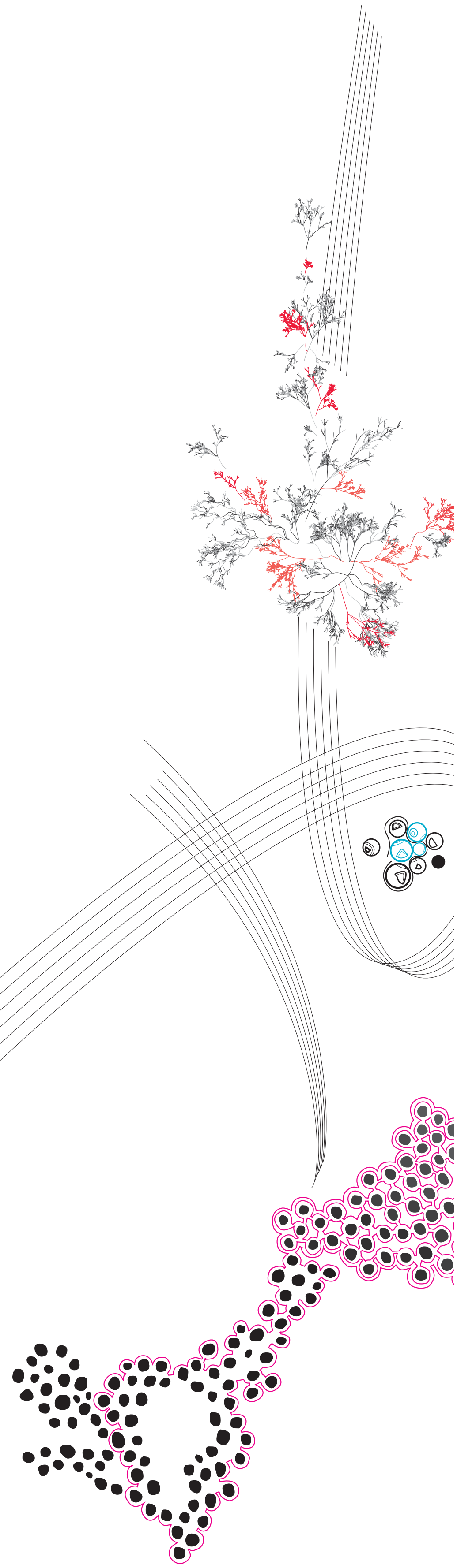

BSc Thesis Applied Mathematics and Applied Physics

Towards quantum and Brownian generalizations of classical geometric concepts

Niels Apeldoorn

Supervisors:
F.P. Schuller
G.H.L.A. Brocks

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Department of Applied Mathematics
Faculty of Electrical Engineering,
Mathematics and Computer Science

UNIVERSITY OF TWENTE.

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Abstract

This work is a proof of concept of the idea that geometry can be reformulated based purely on quantum mechanical theory, rather than classical theory. The action plays a fundamental role in the fields of classical mechanics, quantum mechanics and brownian motion. We use the action to calculate probability distributions for the geometric concepts of area generalized in quantum mechanics and brownian motion. For this we use constrained path integrals to find the probability amplitude for a given area enclosed by a closed curve. We have found that the algebraic area enclosed by a curve in brownian motion is symmetrically distributed around zero and that this distribution converges to the classical result in the classical limit. The concept of Brownian area can be applied to quadrances in rational trigonometry to generalize trigonometric concepts and theorems.

Keywords: Path integral, geometry, Brownian motion, quantum mechanics

Contents

1 Introduction

This work is a proof of concept of the idea that the foundations of classical geometry can be reformulated with the aim to be more directly suited for the application of geometry in quantum mechanics and statistical physics. The concepts of geometry, like those of any other mathematical field, are not related to the physical world a priori. In order to make them part of a physical theory, measurement prescriptions for key quantities need to be postulated.

We illustrate this for the physical theory of classical mechanics. Here, the connection between geometry and physics is provided by Newton's first axiom. It says that the experimental way to identify a straight line in physical space is to observe the trajectories of particles under the influence of no force. Thus the first axiom is a physical definition of a straight line. Even more precisely, the first axiom speaks of *uniform* straight motion, which means that the distance covered by a force-free particle is proportional to the time that passed. This additionally provides a measurement prescription for an affine measure of length. Thus Newton's first axiom does not tell, what particles under the influence of no force will do, but rather what their motion reveals about the underlying geometry. It is a measurement prescription for the geometry of physical space. The second axiom then says, what particles, possibly under the influence of forces, will do, given sufficient initial conditions.

Realizing that the first axiom is required to import the mathematical theory of Euclidean geometry into the physical theory of mechanics is the key to understanding the idea of the present work. For it was Heisenberg's crucial insight that the very concept of a particle trajectory must be abolished entirely in any theory that could explain the phenomena that are now described by contemporary quantum theory in its various formulations. But if particle trajectories no longer play a role in quantum theory, not even as an auxiliary mathematical concept, Newton's first axiom can no longer provide the mechanism that relates the concepts of classical geometry to their application in physics.

Together with the abolition of particle trajectories in quantum mechanics comes another key declaration of quantum mechanics, namely that one can generically not predict any more certain measurement results, but only the spectrum of possible results and definite probabilities for each of those results to be obtained by some individual measurement. The required probability theory is somewhat more refined than classical probability theory. The latter directly deals with real probabilities, but quantum mechanics requires to employ the refined concept of complex *probability amplitudes*, which only upon measurement are converted into real probabilities that arise as the squared absolute value of the probability amplitudes. It is important to realize that probability amplitudes are more fundamental than probabilities, since the central quantum mechanical equation, the Schrödinger equation for probability amplitudes, is only valid until a measurement is taken and hence cannot be rewritten as an equation for the probabilities. The probabilistic nature of quantum mechanical predictions is the final nail in the coffin for a possible import of classical geometry into quantum mechanics by way of any measurement postulate that would replace Newton's first axiom. It is the central tenet of this thesis that the existing postulates of quantum mechanics support the import of a quantum version of geometry.

The situation is somewhat simpler for stochastic processes, such as the prototypical and somewhat universal Brownian motion. This process describes the movement of a classical particle due to thermal fluctuations and collisions in a medium. The central equation here is the heat equation, which arises as an analytic continuation of the Schrödinger equation but, in contrast to the latter, now determines a real probability density rather than a probability amplitude density. Since results of quantum mechanical calculations can often be easily converted to corresponding results on Brownian motion, and vice versa, we discuss the Brownian motion of particles alongside quantum mechanics as another physical theory that requires another type of geometry than the classical one. The difference between quantum, Brownian and classical theory is the way in which and how much of the information contained in the action is extracted. This makes the transition from the general formulation of quantum mechanics to classical mechanics a natural specification. These relations between quantum mechanics, Brownian motion and classical mechanics, all connected by the action, have been graphically depicted in figure 1. In chapter 2 we review the theory of actions and how they are used in classical mechanics, quantum mechanics and Brownian motion.

