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MASTER THESIS

Input – to – State Stability for bilinear systems

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Preface

During the past seven months I have been working on this thesis to conclude my masters in Applied Mathematics and with it my time as a student. Working on my thesis took place during a weird time, Covid-19 came around which forced me to work completely from home only a month after I started. This definitely was not how I imagined working on my thesis and it also would not have been my preference. Luckily I did manage to reach the point that I can present this to you.

Therefore, I also want to thank Felix who helped me reach this point by having the patience to help me with theory I did not have knowledge about and with all the other support. Our meetings were really useful and always gave me new ideas. Also thanks for being readily available email and giving a lot of feedback.

Furthermore, I want to thank my parents for supporting me during my studies and with the choices I made during that time. Also I want to thank Sander for being there for me when I needed it and to keep me motivated, especially when I struggled with working from home. Also I want to thank Dieuwertje, Pascal and Steven for answering silly questions and having fun chats.

Last but not least, I want to thank the members of the assessment committee for taking the time and effort to read and evaluate my thesis.

Abstract

This thesis studies Input–to–State Stability (ISS) for bilinear systems. The purpose of this thesis is to compare different notions of ISS, for example ISS itself, integral ISS and small–gain ISS for linear and bilinear systems. This is started with discussing the different notions of ISS with regard to finite dimensional linear and bilinear systems. After that, infinite dimensional systems are considered. For the infinite dimensional systems it makes a big difference regarding stability whether all operators involved in the system are bounded or not. Therefore, two situations for the input operators are discussed. The main result of this thesis is a condition under which the infinite dimensional bilinear systems with an unbounded input operator are integral Input–to–State Stable. The thesis is concluded with examples of infinite dimensional systems based on partial differential equations.

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Chapter 1

Introduction

In the study of dynamical systems the notion of stability is omnipresent. There are different notions of stability that can be used in different situations or for different types of systems. In this thesis the focus will be on different notions of Input–to–State Stability (ISS).

ISS was introduced by Eduardo Sontag in 1989 in [16] and is a combination of Lyapunov state–space stability and Zames–like external stability based on H^∞, which uses optimisation and input–output methods. For linear state–space systems the stability properties are already well known, therefore ISS is mainly used for nonlinear systems or linear systems with nonlinear (unknown) perturbations or inputs. ISS notions combine internal stability and robustness with respect to the outputs. Based on the notion of ISS multiple slightly weaker notions are developed. The most prominent one is integral Input–to–State Stable (iISS) which first appeared in [17]. This notion of ISS is useful since it does give a property of stability, but is less strict than ISS. Small gain ISS is a notion that lies in between ISS and iISS and was introduced in 1994 by Jiang, Teel and Praly in [12]. The notion of small–gain ISS is older than the notion of iISS; first the notion of ISS was defined, then it was weakened to small–gain ISS and then relaxed to iISS. Finally we mention the notion of strong iISS, which combines iISS and small–gain ISS and can for example be found in [8]. Strong iISS is a weaker notion than ISS, but stronger than iISS and small–gain ISS. It can also be shown that a system is ISS with a Lyapunov type argument. This was already mentioned when ISS was first defined in [16], but the necessity was not proven until 1995 in [18]. An overview of ISS theory is given in [1].

ISS was at first used for finite dimensional systems and only around 2008 a corresponding concept of ISS for infinite dimensional systems was developed. An example of early research regarding ISS for infinite dimensional systems is given in [11]. The development of ISS for infinite dimensional systems is motivated by studying stability for Partial Differential Equations (PDE's) in the context of control theory, but up until today there is still a lot unknown about ISS for these systems. In [14] an overview of the research of infinite dimensional systems up until 2019 can be found.

The notions of ISS are typically applied to systems in state–space representation. In a state–space system often x is used for the state and u for the input. An example of a state–space system is

$$\dot{x}(t) = f(x(t), u(t)), \qquad t \geq 0,$$

with some initial condition x_0 and an input u. There might be certain requirements on the function f such that the solution x exists for all times t. In the special case that

$$\dot{x}(t) = f(x(t), u(t)) = Ax(t) + Bu(t), \qquad t \geq 0,$$

with some initial condition x_0, an input u and with matrices A and B being mappings between suitable spaces, the system reduces to a linear system. Suitable spaces are for example $\mathbb{R}^n$ and $\mathbb{R}^m$.

In this thesis the focus will lie on ISS for bilinear systems. In bilinear systems the function f that was mentioned above will have a multiplication of the input u and the state x. The form of multiplication depends on the dimensions of x and u, but it will look like

$$\dot{x}(t) = Ax(t) + u(t)Bx(t), \qquad t \geq 0,$$

with some initial condition x_0 and an input u. In practice such a multiplication can occur in linear systems with a feedback control of the form $u = Kx$ and a multiplicative disturbance or change in time v, but the applications are broader. Bilinear systems are used in both engineering and science, for example in chemical engineering and biology. An example of a bilinear system is a system for chemotherapy (Example 6.2 of [6]). Here the state is in $\mathbb{R}^2$ with x_1 being the number of tissue cells in phases where they grow and synthesise, phase 1. x_2 is the number of cells preparing for cell division or dividing, phase 2. We have the following system:

$$\dot{x}(t) = \begin{bmatrix} -a_1 & 2a_2 \\ a_1 & -a_2 \end{bmatrix} x(t) + u(t) \begin{bmatrix} 0 & -2a_2 \\ 0 & 0 \end{bmatrix} x(t).$$

In this system a_1 is the mean transit time of cells from phase 1 to phase 2. Cells in phase 2 either divide and two daughter cells proceed to phase 1 at rate $2a_2$, or they are killed by a chemotherapeutic agent at rate $2ua_2$ in which $u \in [0, 1]$. An overview of the theory about bilinear systems and a short history is given in [6].

If one considers ISS for bilinear system you can see that most bilinear systems are not ISS, but that they are iISS. This will be shown in Chapter 5. In the paper where iISS is introduced also the example of bilinear systems is given [17] and the idea of the notion of iISS is related to the bilinear systems which are not ISS, but do have some stability property. We will consider different types of ISS to show when linear and bilinear systems have certain stability properties.

In this thesis at first some general definitions and assumptions will be discussed in Chapter 2. These notions are used in the rest of the thesis. After that the different notions of ISS are given in Chapter 3. In Chapter 4 these ISS notions will be shown for finite dimensional linear systems. Subsequently the different notions of ISS will be shown for finite dimensional bilinear systems in Chapter 5. After that the switch will be made to infinite dimensional systems. The different notions for ISS for infinite dimensional systems are discussed in Chapter 6. In this chapter first linear systems are discussed, followed by bilinear systems. For infinite dimensional systems it is relevant whether all operators involved in the system are bounded or not, therefore this will also be discussed in separate sections. The main result of this paper is Theorem 6.21, which is adopted from recent results in [9] and [10]. The thesis will be concluded with some examples in Chapter 7 and a conclusion in Chapter 8.

Chapter 2

General definitions and assumptions

Before starting with the the main part of the thesis some general definitions that might not be known to all readers are stated. Next to that some assumptions are stated.

Definition 2.1. *A function* $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$*, in which* $\mathbb{R}_{\geq 0} := \{s \in \mathbb{R} | s \geq 0\}$*, which is continuous, strictly increasing and satisfies* $\alpha(0) = 0$ *is called of class* $\mathcal{K}$*. A function of class* $\mathcal{K}$ *which is unbounded is said to be of class* $\mathcal{K}_\infty$*. In addition, a function* $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *with* $\beta(\cdot, t) \in \mathcal{K}_\infty$ *for each* $t \geq 0$ *and* $\beta(r, t)$ *goes to zero as* t *goes to* ∞ *is said to be of class* $\mathcal{KL}$*.*

A function $V : \mathbb{C}^n \to \mathbb{R}$, $n \in \mathbb{N}$, is *positive definite* if $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$. Such a function V is *proper* if $\lim_{\|x\| \to \infty} V(x) = \infty$. Finally a function $V : \mathbb{C}^n \to \mathbb{R}$ is called *smooth* if all the derivatives exist and are continuous.

As said before we will mainly use state–space systems. These systems have a state named x and an input named u. For any fixed $t \geq 0$, $x(t)$ and $u(t)$ are supposed to lie in normed spaces X and U respectively. For example if $X = \mathbb{C}^n$ we will use the Euclidean 2-norm for $x \in X$, so

$$\|x\|_2 = \sqrt{|x_1|^2 + |x_2|^2 + \ldots + |x_n|^2},$$

however a more general norm $\|x\|_X$ is also allowed, so that different spaces X are allowed, for example

$$X = L^p(a, b) := \{\text{measurable functions } f(t) | (\int_a^b |f(t)|^p dt)^{\frac{1}{p}} < \infty\},$$

equipped with its natural norm. For $u \in U^{\mathbb{R}_{\geq 0}}$, which is the set of functions $u : \mathbb{R}_{\geq 0} \to U$. We have the function norm which is defined as $\|u\|_{\infty[0,t]} := \operatorname{ess\,sup}_{t \in [0,t]} \|u(t)\|_U$. We also have the norm of elements of U, $u(t)$, this is again the Euclidean 2-norm, or a general norm $\|u(t)\|_U$. Here the essential supremum is defined as follows: for $f : X \to \mathbb{R}$ and Lebesgue measure μ we define

$$\operatorname{ess\,sup} f := \inf\{a \in \mathbb{R} : \mu(\{x : f(x) < a\}) = 0\}.$$

This means that the function is bounded by the essential supremum almost everywhere, so at some individual points the function might be higher than the essential supremum. For this norm two functions are identified to be equal if they are equal almost everywhere. Often u is piecewise continuous, in that case also the supremum can be used instead of the essential supremum. Finally we have the following notation for u, $u|_{[0,t]} := \{u(s)|s \in [0,t]\}$.

An inequality that is useful in getting estimates that we need is the integral version of Gronwall's inequality. The version we will use is Theorem 1.3.1 of [2].

Lemma 2.2. *Let u and f be continuous and nonnegative functions defined on $I = [a, b]$, so $u(t) \geq 0$ for all $t \in I$ and $f(t) \geq 0$ for all $t \in I$. Furthermore let α be a continuous, nonnegative and nondecreasing function defined on I, so $\alpha(t) \geq 0$ for all $t \in I$. If*

$$u(t) \leq \alpha(t) + \int_a^t f(s)u(s)ds, \qquad t \in I, \tag{2.1}$$

then,

$$u(t) \leq \alpha(t) \exp(\int_a^t f(s)ds), \qquad t \in I. \tag{2.2}$$

When we are going to discuss infinite dimensional systems we will use semigroups and operators. An *operator* is a mapping that maps elements from some normed space X to another normed space Y and are assumed to be linear. For an operator $A : X \to Y$ the operator norm is as follows:

$$\|A\| = \sup_{x \in X, x \neq 0} \frac{\|Ax\|_Y}{\|x\|_X}.$$

If $\|A\|$ is finite, the operator is called bounded. For operators $A : X \to Y$ and $B : Z \to X$ it holds that $\|AB\| \leq \|A\|\|B\|$. Typically all normed spaces are assumed to be Banach spaces. All matrix norms will be induced by the considered vector space norm. Moreover, we define $\mathcal{L}(X, Y)$ as the space of bounded linear operators from X to Y and $\mathcal{L}(X)$ as the space of bounded linear operators from X to X. Now we can define a semigroup:

Definition 2.3. *An operator-valued function T from $\mathbb{R}_{\geq 0}$ to $\mathcal{L}(X)$ is called a strongly continuous semigroup if it satisfies the following properties:*

1. *$T(t + s) = T(t)T(s)$, for all $t, s \geq 0$,*
2. *$T(0) = I$,*
3. *For all $x_0 \in X$, we have that $\|T(t)x_0 - x_0\|_X$ converges to zero, when $t \to 0^+$.*

A semigroup T is linked with a special operator, namely the generator, A. We will often use that an operator generates a semigroup and therefore the definition of a generator is given here.

Definition 2.4. *The infinitesimal generator A of a strongly continuous semigroup on a Hilbert space X is defined by*

$$Ax = \lim_{t \to 0^+} \frac{1}{t}(T(t) - I)x,$$

whenever the limit exists. The domain of A, $D(A)$, is the set of elements x in X for which the limit exists.

We remark that all integrals in this thesis refer to the Lebesgue integral, but most of the time it suffices to consider the Riemann integrals.

Finally we will also need the Banach Fixed Point Theorem.

Theorem 2.5. *Let X be a Banach space, $D \subseteq X$ closed and $F : D \to D$ a contraction, which means that T is Lipschitz continuous with Lipschitz constant $L < 1$:*

$$\|F(u) - F(v)\|_X \leq L\|u - v\|_X \quad \forall u, v \in D.$$

Then F has a unique fixed point $\bar{u} \in D$, which means that $F(\bar{u}) = \bar{u}$.

Chapter 3

Definition ISS

The notions of ISS will be defined for state–space systems, which we specify below. These are defined in a general way, so that both finite and infinite dimensional systems are covered. Moreover, this definition contains linear and nonlinear systems.

Definition 3.1. *Let X and U be Banach spaces with corresponding norms. Let $\phi : \mathbb{R}_{\geq 0} \times D \to X$ in which $D \subset X \times U^{\mathbb{R}_{\geq 0}}$. We call (X, U, ϕ) a system if it satisfies the following properties for all $t, h \in \mathbb{R}_{\geq 0}$ and for all $(x, u), (x, u_2) \in D$:*

1. $\phi(0, x, u) = x$
2. $(\phi(t, x, u), u(t + \cdot)) \in D$ *and* $\phi(t + h, x, u) = \phi(h, \phi(t, x, u), u(t + \cdot))$
3. $(x, u|_{[0,t]}) \in D$ *and* $u|_{[0,t]} = u_2|_{[0,t]}$ *implies that* $\phi(t, x, u) = \phi(t, x, u_2)$

In this system we name X the state–space and U the input space. The inputs u are functions of time, so $u : \mathbb{R}_{\geq 0} \to U$.

Systems defined in this way can also be found in [15]. In this definition 1. represents the initial condition, 2. represents the fact that if the starting conditions are the same, the starting time is independent and 3. represents that if two inputs are the same for a certain time and the state is equal at the beginning of this time, then the system will behave the same, this is also named the causality of the system. Moreover, a system as in Definition 3.1 is defined for all $t \geq 0$. This system is defined in general and usually a more specific form is used. The following systems are a subset of the systems defined in Definition 3.1:

Definition 3.2. *Let $f : X \times U \to X$ be a Lipschitz continuous function, then we can define a system by the solutions to the equations*

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)), \\ x(0) &= x_0, \end{aligned} \tag{3.1}$$

for all $t > 0$, $(x, u) \in D$ with $D = X \times U^{\mathbb{R}_{\geq 0}}$. Here x is the state and u is the input. The input u can be seen as a piecewise continuous function from $\mathbb{R}_{\geq 0}$ to U, so $u(t) \in U$. Then we have that $\phi(t, x_0, u) = x(t)$.

In this a certain set of inputs is allowed, usually $U = \mathbb{C}^m$ and $x(t) \in X = \mathbb{C}^n$, but they are also allowed to be infinite dimensional. We call a system finite dimensional if the dimension of X

is finite and we call a system infinite dimensional if the dimension of X is infinite. For ease of notation the dependence on t is often not written down.

Remark 3.3. *In Definitions 3.1 and 3.2 the state and the input are allowed to be complex-valued, but real-valued state and input are also allowed and from the practical perspective perhaps more useful. However, for this thesis the definition of the system is kept more general and thus also allow complex-valued state, X, and input, U.*

We can now define ISS for systems as in Definition 3.1

Definition 3.4. *A system (X, U, ϕ) as in Definition 3.1 is Input–to–State Stable (ISS) if there exists $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that*

$$\|x(t)\| \leq \beta(\|x_0\|, t) + \gamma(\|u\|_{\infty[0,t]}), \tag{3.2}$$

for all $t \geq 0$, with $(x, u) \in D$ and $x(t) = \phi(t, x, u)$.

ISS can be seen as if the state of a system has to stay small relative to the initial value of the state and the maximum input given. So if no input is given the state converges to zero if time goes to infinity due to β being a $\mathcal{KL}$ function. However, this also means that if the input is nonzero and time goes to infinity the state is bounded by a function depending only on the maximum of the input. Since for some systems this restriction might be too strict we also define a weaker form of ISS, iISS.

Definition 3.5. *A system as in Definition 3.1 is integral Input–to–State Stable (iISS) if there exists $\beta \in \mathcal{KL}$ and $\gamma_1, \gamma_2 \in \mathcal{K}$ such that*

$$\|x(t)\| \leq \beta(\|x_0\|, t) + \gamma_1\Big(\int_0^t \gamma_2(\|u(s)\|_U) ds\Big), \tag{3.3}$$

for all $t \geq 0$, with $(x, u) \in D$ and $x(t) = \phi(t, x, u)$.

In this definition the state is bounded by the integral of the input. This means that the state is only required to decrease in time if the input is decreasing. If one compares with a constant input it is clear that now there is more freedom in how the system responds to inputs. For finite dimensional systems, iISS is weaker than ISS and if a finite dimensional system is ISS it automatically is iISS. This is the case since for ISS the restriction on u is stricter than with iISS. This is also mentioned in [4].

The next notion that is going to be discussed is small–gain ISS.

Definition 3.6. *A system as in Definition 3.1 is small–gain Input–to–State Stable (small–gain ISS) if there exists a $R > 0$, $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that*

$$\|x(t)\| \leq \beta(\|x_0\|, t) + \gamma(\|u\|_{\infty[0,t]}), \tag{3.4}$$

for all $t \geq 0$, with $(x, u) \in D$, $x(t) = \phi(t, x, u)$ and $\|u\|_{\infty[0,t]} < R$.

