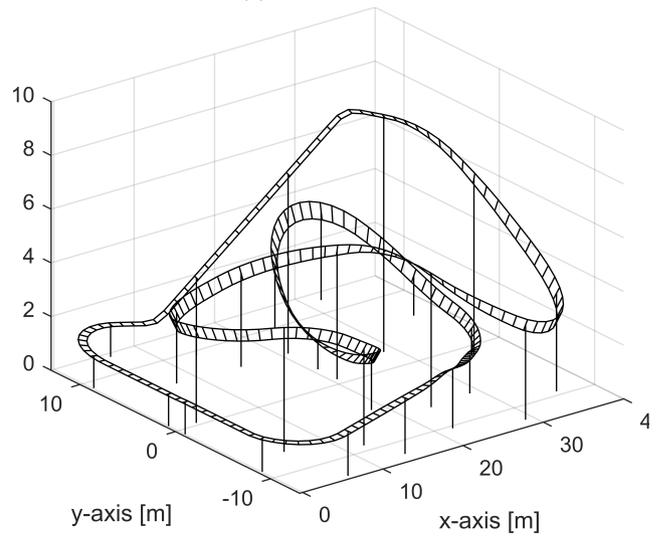
(d) Fourth model

Fig. 16: The four models used in the process of developing the method described in the paper.