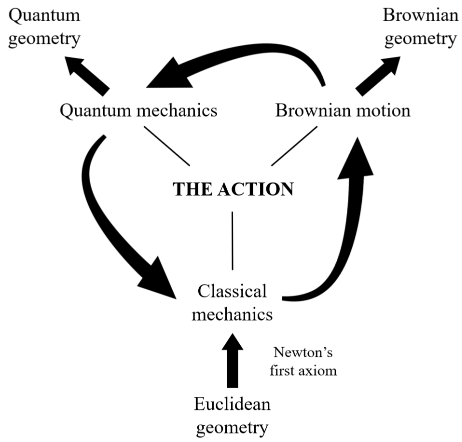

FIGURE 1: Schematic representation of the connections between classical, Euclidean geometry and the concept of quantum and Brownian geometry. The action plays a central role in all three disciplines. We use the description of Euclidean geometry by Newton's first axiom and apply it using quantum mechanical or brownian principles.

How can one now find those avatars of classical geometry that are adapted to quantum mechanics or Brownian motion, respectively? The strategy is, in fact, quite simple. The first step is to connect classical geometry to classical mechanics by Newton's first axiom, which asserts that uniformly traced straight line segments in physical space are curves $\gamma : [t_0, t_1] \to \mathbb{R}^3$ for which the functional

$$S[\gamma] = \frac{m}{2} \int_{t_0}^{t_1} \|\dot{\gamma}(t)\|^2 \, dt$$

is stationary, irrespective of the positive mass m of the classical particle. The second step is to use the path integral formulation of quantum mechanics, which asserts that the very same action functional S determines the probability amplitude density

$$K(x_0, t_0; x_1, t_1) = \int [d\gamma] \exp\left(\frac{i}{\hbar} S[\gamma]\right)$$

for a quantum particle of mass m, which is located at x_0 at time t_0 to be measured at x_1 at time t_1. The integration here is over the space of all piece-wise continuously differentiable curves γ and will be explained in detail in chapters 2 and 3. A similar, entirely real path integral can be written down to directly obtain the real-valued probability density for the corresponding transition of a Brownian particle. In the third step, we may then formulate any classical geometric question using the equations of motions that arise as the stationarity condition $\frac{\delta S}{\delta \gamma} = 0$ from the action functional. Then we pose the corresponding quantum geometric question by appropriately constraining the paths over which we integrate in the path integral in order to obtain conditional probability amplitudes. And similarly, mutatis mutandis, we can obtain conditional probabilities for the corresponding stochastic geometric question. This way, we propose to extract the quantum geometric and Brownian geometric conclusions. The method of constrained path integrals is explained under 3.2.

Geometric shapes and concepts can generally be described as a set of points in space. A triangle, for example, consists of three points. Classically, one would draw the shortest lines between any two of these three points to complete the triangle. A triangle defined by these three points has various properties of geometric interest. These include, the circumference, area and angles of the triangle, which can classically be calculated with the positions of the three points. In quantum mechanics and Brownian motion we can keep this notion as the starting point for a generalized geometry. Starting from fixed points in space, we calculate transition amplitudes from one point to another as a generalization for classical straight lines between the two.

In particular, we will focus on the apparently trivial question which area is enclosed by the trajectory of a force-free particle that starts at some point and ends up at the same point in space a finite time afterwards. The classical result is of course zero, since such a free particle will need to stay at the starting point in order to arrive there again. Quantum mechanically, one finds, after some calculations presented in chapter 4, that the corresponding path integral for the probability amplitude density does not converge and thus that this classical question does not have a quantum analogue. For Brownian motion, however, the path integral converges and yields a probability amplitude that peaks at zero area but also remains non-zero for any non-zero area. Thus the concept of area fails to have a quantum analogue, but well has a stochastic one. We explain how this result on the notion of area gives rise to a complete theory of Brownian trigonometry in chapter 5. These results can be used to further develop the theory of Brownian, and perhaps also quantum, geometry in future research. This has the potential to lead to a new understanding of geometric methods in quantum mechanics, as well as statistical physics and allow for interesting applications in both mathematics and physics.

2 Conceptual review — The universal role of actions

In this section, we will review the role of the action as a fundamental quantity of the fields of classical mechanics, quantum mechanics and Brownian motion. The principles of both quantum mechanics and classical mechanics can be derived from the action. Also, Brownian motion can be described using the same action. The action of particle or system is a description of how the system develops over time. For a particle, the action is denoted as a functional of the piece-wise smooth curve γ: $[t_1, t_2] \to \mathbb{R}^d$, which describes the positions in d-dimensional space as a function of time in the interval $[t_1, t_2]$. Here d is the dimension of the system. The path, or trajectory, of the particle over time is represented by γ in classical mechanics. In quantum mechanics, the position of a particle is not well-defined until a measurement is made, so then γ is only a possible "trajectory". The action is represented as the integral of a Lagrangian $L(q, v)$ evaluated from t_1 to t_2. The Lagrangian is a map L: $\mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$. In the action the arguments of the Lagrangian are the curve γ and its time derivative $\dot{\gamma}$, respectively. Therefore, the action can be expressed as

$$S[\gamma] = \int_{t_1}^{t_2} L(\gamma(t), \dot{\gamma}(t))\, dt. \tag{2.1}$$

The action lies at the foundation of the description of a physical system in classical mechanics, as well as in quantum mechanics, as in the stochastic process of Brownian motion. We will describe the role of the action in each of these separate fields. We will assume, for our purposes, that the system consists of a single particle. In the view of geometry described by the behavior of one physical particle, this is all that we will need.

2.1 The action in classical mechanics

In classical mechanics the action is used to derive the equations of motion of a system. We will describe the role of the action in classical physics and the significance of its stationary points. Consider a curve γ, with $\gamma(t_1) = p_i$ and $\gamma(t_2) = p_f$, and a map

$$a : [-\epsilon, \epsilon] \times [t_1, t_2] \to \mathbb{R}^d\,, \qquad (s, t) \mapsto a(s, t) := a_s(t)$$

with $\epsilon > 0$, such that

$$a_s(t_0) = p_i \qquad \text{and} \qquad a_s(t_1) = p_f \qquad \text{for all } s \in [-\epsilon, \epsilon]\,.$$

Then a is called a variation of the curve γ if $\gamma = a_0$. A curve γ is called a stationary point of the action S if for all variations a of γ the function

$$\delta : [-\epsilon, \epsilon] \to \mathbb{R}\,, \qquad \delta(s) := S[a_s]$$

is stationary at $s = 0$,

$$\delta'(0) = 0\,.$$

One can show that a necessary and sufficient criterion for γ to be a stationary point of S is that the curve γ satisfy the Euler-Lagrange equations [3]:

$$(\partial_i L) \circ (\gamma, \gamma') - ((\partial_{d+i} L) \circ (\gamma, \gamma'))' = 0 \tag{2.2}$$

for $i = 1, \ldots, d$, where

$$(\alpha, \beta) : \mathbb{R} \to \mathbb{R}^d \times \mathbb{R}^d\,, \quad t \mapsto (\alpha(t), \beta(t))\,.$$

The *stationary action principle* then states that the solutions of the classical equations of motion of a system are the stationary points of the action [4]. In classical point mechanics, by convention, the Lagrangian of a particle with mass m in a potential $V\colon \mathbb{R}^d \to \mathbb{R}$ is described by

$$L(\gamma, \dot{\gamma}) = \frac{m}{2}\|\dot{\gamma}\|^2 - V(\gamma). \tag{2.3}$$

Consider a free particle, i.e. $V(\gamma) = 0$. Then the Lagrangian is simply

$$L(\gamma, \dot{\gamma}) = \frac{m}{2}\|\dot{\gamma}\|^2 \tag{2.4}$$

and we recover the classical equations of motion for a free particle from the Euler-Lagrange equations: $m\ddot{\gamma} = 0$. The curve which satisfies this equality represents a straight line, which is, additionally, traversed at a constant velocity. This is also exactly the premise of Newton's first law. On the other hand, when the potential V of a conservative force field $\mathbf{F} = -\nabla V$ is nonzero, the Euler Lagrange equations yield

$$m\ddot{\gamma} = -(\nabla V)(\gamma),$$

which is equivalent to Newton's second law. These laws represent the equations of motion for a single classical particle. Therefore, the equations of motion can be derived from the action and the entire system, in our case the motion of a particle, can be described by the stationary points of the action for the corresponding Lagrangian. For the purpose of describing geometry as the motion of free particles, we will only consider free particles in the remainder of the text, and set $V(\mathbf{r}) = 0$.