This definition is useful for systems that are unstable for large inputs, but that have bounded state if the input is small. This can be seen in the following example:

Example 3.7. *Take the system*

$$\begin{aligned} \dot{x}(t) &= -x(t) + x(t)u(t), \\ x(0) &= x_0, \end{aligned}$$

with $X = U = \mathbb{R}$ and $t \geq 0$. If a constant input $u(t) = 2$, $t \geq 0$, is applied the system takes the form of $\dot{x}(t) = x(t)$ and thus is unstable, but if the input is smaller than 1, then the system is stable. This system is iISS and small–gain ISS which will be proved later on.

There is one more notion of ISS that we will use, namely strong iISS.

Definition 3.8. *A system as in Definition 3.1 is called Strong integral Input–to–State Stable (strong iISS) if it is small–gain ISS and iISS.*

The notions of ISS can also be shown with a Lyapunov type argument.

Definition 3.9. *A function $V : X \to \mathbb{R}_{\geq 0}$ is called an ISS-Lyapunov function for a finite dimensional system as in Definition 3.2, if V is proper and smooth, there exist α_1, $\alpha_2 \in \mathcal{K}_\infty$ such that $\alpha_1(x) \leq V(x) \leq \alpha_2(x)$ and there exist class $\mathcal{K}_\infty$ functions γ and α so that*

$$\nabla V(x) f(x,u) \leq -\alpha(x) + \gamma(\|u\|_U) \tag{3.5}$$

for all $t \geq 0$, with $(x,u) \in D$ and $x(t) = \phi(t,x,u)$.

Theorem 3.10. *A finite dimensional system as in Definition 3.2 is ISS if and only if there exists an ISS-Lyapunov function for the system.*

A proof can be found in [18].

A Lyapunov type argument can also be used for small–gain ISS, but then there has to exist an $R > 0$ such that Equation (3.5) holds for $\|u\|_U < R$. There also is a Lyapunov type definition for iISS, this is as follows:

Definition 3.11. *A function $V : X \to \mathbb{R}_{\geq 0}$ is called an iISS-Lyapunov function for a finite dimensional system as in Definition 3.2, if V is proper and smooth, there exist α_1, $\alpha_2 \in \mathcal{K}_\infty$ such that $\alpha_1(x) \leq V(x) \leq \alpha_2(x)$ and there exist class $\mathcal{K}_\infty$ function γ and $\alpha : [0,\infty) \to [0,\infty)$ so that*

$$\nabla V(x) f(x,u) \leq -\alpha(x) + \gamma(\|u\|_U) \tag{3.6}$$

for all $x \in X$ and $u \in U$.

Theorem 3.12. *A finite dimensional system as in Definition 3.2 is iISS if and only if there exists an iISS-Lyapunov function for the system.*

A proof can be found in [3] (Proof of Theorem 1).

The difference between Definitions 3.9 and 3.11 is quite small and therefore might not clear at first sight, but the difference is that in Definition 3.9 the $\alpha \in \mathcal{K}_\infty$ and that in Definition 3.11 $\alpha : [0,\infty) \to [0,\infty)$, this also shows that Definition 3.11 is weaker.

For systems as in Definition 3.2, with $X = \mathbb{R}^n$ and $U = \mathbb{R}^m$, the relation between the different types of ISS is shown in a clear figure in Figure 3.1. There are more notions of stability for state–space systems, but in this figure the ones that are used in this thesis are mentioned.

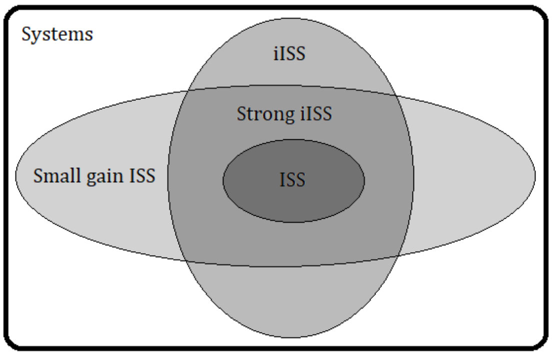

Figure 3.1: Relation between different types of ISS for systems as in Definition 3.2 with $X = \mathbb{R}^n$ and $U = \mathbb{R}^m$ [4]

Chapter 4

Linear systems

To get a feeling for the notions of ISS we will first have a look at finite dimensional linear systems. Linear systems are systems that are as in Definition 3.2, with $f(x,u) = Ax + Bu$, where $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times m}$ are matrices fitting with the dimension of $X = \mathbb{C}^n$ and $U = \mathbb{C}^m$. Thus for a linear system we have

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ x(0) &= x_0. \end{aligned} \tag{4.1}$$

This is a system of the type of Definition 3.2 if u satisfies the conditions mentioned. Again the dependence on time is often omitted. This is the system that will be used in this chapter. The differential equation in this system can be solved using an integrating factor to give the exact solution $x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$. In Section 6.1.1 this derivation will be shown for the infinite dimensional case.

We will show that if A is Hurwitz that then the linear system is ISS. To show that linear systems are ISS it is needed to find an estimate of $\|e^{At}\|$. We will therefore prove the following lemma:

Lemma 4.1. *Let $A \in \mathbb{C}^{n\times n}$ be a matrix. Then for every $\omega > \max\{Re(\lambda) : \lambda$ eigenvalue of $A\}$, there exists a constant $M \geq 1$ such that*

$$\|e^{At}\| \leq Me^{\omega t} \tag{4.2}$$

holds for all $t \geq 0$. Moreover, if A is not Hurwitz, then $\|e^{At}\|$ does not converge to 0 for $t \to \infty$.

Proof. We know that for every matrix A there exists an invertible matrix $T \in \mathbb{C}^{n\times n}$ and a matrix $J \in \mathbb{C}^{n\times n}$ in Jordan normal form such that $A = TJT^{-1}$. Thus, using the definition of the matrix exponential,

$$\begin{aligned} \|e^{At}\| &= \|e^{TJT^{-1}t}\| \\ &= \|Te^{Jt}T^{-1}\| \\ &\leq \|T\|\|T^{-1}\|\|e^{Jt}\|, \end{aligned}$$

where we used the sub-multiplicity of the matrix norm. Here we see that $\|T\|\|T^{-1}\| \geq 1$ since $1 = \|I\| = \|T \cdot T^{-1}\| \leq \|T\|\|T^{-1}\|$. Now we want to further estimate $\|e^{Jt}\|$. To do this we will first look at a certain block J_i of the Jordan matrix J,

$$\|e^{J_i t}\| = \|e^{t(J_i - \lambda_i I) + t\lambda_i I}\| = \|e^{t(J_i-\lambda_i I)} e^{t\lambda_i I}\| \leq e^{t\lambda_i}\|e^{J_i - \lambda_i I}\|.$$

This holds since $t\lambda_i I$ is a diagonal matrix. To make notation a bit easier we will now write $J_i - \lambda_i I = J_i^0$, since the diagonal is set to zero. Due to the form of this matrix we know that if $J_i^0 \in \mathbb{R}^{h_i \times h_i}$, $h_i \leq n$, then $(J_i^0)^{h_i} = 0$. Since a matrix exponential can be written as its Taylor series we have the following:

$$\|e^{J_i^0 t}\| = \left\| \sum_{m=0}^{\infty} \frac{(J_i^0 t)^m}{m!} \right\| \leq \sum_{m=0}^{\infty} \left\| \frac{(J_i^0 t)^m}{m!} \right\| = \sum_{m=0}^{h_i - 1} \left\| \frac{(J_i^0 t)^m}{m!} \right\| \leq \sum_{m=0}^{h_i-1} \frac{t^m}{m!} \|J_i^0\|^m.$$

Now we will show that for every $\varepsilon > 0$, there exists an $M_\varepsilon \geq 1$ such that $\|e^{J_i^0 t}\| \leq M_\varepsilon e^{\varepsilon t}$. Consider a single term of the summation $\sum_{m=0}^{h_i-1} \frac{t^m}{m!}\|J_i^0\|^m$. Then we want to get the following estimate:

$$\begin{aligned} \frac{t^m}{m!}\|J_i^0\|^m &\leq C e^{\varepsilon t} \\ \frac{t^m \|J_i^0\|^m}{m! e^{\varepsilon t}} &\leq C. \end{aligned} \tag{4.3}$$

Define $f_{m,J}(t) := \frac{t^m \|J_i^0\|^m}{m! e^{\varepsilon t}}$. We know that $f_{m,J}(0) = 0$, $\lim_{t\to\infty} f_{m,J}(t) = 0$ and $f_{m,J}(0) \geq 0$ if $t, m \geq 0$. Thus this function reaches a maximum somewhere and if we choose C to be this maximum we know that Equation (4.3) is satisfied. The maximum is attained where the derivative equals zero, if the derivative is zero more than once there might be a minimum as well, or it might be that there are local extrema, so then there has to be checked which of the extreme values is the maximum. The derivative is as follows:

$$\begin{aligned} \frac{d}{dt} f_{m,J}(t) &= 0 \\ \frac{\|J_i^0\|^m}{m!} m t^{m-1} e^{-\varepsilon t} - \frac{\|J_i^0\|^m}{m!} t^m \varepsilon e^{-\varepsilon t} &= 0 \\ \frac{\|J_i^0\|^m}{m!} t^{m-1} e^{-\varepsilon t}(m - t\varepsilon) &= 0 \\ t &= \frac{m}{\varepsilon}. \end{aligned}$$

So $C = f_{m,J}(\frac{m}{\varepsilon}) = \frac{m^m \|J_i^0\|^m}{\varepsilon^m m! e^m}$ and indeed Equation (4.3) holds. Therefore, we now know the following:

$$\begin{aligned} \|e^{J_i t}\| &\leq e^{t\lambda_i}\left(\sum_{m=0}^{h_i-1} \frac{t^m}{m!}\|J_i^0\|^m\right) \\ &\leq e^{t\lambda_i}\left(\sum_{m=0}^{h_i-1} \left(\frac{m^m}{\varepsilon^m m! e^m}\|J_i^0\|^m e^{\varepsilon t}\right)\right) \\ &= e^{t(\lambda_i + \varepsilon)} \sum_{m=0}^{h_i-1} \left(\frac{m^m}{\varepsilon^m m! e^m}\|J_i^0\|^m\right) \\ &\leq e^{t(\lambda_i+\varepsilon)} M_{\varepsilon,i}. \end{aligned}$$

Here $M_{\varepsilon,i}$ is larger or equal then 1.

Finally we need to show the relation between $\|e^{Jt}\|$ and $\|e^{J_i t}\|$. We will do this using the 2-norm for matrices. Due to the block structure of the matrix J that is in Jordan normal form, we know that

$$J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{bmatrix} \text{ and hence, } e^J = \begin{bmatrix} e^{J_1} & & & \\ & e^{J_2} & & \\ & & \ddots & \\ & & & e^{J_k} \end{bmatrix}.$$

If we now want the 2-norm of e^J we have that this is the same as the largest singular value. The singular values of e^J are the eigenvalues of $e^J \cdot (e^J)^* = e^J \cdot e^{J^*}$. Substituting this in the former equation gives:

$$\begin{aligned} e^J \cdot e^{J^*} &= \begin{bmatrix} e^{J_1} & & & \\ & e^{J_2} & & \\ & & \ddots & \\ & & & e^{J_k} \end{bmatrix} \cdot \begin{bmatrix} e^{J_1^*} & & & \\ & e^{J_2^*} & & \\ & & \ddots & \\ & & & e^{J_k^*} \end{bmatrix} \\ &= \begin{bmatrix} e^{J_1}e^{J_1^*} & & & \\ & e^{J_2}e^{J_2^*} & & \\ & & \ddots & \\ & & & e^{J_k}e^{J_k^*} \end{bmatrix}. \end{aligned}$$

So the eigenvalues of $e^J \cdot e^{J^*}$ are in the combination of the eigenvalues of $e^{J_1} \cdot e^{J_1^*}$, $e^{J_2} \cdot e^{J_2^*}$, ..., $e^{J_k} \cdot e^{J_k^*}$. These are the singular values of e^{J_1}, e^{J_2}, ..., e^{J_k}, since the norm of e^J is the largest singular value we now know that $\|e^J\| = \max_i \|e^{J_i}\|$. So we know that for every $\omega > \max\{Re(\lambda) : \lambda \text{ eigenvalue of } A\}$ there exists a constant $M \geq 1$ such that

$$\begin{aligned} \|e^{At}\| \leq \|T\|\|T^{-1}\|\|e^{Jt}\| &= \|T\|\|T^{-1}\| \max_i \|e^{J_i t}\| \\ &\leq \|T\|\|T^{-1}\| \max_i e^{t(\lambda_i+\varepsilon)} M_{\varepsilon,i} \\ &\leq M e^{\omega t}. \end{aligned}$$

Since the matrix 2-norm is a vector induced matrix norm, this is equivalent with other vector induced matrix norms and can be estimated with a constant. Thus this also holds for other vector induced matrix norms, which are all the matrix norms considered in this thesis.

Now we still have to show that if A is not Hurwitz, then $\|e^{At}\|$ does not converge to 0 if t goes to infinity. We will do this using the eigenvalues and corresponding eigenvectors of A. We use the 2-norm, thus

$$\|e^{At}\|_2 = \sup_{x \in \mathbb{C}^n, x \neq 0} \frac{\|e^{At}x\|_2}{\|x\|_2},$$

so if we have a certain $x \in \mathbb{C}^n$ for which this does not converge to zero we know the norm does not converge to zero. Take v the normalised eigenvector corresponding with the eigenvalue λ that has non negative real part, so $Re(\lambda) \geq 0$. Then we have the following:

$$\|e^{At}\| \geq \|e^{At}v\|_2 = \|e^{\lambda t}v\|_2 = e^{\lambda t}\|v\|_2 = e^{\lambda t}.$$

Since the real part of λ is non-negative we see that this will not converge to zero if t goes to ∞. $\square$

Now we can use the previous lemma to prove the following theorem:

Theorem 4.2. *For linear systems of the form* (4.1) *the following are equivalent:*

1. *A is Hurwitz,*
2. *The system is ISS,*
3. *The system is iISS,*
4. *The system is small–gain ISS,*
5. *The system is strong iISS.*

Proof. Because the linear system is finite dimensional we have that $2 \Rightarrow 4$. Also we have by definition that $3 \wedge 4 \Rightarrow 5$, $5 \Rightarrow 4$ and $5 \Rightarrow 3$. If A is not Hurwitz we see by Lemma 4.1 that then $\|e^{At}\|$ will not converge to zero. Therefore, when the input is zero, $x(t)$ will not converge as t goes to infinity and the system is not iISS nor ISS. Thus $2 \Rightarrow 1$, $3 \Rightarrow 1$ and $4 \Rightarrow 1$. Now the only things left to show are $1 \Rightarrow 2$ and $1 \Rightarrow 3$.

To show this we will use the solution of the differential equation,

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau.$$

We will look at the norm to estimate this in such a way that we get Equation (3.2). Since A is Hurwitz we will use that according to Lemma 4.1 there exist $M \geq 1$ and $\omega < 0$ such that $\|e^{At}\| \leq Me^{\omega t}$. Therefore,

$$\begin{aligned}\|x(t)\| &= \|e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau\| \\ &\leq \|e^{At}\|\|x_0\| + \int_0^t \|e^{a(t-\tau)}\|\|B\|\|u(\tau)\|d\tau \\ &\leq Me^{\omega t}\|x_0\| + M\|B\|\int_0^t e^{\omega(t-\tau)}\|u(\tau)\|d\tau.\end{aligned}$$

The next step is to use the Hölder's inequality. Note that $\int_0^t e^{\omega(t-\tau)}\|u(\tau)\|d\tau$ is actually $\|e^{\omega(t-\tau)}\|u(\tau)\|\|_1$, thus it can be estimated. We will use the infinity-norm for the part with u and the 1-norm for the part with the exponential,

$$\begin{aligned}\|x(t)\| &\leq Me^{\omega t}\|x_0\| + M\|B\|\int_0^t e^{\omega(t-\tau)}\|u(\tau)\|d\tau \qquad (4.4)\\ &\leq Me^{\omega t}\|x_0\| + M\|B\|\int_0^t e^{\omega(t-\tau)}d\tau\|u\|_{\infty[0,t]} \\ &= Me^{\omega t}\|x_0\| + M\|B\|(-\frac{1}{\omega} + \frac{1}{\omega}e^{\omega t})\|u\|_{\infty[0,t]} \\ &\leq Me^{\omega t}\|x_0\| - M\|B\|\frac{1}{\omega}\|u\|_{\infty[0,t]}.\end{aligned}$$

Where the last step holds since $\omega < 0$. Thus Equation (3.2) is satisfied with $\beta(\|x_0\|, t) = Me^{\omega t}\|x_0\|$ and $\gamma(s) = -M\|B\|\frac{1}{\omega}s$. Therefore, $1 \Rightarrow 2$.

To show $1 \Rightarrow 3$, we take a look at Equation (4.4), the term $e^{\omega(t-\tau)}$ can be estimated with 1. Therefore, already Equation (3.3) is satisfied with $\beta(\|x_0\|, t) = Me^{\omega t}\|x_0\|$, $\gamma_1(s) = M\|B\|s$ and $\gamma_2 = s$. Thus $1 \Rightarrow 3$ and the proof is completed. □

Chapter 5

Bilinear systems

After having looked at finite dimensional linear systems we will now take a look at ISS for bilinear systems. The class of finite dimensional bilinear systems is a subset of the systems as in Definition 3.2 and is defined as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \sum_{i=1}^{m} u_i(t)B_i x(t), \\ x(0) &= x_0. \end{aligned} \tag{5.1}$$

with $x \in X \subseteq \mathbb{C}^n$, $u \in U \subseteq \mathbb{C}^m$, $A \in \mathbb{C}^{n\times n}$ and $B_i \in \mathbb{C}^{n\times n}$, $i \in \{1, ..., m\}$. Without loss of generality we will assume that there is only one input and that therefore there also is just one B matrix and the summation vanishes, this gives the system

$$\begin{aligned} \dot{x}(t) = Ax(t) + u(t)Bx(t), \\ x(0) = x_0. \end{aligned} \tag{5.2}$$

If you take $A = -1$ and $B = 1$ as in Example 3.7, it can already be seen that this system is not ISS, since if the constant input $u = 2$ is applied, the system becomes $\dot{x} = x$, which is unstable. However, we would like to show that the system (5.2) is iISS.