In conclusion, one can say that the stationary points of the action contain the only relevant information for a physical description of the classical system. All curves which are not stationary points can be disregarded in classical mechanics. However, when we go beyond classical mechanics, these other curves should be considered in order to obtain a complete description. In the following subsections we will consider the interpretation of the action in Brownian motion and in quantum mechanics, when classical principles no longer apply.

2.2 The action in quantum mechanics

Classical mechanics makes use of only a limited amount of information contained in the action. In the classical limit this is satisfactory, since the contribution of the information from the action outside the stationary points can be neglected. However, when these can no longer be neglected, we find ourselves in the field of quantum mechanics. In quantum mechanics we will have to make use of all the information of the system contained in the action for a given Lagrangian. With this a probability amplitude can be determined, which corresponds to a certain path γ through space-time.

In quantum mechanics the classical action, in units of $\hbar$, describes the phase of the probability amplitude of a certain particle trajectory γ:

$$\phi[\gamma] \sim \exp\left[\frac{i}{\hbar}S[\gamma]\right]. \tag{2.5}$$

The square of the absolute value of this probability amplitude represents the probability density that a particle follows the specific path. Let x_i denote the position $\gamma(t_i)$ with $i = 0, 1, 2, ..., N$ and $t_i < t_{i+1} = t_i + \epsilon$ with $t_i \in [t_0, t_N)$. A probability amplitude can be calculated for the path of the particle to be measured within a specified region of space-time R. The trajectory consisting of N line segments connecting the x_i's is a reasonable approximation of a curve γ as N tends to infinity. With $\sigma_{i,i+1}$ we denote the straight line segment from x_i to x_{i+1}. The velocity at which this line segment is traversed is constant and given by $\dot{\sigma}_{i,i+1} = \dot{\gamma}(t_i)$. The transition amplitude from x_i to x_{i+1} is then

$$\phi[\sigma_{i,i+1}] \sim \exp\left[\frac{i}{\hbar}S[\sigma_{i,i+1}]\right]. \tag{2.6}$$

In quantum mechanics the probability amplitude of transitioning from x_i to x_{i+2} is equal to the probability amplitude of the transition from x_i to x_{i+1} times that of the transition from x_{i+1} to x_{i+2}, integrated over all possible values of x_{i+1}. Applying this rule together with the postulated expression for the probability amplitude we find the probability amplitude that the path of the particle is in the region R

$$\phi(R) = \lim_{N\to\infty}\int_R \exp\left[\frac{i}{\hbar}\sum_{i=0}^{N-1} S[\sigma_{i,i+1}]\right]\cdots\frac{dx_i}{A}\frac{dx_{i+1}}{A}\cdots. \tag{2.7}$$

Note that the sum $\sum_{i=0}^{N-1} S[\sigma_{i,i+1}]$ is equal to the action $S[\gamma]$ of the complete curve as N tends to infinity. Furthermore, a normalization constant A has been introduced for each separate point in time, such that the probability of a certain event returns unity. [1]

Similarly, a propagator can be established which *propagates* a particle from x_0 at time t_0 to x_1 at time t_1 with $t_1 > t_0$. It represents the probability amplitude of finding a particle at position x_1 at time t_1, given that it was position x_0 at a previous time t_0. This propagator is equal to the sum of the contributions, i.e. the probability densities, of all paths between x_0 and x_1 and is denoted by

$$K(x_0, t_0; x_1, t_1) = \sum_\gamma \phi[\gamma] \quad \text{with } \gamma(t_0) = x_0 \text{ and } \gamma(t_1) = x_1.$$

The magnitude of the contributions are equal, but the phase contributions depend on the classical action. Therefore, the propagator can be expressed as

$$K(x_0, t_0; x_1, t_1) = \mathcal{N}\sum_\gamma \exp\left[\frac{i}{\hbar}S[\gamma]\right],$$

where $\mathcal{N}$ is a normalization constant and the summation is over all paths. The summation over 'all paths' is an infinite summation over a set containing every conceivable curve connecting x_0 to x_1. For this the path integral has been introduced, which represents exactly this operation. It allows us to express the propagator as

$$K(x_0, t_0; x_1, t_1) = \int [d\gamma] \exp\left[\frac{i}{\hbar}S[\gamma]\right]. \tag{2.8}$$

[2]

The path integral is, as the name suggests, analogous to an ordinary integral, except that the integration operation is now carried out over all paths in a certain region, instead of over all numbers in a certain interval. How one could actually attempt to perform this operation will be discussed in section 3.

Now we have established a formulation of quantum mechanics. In theory, all quantum mechanical systems can be described using the probability amplitude $\phi[\gamma]$, derived from the action corresponding to the system, as well as the propagator $K(x_0, t_0; x_1, t_1)$. A wave function can be constructed using the path integral.

$$\psi(x_k, t) = \lim_{\epsilon \to 0} \int_{R'} \exp\left[\frac{i}{\hbar} \sum_{i=-\infty}^{k-1} S[\sigma_{i,i+1}]\right] \frac{dx_{k-1}}{A} \frac{dx_{k-2}}{A} \cdots . \tag{2.9}$$

Here R' is a specified region of space-time, in time completely located before $t = t_k$. In his original paper [1], Feynman demonstrates that this wave function satisfies the Schrödinger equation, if we set

$$A = \sqrt{\frac{2\pi\hbar\epsilon i}{m}}. \tag{2.10}$$

Schrödinger's differential equation representation and Heisenberg's matrix representation may be more familiar representations of quantum mechanics, but these representations are entirely equivalent to Feynman's path integral representation. More parallels to the path integral can be found in the subject of Brownian motion, which will be discussed next.

2.3 The action in Brownian motion

In Brownian motion one deals with classical particles, but they behave in a probabilistic manner, similar to quantum particles. This leads to the Brownian analogue to Feynman's path integral; the Wiener integral. At the foundation of this integral we again find a quantity, which resembles the action of a classical system. We will derive this integral in order to describe the Brownian motion of a classical particle, which satisfies the diffusion equation

$$\frac{\partial u(\mathbf{r}, t)}{\partial t} = \nabla \cdot [D(u, \mathbf{r}) \nabla u(\mathbf{r}, t)].$$

The probability of a particle being located at position $\mathbf{r}$ at time t is represented by $u(\mathbf{r}, t)$ and $D(u, \mathbf{r})$ is the diffusion coefficient at that position. The diffusion coefficient represents the rate at which the average area within which the particle has diffused increases over time. The diffusion coefficient depends on the medium and the temperature. For Brownian motion we assume that the diffusion coefficient is constant, for which the diffusion equation becomes identical to the heat equation

$$\frac{\partial u(\mathbf{r}, t)}{\partial t} = D \nabla^2 u(\mathbf{r}, t). \tag{2.11}$$

In Brownian motion the random movement of a classical particle is described. The probability that a particle reaches a point x within a sufficiently small radius dx starting from x_0 after sufficiently large number of steps N of sufficiently small length ℓ in three dimensions is

$$P(x_0, 0; x, \epsilon) dx = \left(\frac{3}{2\pi N \ell^2}\right)^{\frac{3}{2}} \exp\left[-\frac{3(x - x_0)^2}{2N\ell^2}\right] dx. \tag{2.12}$$

[2] (p.16). We assume that each step in the process takes the same amount of time, such that the total time is N times the duration of one step. This total duration is then ϵ. The probability distribution $P(x_0, 0; x, \epsilon)$ satisfies the heat equation, if we let $D = \frac{N\ell^2}{6\epsilon}$. This is a transition probability density, for which holds that

$$P(x_0, t_0; x_2, t_2) = \int P(x_0, t_0; x_1, t_1) P(x_1, t_1; x_2, t_2) dx_1,$$

similar to the probability amplitude in quantum mechanics. Then the probability that a particle's trajectory lies in a region R of space can then be determined by applying this rule in combination with the probability $P(x_0, 0; x, \epsilon)dx$. this yields

$$P(R) = \lim_{n\to\infty} \left(\frac{1}{4\pi D\epsilon}\right)^{\frac{3n}{2}} \int_R \exp\left[-\sum_{i=0}^{n-1} \frac{(x_{i+1}-x_i)^2}{4D\epsilon}\right] \cdots dx_i dx_{i+1} \cdots . \tag{2.13}$$

If we denote $\frac{(x_{i+1}-x_i)^2}{4D\epsilon^2}$ with the action $S'[\sigma_{i,i+1}]$ and let $A = (4\pi D\epsilon)^{\frac{3}{2}}$, then we can write

$$P(R) = \lim_{n\to\infty} \int_R \exp\left[-\sum_{i=0}^{n-1} S'[\sigma_{i,i+1}]\epsilon\right] \cdots \frac{dx_i}{A}\frac{dx_{i+1}}{A} \cdots = \int [d\gamma] \exp\left[-S'[\gamma]\right], \tag{2.14}$$

where the curve γ consists of the points x_i; $\gamma(t_i) = x_i$. This is known as the Wiener integral. $S'[\gamma]$ represents the limit as n tends to infinity of $\sum_{i=0}^{n-1} S'[\sigma_{i,i+1}]$. This expression corresponds to the action

$$S'[\gamma] = \frac{1}{4D} \int_0^{n\epsilon} \dot{\gamma}(t)^2 dt, \tag{2.15}$$

if $n\epsilon$ is kept constant as the limit is taken. This is very similar to the action which follows from the classical Lagrangian for a free particle.

The similarities between the Wiener integral and Feynman's path integral are evident. If we make the correspondence

$$D \to \frac{i\hbar}{2m} \quad ,$$

the Schrödinger equation, with the potential $V = 0$, becomes equivalent to the heat equation and the propagator K becomes identical to the probability that a particle goes from x_0 to a position x by Brownian motion in a time t. The conversion from Brownian motion to quantum mechanics can, aside from the different constants, be viewed as a Wick rotation. With this we mean a transition from ordinary time to imaginary time, i.e. $t \to iT$, or equivalently the other way around. The Wick rotation is a very useful tool for transforming a non-convergent path integral into a path integral that can actually be solved. However,

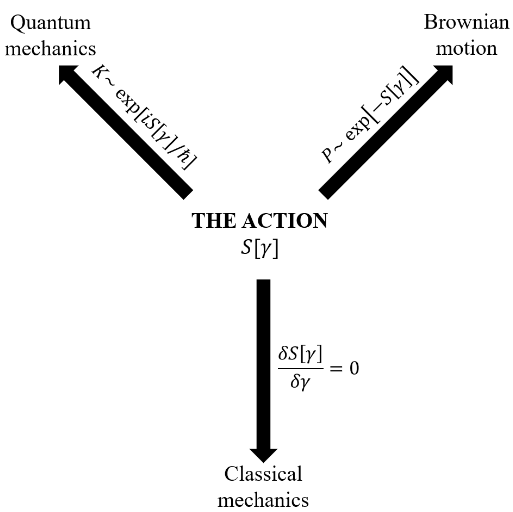

FIGURE 2: Schematic representation of the role of the action in quantum mechanics, classical mechanics and Brownian motion

by performing the rotation in order to solve a problem, the solution can also be interpreted as a solution for Brownian motion, rather than a quantum mechanical result.

We have seen that the three fields of classical mechanics, quantum mechanics and Brownian motion are all linked by the action S.The ways in which the theories emerge from the action are shown in figure 2.

3 Technical review — Path integral methods

In the previous section we introduced the concept of path integrals and how they are used to compute probabilities. Now we will consider the actual calculation of such a probability distribution. The calculation of the free particle propagator will be presented in this chapter. Furthermore, constrained path integrals will be introduced, which is an important tool to be used in chapter 4. The constraints for enclosed area of a curve will be derived.

Our starting point is the propagator $K(x_0, t_0; x_N, T)$. Without loss of generality we can let $t_0 = 0$. For the evaluation we will consider the expression

$$K(x_0, 0; x_N, T) = \lim_{N\to\infty} \int_{\mathbb{R}^{N-1}} \frac{1}{A} \exp\left[\frac{i}{\hbar}\sum_{j=0}^{N-1} S[\sigma_{j,j+1}]\right] \frac{dx_1}{A}\frac{dx_2}{A}\cdots\frac{dx_{N-1}}{A}. \tag{3.1}$$

This is equivalent to equation (2.8). Note that there is no integration over x_0 or x_N, since the endpoints of the curves are fixed by the arguments of the propagator. All results that are derived for quantum mechanics can also be translated to Brownian motion, due to the parallels between the two theories. In this research we want to use path integrals in the quantum generalization of classical of geometry. Classical straight lines are the solutions to the classical equations of motion for free particles. Therefore, straight 'quantum' lines will also have to be considered in the absence of a potential. Thus, in the following we will only consider the free particle Lagrangian (2.4).