Proposition 5.1. *Bilinear systems are iISS if and only if A is Hurwitz. In this case,*

$$\beta(s,t) = (1 + Me^{\omega t}s)^2 - 1, \gamma_1(s) = e^{2s} - 1 \text{ and } \gamma_2(s) = M|s|\|B\|.$$

In which $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|e^{At}\| \leq Me^{\omega t}$ for all $t \geq 0$.

Proof. To proof '$\Leftarrow$' we use the idea of Theorem 4.2 of [13]. To start with we rewrite and integrate both sides of $\dot{x} = Ax + uBx$,

$$\begin{aligned} \dot{x}(\tau) &= Ax(\tau) + u(\tau)Bx(\tau) \\ e^{-A\tau}\dot{x}(\tau) &= e^{-A\tau}Ax(\tau) + e^{-A\tau}u(\tau)Bx(\tau) \\ e^{-A\tau}\dot{x}(\tau) - e^{-A\tau}Ax(\tau) &= e^{-A\tau}u(\tau)Bx(\tau) \\ \frac{d}{d\tau}(e^{-A\tau}x(\tau)) &= e^{-A\tau}u(\tau)Bx(\tau). \end{aligned}$$

Now we will integrate both sides to get rid of the derivative,

$$\begin{aligned}\int_0^t \frac{d}{d\tau}(e^{-A\tau}x(\tau))d\tau &= \int_0^t e^{-A\tau}u(\tau)Bx(\tau)d\tau \\ e^{-At}x(t) - x_0 &= \int_0^t e^{-A\tau}u(\tau)Bx(\tau)d\tau \\ x(t) &= e^{At}x_0 + \int_0^t e^{A(t-\tau)}u(\tau)Bx(\tau)\tau.\end{aligned}$$

This almost looks like the solution of a linear system, only now there still is an $x(\tau)$ on the right side, therefore the estimation will be done a bit different, but with the help of Lemma 2.2 it is possible to get rid of this term. To get an estimate like in Equation (3.3) we take the norm. This gives,

$$\begin{aligned}\|x(t)\| &= \|e^{At}x_0 + \int_0^t e^{A(t-\tau)}u(\tau)Bx(\tau)\tau\| \\ \|x(t)\| &\leq \|e^{At}\|\|x_0\| + \int_0^t \|e^{A(t-\tau)}u(\tau)Bx(\tau)\|d\tau.\end{aligned}$$

Now we apply Lemma 4.1, multiply the entire inequality with $e^{-\omega t}$ and define $z(t) = e^{-\omega t}x(t)$ to rearrange this inequality. This gives,

$$\|x(t)\| \leq Me^{\omega t}\|x_0\| + \int_0^t Me^{\omega(t-\tau)}|u(\tau)|\|B\|\|x(\tau)\|d\tau \tag{5.3}$$

$$\begin{aligned}e^{-\omega t}\|x(t)\| &\leq M\|x_0\| + \int_0^t Me^{-\omega\tau}|u(\tau)|\|B\|\|x(\tau)\|d\tau \\ \|z(t)\| &\leq M\|x_0\| + \int_0^t M|u(\tau)|\|B\|\|z(\tau)\|d\tau.\end{aligned}$$

This inequality now is in the right form to apply Lemma 2.2. After that again $z(t) = e^{-\omega t}x(t)$ is used to go back from z to x,

$$\begin{aligned}\|z(t)\| &\leq M\|x_0\| \exp\big(\int_0^t M|u(\tau)\|B\|d\tau\big) \\ e^{-\omega t}\|x(t)\| &\leq M\|x_0\| \exp\big(\int_0^t M|u(\tau)|\|B\|d\tau\big)\end{aligned}$$

$$\|x(t)\| \leq e^{\omega t}M\|x_0\| \exp\big(\int_0^t M|u(\tau)|\|B\|d\tau\big). \tag{5.4}$$

At this point at first it seems to make sense to apply $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$, however, Equation (3.3) requires $\gamma_1, \gamma_2 \in \mathcal{K}$, thus $\gamma_1(0) = 0$, this is not satisfied if $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ is used. Therefore, we have a little trick to apply the function $\alpha(r) = \ln(1 + r)$ to both sides. This function has the following property:

$$\begin{aligned}\ln(1 + ae^b) &\leq \ln((1 + a)e^b) \\ &= \ln(1 + a) + b.\end{aligned}$$

So if we now use $a = Me^{\omega t}\|x_0\|$ and $b = \int_0^t M|u(\tau)|\|B\|d\tau$ we get the following:

$$\begin{aligned}\alpha(\|x(t)\|) &\leq \alpha(e^{\omega t}M\|x_0\| \exp\big(\int_0^t M|u(\tau)|\|B\|d\tau)) \\ \alpha(\|x(t)\|) &\leq \ln(1 + Me^{\omega t}\|x_0\|) + \int_0^t M|u(\tau)|\|B\|d\tau \\ \|x(t)\| &\leq \alpha^{-1}\left(\ln(1 + Me^{\omega t}\|x_0\|) + \int_0^t M|u(\tau)|\|B\|d\tau\right).\end{aligned}$$

Now use $\alpha^{-1}(a + b) \leq \alpha^{-1}(2a) + \alpha^{-1}(2b)$ to estimate $\|x(t)\|$ with a sum of a term with x_0 and a term with u. Also use that $\alpha^{-1}(s) = e^s - 1$,

$$\begin{aligned}\|x(t)\| &\leq \alpha^{-1}(2\ln(1 + Me^{\omega t}\|x_0\|)) + \alpha^{-1}(2\int_0^t M|u(\tau)|\|B\|d\tau) \\ \|x(t)\| &\leq (1 + Me^{\omega t}\|x_0\|)^2 - 1 + \exp\big(2\int_0^t M|u(\tau)|\|B\|d\tau) - 1.\end{aligned}$$

So now we have Equation (3.3) with $\beta(s,t) = (1 + Me^{\omega t}s)^2 - 1$, $\gamma_1(s) = e^{2s} - 1$ and $\gamma_2(s) = M|s|\|B\|$. Thus bilinear systems are iISS.

If A is not Hurwitz, then the system with zero input is not stable, thus you will never be able to find a suitable β function.

□

We have just shown that bilinear systems are iISS. This is done by establishing an estimate for $\|x(t)\|$. Now the question arises how strict this estimate is. To discuss this let us first write down the estimate we made:

$$\begin{aligned}\|x(t)\| &\leq \beta(\|x_0\|, t) + \gamma_1\big(\int_0^t \gamma_2(\|u(s)\|_U)ds) \\ &= (1 + Me^{\omega t}\|x_0\|)^2 - 1 + \exp\big(2M\|B\| \int_0^t \|u(s)\|_U ds\big) - 1.\end{aligned}$$

If we now look at $t = 0$ we get the following:

$$\|x(0)\| \leq (1 + M\|x_0\|)^2 - 1 = 2M\|x_0\| + M^2\|x_0\|^2.$$

Recalling that $M \geq 1$ we see that the estimate is more than twice as big as actually needed. Also note that if from a certain time t_0 the input is zero, the state will go to zero if time goes to infinity. However if we look at the 'γ-part' of the estimate this can only increase and will never decrease. Therefore this will also give a big estimate for times after this so called t_0.

It is hard to get a better feeling for these estimates by just looking at the formulas. Therefore we will look at an example.

Example 5.2. *In this example we will consider the system simple*

$$\begin{aligned}\dot{x}(t) &= -0.1x(t) + u(t)x(t), \\ x(0) &= x_0.\end{aligned}$$

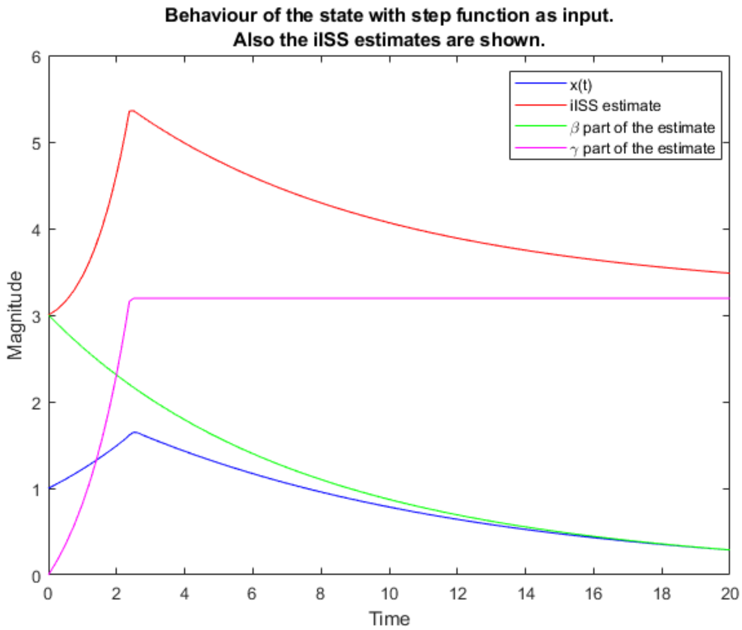

Figure 5.1: Behaviour of the state for the system $\dot{x}(t) = -0.1x(t) + u(t)x(t)$ with $x_0 = 1$ and input u a step-function that is 0.4 until $t = 2.5$ and 0 after $t = 2.5$. Also the iISS estimate is plotted, next to the total estimate also the 'β-part' and the 'γ-part' are shown.

In which we have $X = U = \mathbb{R}$. Using Matlab *the differential equation is solved for a certain input u and the solution $x(t)$ is plotted. To get an idea of how strict the estimates for iISS are we plotted $x(t)$, the total estimate, the 'β-part' of the estimate and the 'γ-part' of the estimate. This can be seen in Figure 5.1. In this case we used $x_0 = 1$ and for u a stepfunction that first is 0.4 and becomes equal to zero at $t = 2.5$.*

In Figure 5.1 it can be seen that the 'β-part' of the estimate is quite high at the beginning. Also it can be seen that when the time continuous the 'β-part' of the estimate goes to zero by which the 'γ-part' becomes important. This will always be the case, since the β-function has to decrease in time. Finally it can be seen that the 'γ-part' never decreases, so that even when the input becomes equal to zero and state is still allowed to have some off set.

Now that it is known that bilinear systems in general are iISS, but not ISS, the question now arises whether they are small–gain ISS. If we look at Example 3.7, $\dot{x} = -x + xu$ and constant inputs are considered. Then it can be seen that this system is stable if $u < 1$. So this could be a bound for the small–gain ISS. To show that bilinear systems are small–gain ISS we will use a Lyapunov method. Now first we will show using a Lyapunov function that all bilinear systems are iISS and then we will extend this proof to show that they are also small–gain ISS.

We have the system $\dot{x} = Ax + uBx$ and for this system a first guess for the Lyapunov function is the function $V(x) = x^*Px$ with P the positive symmetric solution of $A^*P + PA = -I$. This matrix P exists if A is Hurwitz and that is also a criterion for the system to be stable. Now we will show that with this choice of V Equation (3.6) is not satisfied and that this standard choice is not a Lyapunov function for bilinear systems as in Definition 3.11. This goes as follows,

$$\begin{aligned}\nabla V(x)f(x,u) &= 2x^*P\cdot(Ax+uBx)\\ &= 2x^*PAx+2x^*PuBx\\ &= -\|x\|^2+2ux^*PBx\\ &\leq -\|x\|^2+2|ux^*PBx|\\ &\leq -\|x\|^2+2|u|\cdot\|PB\|\cdot\|x\|^2\\ &\leq -\|x\|^2+\|x\|^4+4|u|^2\cdot\|PB\|^2.\end{aligned}$$

However, $-\|x\|^2+\|x\|^4$ is positive for larger x, thus we can never find a positive α function, such that this is smaller than $-\alpha$. Therefore this choice of V is not a Lyapunov function for bilinear systems as in Definition 3.11. In [17] a dissipation Lyapunov iISS theorem is given and with this theorem it is possible to prove that a bilinear system is iISS using the Lyapunov function x^*Px.

Theorem 5.3. *If we have a finite dimensional system as in Definition 3.2. There exists a positive-definite proper smooth function $V:\mathbb{C}^n\to\mathbb{R}_{\geq 0}$, such that there exists $\alpha_1,\alpha_2\in\mathcal{K}_\infty$ such that $\alpha_2(\|x\|)\leq V(x)\leq\alpha_2(\|x\|)$. If moreover, there exists a constant $q>0$ and $\gamma_1,\gamma_2\in\mathcal{K}_\infty$ so that*

$$\nabla V(x)f(x,u)\leq(\gamma_1(\|u\|)-q)V(x)+\gamma_2(\|u\|) \tag{5.5}$$

for all $x\in\mathbb{C}^n$ and $u\in U$, then the system is iISS.

Proof. In this proof we will show that when a Lyapunov function as in Theorem 5.3 exists, that then also Equation (3.3) is satisfied and thus the system is iISS. To do this we will start with the estimation of $V(x)$ and we want to get an estimation of $\|x\|$. The steps made in this proof are quite similar to the steps in the proof of Proposition 5.1. First we have an equation with a derivative of V and we want to get an equation with V. Also there exists $\gamma\in\mathcal{K}_\infty$ such that $\gamma_1(s)\leq\gamma(s)$ for all $s\geq 0$ and $\gamma_2(s)\leq\gamma(s)$ for all $s\geq 0$. This gives the following estimation,

$$\begin{aligned}\nabla V(x(t))f(x(t),u(t)) &\leq (\gamma_1(\|u(t)\|)-q)V(x(t))+\gamma_2(\|u(t)\|)\\ \frac{d}{dt}V(x(t)) &\leq (\gamma(\|u(t)\|)-q)V(x(t))+\gamma(\|u(t)\|).\end{aligned}$$

Now we multiply both sides with e^{qt} and get,

$$\begin{aligned}e^{qt}\nabla V(x(t))f(x(t),u(t)) &\leq e^{qt}(\gamma(\|u(t)\|)-q)V(x(t))+e^{qt}\gamma(\|u(t)\|)\\ e^{qt}\frac{d}{dt}V(x(t))+e^{qt}qV(x(t)) &\leq e^{qt}\gamma(\|u(t)\|)V(x)+e^{qt}\gamma(\|u(t)\|)\\ \frac{d}{dt}(V(x(t))e^{qt}) &\leq e^{qt}\gamma(\|u(t)\|)V(x(t))+e^{qt}\gamma(\|u(t)\|)\\ V(x(t))e^{qt}-V(x(0)) &\leq \int_0^t e^{q\tau}\gamma(\|u(\tau)\|)(V(x(\tau))+1)d\tau\\ V(x(t))e^{qt} &\leq V(x_0)+\int_0^t e^{q\tau}\gamma(\|u(\tau)\|)(V(x(\tau))+1)d\tau.\end{aligned}$$

We will substitute $z(t) = e^{qt}V(x(t))$,

$$
\begin{aligned}
z(t) &\leq V(x_0) + \int_0^t \gamma(\|u(\tau)\|)(z(\tau) + e^{q\tau})d\tau \\
z(t) &\leq V(x_0) + \int_0^t \gamma(\|u(\tau)\|)e^{q\tau}d\tau + \int_0^t \gamma(\|u\|)z(\tau)d\tau.
\end{aligned}
$$

This is now in the correct form to apply Lemma 2.2, so we get the following:

$$
\begin{aligned}
z(t) &\leq V(x_0) + \int_0^t \gamma(\|u(\tau)\|)e^{q\tau}d\tau + \int_0^t \gamma(\|u(\tau)\|)z(\tau)d\tau \\
z(t) &\leq (V(x_0) + \int_0^t \gamma(\|u(\tau)\|)e^{q\tau}d\tau) \exp(\int_0^t \gamma(\|u(\tau)\|)d\tau) \\
e^{qt}V(x(t)) &\leq (V(x_0) + \int_0^t \gamma(\|u(\tau)\|)e^{q\tau}d\tau) \exp(\int_0^t \gamma(\|u(\tau)\|)d\tau) \\
V(x(t)) &\leq e^{-qt}V(x_0)\exp(\int_0^t \gamma(\|u(\tau)\|)d\tau) + \int_0^t \gamma(\|u(\tau)\|)e^{-q(t-\tau)}d\tau \exp(\int_0^t \gamma(\|u(\tau)\|)d\tau).
\end{aligned}
$$

Here we substituted $z(t) = e^{qt}V(x(t))$ to get back to x. Now we note that $e^{-q(t-\tau)} \leq 1$ and to simplify we substitute $h(t) = \int_0^t \gamma(\|u(\tau)\|)d\tau$. What we want in the end is an estimate of $\|x\|$ with a β function depending on x_0 and t and a γ_1 function depending on u. Right now we still have the product of x_0 and a term with u. We apply a little trick to get this to a sum and to make sure that in the end we get $\gamma_1(0) = 0$. This trick is as follows:

$$
\begin{aligned}
e^{-qt}V(x_0)\exp(h(t)) &= e^{-qt}V(x_0) + e^{-qt}V(x_0)(\exp(h(t)) - 1) \\
&\leq e^{-qt}V(x_0) + \frac{1}{2}e^{-2qt}V(x_0)^2 + \frac{1}{2}(\exp(h(t)) - 1)^2.
\end{aligned}
$$

Here we used that $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$. We will now use all of this to rewrite the estimation of $V(x(t))$,

$$
\begin{aligned}
V(x(t)) &\leq e^{-qt}V(x_0)\exp(\int_0^t \gamma(\|u(\tau)\|)d\tau) + \int_0^t \gamma(\|u(\tau)\|)e^{-q(t-\tau)}d\tau \exp(\int_0^t \gamma(\|u(\tau)\|)d\tau) \\
V(x(t)) &\leq e^{-qt}V(x_0)\exp(\int_0^t \gamma(\|u(\tau)\|)d\tau) + \int_0^t \gamma(\|u(\tau)\|)d\tau \exp(\int_0^t \gamma(\|u(\tau)\|)d\tau) \\
V(x(t)) &\leq e^{-qt}V(x_0)\exp(h(t)) + h(t)\exp(h(t)) \\
V(x(t)) &\leq e^{-qt}V(x_0) + \frac{1}{2}e^{-2qt}V(x_0)^2 + \frac{1}{2}(\exp(h(t)) - 1)^2 + h(t)\exp(h(t)).
\end{aligned}
$$

For the next steps we will use that there exist $\alpha_1(s), \alpha_2(s) \in \mathcal{K}_\infty$ such that $\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$. Also we define $\mathcal{K}_\infty$ functions θ_1 and θ_2 to be $\theta_1(r) = r + \frac{1}{2}r^2$ and $\theta_2(r) = \frac{1}{2}(r-1)^2 + re^r$.