3.1 Practical evaluation of elementary path integrals

From the free particle Lagrangian we will first derive an expression for $S[\sigma_{j,j+1}]$. We let $t_{j+1} - t_j = \frac{T}{N}$ tend to zero, then, with the Lagrangian (2.4), the action can be rewritten by applying first order approximations of the time derivative and time integral:

$$\begin{aligned} S_{\text{free}}[\sigma_{j,j+1}] &= \lim_{N\to\infty} \int_{t_j}^{t_j+\frac{T}{N}} L(\gamma(t), \dot{\gamma}(t))dt \\ &= \lim_{N\to\infty} \int_{t_j}^{t_j+\frac{T}{N}} \frac{m}{2}\|\dot{\gamma}(t)\|^2 dt \\ &= \lim_{N\to\infty} \frac{T}{N}\left[\frac{m}{2}\|\dot{\gamma}(t)\|^2\right]_{t=t_j} \\ &= \lim_{N\to\infty} \frac{mT}{2N}\frac{\|\gamma(t_{j+1}) - \gamma(j_i)\|^2}{\left(\frac{T}{N}\right)^2} \\ &= \lim_{N\to\infty} \frac{mN}{2T}\|x_{j+1} - x_j\|^2. \end{aligned} \tag{3.2}$$

We insert this expression into eq. (3.1) to obtain

$$K(x_0, 0; x_N, T) = \lim_{N\to\infty} \int_{\mathbb{R}^{N-1}} \frac{1}{A} \exp\left[\frac{i}{\hbar}\sum_{j=0}^{N-1} \frac{mN}{2T}\|x_{j+1} - x_j\|^2\right] \frac{dx_1}{A}\frac{dx_2}{A}\cdots\frac{dx_{N-1}}{A}. \tag{3.3}$$

This expression consists of $N - 1$ Gaussian integrals. If this $(N - 1)$-dimensional integral can be expressed as a function of N, we can take the limit of that expression as N goes to infinity to obtain the propagator. A useful theorem for the evaluation of path integrals of this form is the following

Theorem 3.1. *Let* $\mathbf{A} \in \mathbb{R}^{n\times n}$ *be a symmetric matrix with* $\det \mathbf{A} \neq 0$ *and* $\mathbf{x} \in \mathbb{R}^n$*. Then*

$$\int_{\mathbb{R}^n} \exp\left[i\mathbf{x}^T\mathbf{A}\mathbf{x}\right] d\mathbf{x} = \sqrt{\frac{(i\pi)^n}{\det \mathbf{A}}}.$$

Proof. From the properties of $\mathbf{A}$ it follows that there exists, by the *Spectral Theorem for Symmetric Matrices*, an invertible matrix $\mathbf{Q}$, such that $\mathbf{Q}\mathbf{Q}^T = I$ and $D = \mathbf{Q}^T b\mathbf{Q}$ is a diagonal matrix with the eigenvalues of $\mathbf{A}$ on the diagonal. We use that $\det \mathbf{AB} = \det(\mathbf{A})\det(\mathbf{B})$, which implies that $\det(\mathbf{Q}) = \det(\mathbf{Q}^T) = 1$ and that $\det(\mathbf{A}) = \det(\mathbf{D}) = \prod_{i=1}^d d_i$, where d_k represents the element at position (k, k) in $\mathbf{D}$. The path integral reduces to

$$\begin{aligned}\prod_{k=1}^n \int_{-\infty}^{\infty} dx_k \exp\left[i\mathbf{x}^T\mathbf{A}\mathbf{x}\right] &= \prod_{k=1}^n \int_{-\infty}^{\infty} dx_k \exp\left[i\mathbf{x}^T\left(\mathbf{Q}^{-1}\right)^T\mathbf{D}\mathbf{Q}^{-1}\mathbf{x}\right]\\ &= \prod_{k=1}^n \int_{-\infty}^{\infty} dx_k \exp\left[i\left(\mathbf{Q}^{-1}\mathbf{x}\right)^T\mathbf{D}\mathbf{Q}^{-1}\mathbf{x}\right]\end{aligned}$$

We introduce the new vector $\tilde{\mathbf{x}} = \mathbf{Q}^{-1}\mathbf{x}$ and then we can rewrite the integral using the Jacobian matrix $\mathbf{J} = \frac{\partial(x_1,x_2,\ldots,x_n)}{\partial(\tilde{x}_1,\tilde{x}_2,\ldots,\tilde{x}_n)}$

$$\prod_{k=1}^n \int_{-\infty}^{\infty} dx_k \exp\left[i\left(\mathbf{Q}^{-1}\mathbf{x}\right)^T\mathbf{D}\mathbf{Q}^{-1}\mathbf{x}\right] = \prod_{k=1}^n \int_{-\infty}^{\infty} d\tilde{x}_k |\det \mathbf{J}| \exp\left[i\tilde{\mathbf{x}}^T\mathbf{D}\tilde{\mathbf{x}}\right]$$

By definition, $\mathbf{J} = \mathbf{Q}^{-1} = \mathbf{Q}^T$, so, as established above, $\det \mathbf{J} = \det\ \mathbf{Q}^T = 1$, which means that

$$\begin{aligned}\prod_{k=1}^n \int_{-\infty}^{\infty} d\tilde{x}_k |\det \mathbf{J}| \exp\left[i\tilde{\mathbf{x}}^T\mathbf{D}\tilde{\mathbf{x}}\right] &= \prod_{k=1}^n \int_{-\infty}^{\infty} d\tilde{x}_k \exp\left[i\tilde{\mathbf{x}}^T\mathbf{D}\tilde{\mathbf{x}}\right]\\ &= \prod_{k=1}^n \int_{-\infty}^{\infty} d\tilde{x}_k \exp\left[id_k\tilde{x}_k^2\right].\end{aligned}$$

This is the product of n Gaussian integrals, which have solutions

$$\sqrt{\frac{i\pi}{d_k}},$$

such that the solution to the product of integrals becomes

$$\prod_{k=1}^n \int_{-\infty}^{\infty} d\tilde{x}_k \exp\left[id_k\tilde{x}_k^2\right] = \prod_{k=1}^n \sqrt{\frac{i\pi}{d_k}} = \sqrt{\frac{(i\pi)^n}{\prod_{k=1}^n d_k}} = \sqrt{\frac{(i\pi)^n}{\det \mathbf{A}}}.$$

Thereby proving the result. □

The paths over which the integration is performed can be decomposed into the classical trajectory and a curve which is varied. This is achieved by letting

$$\gamma(t) = \gamma_{cl}(t) + \beta(t), \tag{3.4}$$

where $\gamma_{cl}(t)$ is the classical trajectory between the endpoints obeying the Euler-Lagrange equations. Hence it follows that $\beta(t)$ must obey

$$\beta(0) = 0, \quad \beta(T) = 0.$$

This enables us to write

$$S[\gamma] = S[\gamma_{cl} + \beta] = \int_0^T L(\gamma_{cl}(t) + \beta(t), \dot{\gamma}_{cl}(t) + \dot{\beta}(t))dt.$$

For quadratic Lagrangians this action can be expressed in terms of $S[\beta]$ and $S[\gamma_{cl}]$. This can simplify the calculation, since the expression of $S[\gamma_{cl}]$ is known and β now has the useful property that it equals zero at the endpoints, which removes cross terms, as we will see below. This method can be applied to the free particle Lagrangian, which is of the quadratic form, in one dimension. This will be demonstrated here.