This gives

$$\begin{aligned}
V(x(t)) &\leq e^{-qt}V(x_0) + \frac{1}{2}e^{-2qt}V(x_0)^2 + \frac{1}{2}(\exp(h(t)) - 1)^2 + h(t)\exp(h(t)) \\
\alpha_1(\|x(t)\|) &\leq e^{-qt}V(x_0) + \frac{1}{2}e^{-2qt}V(x_0)^2 + \frac{1}{2}(\exp(h(t)) - 1)^2 + h(t)\exp(h(t)) \\
\alpha_1(\|x(t)\|) &\leq \theta_1(e^{-qt}V(x_0)) + \theta_2(h(t)) \\
\alpha_1(\|x(t)\|) &\leq \theta_1(e^{-qt}\alpha_2(\|x_0\|)) + \theta_2(h(t)) \\
\|x(t)\| &\leq \alpha_1^{-1}(\theta_1(e^{-qt}\alpha_2(\|x_0\|)) + \theta_2(h(t))) \\
\|x(t)\| &\leq \alpha_1^{-1}(2\theta_1(e^{-qt}\alpha_2(\|x_0\|))) + \alpha_1^{-1}(\theta_1(h(t)) \\
\|x(t)\| &\leq \alpha_1^{-1}(2\theta_1(e^{-qt}\alpha_2(\|x_0\|))) + \alpha_1^{-1}(\theta_1(\int_0^t \gamma(\|u(\tau)\|)d\tau)).
\end{aligned}$$

So $\beta(\|x_0\|, t) = \alpha_1^{-1}(2\theta_1(e^{-qt}\alpha_2(\|x_0\|)))$, $\tilde{\gamma}_1(s) = \alpha_1^{-1}(\theta_1(s))$ and $\tilde{\gamma}_2(s) = \gamma(s)$. Thus Equation (3.3) is satisfied and the system is iISS. □

What is interesting to observe is that these different theorems need different Lyapunov functions. For example where $V(x) = x^*Px$ was not sufficient to show that bilinear systems are iISS regarding Equation (3.6), this Lyapunov function will work to show that bilinear systems are iISS using Theorem 5.3, which will be shown now and which will give a start for showing small–gain ISS.

Proposition 5.4. *If A is Hurwitz, then the Lyapunov function $V(x) = x^*Px$, with P the solution of $AP + PA^* = -I$, is a Lyapunov function for bilinear systems as in Theorem 5.3, thus bilinear systems are iISS.*

Proof. To show this we will use the Lyapunov function $V(x) = x^*Px$ with P the solution of $PA + A^*P = -I$,

$$\begin{aligned}
\nabla V(x)f(x, u) &= 2x^*PAx + 2ux^*PBx \\
&= -\|x\|^2 + 2ux^*PBx \\
&\leq -\|x\|^2 + 2|u|\|x\|^2\|PB\| \\
&\leq (2|u|\|PB\| - 1)\|x\|^2 \\
&\leq (r2|u|\|PB\| - q)V(x).
\end{aligned}$$

Where $\frac{1}{r}$ is the smallest eigenvalue of P and $\frac{1}{q}$ is the largest eigenvalue of P. The last step needs some extra explanation, since P is positive definite and symmetric, P has only positive eigenvalues. Then it holds that $\frac{1}{r}\|x\|^2 \leq x^*Px \leq \frac{1}{q}\|x\|^2$, thus $\|x\|^2 \leq rV(x)$ and $-\|x\|^2 \leq -qV(x)$. Thus the requirements from Theorem 5.3 are satisfied and the bilinear system is iISS. □

We had already proved that bilinear systems are iISS, but above we have shown that it is also possible to prove this with a Lyapunov function. We can continue from here for small–gain ISS.

Proposition 5.5. *A bilinear system is small–gain ISS if and only if A is Hurwitz. In that case, R can be chosen to be $R = \frac{1}{2\|B\|\|P\|}$. In which P is the positive solution of the Lyapunov equation $PA + A^*P = -I$.*

Proof. Take $V(x) = x^* P x$, with P the solution of $PA + A^* P = -I$. Then we assume that there is an $R > 0$ such that $\|u\|_\infty < R$. This R we can still choose, but we will already assume there is one. This gives the following estimation:

$$\begin{aligned} \nabla V(x) f(x,u) &\leq -\|x\|^2 + 2|u| \cdot \|PB\| \cdot \|x\|^2 \\ &\leq -\|x\|^2 + 2R\|P\| \cdot \|B\| \cdot \|x\|^2 \\ &= (-1 + 2R\|P\| \cdot \|B\|)\|x\|^2. \end{aligned} \tag{5.6}$$

To prove the system is small–gain ISS we need to show that Equation (3.5) is satisfied. Thus we need that $(1 - 2R\|P\|\|B\|)s^2 = \alpha(s) \in \mathcal{K}_\infty$. This is the case if $1 - 2R\|P\|\|B\| > 0$, thus if $R < \frac{1}{2\|B\|\|P\|}$. In fact, equality will also work since it already holds that $\|u\| < R$. □

This chapter can be summarised with the following theorem for bilinear systems:

Theorem 5.6. *For bilinear systems $\dot{x} = Ax + uBx$, the following are equivalent:*

1. *The system is iISS,*
2. *The matrix A is Hurwitz,*
3. *The system is strong iISS,*
4. *The system is small–gain ISS.*

Proof. $1 \Leftrightarrow 2$ follows from Proposition 5.1. $2 \Leftrightarrow 4$ follows from Proposition 5.5. Because of the definition of strong iISS we have that $3 \Rightarrow 1 \wedge 4$ and because 2 implies both 1 and 4 we have that $2 \Rightarrow 3$. Thus the proof is completed. □

Chapter 6

Infinite dimensional systems

After having reviewed the ISS properties of finite dimensional linear and bilinear systems we will have a look at infinite dimensional systems. At first linear systems will be considered and after that bilinear systems will be considered.

Infinite dimensional systems are formulated using more general operators instead of matrices. For these operators we distinguish bounded and unbounded operators which are defined below.

Definition 6.1. *An operator $B : D(B) \to X$ with $D(B) \subseteq X$, is called bounded if*

$$\|B\| = \sup_{x \in D(B), x \neq 0} \frac{\|Bx\|}{\|x\|} < \infty.$$

An operator that is not bounded is called unbounded.

If we make the step from finite dimensional systems to infinite dimensional systems often the requirement that A is Hurwitz is replaced by the requirement that A should generate an exponentially stable semigroup. For finite dimensional matrix operators this requirement is equivalent and in that case we have that $T(t) = e^{At}$. Using Lemma 4.1 it is easy to show that this indeed is a semigroup. However, if we make the switch to infinite dimensional systems it is not sufficient that all eigenvalues of A are negative in order for A to generate an exponentially stable semigroup. This can for example be seen in [7].

6.1 Linear systems

There are two categories of infinite dimensional linear systems. The difference is whether B is bounded or not. The 'easy' case is when B is bounded, therefore this will be treated first. After that the case where B is unbounded will be treated, what is done for the systems with unbounded B will also hold for the case B is bounded.

6.1.1 Bounded B

At first we define general linear systems with bounded B. This can be seen as an example of the system class defined in Definition 3.1.

Definition 6.2. *Let $A : D(A) \to X$ with $D(A) \subseteq X$ and $B \in \mathcal{L}(U, X)$, let A generate a semigroup T, we call*

$$\phi(t, x_0, u) = x(t) = T(t)x_0 + \int_0^t T(t-s)Bu(s)ds \tag{6.1}$$

a mild solution related to the formal equation

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{6.2}$$

with $(x_0, u) \in D = X \times U^{\mathbb{R}\geq 0}$ and u piecewise continuous. We call (X, U, ϕ) a (infinite dimensional) linear system.

Remark 6.3. *Let us show that the system from Definition 6.2 indeed satisfies the condition from Definition 3.1. Therefore, we will show that (X, U, ϕ) satisfies the three properties of a system. To show this, we need that ϕ is well defined. This is the case since the integral in Equation* (6.1) *exists in X as X-valued Lebesgue integral as can be seen in [5].*

1. *We have to show that $\phi(0, x, u) = x$, this is the case since*

$$\phi(0, x_0, u) = T(0)x_0 + \int_0^0 T(-s)Bu(s)ds = x_0.$$

2. *We have to show that $(\phi(t, x, u), u(t + \cdot)) \in D$ and $\phi(t + h, x, u) = \phi(h, \phi(t, x, u), u(t + \cdot))$, this is the case since at first $x(t) = \phi(t, x_0, u) \in X$, which holds because the integral exists in X, thus $(\phi(t, x_0, u), u(t + \cdot)) \in D$ and secondly*

$$\begin{aligned}\phi(t+h, x, u) &= T(t+h)x_0 + \int_0^{t+h} T(t+h-s)Bu(s)ds \\ &= T(h)(T(t)x_0 + \int_0^t T(t-s)Bu(s)ds) + \int_t^{t+h} T(t+h-s)Bu(s)ds \\ &= \phi(h, \phi(t, x_0, u), u(t + \cdot)).\end{aligned}$$

3. *We have to show that $(x, u|_{[0,t]}) \in D$ and $u|_{[0,t]} = u_2|_{[0,t]}$ implies that $\phi(t, x, u) = \phi(t, x, u_2)$, this is the case since to determine $x(t)$ only $u(s)|_{[0,t]}$ is used, so if the input on this interval is the same, then $x(t)$ will be the same.*

Therefore, the system in Definition 6.2 is indeed a system as defined in Definition 3.1.

In practice ystems are often written in Ordinary Differential Equation (ODE) form, this is also possible for systems as in Definition 6.2 and can be seen in the next remark.

Remark 6.4. *The ODE stated below will be understood in the sense of Definition 6.2, which gives its mild solution. The ODE is as follows:*

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t), \\ x(0) &= x_0.\end{aligned}$$

Where $x \in X$, $u \in U$ and $u : [0, \infty) \to U$ is piecewise continuous. Next to that $B : U \to X$ and $A : D(A) \to X$ has to generate a semigroup. (6.1).

We will now show how to solve the ODE in Remark 6.4 to get the mild solution stated in Equation (6.1) using an integrating factor. This goes as follows,

$$\begin{aligned}
\frac{dx}{dt}(t) &= Ax(t) + Bu(t) \\
\frac{dx}{dt} - Ax(t) &= Bu(t) \\
e^{-At}\frac{dx}{dt} - e^{-At}Ax(t) &= e^{-At}Bu(t) \\
\frac{d}{dt}(e^{-At}x(t)) &= e^{-At}Bu(t) \\
\int_0^t \frac{d}{d\tau}(e^{-A\tau}x(\tau))d\tau &= \int_0^t e^{-A\tau}Bu(\tau)d\tau \\
e^{-At}x(t) - x(0) &= \int_0^t e^{-A\tau}Bu(\tau)d\tau \\
x(t) &= e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau
\end{aligned}$$

So indeed the solution of the ODE in Remark 6.4 is Equation (6.1). For the finite dimensional linear systems and the infinite dimensional linear system with bounded B this gives the exact solution. However, if B is unbounded the solution given here cannot be seen as an exact solution, but is a mild solution.

For the situation with the linear system with bounded B the proof to show that the system is ISS is similar to what is done in Theorem 4.2. It does still hold that ISS implies small–gain ISS by definition.

Theorem 6.5. *For infinite dimensional linear systems as in Definition 6.2 the following statements are equivalent:*

1. *A generates an exponentially stable semigroup,*
2. *The system is ISS,*
3. *The system is iISS,*
4. *The system is small–gain ISS,*
5. *The system is strong iISS.*

Proof. First we will show $2 \Rightarrow 1$. We already know that A generates a semigroup T, we only have to show that this semigroup is exponentially stable. If we set u equal to zero we know from the ISS or iISS property that for all $t \geq 0$,

$$\|x(t)\| = \|T(t)x_0\| \leq \beta(\|x_0\|, t).$$

If we use a slightly different, but equivalent definition of the operator norm that we used before, we get the following for every $t \geq 0$:

$$\|T(t)\| = \sup_{\|x_0\| \leq 1} \|T(t)x_0\| \leq \sup_{\|x_0\| \leq 1} \beta(\|x_0\|, t) \leq \beta(1, t).$$

Where the last inequality holds since β is increasing in $\|x_0\|$. Because β is decaying in t we know that there is a $t_0 \geq 0$ such that $\beta(1, t) < 1$ for all $t \geq t_0$. Therefore we know that there exists a $t_0 \geq 0$ such that $\|T(t)\| < 1$ for all $t \geq t_0$. Now we also know that there exists an $\omega < 0$

such that $\|T(t_0)\| = e^{\omega t_0}$. Moreover we know by the uniform boundedness principle that for every compact interval in $[0, \infty)$ we have that $\|T(\cdot)\|$ is bounded by a constant. This will also hold for the interval $[0, t_0]$. Thus there exists a $C \geq 0$ such that for any $r \in [0, t_0]$ we have that $\|T(r)\| \leq C$. Now we will combine this together with the properties of a semigroup to show that T is exponentially stable. To do we use that for every $t \geq 0$ there exists an $r \in [0, t_0]$ and an $n \in \mathbb{N}$ such that $t = nt_0 + r$. This gives,

$$\begin{aligned}\|T(t)\| &= \|T(nt_0 + r)\| = \|T(t_0)^n T(r)\| \\ &\leq \|T(t_0)\|^n \|T(r)\| \leq Ce^{n\omega t_0} = Ce^{n\omega \frac{t-r}{n}} \\ &= Ce^{\omega t} e^{-\omega r} = Me^{\omega t},\end{aligned}$$

in which $M = Ce^{-\omega r}$. Therefore the semigroup is exponentially stable and thus, $2 \Rightarrow 1$, $3 \Rightarrow 1$ and $4 \Rightarrow 1$. By the definition of small–gain ISS we have that $2 \Rightarrow 4$ and by the definition of strong iISS we have that $5 \Rightarrow 3$ and that $3 \wedge 4 \Rightarrow 5$. Thus we now only have have $1 \Rightarrow 2$ and $1 \Rightarrow 3$ left to show.

First we will show $1 \Rightarrow 2$. We will do this by estimating the norm of $x(t)$ to get Equation (3.2). To do this we will use the mild solution defined in Definition 6.2. Next to that we will use that for an exponentially stable semigroup T there exist $M \geq 1$ and $\omega < 0$ such that $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$. This gives,

$$\begin{aligned}\|x(t)\| &= \|T(t)x_0 + \int_0^t T(t-s)Bu(s)ds\| \\ &\leq \|T(t)x_0\| + \|\int_0^t T(t-s)Bu(s)ds\| \\ &\leq \|T(t)\|\|x_0\| + \int_0^t \|T(t-s)\|\|B\|\|u(s)\|ds \\ &\leq Me^{\omega t}\|x_0\| + \int_0^t M\|B\|e^{\omega(t-s)}\|u(s)\|ds.\end{aligned}$$

Next we will apply Hölder's inequality on the integral part with the infinity-norm on u and the 1-norm on the other terms,

$$\begin{aligned}\|x(t)\| &\leq Me^{\omega t}\|x_0\| + \int_0^t M\|B\|e^{\omega(t-s)}\|u(s)\|ds \\ &\leq Me^{\omega t}\|x_0\| + M\|B\| \int_0^t e^{\omega(t-s)}ds\|u\|_{\infty[0,t]} \\ &\leq Me^{\omega t}\|x_0\| + M\|B\|(-\frac{1}{\omega} + \frac{1}{\omega}e^{\omega t})\|u\|_{\infty[0,t]} \\ &\leq Me^{\omega t} - M\|B\|\frac{1}{\omega}\|u\|_{\infty[0,t]}.\end{aligned}$$

Where the last step holds since $\omega < 0$. Therefore, Equation (3.2) is satisfied with

$$\beta(\|x_0\|, t) = M^{\omega t}\|x_0\| \text{ and } \gamma(s) = -M\|B\|\frac{1}{\omega}s,$$

thus $1 \Rightarrow 2$.