Our goal is to evaluate the propagator $K(x_0, 0; x_N, T)$. Therefore we need to calculate

$$\begin{aligned} K(x_0, 0; x_N, T) &= \int [d\gamma] \exp\left[\frac{i}{\hbar} S[\gamma]\right] \\ &= \lim_{N\to\infty} \int_{\mathbb{R}^{N-1}} \frac{1}{A} \exp\left[\frac{i}{\hbar} \sum_{j=0}^{N-1} \frac{mN}{2T} (x_{i+1} - x_i)^2\right] \frac{dx_1}{A} \frac{dx_2}{A} \cdots \frac{dx_{N-1}}{A}. \end{aligned} \tag{3.5}$$

For the free particle Lagrangian it is know that the solution to the Euler-Lagrange equations is a straight line

$$\gamma_{cl}(t) = \frac{x_N - x_0}{T} t + x_0. \tag{3.6}$$

We apply the decomposition method to obtain

$$S[\gamma] = \frac{m}{2} \int_0^T (\dot{\gamma}_{cl}(t) + \dot{\beta}(t))^2 dt = \frac{m}{2} \left(\int_0^T \dot{\gamma}_{cl}(t)^2 dt + \int_0^T \dot{\beta}(t)^2 dt + \int_0^T 2\dot{\gamma}_{cl}(t)\dot{\beta}(t) dt \right). \tag{3.7}$$

We can recognize $S[\gamma_{cl}]$ and $S[\beta]$, respectively, in the first two terms. Furthermore, since

$$\dot{\gamma}_{cl} = \frac{x_N - x_0}{T} \tag{3.8}$$

is independent of time t, the third integral, over cross terms, reduces to zero:

$$\int_0^T 2\dot{\gamma}_{cl}(t)\dot{\beta}(t) dt = 2\frac{x_N - x_0}{T} \left[\beta(T) - \beta(0)\right]_0^T = 0,$$

since $\beta(0) = \beta(T) = 0$. Therefore, for the free particle Lagrangian

$$S[\gamma] = S[\gamma_{cl}] + S[\beta]$$

with

$$S[\gamma_{cl}] = \frac{m}{2} \int_0^T \left(\frac{x_N - x_0}{T}\right)^2 dt = \frac{m}{2T}(x_N - x_0)^2. \tag{3.9}$$

Thus, we can write

$$\begin{aligned} K(x_0, 0; x_N, T) &= \exp\left[\frac{m}{2T}(x_N - x_0)^2\right] \int [d\beta] \exp\left[\frac{i}{\hbar} S[\beta]\right] \\ &= \exp\left[\frac{m}{2T}(x_N - x_0)^2\right] \lim_{N\to\infty} A^{-N} \times \\ &\quad \int_{\mathbb{R}^{N-1}} \exp\left[\frac{i}{\hbar} \sum_{j=0}^{N-1} \frac{mN}{2T} (z_{i+1} - z_i)^2\right] dz_1 dz_2 \cdots dz_{N-1}. \end{aligned} \tag{3.10}$$

Now the z_i's represent the points of the curve β; $z_i = \beta(t_i)$ with $z_0 = z_N = 0$. The sum in the exponential can be expanded as follows

$$\frac{1}{\hbar} \sum_{j=0}^{N-1} \frac{mN}{2T} (z_{i+1} - z_i)^2 = \frac{mN}{2T\hbar} \left(z_0^2 - 2z_0 z_1 + 2z_1^2 - 2z_1 z_2 + 2z_2^2 + \cdots + 2z_{N-1}^2 - 2z_{N-1} z_N + z_N^2\right).$$

Recall that $z_0 = z_N = 0$, such that we can write this sum as a matrix vector product $\mathbf{z}^T\mathbf{M}\mathbf{z}$ with the $N-1 \times N-1$ symmetric matrix

$$\mathbf{M} = \frac{mN}{2T\hbar}\begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & \dots & -1 & 2 \end{bmatrix}$$

and the vector

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{N-1} \end{bmatrix}.$$

We can express the determinant D_{N-1} of the matrix $\frac{2T\hbar}{mN}\mathbf{M}$ with the recursion relation

$$D_n = 2D_{n-1} - D_{n-2}, \tag{3.11}$$

where n represents the number of rows, which is equal to the number of columns, of the matrix. With the condition that $D_1 = 2$ and $D_2 = 3$ we obtain the explicit expression $D_n = n + 1$ The determinant of $\mathbf{M}$ is then

$$\det \mathbf{M} = N\left(\frac{mN}{2T\hbar}\right)^{N-1}. \tag{3.12}$$

By Theorem 3.1, then

$$K(x_0, 0; x_N, T) = \exp\left[\frac{im}{2\hbar T}(x_N - x_0)^2\right] \lim_{N\to\infty} A^{-N}\sqrt{\left(\frac{2T\hbar}{mN}\right)^{N-1}\frac{(i\pi)^{N-1}}{N}}. \tag{3.13}$$

With $A = \sqrt{\frac{2\pi\hbar Ti}{mN}}$, see (2.10), the expression for the propagator becomes

$$K(x_0, 0; x_N, T) = \sqrt{\frac{m}{2\hbar\pi iT}}\exp\left[\frac{im}{2\hbar T}(x_N - x_0)^2\right]. \tag{3.14}$$

[5] In d dimensions the integration of the Gaussians over $N-1$ variables is performed d times, once for each dimension. The expression can then easily be generalized to d dimensions, where

$$K(x_0, 0; x_N, T) = \left(\frac{m}{2\hbar\pi iT}\right)^{\frac{d}{2}}\exp\left[\frac{im}{2\hbar T}\|x_N - x_0\|^2\right]. \tag{3.15}$$

This propagator contains the contributions from all curves from x_0 to x_N over a time T. In the case of quantum geometry we want to consider only the contributions from curves satisfying an imposed (geometric) criterion. For this so-called *constrained* path integrals can be used.

3.2 Constrained path integrals

In the description of path integrals we have constrained the paths by fixing their endpoints in order to determine the propagator K. However, there are also other constraints that may be imposed on the piece-wise smooth curves γ by means of an additional functional $S_{\text{constr}}[\gamma]$, which we call the constraint functional. Suppose we want to consider only the curves for which this functional returns a value $c \in \mathbb{R}$. Then the propagator associated with only these curves can be expressed as

$$K_c = \int [d\gamma]\delta(S_{\text{constr}}[\gamma] - c) \exp\left[\frac{i}{\hbar} S_{\text{free}}\right]. \tag{3.16}$$

[2](p. 156) Here the free particle action is

$$S_{\text{free}}[\gamma] = \int_{t_1}^{t_2} \frac{m}{2} \|\dot{\gamma}(t)\|^2 dt \tag{3.17}$$

This gives us only the contributions of curves for which $S_{\text{constr}}[\gamma] = c$, since $\delta(S_{\text{constr}}[\gamma] - c) = 0$ whenever $S_{\text{constr}}[\gamma] \neq c$. To evaluate this path integral, we represent the Dirac δ-distribution as a Fourier integral and write

$$K_c = \int_{-\infty}^{\infty} e^{-i\lambda A} K_\lambda d\lambda, \tag{3.18}$$

where K_λ is the propagator

$$K_\lambda = \int [d\gamma] \exp\left[\frac{i}{\hbar} S_\lambda[\gamma]\right] \tag{3.19}$$

with the constrained action

$$S_\lambda[\gamma] = S_{\text{free}}[\gamma] + \hbar\lambda S_{\text{constr}}[\gamma]. \tag{3.20}$$

Now let us address an example of these constraint functionals. We will constrain the enclosed area of curves starting and ending at the same point. Stokes' theorem states for a given vector field $\vec{F}$ and a given region A with boundary ∂A

$$\int_A (\vec{\nabla} \times \vec{F}) \cdot d\vec{n} = \oint_{\partial A} \vec{F} \cdot d\vec{r}. \tag{3.21}$$

In Cartesian coordinates the curl of a vector field $\vec{F}$ can be computed as a function of the Cartesian components F_x, F_y and F_z with $\vec{F} = F_x\vec{e}_1 + F_y\vec{e}_2 + F_z\vec{e}_3$.