Finally we will show that 1 ⇒ 3. This goes similar as before, only Hölder's inequality will not be applied. So we already know the following:

$$\|x(t)\| \leq Me^{\omega t}\|x_0\| + M\|B\| \int_0^t e^{\omega(t-s)}\|u(s)\|ds.$$

Now we use that $e^{\omega(t-s)} \leq 1$ and then we get that the system is iISS with $\beta(\|x_0\|, t) = M^{\omega t}\|x_0\|$, $\gamma_1(s) = M\|B\|s$ and $\gamma_2(s) = s$. Thus 1 ⇒ 3, which completes the proof. ☐

Remark 6.6. *For iISS different choices of γ_1 and γ_2 are allowed if further estimations are made. Therefore, we will apply Hölder's inequality with a general q-norm for u and a p-norm for the other terms satisfying $\frac{1}{p} + \frac{1}{q} = 1$. This gives,*

$$\begin{aligned}\|x(t)\| &\leq Me^{\omega t}\|x_0\| + M\|B\|(\int_0^t e^{p\omega(t-s)}ds)^{\frac{1}{p}}(\int_0^t \|u(s)\|^q ds)^{\frac{1}{q}} \\ &\leq Me^{\omega t}\|x_0\| + M\|B\|(-\frac{1}{p\omega} + \frac{1}{p\omega}e^{p\omega t})^{\frac{1}{p}}(\int_0^t \|u(s)\|^q ds)^{\frac{1}{q}} \\ &\leq Me^{\omega t}\|x_0\| + M\|B\|(-\frac{1}{p\omega})^{\frac{1}{p}}(\int_0^t \|u(s)\|^q ds)^{\frac{1}{q}}.\end{aligned}$$

Where the last step holds since $\omega < 0$. Thus Equation (3.3) *is also satisfied with $\beta(\|x_0\|, t) = M^{\omega t}\|x_0\|$, $\gamma_1(s) = M\|B\|(-\frac{1}{p\omega})^{\frac{1}{p}} s^{\frac{1}{q}}$ and $\gamma_2(s) = s^q$.*

6.1.2 Unbounded B

To discuss all the results regarding infinite dimensional systems with unbounded operator B we need another notion that is not yet discussed. This is an extension of X, named X_{-1}. To define this we use the resolvent operator based on A. We take a $\lambda \in \mathbb{C}$ in the resolvent set of A, then we say that all x in the completion of X with $\|(\lambda I - A)^{-1}x\|_X < \infty$ are in X_{-1}, this set is independent of λ. Now we can define the operator A_{-1} as an operator that has the same operation as A, but with $D(A_{-1}) = X$, so $A_{-1} : X \to X_{-1}$. If A generates a semigroup, then the corresponding semigroup T_{-1} generated by A_{-1} has the same action as T and has $T_{-1} : X_{-1} \to X_{-1}$.

Now this is made clear we can define infinite dimensional linear systems with unbounded B.

Definition 6.7. *Let $A : D(A) \to X_{-1}$ with $D(A) \subseteq X$ and $B \in \mathcal{L}(U, X_{-1})$, let A generate a semigroup T on X. We call*

$$\phi(t, x_0, u) = x(t) = T(t)x_0 + \int_0^t T(t-s)Bu(s)ds \tag{6.3}$$

a mild solution, with values in X_{-1}, related to the formal equation

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{6.4}$$

with u piecewise continuous and $(x_0, u) \in D \subseteq X \times U^{\mathbb{R}_{\geq 0}}$ such that $x \in X$ for all $t \geq 0$. We call (X, U, ϕ) a (infinite dimensional) unbounded linear system.

Remark 6.8. *Let us show that the system from Definition 6.7 indeed satisfies the conditions from Definition 3.1. Therefore, we will show that (X, U, ϕ) satisfies the three properties of a system. To show this, we need that ϕ is well defined. This is the case since the integral in Equation* (6.3) *exists in X_{-1} as X_{-1}-valued Lebesgue integral is bounded as can be seen in [5].*

1. *We have to show that $\phi(0, x, u) = x$, this is the case since*

$$\phi(0, x_0, u) = T(0)x_0 + \int_0^0 T(-s)Bu(s)ds = x_0.$$

2. *We have to show that $(\phi(t, x, u), u(t + \cdot)) \in D$ and $\phi(t + h, x, u) = \phi(h, \phi(t, x, u), u(t + \cdot))$, this is the case since at first $(\phi(t, x_0, u), u(t + \cdot)) \in D$ by the definition of D and secondly*

$$\begin{aligned}\phi(t + h, x, u) &= T(t + h)x_0 + \int_0^{t+h} T(t + h - s)Bu(s)ds \\ &= T(h)(T(t)x_0 + \int_0^t T(t - s)Bu(s)ds) + \int_t^{t+h} T(t + h - s)Bu(s)ds \\ &= \phi(h, \phi(t, x_0, u), u(t + \cdot)).\end{aligned}$$

3. *We have to show that $(x, u|_{[0,t]}) \in D$ and $u|_{[0,t]} = u_2|_{[0,t]}$ implies that $\phi(t, x, u) = \phi(t, x, u_2)$, this is the case since to determine $x(t)$ only $u(s)|_{[0,t]}$ is used, so if the input on this interval is the same, then $x(t)$ will be the same.*

Therefore, the system in Definition 6.7 is indeed a system as defined in Definition 3.1.

In practice systems are often written in ODE form, this is also possible for systems as in Definition 6.7 and can be seen in the next remark.

Remark 6.9. *The ODE stated below will be understood in the sense of Definition 6.7, which gives its mild solution. The ODE is as follows:*

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t), \\ x(0) &= x_0.\end{aligned}$$

Where $x \in X_{-1}$, $u \in U$ and $u : [0, \infty) \to U$ is piecewise continuous. Next to that $B : U \to X_{-1}$ and $A : D(A) \to X_{-1}$ has to generate a semigroup.

Under a certain condition these systems are iISS, however, the condition that we will show only holds for 'diagonal' systems, therefore we will now define what 'diagonal' systems are.

Definition 6.10. *We call an operator $A : D(A) \to X$ with $D(A) \subseteq X$ diagonal, if A possesses an orthonormal basis of X consisting of eigenvectors $(\mathrm{e}_n)_{n\in\mathbb{N}}$ with eigenvalues $(a_n)_{n\in\mathbb{N}}$ lying in a sector in the open left half-plane $\mathbb{C}_-$, where the vertex of the sector is at zero and the opening angle is less than π.*

For a diagonal operator $A : D(A) \to X$ with $D(A) \subseteq X$, we therefore have that for any $x \in X$ there exists a unique sequence of coefficients $(x_n) \in \ell^2$ such that $x = \sum_{n\in\mathbb{N}} x_n \mathrm{e}_n$. Moreover, it follows that

$$D(A) = \{x = \sum x_n\mathrm{e}_n : \sum a_n x_n \mathrm{e}_n \text{ converges } \} = \{x = \sum x_n \mathrm{e}_n : (a_n x_n) \in \ell^2\}.$$

For infinite dimensional linear systems with unbounded B the following result is proved in [10].

Theorem 6.11. *Let $U = \mathbb{C}$, and assume that the operator A is diagonal as defined in Definition 6.10. Let X_{-1} be defined as mentioned in the beginning of Section 6.1.2 and $B \in \mathcal{L}(U, X_{-1})$. Then the system as in Definition 6.7 is iISS.*

For the proof see [10]. Since we will encounter a similar proof for bilinear systems, see Section 6.2.2, we refrain from including it here.

6.2 Bilinear systems

6.2.1 Bounded B

At first we define general linear systems with bounded B. This can be seen as yet another example of Definition 3.1.

Definition 6.12. *Let $A : D(A) \to X$ with $D(A) \subseteq X$ and $B \in \mathcal{L}(U, X)$, let A generate a semigroup T. We call a function $x : [0, \infty) \to X$ solving*

$$x(t) = T(t)x_0 + \int_0^t T(t-s)u(s)Bx(s)ds \tag{6.5}$$

for all $t \geq 0$ a mild solution related to the formal equation

$$\dot{x}(t) = Ax(t) + u(t)Bx(t) \tag{6.6}$$

with $(x_0, u) \in D = X \times U^{\mathbb{R}_{\geq 0}}$ and u piecewise continuous. We call this solution $x(t) = \phi(t, x_0, u)$. We call (X, U, ϕ) an (infinite dimensional) bilinear system. With Theorem 2.5 it can be shown that a unique solution $x(t)$ of Equation (6.5) exists.

Remark 6.13. *Let us show that the system from Definition 6.12 indeed satisfies the conditions from Definition 3.1. Therefore, we will show that (X, U, ϕ) satisfies the three properties of a system. To show this, we need that ϕ is well defined. This is the case since the integral in Equation 6.5 exists in X as X-valued Lebesgue integral as can be seen in [5].*

1. *We have to show that $\phi(0, x, u) = x$, this is the case since*

$$\phi(0, x_0, u) = T(0)x_0 + \int_0^0 T(-s)u(s)Bx(s)ds = x_0.$$

2. *We have to show that $(\phi(t, x, u), u(t + \cdot)) \in D$ and $\phi(t + h, x, u) = \phi(h, \phi(t, x, u), u(t + \cdot))$, this is the case since at first $\phi(t, x_0, u) \in X$, which holds because the integral exists in X, thus $(\phi(t, x_0, u), u(t + \cdot)) \in D$ and secondly*

$$\begin{aligned}\phi(t+h, x, u) &= T(t+h)x_0 + \int_0^{t+h} T(t+h-s)u(s)Bx(s)ds \\ &= T(h)(T(t)x_0 + \int_0^t T(t-s)u(s)Bx(s)ds) + \int_t^{t+h} T(t+h-s)u(s)Bx(s)ds \\ &= \phi(h, \phi(t, x_0, u), u(t + \cdot)).\end{aligned}$$

   In which we used that Equation (6.5) has a unique solution x.

3. *We have to show that $(x, u|_{[0,t]}) \in D$ and $u|_{[0,t]} = u_2|_{[0,t]}$ implies that $\phi(t, x, u) = \phi(t, x, u_2)$, this is the case since to determine $x(t)$ only $u(s)|_{[0,t]}$ is used, so if the input on this interval is the same, then $x(t)$ will be the same.*

Therefore, the system in Definition 6.12 is indeed a system as defined in Definition 3.1.

In Equation (6.5) $x(t)$ is mentioned both at the left and right hand side. When using Definition 6.12, Theorem 2.5 should be used to show that there exists a $x(t)$ satisfying Equation (6.5). Now we will sketch the idea of how to show this. We will take a certain interval $I = [0, \delta]$ and have $t \in I$ and take $x, y : I \to X$. To apply Theorem 2.5 we will fix x_0 and u. Moreover, we will take $F(x)$ to be the right hand side of Equation (6.5), in which $x : [0, \delta] \to X$. We want to find a unique solution $x : [0, \delta] \to X$ that satisfies Equation (6.5). To do this we have to make δ small enough such that F becomes a contraction on the set of continuous functions from $[0, \delta]$ to X.

$$\begin{aligned}
\|F(x)(t) - F(y)(t)\| &= \|T(t)x_0 + \int_0^t T(t-s)u(s)Bx(s)ds - T(t)x_0 - \int_0^t T(t-s)u(s)By(s)ds\| \\
&\leq \| \int_0^t T(t-s)u(s)B(x(s) - y(s))ds\| \\
&\leq \int_0^t \|T(t-s)u(s)B(x(s) - y(s))\|ds \\
&\leq \int_0^t \|T(t-s)\|\|u(s)\|\|B\|\|x(s) - y(s)\|ds.
\end{aligned}$$

Now take $M \geq 1, \omega \in \mathbb{R}$ such that $\|T(t)\| \leq Me^{\omega t}$. This gives,

$$\begin{aligned}
&\leq \int_0^t Me^{\omega(t-s)}\|u(s)\|\|B\|\|x(s) - y(s)\|ds \\
&\leq \|B\|Me^{\omega t} \sup_{s\in[0,t]} (\|u(s)\|\|x(s) - y(s)\|_X) \int_0^t e^{-\omega s}ds \\
&\leq \|B\|Me^{\omega\delta} \sup_{s\in[0,\delta]} \|u(s)\| \sup_{s\in[0,\delta]} \|x(s) - y(s)\|_X \frac{1}{\omega}(1 - e^{-\omega\delta}).
\end{aligned}$$

The constant, $\|B\|Me^{\omega\delta} \sup_{s\in[0,\delta]} \|u(s)\| \frac{1}{\omega}(1 - e^{-\omega\delta})$, goes to zero if δ goes to zero. Therefore, the interval can be made such that the constant is smaller than 1. Then according to Theorem 2.5 there exists a solution $x(t)$ for t in the interval. Moreover, the interval can be moved and therefore it can be shown that a solution exists for all $t \geq 0$.

In practice systems are often written in ODE form, this is also possible for systems as in Definition 6.12 and can be seen in the next remark.

Remark 6.14. *The ODE stated below will be understood in the sense of Definition 6.12, which gives its mild solution. The ODE is as follows:*

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + u(t)Bx(t), \\
x(0) &= x_0.
\end{aligned}$$

Where $x \in X$, $u \in U$ and $u : [0, \infty) \to U$ is piecewise continuous. Next to that $B : U \to X$ and $A : D(A) \to X$ has to generate a semigroup.

We will show that infinite dimensional bilinear systems with bounded B are iISS. To show this, the proof of 5.1 should be extended.

Theorem 6.15. *An infinite dimensional bilinear system as in Definition 6.12 is iISS if and only if A generates an exponentially stable semigroup T.*

Proof. The proof is similar to the proof of Proposition 5.1. Only the beginning is a bit different, so we will show how to get to Equation (5.3) and from there the same steps can be applied, so they will not be repeated.

To get to Equation (5.3) we will start with the mild solution from Definition 6.12. We will take the norm and again use that for an exponentially stable semigroup T there exist $M \geq 1$ and $\omega < 0$ such that $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$. This gives,

$$\begin{aligned}
\|x(t)\| &= \|T(t)x_0 + \int_0^t T(t-s)u(s)Bx(s)ds\| \\
&\leq \|T(t)x_0\| + \|\int_0^t T(t-s)u(s)Bx(s)ds\| \\
&\leq \|T(s)\|\|x_0\| + \int_0^t \|T(t-s)\||u(s)|\|B\|\|x(s)\|ds \\
&\leq Me^{\omega t}\|x_0\| + \int_0^t Me^{\omega(t-s)}|u(s)\|B\|\|x(s)\|ds.
\end{aligned}$$

This is equal to Equation (5.3) and the proof can be continued like the proof in Proposition 5.1. Therefore, the system is iISS with $\beta(s,t) = (1 + Me^{\omega t}s)^2 - 1$, $\gamma_1(s) = e^{2s} - 1$ and $\gamma_2(s) = M|s|\|B\|$.

Finally, if the system is iISS then A generates an exponentially stable semigroup. This follows from the proof of Theorem 6.5, since if u is equal to zero the linear and bilinear system are identical. □

Next we will show that these infinite dimensional bilinear systems with bounded B are also small–gain ISS.

Theorem 6.16. *An infinite dimensional bilinear system as in Definition 6.12 with bounded B is small–gain ISS if and only if A is exponentially stable. In that case, R can be chosen to be $R = \frac{-\omega}{M\|B\|}$. In which $\omega < 0$ and $M \geq 1$ are such that $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$.*

Proof. We will only show '⇐', the other way is clear from what is mentioned before. For the finite dimensional case we used a Lyapunov argument to show small–gain ISS. However this cannot be extended to the infinite dimensional case. Therefore we will show small–gain ISS using Definition 3.6. This also is the reason why there is a different bound as in the finite dimensional case.

In the proof of Theorem 6.15 we already showed for the infinite dimensional case how to reach Equation (5.3). In the proof of Proposition 5.1 we showed how to go from Equation (5.3) to Equation (5.4). Therefore we will now continue from Equation (5.4) to show that infinite dimensional bilinear systems are small–gain ISS. In this we used that there exist $\omega < 0$ and $M \geq 1$ such that $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$. Furthermore we will use that there exists a $R > 0$ for which we will take $|u(t)| < R$ for all $t \geq 0$. This gives the following:

$$\begin{aligned}\|x(t)\| &\leq e^{\omega t} M \|x_0\| \exp(\int_0^t M|u(\tau)| \|B\| d\tau) \\ &\leq e^{\omega t} M \|x_0\| e^{MR\|B\|t} \\ &= Me^{(\omega + MR\|B\|)t} \|x_0\|.\end{aligned}$$

Which gives an estimate as in Definition 3.6 if $\omega + MR\|B\| < 0$. This means that the system is indeed small–gain ISS with $R = \frac{-\omega}{M\|B\|}$. However, it might be that there is a bigger R that is also allowed. □

The results given above can again be combined into one theorem.

Theorem 6.17. *For infinite dimensional bilinear systems with bounded B, systems as in Definition 6.12, the following are equivalent:*

1. *A generates a semigroup that is exponentially stable,*
2. *The system is iISS,*
3. *The system is strong iISS,*
4. *The system is small–gain ISS.*

Proof. $1 \Leftrightarrow 2$ follows from Theorem 6.15. $1 \Leftrightarrow 4$ follows from Theorem 6.16. Because of the definition of strong iISS we have that $3 \Rightarrow 2 \wedge 4$ and because 2 implies both 1 and 4 we have that $2 \Rightarrow 3$. □

6.2.2 Unbounded B

What starts to get more complicated is the same idea for the infinite dimensional bilinear system, but now with unbounded B. This does change the definition of the system slightly, so the system will be defined again.

Definition 6.18. *Let $A : D(A) \to X$ with $D(A) \subseteq X$ and $B \in \mathcal{L}(U, X_{-1})$, let A generate a semigroup T. We call*

$$x(t) = T(t)x_0 + \int_0^t T(t-s)u(s)Bx(s)ds \tag{6.7}$$

a mild solution related to the formal equation

$$\dot{x}(t) = Ax(t) + u(t)Bx(t) \tag{6.8}$$

with u piecewise continuous and $(x_0, u) \in D \subseteq X \times U^{\mathbb{R}_{\geq 0}}$ such that $x \in X$ for all $t \geq 0$. We call (X, U, ϕ) a (infinite dimensional) unbounded linear system. Due to Theorem 2.5 a unique solution $x(t)$ of Equation (6.7) *exists. We call this solution $\phi(t, x_0, u)$.*

Remark 6.19. *Let us show that the system from Definition 6.18 indeed satisfies the conditions from Definition 3.1. Therefore, we will show that (X, U, ϕ) satisfies the three properties of a system. To show this, we need that ϕ is well defined. This is the case since the integral in Equation* (6.7) *exists in X_{-1} as X_{-1}-valued Lebesgue integral as can be seen in [5].*

1. *We have to show that $\phi(0, x, u) = x$, this is the case since*

$$\phi(0, x_0, u) = T(0)x_0 + \int_0^0 T(-s)u(s)Bx(s)ds = x_0.$$

2. *We have to show that* $(\phi(t,x,u),u(t+\cdot)) \in D$ *and* $\phi(t+h,x,u) = \phi(h,\phi(t,x,u),u(t+\cdot))$, *this is the case since at first* $(\phi(t,x_0,u),u(t+\cdot)) \in D$ *by the definition of* D *and secondly*

$$\begin{aligned}\phi(t+h,x,u) &= T(t+h)x_0 + \int_0^{t+h} T(t+h-s)u(s)Bx(s)ds \\ &= T(h)(T(t)x_0 + \int_0^t T(t-s)u(s)Bx(s)ds) + \int_t^{t+h} T(t+h-s)u(s)Bx(s)ds \\ &= \phi(h,\phi(t,x_0,u),u(t+\cdot)).\end{aligned}$$

In which we used that Equation (6.7) has a unique solution x.

3. *We have to show that* $(x,u|_{[0,t]}) \in D$ *and* $u|_{[0,t]} = u_2|_{[0,t]}$ *implies that* $\phi(t,x,u) = \phi(t,x,u_2)$, *this is the case since to determine* $x(t)$ *only* $u(s)|_{[0,t]}$ *is used, so if the input on this interval is the same, then* $x(t)$ *will be the same.*

Therefore, the system in Definition 6.18 is indeed a system as defined in Definition 3.1.

In Equation (6.7) $x(t)$ is mentioned both at the left and right hand side. When using Definition 6.18, Theorem 2.5 should be used to show that there exists a function x satisfying Equation (6.7). This can be done similarly to the bounded case if the extended semigroup $T_{-1}(t) : X_{-1} \to X_{-1}$ is considered and if it is allowed that $x \in X_{-1}$. This results in a generalised solution $x : [0,\infty) \to X_{-1}$. When it is needed that $x \in X$ this needs to be shown separately, an example of how this is done can be seen in the proof of Theorem 6.21.

In practice systems are often written in ODE form, this is also possible for systems as in Definition 6.18 and can be seen in the next remark.

Remark 6.20. *The ODE stated below will be understood in the sense of Definition 6.18, which gives its mild solution. The ODE is as follows:*

$$\begin{aligned}\dot{x}(t) &= Ax(t) + u(t)Bx(t), \\ x(0) &= x_0.\end{aligned}$$

Where $x \in X_{-1}$, $u \in U$ *and* $u : [0,\infty) \to U$ *is piecewise continuous. Next to that* $B : U \to X_{-1}$ *and* $A : D(A) \to X_{-1}$ *has to generate a semigroup.*

We will now show that under certain circumstances this system is also iISS. Because B is unbounded the norm of B does not exist and cannot be used in the estimate for iISS. The proof is based on the proof of Proposition 5.1, but the norm of B is avoided and instead there is a requirement on a combination of A and B just like is done in [10] for the linear systems.

Theorem 6.21. *An infinite dimensional bilinear system as in Definition 6.18 with diagonal* A *and* B *is iISS if* A *generates an exponentially stable semigroup* T *and*

$$f(s) = \sum_{n\in\mathbb{N}} \frac{|b_n|^2}{|a_n|} e^{(a_n-\omega)s}$$

gives $f \in L^p[0,\infty)$ *with some* p *such that* $1 < p < \infty$. *Here,* a_n, $n \in \mathbb{N}$ *are the eigenvalues* A, b_n, $n \in \mathbb{N}$ *are the eigenvalues of* B *and* ω *is such that* $\omega - a_n < 0$ *for all* $n \in \mathbb{N}$. *If the system is iISS we have that*

$$\beta(s,t) = (1 + 2Mse^{\frac{1}{2}\omega t})^2 - 1, \gamma_1(s) = e^{2s} - 1 \text{ and } \gamma_2(s) = \frac{4^q K^q}{2q}|s|^{2q}$$

with q *such that* $\frac{1}{p} + \frac{1}{q} = 1$, $M \geq 1$ *such that* $\|T(t)\| \leq Me^{\omega t}$ *and* $K = \|f\|_{L_p}$.

Proof. As mentioned before this proof is based on the proof of Proposition 5.1, but does not use the norm of B. Therefore, also ideas from the proof of Theorem 6.11 are used, these can be found in [10]. We will again us that there exists an $M \geq 1$ and an $\omega < 0$ such that $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$. Moreover, we assume that ω is chosen such that $a_n - \omega < -\varepsilon$ holds for all n and an $\varepsilon > 0$. We will also use that e_n, $n \in \mathbb{N}$ is an orthonormal basis for X.

Since B is unbounded the first step is to show that the mild solution as in Definition 6.18 exists. The mild solution is

$$x(t) = T(t)x_0 + \int_0^t T(t-s)u(s)Bx(s)ds$$

and we will show that the integral exists as element of X, because $f \in L^p$. This is done as follows:

$$\begin{aligned}\|\int_0^t T(t-s)u(s)Bx(s)ds\|^2 &= \|\int_0^t (\sum_{n\in\mathbb{N}} e^{a_n(t-s)}b_n\mathrm{e}_n x_n(s))u(s)ds\|^2 \\ &\leq \sum_{n\in\mathbb{N}} |b_n|^2 \cdot |\int_0^t e^{a_n(t-s)}x_n(s)u(s)ds|^2 \\ &\leq \sum_{n\in\mathbb{N}} |b_n|^2 \cdot (\int_0^t e^{a_n(t-s)}|x_n(s)| \cdot |u(s)|ds)^2 \\ &\leq \sum_{n\in\mathbb{N}} \frac{|b_n|^2}{|a_n|^2} \cdot (\int_0^t |a_n|e^{a_n(t-s)}|x_n(s)| \cdot |u(s)|ds).\end{aligned}$$

This holds since the inequality occurs if the sum and the integral are interchanged. Another inequality occurs when the norm is taken into the sum and integral. Now we will apply Cauchy-Schwarz with $f(s) = \sqrt{|a_n|e^{a_n(t-s)}}|x_n(s)| \cdot |u(s)|$ and $g(s) = \sqrt{|a_n|e^{a_n(t-s)}}$ and then one of the integrals is always smaller than 1, so an upper bound is 1. Also the sum and the integral will be interchanged:

$$\begin{aligned}\|\int_0^t T(t-s)u(s)Bx(s)ds\|^2 &\leq \sum_{n\in\mathbb{N}} \frac{|b_n|^2}{|a_n|^2}(\int_0^t |a_n|e^{a_n(t-s)}|x_n(s)|^2|u(s)|^2ds) \cdot (\int_0^t |a_n|e^{a_n(t-s)}ds)^2 \\ &\leq \int_0^t (\sum_{n\in\mathbb{N}} \frac{|b_n|^2}{|a_n|}e^{a_n(t-s)})\|x(s)\|^2|u(s)|^2ds.\end{aligned}$$

Now we will multiply with $e^{-\omega t}$. Also we will add the term $e^{\omega s}e^{-\omega s}$, which equals 1. This will be rewritten to get f in the estimation:

$$\begin{aligned}\|\int_0^t T(t-s)u(s)Bx(s)ds\|^2 &\leq e^{-\omega t}\int_0^t (\sum_{n\in\mathbb{N}} \frac{|b_n|^2}{|a_n|}e^{a_n(t-s)})e^{\omega s}e^{-\omega s}\|x(s)\|^2|u(s)|^2ds \\ &\leq \int_0^t (\sum_{n\in\mathbb{N}} \frac{|b_n|^2}{|a_n|}e^{a_n(t-s)})e^{\omega s}e^{-\omega s}\|x(s)\|^2|u(s)|^2ds \\ &\leq \int_0^t (\sum_{n\in\mathbb{N}} \frac{|b_n|^2}{|a_n|}e^{(a_n-\omega)(t-s)})e^{-\omega s}\|x(s)\|^2|u(s)|^2ds \\ &= \int_0^t f(t-s)e^{-\omega s}\|x(s)\|^2|u(s)|^2ds.\end{aligned}$$

Now we will apply Hölder's inequality and apply a substitution in the integral over f, which will result in the norm of f that we named K. This gives,

$$\begin{aligned}\|\int_0^t T(t-s)u(s)Bx(s)ds\|^2 &\leq (\int_0^t f(t-s)^p ds)^{\frac{1}{p}} (\int_0^t e^{-q\omega s}\|x(s)\|^{2q}|u(s)|^{2q}ds)^{\frac{1}{q}} \\ &\leq (\int_0^t f(v)^p dv)^{\frac{1}{q}} \sup_{s\in[0,t]} (\|x(s)\|^{2q}|u(s)|^{2q}) \int_0^t e^{-\omega s}ds \\ &\leq K \sup_{s\in[0,t]} (\|x(s)\|^{2q}|u(s)|^{2q}) \frac{1}{\omega}(1-e^{-\omega t}).\end{aligned}$$

This shows that the integral is bounded and is properly defined. Later on we will show that this system is iISS, which implies that the state is bounded. Therefore, the integral exists. Note that this estimate will also work if the boundaries of the integral are changed. The only difference will be that the bound gets smaller. With the same estimate we can show with Theorem 2.5 that there actually exists a unique solution $x(t)$. The constant goes to zero if the interval gets smaller. Therefore, the interval can be constructed such that the constant is smaller than 1. Then according to Theorem 2.5 there exists a solution $x(t)$ for t in the interval. Moreover, the interval can be moved and therefore it can be shown that a solution exist for all $t \geq 0$.

Next we will show the system is iISS. Many of the estimations made on the integral are similar to what has been done above:

$$\begin{aligned}\|x(t)\|^2 &= \|T(t)x_0 + \int_0^t T(t-s)u(s)Bx(s)ds\|^2 \\ &\leq (\|T(t)\|\|x_0\| + \|\int_0^t T(t-s)u(s)Bx(s)ds\|)^2 \\ &\leq 2\|T(t)\|^2\|x_0\|^2 + 2\|\int_0^t T(t-s)u(s)Bx(s)ds\|^2 \\ &\leq 2M^2e^{2\omega t}\|x_0\|^2 + 2\|\int_0^t (\sum_{n\in\mathbb{N}} e^{a_n(t-s)}b_n \mathrm{e}_n x_n(s))u(s)ds\|^2 \\ &\leq 2M^2e^{2\omega t}\|x_0\|^2 + 2\sum_{n\in\mathbb{N}} |b_n|^2 \cdot |\int_0^t e^{a_n(t-s)}x_n(s)u(s)ds|^2 \\ &\leq 2M^2e^{2\omega t}\|x_0\|^2 + 2\sum_{n\in\mathbb{N}} |b_n|^2 \cdot (\int_0^t e^{a_n(t-s)}|x_n(s)|\cdot|u(s)|ds)^2 \\ &= 2M^2e^{2\omega t}\|x_0\|^2 + 2\sum_{n\in\mathbb{N}} \frac{|b_n|^2}{|a_n|^2} \cdot (\int_0^t |a_n|e^{a_n(t-s)}|x_n(s)|\cdot|u(s)|ds)^2.\end{aligned}$$

Now we will apply Cauchy-Schwarz with $f = \sqrt{|a_n|e^{a_n(t-s)}}|x_n(s)|\cdot|u(s)|$ and $g = \sqrt{|a_n|e^{a_n(t-s)}}$. This gives the following:

$$\begin{aligned}\|x(t)\|^2 &\leq 2M^2e^{2\omega t}\|x_0\|^2 + 2\sum_{n\in\mathbb{N}} \frac{|b_n|^2}{|a_n|^2}(\int_0^t |a_n|e^{a_n(t-s)}|x_n(s)|^2|u(s)|^2ds) \cdot (\int_0^t |a_n|e^{a_n(t-s)}ds) \\ &\leq 2M^2e^{\omega t}\|x_0\|^2 + 2\int_0^t (\sum_{n\in\mathbb{N}} \frac{|b_n|^2}{|a_n|}e^{a_n(t-s)})\|x(s)\|^2|u(s)|^2ds.\end{aligned}$$

The next step is to multiply the entire inequality with $e^{-\omega t}$ and substitute $z(t) = e^{-\omega t}\|x(t)\|^2$,

$$\|x(t)\|^2 \leq 2M^2 e^{\omega t}\|x_0\|^2 + 2\int_0^t (\sum_{n\in\mathbb{N}} \frac{|b_n|^2}{|a_n|} e^{a_n(t-s)})\|x(s)\|^2|u(s)|^2 ds$$
$$e^{-\omega t}\|x(t)\|^2 \leq 2M^2\|x_0\|^2 + 2e^{-\omega t}\int_0^t (\sum_{n\in\mathbb{N}} \frac{|b_n|^2}{|a_n|} e^{a_n(t-s)})\|x(s)\|^2|u(s)|^2 ds$$
$$e^{-\omega t}\|x(t)\|^2 \leq 2M^2\|x_0\|^2 + 2\int_0^t e^{-\omega(t-s)}(\sum_{n\in\mathbb{N}} \frac{|b_n|^2}{|a_n|} e^{a_n(t-s)})e^{-\omega s}\|x(s)\|^2|u(s)|^2 ds$$
$$z(t) \leq 2M^2\|x_0\|^2 + 2\int_0^t (\sum_{n\in\mathbb{N}} \frac{|b_n|^2}{|a_n|} e^{(a_n-\omega)(t-s)})z(s)|u(s)|^2 ds.$$

Next we will use a substitution of the variables, namely $v = t - s$ and later on $s = t - v$ to go back. Moreover, we will define $f(s) = \sum_{n\in\mathbb{N}} \frac{|b_n|^2}{|a_n|} e^{(a_n-\omega)s}$ and assume that $f \in L^p$. This implies that there exists a $K > 0$ such that $\|f\|_{L^p} \leq K$. This is useful after Hölder's inequality is applied. Since we choose ω such that $a_n - \omega < 0$ it seems to be reasonable, to assume that $f \in L^p$. The estimation is as follows:

$$z(t) \leq 2M^2\|x_0\|^2 + 2\int_0^t (\sum_{n\in\mathbb{N}} \frac{|b_n|^2}{|a_n|} e^{(a_n-\omega)(t-s)})z(s)|u(s)|^2 ds$$
$$z(t) \leq 2M^2\|x_0\|^2 + 2\int_0^t \frac{|b_n|^2}{|a_n|} e^{(a_n-\omega)v})z(t-v)|u(t-v)|^2 dv.$$

Now we will apply Hölder's inequality and use f like defined above. This gives,

$$z(t) \leq 2M^2\|x_0\|^2 + 2(\int_0^t f(v)^p dv)^{\frac{1}{p}}(\int_0^t (z(t-v)|u(t-v)|^2)^q dv)^{\frac{1}{q}}$$
$$z(t) \leq 2M^2\|x_0\|^2 + 2K(\int_0^t (z(s)|u(s)|^2)^q ds)^{\frac{1}{q}}$$
$$z(t)^q \leq (2M^2\|x_0\|^2 + 2K(\int_0^t (z(s)|u(s)|^2)^q ds)^{\frac{1}{q}})^q.$$

Now the inequality $(a+b)^q \leq (2a)^q + (2b)^q$ is applied for $a, b \geq 0$ and $q \geq 1$. These requirements are satisfied due to the norms and the fact that when applying Hölder's inequality you have that $p, q \geq 1$. Also Lemma 2.2 will be applied here:

$$z(t)^q \leq (4M^2\|x_0\|^2)^q + (4K(\int_0^t (z(s)|u(s)|^2)^q ds)^{\frac{1}{q}})^q$$
$$z(t)^q \leq 4^q M^{2q}\|x_0\|^{2q} + \int_0^t 4^q K^q z(s)^q|u(s)|^{2q} ds$$
$$z(t)^q \leq 4^q M^{2q}\|x_0\|^{2q} \exp(\int_0^t 4^q K^q|u(s)|^{2q} ds).$$

Now we will substitute $z(t) = e^{-\omega t}\|x(t)\|^2$ to again get x in there. This gives,

$$\begin{aligned}
(e^{-\omega t}\|x(t)\|^2)^q &\leq 4^q M^{2q}\|x_0\|^{2q}\exp(\int_0^t 4^q K^q |u(s)|^{2q} ds) \\
e^{-q\omega t}\|x(t)\|^{2q} &\leq 4^q M^{2q}\|x_0\|^{2q}\exp(\int_0^t 4^q K^q |u(s)|^{2q} ds) \\
\|x(t)\|^{2q} &\leq 4^q M^{2q}\|x_0\|^{2q} e^{q\omega t}\exp(\int_0^t 4^q K^q |u(s)|^{2q} ds) \\
\|x(t)\| &\leq 2M\|x_0\| e^{\frac{1}{2}\omega t}\exp(\frac{4^q K^q}{2q}\int_0^t |u(s)|^{2q} ds).
\end{aligned}$$

From now on we continue like in the proof of Proposition 5.1. So we again will apply $\alpha(r) = \ln(1+r)$ with the same properties and inverse as mentioned in the proof of Proposition 5.1:

$$\begin{aligned}
\alpha(\|x(t)\|) &\leq \alpha(2M\|x_0\| e^{\frac{1}{2}\omega t}\exp(\frac{4^q K^q}{2q}\int_0^t |u(s)|^{2q} ds)) \\
\alpha(\|x(t)\|) &\leq \ln(1 + 2M\|x_0\| e^{\frac{1}{2}\omega t}) + \frac{4^q K^q}{2q}\int_0^t |u(s)|^{2q} ds \\
\|x(t)\| &\leq \alpha^{-1}(\ln(1 + 2M\|x_0\| e^{\frac{1}{2}\omega t}) + \frac{4^q K^q}{2q}\int_0^t |u(s)|^{2q} ds) \\
\|x(t)\| &\leq \alpha^{-1}(2\ln(1 + 2M\|x_0\| e^{\frac{1}{2}\omega t})) + \alpha^{-1}(2\frac{4^q K^q}{2q}\int_0^t |u(s)|^{2q} ds) \\
\|x(t)\| &\leq (1 + 2M\|x_0\| e^{\frac{1}{2}\omega t})^2 - 1 + \exp(2\int_0^t \frac{4^q K^q}{2q}|u(s)|^{2q} ds) - 1.
\end{aligned}$$

Now we have Equation (3.3) with

$$\beta(s,t) = (1 + 2Mse^{\frac{1}{2}\omega t})^2 - 1, \gamma_1(s) = e^{2s} - 1 \text{ and } \gamma_2(s) = \frac{4^q K^q}{2q}|s|^{2q}.$$

Thus the infinite dimensional bilinear system is iISS if $f \in L^p$ and A generates an exponentially stable semigroup T. □

Remark 6.22. *The condition of $f \in L^p[0,\infty)$ is not a necessary condition. There are systems that do not satisfy this condition, but that are iISS. An example can be found in [9].*

Chapter 7

Examples

7.1 Linear system

At first we will start with an example of a linear infinite dimensional system. This is based on the PDE of the heat equation with Neumann boundary control on one side, and is as follows:

$$\begin{aligned}\frac{\partial}{\partial t}x(\xi,t) &= \frac{\partial^2}{\partial \xi^2}x(\xi,t) - \pi^2 x(\xi,t),\\ \frac{\partial}{\partial \xi}x(0,t) &= 0, \frac{\partial}{\partial \xi}x(1,t) = u(t),\\ x(\xi,0) = x_0(\xi) &\qquad \text{with } \xi \in (0,1), t > 0,\end{aligned}$$

where u is piecewise continuous.