$$\vec{\nabla} \times \vec{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\vec{e}_1 + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\vec{e}_2 + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\vec{e}_3. \tag{3.22}$$

Requiring that $\vec{\nabla} \times \vec{F} = \vec{e_3}$ we obtain the system of partial differential equations

$$\begin{aligned} \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} &= 0 \\ \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} &= 0 \\ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} &= 1. \end{aligned}$$

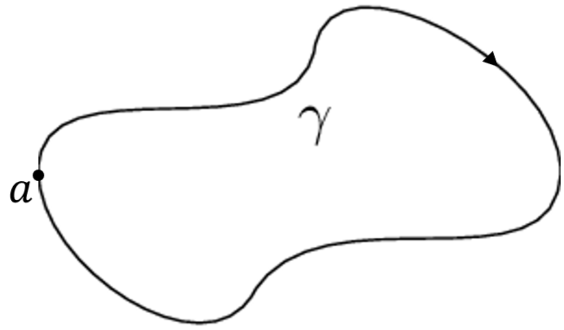

FIGURE 3: A two dimensional closed curve γ starting and ending at the same point a.

To find a simple explicit solution we can set $F_z = 0$. This also corresponds to the fact that we will be working in a single plane in three-dimensional space, in this case the xy-plane. The system of equations can then be rewritten to

$$\begin{aligned} \frac{\partial F_y}{\partial z} &= 0 \\ \frac{\partial F_x}{\partial z} &= 0 \\ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} &= 1, \end{aligned}$$

where the first two equations guarantee that the vector field is independent of z. Any solution of the form

$$F_x = Cy, \quad F_y = (C+1)x, \quad F_z = 0$$

will satisfy the equality. For symmetry reasons, we will use $F_x = -\frac{y}{2}$, $F_y = \frac{x}{2}$, $F_z = 0$. From this it follows that an explicit expression for a vector field with $\vec{\nabla} \times \vec{F} = \vec{e}_3$ is

$$\vec{F} = \tfrac{1}{2}(-y\vec{e}_1 + x\vec{e}_2). \tag{3.23}$$

This result can be used to construct a constraint functional for the area enclosed by a two dimensional closed curve γ:

$$[0,T] \to \mathbb{R}^2\,, \quad \gamma(0) = \gamma(T)$$

in the xy-plane, see figure 3. If the curve γ encloses the region A with area $\mathcal{A}$, then this area is equal to

$$\mathcal{A} = \int_A \vec{e}_3 \cdot d\vec{n} = \int_A (\vec{\nabla} \times \vec{F}) \cdot d\vec{n}. \tag{3.24}$$

Therefore, by Stokes' theorem the desired area constraint functional can be expressed as

$$S_{\text{area}}[\gamma] = \oint\limits_\gamma \vec{F} \cdot d\vec{r} = \oint\limits_\gamma \tfrac{1}{2}(-y\vec{e}_1 + x\vec{e}_2) \cdot d\vec{r} \tag{3.25}$$

We can decompose γ into its Cartesian components

$$\gamma = \gamma_1\vec{e}_1 + \gamma_2\vec{e}_2, \tag{3.26}$$

such that we can write

$$S_{\text{area}}[\gamma] = \oint\limits_\gamma \tfrac{1}{2}(-y\vec{e}_1 + x\vec{e}_2) \cdot d\vec{r} = \int_0^T \tfrac{1}{2}(\gamma_1(t)\dot{\gamma}_2(t) - \gamma_2(t)\dot{\gamma}_1(t))dt. \tag{3.27}$$

4 Proof of concept – Classical, quantum and Brownian area

In this section, we will attempt to generalize the concept of a two-dimensional area using the path integrals in both quantum mechanics and Brownian motion. To achieve this we will use the tools developed in the previous section. We will start by calculating the probability amplitude, respectively the probability, for a closed curve to enclose an algebraic area $\mathcal{A}$ for a quantum particle and a Brownian particle. For this we will apply the area constraint described in the previous section in a constrained path integral.

4.1 Classical area

Classically, the area of a two-dimensional region in space is a well defined quantity. There are well-known formulas which give you exactly the area of, for example, a triangle or a square. For any arbitrary shape, the area can be obtained through integration over that region of space. An area can be described by the trajectories of a particle. For an area with straight lines as sides this means, classically, that each side is a trajectory of a particle with a different initial position and a different (constant) velocity, assuming there is no external potential present. In this section we focus on the apparently trivial question which area is enclosed by the trajectory of a force-free particle that starts at some point and ends up at the same point in space a finite time afterwards. The classical result is of course zero, since such a free particle will need to stay at the starting point in order to arrive there again. However, if the particle trajectory, like in quantum mechanics and Brownian motion, is not restricted to a straight line, a finite area can be enclosed. To obtain this area one can use the area functional (3.27), as described under 3.2. This calculation of area by integration is still classical. However, the curves that are used to compute the areas do not correspond to classical free particle trajectories. Therefore, we can consider these curves as quantum and Brownian paths and develop the notion of quantum and Brownian area by applying path integrals, which will be done below.

4.2 Quantum area

To derive a quantum mechanical generalization of the concept of area we consider the area $\mathcal{A}$ enclosed by a curve for which we compute a probability amplitude $K_{\mathcal{A}}$. We want to constrain the curves of the path integrals to have an area $\mathcal{A}$, with $a := \gamma(0) = \gamma(T)$. This constraint can be expressed, following from the derivations above, as

$$S_{\text{area}}[\gamma] = \frac{1}{2}\int_0^T \gamma_1(t)\dot{\gamma}_2(t) - \gamma_2(t)\dot{\gamma}_1(t)dt = \mathcal{A}. \tag{4.1}$$

Then the propagator of curves obeying this constrained can be expressed as

$$K_{\mathcal{A}} = \int_{-\infty}^{\infty} e^{-i\lambda\mathcal{A}} K_\lambda d\lambda, \tag{4.2}$$

where K_λ is the propagator

$$K_\lambda = \int [d\gamma] \exp\left[\frac{i}{\hbar} S_\lambda\right] \tag{4.3}$$

with the constrained action

$$S_\lambda = S_{\text{free}} + \hbar\lambda S_{\text{area}} = \frac{m}{2}\int_0^T \big(\dot{\gamma}_1(t)^2 + \dot{\gamma}_2(t)^2 + \frac{\lambda\hbar}{m}(\gamma_1(t)\dot{\gamma}_2(t) - \gamma_2(t)\dot{\gamma}_1(t))\big)dt. \tag{4.4}$$

We introduce discrete points in time t_k with $t_k - t_{k-1} = \epsilon$, $t_0 = 0$, $t_N = T$ and we define $\gamma_1(t_k) = x_k$ and $\gamma_2(t_k) = y_k$. Then for this particular action

$$S_\lambda[(x_i, y_i; x_{i+1}, y_{i+1})] = \frac{m}{2\epsilon}\left((x_j - x_{j-1})^2 + (y_j - y_{j-1})^2 + \frac{\hbar\lambda\epsilon}{m}\left((x_{j-1}y_j - x_j y_{j-1})\right)\right). \tag{4.5}$$