From this PDE a state–space system can be generated. The state is still allowed to change in time, so we will have that $X = L^2(0,1)$, by which the 'location' is included in the state. At first we will assume $u = 0$ and determine A. After that B will be determined. If u is set to be identical to zero we can derive the form of A as follows:

$$\begin{aligned}Af &= \frac{\partial^2}{\partial \xi^2}f - \pi^2 If,\\ D(A) &= \{f \in H^2[0,1] : \frac{\partial^2}{\partial \xi^2}f \in X, \frac{\partial}{\partial \xi}f(0) = 0, \frac{\partial}{\partial \xi}f(1) = 0\},\end{aligned}$$

where H^2 refers to the Sobolev space. As you can see now on both boundaries there is the requirement that the derivative is zero, this would be the case if u is equal to zero. This A has a convenient property, namely that the eigenfunctions form an orthonormal basis for X. These eigenfunctions are as follows: $\mathrm{e}_0(\xi) = 1$, $\mathrm{e}_n(\xi) = \sqrt{2}\cos(n\pi\xi)$, $n \in \mathbb{N}$. We will first show that these are indeed eigenfunctions of A and after that we will show that this is an orthonormal basis for X. To show that they are eigenfunctions we will determine $A\mathrm{e}_0$ and $A\mathrm{e}_n$. We will note that this becomes equal to the corresponding eigenfunction multiplied by a constant, the eigenvalue. This can be seen here,

$$\begin{aligned}
A\mathrm{e}_0 &= \frac{\partial^2}{\partial \xi^2} 1 - \pi^2 \\
&= -\pi^2 \\
&= -\pi^2 \mathrm{e}_0, \\
A\mathrm{e}_n &= \frac{\partial^2}{\partial \xi^2} \sqrt{2}\cos(n\pi\xi) - \pi^2\sqrt{2}\cos(n\pi\xi) \\
&= -\sqrt{2}n^2\pi^2\cos(n\pi\xi) - \pi^2\sqrt{2}\cos(n\pi\xi) \\
&= (-n^2\pi^2 - \pi^2)\mathrm{e}_n.
\end{aligned}$$

So these are indeed eigenfunctions of A with the eigenvalues $a_n = -\pi^2(n^2+1)$. The next step is to show that these eigenfunctions form an orthonormal basis. To do this we will show that the inverse of $A + (1+\pi^2)I$ is

$$\begin{aligned}
(Sh)(x) = &\int_0^x \big(\cot(1)\cos(x)\cos(s) + \sin(x)\cos(s)\big)h(s)ds \\
&+ \int_x^t \big(\cot(1)\cos(x)\cos(s) + \sin(s)\cos(x)\big)h(s)ds.
\end{aligned}$$

From A.4.21 in [5] we know that this operator S is compact and self-adjoint. So if we show that S is indeed the inverse of $A + (1+\pi^2)I$, then the eigenfunctions of A form a basis. Now we will show that $(((A + (1+\pi^2)I)S)h)(x) = h(x)$ and that $(S(A + (1+\pi^2)I)h)(x) = h(x)$.

Lemma 7.1. *The inverse of the operator $A + (1+\pi^2)I$ with A being the operator $\frac{\partial^2}{\partial^2\xi} - \pi^2 I$ is*

$$\begin{aligned}
(Sh)(x) = &\int_0^x (\cot(1)\cos(x)\cos(s) + \sin(x)\cos(s))h(s)ds \\
&+ \int_x^t (\cot(1)\cos(x)\cos(s) + \sin(s)\cos(x))h(s)ds.
\end{aligned}$$

Proof. Now we will show that $(((A + (1+\pi^2)I)S)h)(x) = h$ and that $(S(A + (1+\pi^2)I)h)(x) = h(x)$:

$$\begin{aligned}
((A + (1+\pi^2)I)S)h)(x) &= ((\frac{d^2}{dx^2}S)h)(x) + (Sh)(x) \\
&= \frac{d^2}{dx^2}\big(\int_0^x (\cot(1)\cos(x)\cos(s) + \sin(x)\cos(s))h(s)ds + \\
&\quad \int_x^1 (\cot(1)\cos(x)\cos(s) + \sin(s)\cos(x))h(s)ds\big) + (Sh)(x) \\
&= \frac{d^2}{dx^2}\big(\cot(1)\cos(x)\int_0^x \cos(s)h(s)ds + \sin(x)\int_0^x \cos(s)h(s)ds + \\
&\quad \cot(1)\cos(x)\int_x^1 \cos(s)h(s)ds + \cos(x)\int_x^1 \sin(s)h(s)ds\big) + (Sh)(x).
\end{aligned}$$

Now we will differentiate this once using the product rule and rewrite the terms that cancel out,

$$
\begin{aligned}
&= \frac{d}{dx}\big(\cot(1)\cos(x)\cos(x)h(x) - \cot(1)\sin(x)\int_0^x \cos(s)h(s)ds \\
&\quad + \cos(x)\sin(x)h(x) + \cos(x)\int_0^x \cos(s)h(s)ds \\
&\quad - \cot(1)\cos(x)\cos(x)h(x) - \cot(1)\sin(x)\int_x^1 \cos(s)h(s)ds \\
&\quad - \sin(x)\cos(x)h(x) - \sin(x)\int_x^1 \sin(s)h(s)ds\big) + (Sh)(x) \\
&= \frac{d}{dx}\big(- \cot(1)\sin(x)\int_0^x \cos(s)h(s)ds + \cos(x)\int_0^x \cos(s)h(s)ds \\
&\quad - \cot(1)\sin(x)\int_x^1 \cos(s)h(s)ds - \sin(x)\int_x^1 \sin(s)h(s)ds\big) + (Sh)(x).
\end{aligned}
$$

We will differentiate again and use that $\cos(x)^2 + \sin(x)^2 = 1$. This gives,

$$
\begin{aligned}
&= - \cot(1)\sin(x)\cos(x)h(x) - \cot(1)\cos(x)\int_0^x \cos(s)h(s)ds \\
&\quad \cos(x)\cos(x)h(x) - \sin(x)\int_0^x \cos(s)h(s)ds \\
&\quad \cot(1)\sin(x)\cos(x)h(x) - \cot(1)\cos(x)\int_x^1 \cos(s)h(s)dsds \\
&\quad + \sin(x)\sin(x)h(x) - \cos(x)\int_x^1 \sin(s)h(s)ds + (Sh)(x) \\
&= -(Sh)(x) + h(x) + (Sh)(x) \\
&= h(x).
\end{aligned}
$$

Thus this operation indeed is equal to the identity operation. Now we will show it the other way around. This will be done with integrating by parts twice:

$$
\begin{aligned}
(S(A + (1+\pi^2)I)h)(x) &= (S\frac{d^2}{dx^2}h)(x) + (Sh)(x) \\
&= \cot(1)\cos(x)\int_0^x \cos(s)\frac{d^2}{ds^2}h(s)ds + \sin(x)\int_0^x \cos(s)\frac{d^2}{ds^2}h(s)ds \\
&\quad + \cot(1)\cos(x)\int_x^1 \cos(s)\frac{d^2}{ds^2}h(s)ds + \cos(x)\int_x^1 \sin(s)\frac{d^2}{ds^2}ds + (Sh)(x) \\
&= \cot(1)\cos(x)\big(\cos(s)\frac{d}{ds}h(s)\Big|_{s=0}^{x} + \int_0^x \sin(s)\frac{d}{ds}h(s)ds\big) \\
&\quad + \sin(x)\big(\cos(s)\frac{d}{ds}h(s)\Big|_{s=0}^{x} + \int_0^x \sin(s)\frac{d}{ds}h(s)ds\big) \\
&\quad + \cot(1)\cos(x)\big(\cos(s)\frac{d}{ds}h(s)\Big|_{s=x}^{1} + \int_0^x \sin(s)\frac{d}{ds}h(s)ds\big) \\
&\quad + \cos(x)\big(\sin(s)\frac{d}{ds}h(s)\Big|_{s=x}^{1} - \int_x^1 \cos(s)\frac{d}{ds}h(s)ds\big) + (Sh)(x).
\end{aligned}
$$

Now we will use that $h \in D(A)$ and that thus $\frac{d}{ds}h(s)\Big|_{s=0} = \frac{d}{ds}h(s)\Big|_{s=1} = 0$. This will make some terms cancel out. This goes as follows,

$$\begin{aligned}
&= \cot(1)\cos(x)\cos(x)\frac{d}{ds}h(s)\Big|_{s=x} + \cot(1)\cos(x)\int_0^x \sin(s)\frac{d}{ds}h(s)ds \\
&\quad + \sin(x)\cos(x)\frac{d}{ds}h(s)\Big|_{s=x} + \sin(x)\int_0^x \sin(s)\frac{d}{ds}h(s)ds \\
&\quad - \cot(1)\cos(x)\cos(x)\frac{d}{ds}h(s)\Big|_{s=x} + \cot(1)\cos(x)\int_x^1 \sin(s)\frac{d}{ds}h(s)ds \\
&\quad - \cos(x)\sin(x)\frac{d}{ds}h(s)\Big|_{s=x} - \cos(x)\int_x^1 \cos(s)\frac{d}{ds}h(s)ds + (Sh)(x) \\
&= \cot(1)\cos(x)\int_0^x \sin(s)\frac{d}{ds}h(s)ds + \sin(x)\int_0^x \sin(s)\frac{d}{ds}h(s)ds \\
&\quad + \cot(1)\cos(x)\int_x^1 \sin(s)\frac{d}{ds}h(s)ds - cos(x)\int_x^1 \cos(s)\frac{d}{ds}h(s)ds + (Sh)(x).
\end{aligned}$$

We will integrate by parts again and use that $\cot(1) = \frac{\cos(1)}{\sin(1)}$,

$$\begin{aligned}
&= \cot(1)\cos(x)\big(\sin(s)h(s)\Big|_{s=0}^{x} - \int_0^x \cos(s)h(s)ds\big) \\
&\quad + \sin(x)\big(\sin(s)h(s)\Big|_{s=0}^{x} - \int_0^x \cos(s)h(s)ds\big) \\
&\quad + \cot(1)\cos(x)\big(\sin(s)h(s)\Big|_{s=x}^{1} - \int_x^1 \cos(s)h(s)ds\big) \\
&\quad - \cos(x)\big(\cos(s)h(s)\Big|_{s=x}^{1} + \int_x^1 \sin(s)h(s)ds\big) + (Sh)(x) \\
&= -(Sh)(x) + \cot(1)\cos(x)\sin(x)h(x) - 0 + \sin(x)\sin(x)h(x) - 0 \\
&\quad + \cot(1)\cos(x)\sin(1)h(1) - \cot(1)\cos(x)\sin(x)h(x) \\
&\quad - \cos(x)\cos(1)h(1) + \cos(x)\cos(x)h(x) + (Sh)(x) \\
&= h(x).
\end{aligned}$$

Now we have shown that S indeed is the inverse of $A + (1+\pi^2)I$, with A being the operator $\frac{\partial^2}{\partial^2\xi} - \pi^2 I$. □

Therefore as stated above we know that the eigenvectors of A from a basis for X. Next we will show that this A generates an exponentially stable semigroup. To do this we will assume that for every $x \in X$ there exists $x_n \in \mathbb{R}$, $n \in \mathbb{N}$ such that $x = \sum_{n\in\mathbb{N}} x_n \mathrm{e}_n$. The semigroup that is generated by A is as follows: $T(t)x = \sum_{n\in\mathbb{N}} e^{a_n t} x_n \mathrm{e}_n$. First we will show that this indeed is a semigroup, after that we will show that this is the semigroup that is generated by A.

To show that this is a semigroup we need to show the three properties from definition 2.3.

1. $T(t+s) = T(t)T(s)$, for all $t, s \geq 0$ is satisfied since

$$\begin{aligned} T(t)T(s)x &= T(t)\sum_{n\in\mathbb{N}} e^{a_n s}x_n \mathrm{e}_n \\ &= \sum_{n\in\mathbb{N}} e^{a_n s}x_n T(t)\mathrm{e}_n \\ &= \sum_{n\in\mathbb{N}} e^{a_n s}x_n e^{a_n t}\mathrm{e}_n \\ &= \sum_{n\in\mathbb{N}} e^{a_n (t+s)}x_n \mathrm{e}_n \\ &= T(t+s)x. \end{aligned}$$

2. $T(0) = I$ is satisfied since

$$T(0)x = \sum_{n\in\mathbb{N}} e^{a_n 0}x_n \mathrm{e}_n = \sum_{n\in\mathbb{N}} x_n \mathrm{e}_n = Ix$$

3. For all $x \in X$, we have that $\|T(t)x - x\|_X$ converges to zero, when $t \to 0^+$. This takes a bit longer to show. We will assume that $t \leq 1$ which is a fair assumption, since we will take $t \to 0^+$. This gives,

$$\begin{aligned} \|T(t)x - x\|^2 &= \|\sum_{n\in\mathbb{N}}(e^{a_n t}x_n\mathrm{e}_n - x_n\mathrm{e}_n)\|^2 \\ &= \|\sum_{n\in\mathbb{N}}(e^{a_n t} - 1)x_n\mathrm{e}_n)\|^2 \\ &= \sum_{n\in\mathbb{N}} |e^{a_n t} - 1|^2|x_n|^2 \\ &= \sum_{n=1}^{N} |e^{a_n t} - 1|^2|x_n|^2 + \sum_{n=N+1}^{\infty} |e^{a_n t} - 1|^2|x_n|^2 \end{aligned}$$

Now we define $K := \sup_{0\leq t\leq 1 \& n\geq 1} |e^{a_n t} - 1|^2$. Moreover, for any $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that

$$\sum_{n=N+1}^{\infty} |x_n|^2 \leq \frac{\varepsilon}{2K}.$$

After this we choose $\delta \leq 1$ such that $\sup_{1\leq n\leq N} |e^{a_n t} - 1|^2 \leq \frac{\varepsilon}{2\|z\|^2}$ for all $0 \leq t \leq \delta$. This δ depends both on ε and N. Now we can continue with the estimate:

$$\begin{aligned} \|T(t)x_0 - x_0\|^2 &= \sum_{n=1}^{N} |e^{a_n t} - 1|^2|x_n|^2 + \sum_{n=N+1}^{\infty} |e^{a_n t} - 1|^2|x_n|^2 \\ &\leq \sup_{1\leq n\leq N} |e^{a_n t} - 1|^2 \sum_{n=1}^{N} |x_n|^2 + \sup_{0\leq t\leq 1\& n\geq 1} |e^{a_n t} - 1|^2 \sum_{n=N+1}^{\infty} |x_n|^2 \\ &\leq \frac{\varepsilon}{2\|z\|^2} + K\frac{\varepsilon}{2K} \\ &\leq \varepsilon. \end{aligned}$$

Thus we have that $\|T(t)x - x\|_X$ converges to zero, when $t \to 0^+$.

Now it is known that T is a semigroup we still have to show that A is its generator. In Definition 2.4 it is defined what the generator of the semigroup is. We have to show that

$$Ax = \lim_{h\to 0^+} \frac{1}{h}(T(t)x - x) \text{ exists } \Leftrightarrow x \in D(A).$$

We will start with showing '$\Rightarrow$'. As before we have that for every $x \in X$ there exists $x_n \in \mathbb{R}$, $n \in \mathbb{N}$ such that $x = \sum_{n\in\mathbb{N}} x_n \mathrm{e}_n$. Moreover, we have that $x \in D(A) \Leftrightarrow \sum_{n\in\mathbb{N}} |a_n x_n|^2 < \infty$. Now we will work out the limit, but not use the fact that we have the solution Ax already,

$$\begin{aligned}\lim_{h\to 0^+} \frac{1}{h}(T(t)x - x) \text{ exists } &\Leftrightarrow \exists y \in X \text{ s.t. } \lim_{h\to 0^+} (\frac{1}{h}(T(t)x - x) - y) = 0 \\ &\Leftrightarrow \exists y \in X \text{ s.t. } \lim_{h\to 0^+} \|\frac{1}{h}(T(t)x - x) - y\|^2 = 0 \\ &\Leftrightarrow \exists y \in X \text{ s.t. } \lim_{h\to 0^+} \sum_{n\in\mathbb{N}} |\frac{1}{h}(e^{a_n h} - 1)x_n - y_n|^2 = 0\end{aligned}$$

in which $y = \sum_{n\in\mathbb{N}} y_n \mathrm{e}_n$. Since all terms in the sum are positive we know that if the sum goes to zero all individual terms need to go to zero. Therefore, we can continue,

$$\begin{aligned}\lim_{h\to 0^+} \frac{1}{h}(T(t)x - x) \text{ exists } &\Leftrightarrow \exists y \in X \text{ s.t. } \lim_{h\to 0^+} |\frac{1}{h}(e^{a_n h} - 1)x_n - y_n|^2 = 0 \quad \forall n \in \mathbb{N} \\ &\Leftrightarrow \exists y \in X \text{ s.t. } |\lim_{h\to 0^+} \frac{1}{h}(e^{a_n h} - 1)x_n - y_n|^2 = 0 \quad \forall n \in \mathbb{N}.\end{aligned}$$

Next we will use the definition of the derivative of $e^{a_n t}$ at $t = 0$ to get that $\lim_{h\to 0^+} \frac{1}{h}(e^{a_n h} - 1) = a_n$. This gives,

$$\begin{aligned}\lim_{h\to 0^+} \frac{1}{h}(T(t)x - x) \text{ exists } &\Rightarrow \exists y \in X \text{ s.t. } |a_n x_n - y_n|^2 = 0 \quad \forall n \in \mathbb{N} \\ &\Rightarrow \exists y \in X \text{ s.t. } a_n x_n = y_n \quad \forall n \in \mathbb{N} \\ &\Rightarrow \exists y \in X \text{ s.t. } \sum_{n\in\mathbb{N}} |a_n x_n|^2 = \sum_{n\in\mathbb{N}} |y_n|^2 < \infty.\end{aligned}$$

The sum over y_n is finite since $y \in X$. Therefore, we now have shown '$\Rightarrow$' with the limit equals Ax. The next step is to show '$\Leftarrow$'. We know that $x \in D(A)$ and that therefore $\sum_{n\in\mathbb{N}} |a_n x_n|^2 < \infty$. Now we will show that $\lim_{h\to 0^+} \frac{1}{h}(T(t)x - x) = Ax$. We have shown above that

$$\begin{aligned}\lim_{h\to 0^+} \frac{1}{h}(T(t)x - x) \text{ exists } &\Leftrightarrow \exists y \in X \text{ s.t. } \lim_{h\to 0^+} \sum_{n\in\mathbb{N}} |\frac{1}{h}(e^{a_n h} - 1)x_n - y_n|^2 = 0 \\ &\Leftrightarrow \lim_{h\Rightarrow 0^+} \sum_{n\in\mathbb{N}} |\frac{1}{h}(e^{a_n h} - 1)x_n - a_n x_n|^2 = 0.\end{aligned}$$

In which we will already use that $y = Ax = \sum_{n\in N} a_n x_n \mathrm{e}_n$. Without loss of generality we will now assume that $h \in [0, 1]$, since we take the limit of h to 0^+, we can assume that h is not big. Now we would like to switch around the limit and the sum, therefore we will show that the sum converges absolutely and uniformly for $h \in [0, 1]$. This will be done by estimating the sum,

$$\begin{aligned}\sum_{n\in\mathbb{N}} |\frac{1}{h}(e^{a_n h} - 1)x_n - a_n x_n|^2 &\leq \sum_{n\in\mathbb{N}} (2|\frac{1}{h}(e^{a_n h} - 1)x_n|^2 + 2|a_n x_n|^2) \\ &\leq \sum_{n\in\mathbb{N}} (2|\frac{1}{h}(e^{a_n h} - 1)|^2 |x_n|^2 + 2|a_n|^2 |x_n|^2).\end{aligned}$$

Now we will use the mean value theorem for $f(t) = e^{a_n t}$. By the mean value theorem we know there exists an $c \in [0, h]$ such that $\frac{f(h)-f(0)}{h-0} = f'(c)$. Therefore, we know that

$$\frac{|e^{a_n h} - 1|}{h} = |a_n e^{a_n c}| \leq \sup_{s \in [0,h]} |a_n e^{a_n s}| = |a_n|.$$

Using this to further estimate the sum gives,

$$\sum_{n \in \mathbb{N}} |\frac{1}{h}(e^{a_n h} - 1)x_n - a_n x_n|^2 \leq \sum_{n \in \mathbb{N}} 4|a_n|^2 |x_n|^2 < \infty,$$

where this sum is finite since $x \in D(A)$. Since we estimated the sum independent of h, we know by the Weierstrass M-test that the sum converges absolutely and uniformly for $h \in [0, 1]$. Therefore, the sum and the limit can be switched around. If we then also apply the definition of the derivative like done before, this gives:

$$\begin{aligned}
\lim_{h \Rightarrow 0^+} \sum_{n \in \mathbb{N}} |\frac{1}{h}(e^{a_n h} - 1)x_n - a_n x_n|^2 &= \sum_{n \in \mathbb{N}} \lim_{h \Rightarrow 0^+} |\frac{1}{h}(e^{a_n h} - 1)x_n - a_n x_n|^2 \\
&= \sum_{n \in \mathbb{N}} | \lim_{h \Rightarrow 0^+} \frac{1}{h}(e^{a_n h} - 1)x_n - a_n x_n|^2 \\
&\leq \sum_{n \in \mathbb{N}} \lim_{h \Rightarrow 0^+} |a_n x_n - a_n x_n|^2 = 0.
\end{aligned}$$

This implies that $\lim_{h \to 0^+} \frac{1}{h}(T(t)x - x)$ exists and is equal to Ax, thus we have now shown '$\Leftarrow$' and indeed A is the generator of the semigroup T.

The next step is to determine B. To determine B we will use a general system that has the same Neumann boundary condition as mentioned before and we will use that it is known what the solution of a bilinear system looks like. We will start with a general system in the form

$$\begin{aligned}
\dot{x}(t) &= \mathfrak{A}x(t), \\
\mathfrak{B}x(t) &= u(t), \\
x(0) &= x_0.
\end{aligned}$$

Here we have $\mathfrak{A}$ defined like A was defined before, but with a slightly different domain, so

$$\begin{aligned}
\mathfrak{A}f &= \frac{\partial^2}{\partial \xi^2} f - \pi^2 I f, \\
D(\mathfrak{A}) &= \{f \in H^2[0, 1] : \frac{\partial^2}{\partial \xi^2} f \in X, \frac{\partial}{\partial \xi} f(0) = 0\},
\end{aligned}$$

where H^2 refers to the Sobolev space. So now the boundary condition where u was mentioned is not in $D(\mathfrak{A})$. Furthermore, $\mathfrak{B}$ is the operator that takes the spatial derivative and then substitutes $\xi = 1$, so $\mathfrak{B}f = \frac{\partial}{\partial \xi} f\big|_{\xi=1}$. In this way you can see that the domain of $\mathfrak{A}$ extended with the requirement that f should be in the kernel of $\mathfrak{B}$ gives exactly the domain of A. Thus $\mathfrak{A}|_{\ker(\mathfrak{B})} = A$.

Now we define $z = x - B_0 u$, with $B_0 f(\xi) = \xi f(\xi)$ and we will have a look at $\frac{d}{dt} z = \dot{z}$. This might seem a bit unusual, but this substitution will give us the opportunity to get u and x

together and to get a linear state–space system:

$$\begin{aligned}\dot{z} &= \dot{x} - B_0\dot{u}\\ \dot{z} &= \mathfrak{A}x - B_0\dot{u}\\ \dot{z} &= \mathfrak{A}z + \mathfrak{A}B_0u - B_0\dot{u}.\end{aligned}$$

Using the combination of the generalised solution for the linear system in Definition 6.7 we can get rid of the derivative to z. After that it will be further rewritten to get something similar to the generalised solution of a linear system in x, from which B can be determined. Moreover, we use that $\mathfrak{A}B_0 = 0$, since $B_0 = \xi$ is not in the domain of $\mathfrak{A}$. This gives,

$$\begin{aligned}z(t) &= T(t)z(0) + \int_0^t T(t-s)(\mathfrak{A}B_0u - B_0\dot{u})ds\\ z(t) &= T(t)z(0) - \int_0^t T(t-s)B_0\dot{u}ds\\ x(t) - B_0u(t) &= T(t)(x(0) - B_0u(0)) - \int_0^t T(t-s)B_0\dot{u}ds\\ x(t) &= T(t)x(0) - T(t)B_0u(0) + B_0u(t) - \int_0^t T(t-s)B_0\dot{u}ds.\end{aligned}$$

Next we want to get rid of the $\dot{u}$ and get a u instead. The most intuitive way to do this is by integration by parts. However, that would mean we need to differentiate T, from the theory of semigroups we know that the derivative of $T(t)x$ is $T(t)Ax$, see also Theorem 2.1.13 of [5]. Since we have an unbounded B we will use the extension of T defined at the beginning of Chapter 6. So, we can integrate by parts using the extended space X_{-1}. This goes as follows:

$$\begin{aligned}x(t) &= T(t)x(0) - T(t)B_0u(0) + B_0u(t) - \int_0^t T(t-s)B_0\dot{u}(s)ds\\ x(t) &= T(t)x(0) - T(t)B_0u(0) + B_0u(t) - T(t-s)B_0u(s)|_{s=0}^t - \int_0^t T_{-1}(t-s)A_{-1}B_0u(s)ds\\ x(t) &= T(t)x(0) - T(t)B_0u(0) + B_0u(t) - B_0u(t) + T(t)B_0u(0) - \int_0^t T_{-1}(t-s)A_{-1}B_0u(s)ds\\ x(t) &= T(t)x(0) - \int_0^t T_{-1}(t-s)A_{-1}B_0u(s)ds.\end{aligned}$$

If we look at the general solution of a linear system as in Definition 6.7 we can now conclude that $Bf = -A_{-1}B_0f$. We want to write this again in terms of the eigenfunctions, such that we can say $Bx = \sum_{n\in\mathbb{N}} b_nx_n\mathrm{e}_n$, where $x = \sum_{n\in\mathbb{N}} x_n\mathrm{e}_n$. To do this we will determine the coefficients c_n such that $\xi = \sum_{n\in\mathbb{N}} c_n\mathrm{e}_n$. This will be done by calculating the inner product $\langle\xi, \mathrm{e}_n(\xi)\rangle$, since $c_n = \langle\xi, \mathrm{e}_n(\xi)\rangle$. First we will determine c_0 and then c_n for $n > 0$. This goes as follows:

$$c_0 = \langle\xi, \mathrm{e}_0(\xi)\rangle = \langle\xi, 1\rangle = \int_0^1 \xi d\xi = \frac{1}{2}\xi^2\Big|_{\xi=0}^1 = \frac{1}{2}.$$

Now we will determine c_n for $n > 0$,

$$\begin{aligned} c_n &= \langle \xi, \mathrm{e}_n(\xi)\rangle \\ &= \int_0^1 \xi \cos(n\pi\xi) d\xi \\ &= \xi \frac{1}{n\pi} \sin(n\pi\xi)\Big|_{\xi=0}^1 - \int_0^1 \frac{1}{n\pi} \sin(n\pi\xi) d\xi \\ &= \frac{1}{n\pi} \sin(n\pi) + \frac{1}{(n\pi)^2} \cos(n\pi\xi)\Big|_{\xi=0}^1 \\ &= \frac{1}{(n\pi)^2} \cos(n\pi) - \frac{1}{(n\pi)^2} \\ &= \begin{cases} 0 & n \text{ is even}, \\ -\frac{2}{(n\pi)^2} & n \text{ is odd}. \end{cases} \end{aligned}$$

Now we have that $b_n = a_n c_n$, thus

$$b_n = \begin{cases} -\frac{1}{2}\pi^2 & n = 0 \\ 0 & n \text{ is even}, n \neq 0, \\ 2\frac{n^2+1}{n^2} & n \text{ is odd}. \end{cases}$$

Now we can conclude that

$$\begin{aligned} \sum_{n\in\mathbb{N}} \frac{|b_n|^2}{|a_n|^2} &= \frac{1}{2} + \sum_{n\in\mathbb{N}, n \text{ odd}} \frac{4\frac{(n^2+1)^2}{n^4}}{\pi^4(n^2+1)^2} \\ &= \frac{1}{2} + \sum_{n\in\mathbb{N}, n \text{ odd}} \frac{4}{\pi^4 n^4} < \infty. \end{aligned}$$

Thus $B \in X_{-1}$ and according to Theorem 6.11 the system is iISS.

7.2 Bilinear system

For the bilinear system we want to show a system that has an unbounded B and will be iISS due to Theorem 6.21. This example will not be based as literally on a PDE as the example in Section 7.1, but we will use the same operator $A : D(A) \to X$, with $X = L^2(0, 1)$. B will be picked such that we have a bilinear system that will be iISS.

So we will have a system of the form

$$\dot{x}(t) = Ax(t) + u(t)Bx(t),$$

with $u(t) \in \mathbb{C}$, u piecewise continuous and A the operator defined as follows:

$$\begin{aligned} Af &= \frac{\partial^2}{\partial\xi^2} f - \pi^2 I f, \\ D(A) &= \{f \in H^2[0, 1] : \frac{\partial^2}{\partial\xi^2} f \in X, \frac{\partial}{\partial\xi} f(0) = 0, \frac{\partial}{\partial\xi} f(1) = 0\}, \end{aligned}$$

where H^2 refers to the Sobolev space. We have shown before that regarding the eigenfunctions $\mathrm{e}_0(\xi) = 1$, $\mathrm{e}_n(\xi) = \sqrt{2}\cos(n\pi\xi)$, $n \in \mathbb{N}$, this operator can be seen as a 'diagonal' operator with $a_n = -\pi^2(n^2+1)$. Now we will take the operator B also 'diagonal' with $b_n = \sqrt{|a_n|} = \pi\sqrt{n^2+1}$. If we now look at Theorem 6.21 we see that the requirement is that $f \in L^p(0,1)$ with $f(s) = \sum_{n\in\mathbb{N}} \frac{|b_n|^2}{|a_n|} e^{(a_n-\omega)s}$. In this case the fraction with b_n and a_n will be equal to 1. Thus we will show that $f(s) = \sum_{n\in\mathbb{N}} e^{(-\pi^2(n^2+1)-\omega)s}$ is in $L^p(0,1)$. Where we choose $\omega = -\frac{1}{2}\pi^2$ such that we have $-\pi^2 < \omega < 0$ and we will take $p = 2$. Now we will show that $\|f\|^2_{L^2}$ is finite:

$$\begin{aligned}
\|f\|^2_{L^2} &= \int_0^\infty (\sum_{n\in\mathbb{N}} e^{-\pi^2(n^2+\frac{1}{2})s})^2 ds \\
&= \int_0^\infty \sum_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}} e^{-\pi^2(n^2+\frac{1}{2})s-\pi^2(m^2+\frac{1}{2})s} ds.
\end{aligned}$$

Here we used that $(\sum_{n\in\mathbb{N}} c_n)^2 = \sum_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}} c_n c_m$. Further estimation gives,

$$\begin{aligned}
\|f\|^2_{L^2} &= \sum_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}} \int_0^\infty e^{-\pi^2 s} ds \\
&= \sum_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}} \frac{1}{-\pi^2(n^2+m^2+1)} e^{-\pi^2(n^2+m^2+1)s} \Big|_{s=0}^{\infty} \\
&= \sum_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}} \frac{1}{\pi^2(n^2+m^2+1)}.
\end{aligned}$$

Note that $\sum_{n\in\mathbb{N}} \frac{1}{n^2}$ is finite. Thus for each n this means that the sum over m is converging, since it is always be smaller than $\sum_{n\in\mathbb{N}} \frac{1}{n^2}$. Then the same can be done with the sum over n and thus the complete sum is finite. Therefore, $f \in L^2(0,1)$ and thus the system is iISS according to Theorem 6.21.

Chapter 8

Conclusion

The goal of this thesis was to research ISS for bilinear systems. This resulted in a clear overview of the ISS stability properties of finite dimensional linear and bilinear systems. It is also shown that these stability properties can be extended to infinite dimensional linear and bilinear systems with bounded operators.

For bilinear infinite dimensional systems with unbounded operators we presented a sufficient requirement under which these systems are iISS. Not all bilinear infinite dimensional systems with unbounded operators that are iISS are contained by this requirement. However, this requirement is useful, since it does give a class of systems that are iISS. Further research could be done to check how strict the requirements for the infinite dimensional bilinear system with unbounded operators are. It might be possible to find other requirements that are practically better to check. Further research might also be done regarding a small–gain property for infinite dimensional bilinear systems with unbounded B.

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