We insert this action over the infinitesimal time interval ϵ into the expression for the propagator (3.1)

$$\begin{aligned} K_\lambda = \lim_{N\to\infty} A^{-2N} \prod_{k=1}^{N-1} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx_k dy_k \exp\Bigg[\frac{im}{2\hbar\epsilon}\Bigg(&\sum_{j=1}^{N}(x_j - x_{j-1})^2 + (y_j - y_{j-1})^2 \\ &+\frac{\hbar\lambda\epsilon}{m}(x_{j-1}y_j - x_j y_{j-1})\Bigg)\Bigg]. \end{aligned} \tag{4.6}$$

Let

$$\begin{aligned} \gamma_1(t) &= \gamma_{cl,1} + \beta_1(t) \\ \gamma_2(t) &= \gamma_{cl,2} + \beta_2(t), \end{aligned} \tag{4.7}$$

where $\gamma_{cl}(t)$ is the classical trajectory of a free particle from the start point to the end point. Since we have that $\gamma(0) = \gamma(T) = a$, this trajectory consists only of the point $a = (a_1, a_2)$. As a consequence $\dot\gamma = \dot\beta$ and β has the property that $\beta(0) = \beta(T) = 0$. Substituting this expression for γ into the expression for the probability amplitude $K_{\lambda,N}$ gives

$$K_\lambda = \int [d\beta] \exp\left[\frac{i}{\hbar}S_\lambda\right]$$

with

$$S_\lambda = \frac{m}{2}\int_0^T \dot\beta_1(t)^2 + \dot\beta_2(t)^2 + \frac{\hbar\lambda}{m}((a_1 + \beta_1(t))\dot\beta_2(t) - (a_2 + \beta_2(t))\dot\beta_1(t))dt$$

$$= \frac{m}{2}\int_0^T \dot\beta_1(t)^2 + \dot\beta_2(t)^2 + \frac{\lambda}{m}(\beta_1(t)\dot\beta_2(t) - \beta_2(t)\dot\beta_1(t))dt + \frac{m}{2}\int_0^T \frac{\hbar\lambda}{m}(a_1\dot\beta_2 - a_2\dot\beta_1)dt$$

$$= \frac{m}{2}\int_0^T \dot\beta_1(t)^2 + \dot\beta_2(t)^2 + \frac{\hbar\lambda}{m}(\beta_1(t)\dot\beta_2(t) - \beta_2(t)\dot\beta_1(t))dt + \frac{\hbar\lambda}{2}(a_1(\beta_2(T) - \beta_2(0)) - a_2(\beta_1(T) - \beta_1(0)).$$

Using the property that $\beta(T) = \beta(0) = 0$, we find that without loss of generality we can consider the probability amplitude for curves starting and ending in the origin

$$K_\lambda = \int [d\beta] \exp\left[\frac{i}{\hbar}\frac{m}{2}\int_0^T \dot\beta_1(t)^2 + \dot\beta_2(t)^2 + \frac{\hbar\lambda}{m}(\beta_1(t)\dot\beta_2(t) - \beta_2(t)\dot\beta_1(t))dt\right].$$

We discretize this expression in the same way as we did for γ, letting $\beta_1(t_k) = x_k$ and $\beta_2(t_k) = y_k$, such that the amplitude reduces to the expression in (4.6), but now we have that $x_0 = x_N = y_0 = y_N = 0$. We expand the expression as follows

$$\begin{aligned} A^{-2N} \prod_{k=1}^{N-1} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx_k dy_k \exp\Bigg[\frac{im}{2\hbar\epsilon}\Bigg(&\sum_{j=1}^{N-1} 2x_j^2 - x_j x_{j-1} - x_j x_{j+1} + 2y_j^2 - y_j y_{j-1} - y_j y_{j+1} \\ &+\frac{\hbar\lambda\epsilon}{2m}x_{j-1}y_j + \frac{\hbar\lambda\epsilon}{2m}x_j y_{j+1} - \frac{\hbar\lambda\epsilon}{2m}x_j y_{j-1} - \frac{\hbar\lambda\epsilon}{2m}x_{j+1}y_j\Bigg)\Bigg]. \end{aligned}$$

We introduce the composite vector $\mathbf{z} = (x_1, y_1, x_2, y_2, ..., x_{N-1}, y_{N-1})^T$, such that, using that $x_0 = x_N = y_0 = y_N = 0$, the summation can be written as a matrix vector product

$$K_{\lambda,N} = A^{-2N} \prod_{k=1}^{N-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx_k dy_k \exp\left[i\mathbf{z}^T \tilde{\mathbf{M}} \mathbf{z}\right]. \tag{4.8}$$

with $\tilde{\mathbf{M}}$ a $(2N-2) \times (2N-2)$ symmetric matrix.

$$\tilde{\mathbf{M}} = \frac{m}{2\hbar\epsilon}\mathbf{M} = \frac{m}{2\hbar\epsilon}\begin{bmatrix} 2 & 0 & -1 & \frac{\hbar\lambda\epsilon}{2m} & 0 & 0 & \cdots \\ 0 & 2 & -\frac{\hbar\lambda\epsilon}{2m} & -1 & 0 & 0 & \cdots \\ -1 & -\frac{\hbar\lambda\epsilon}{2m} & 2 & 0 & -1 & \frac{\hbar\lambda\epsilon}{2m} & \cdots \\ \frac{\hbar\lambda\epsilon}{2m} & -1 & 0 & 2 & -\frac{\hbar\lambda\epsilon}{2m} & -1 & \cdots \\ 0 & 0 & -1 & -\frac{\hbar\lambda\epsilon}{2m} & 2 & 0 & \cdots \\ 0 & 0 & \frac{\hbar\lambda\epsilon}{2m} & -1 & 0 & 2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

We are interested in the determinant of this matrix. From linear algebra we have the result that, for a square block matrix $\begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{D} & \mathbf{E} \end{bmatrix}$, with $\mathbf{B}$ invertible, the determinant is given by

$$\begin{vmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{D} & \mathbf{E} \end{vmatrix} = \det \mathbf{B} \times \det\left(\mathbf{E} - \mathbf{D}\mathbf{B}^{-1}\mathbf{C}\right).$$

Applying this to our matrix $\mathbf{M}$ we can recognize that for any value of N the matrix can be written as

$$\mathbf{M} = \mathbf{M}_{2N-2} = \begin{bmatrix} \mathbf{B} & \mathbf{C} \quad \mathbf{0} \\ \mathbf{D} & \\ \mathbf{0} & \mathbf{M}_{2N-4} \end{bmatrix}$$

with $\mathbf{B} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} -1 & \frac{\hbar\lambda\epsilon}{2m} \\ -\frac{\hbar\lambda\epsilon}{2m} & -1 \end{bmatrix}$ and $\mathbf{D} = \begin{bmatrix} -1 & -\frac{\hbar\lambda\epsilon}{2m} \\ \frac{\hbar\lambda\epsilon}{2m} & -1 \end{bmatrix}$. The determinant is then

$$\det \mathbf{M}_{2N-2} = \det \mathbf{B} \times \det\left(\mathbf{M}_{2N-4} - \begin{bmatrix} \mathbf{D} \\ \mathbf{0} \end{bmatrix} \mathbf{B}^{-1} \begin{bmatrix} \mathbf{C} & \mathbf{0} \end{bmatrix}\right).$$

In this case

$$\begin{bmatrix} \mathbf{D} \\ \mathbf{0} \end{bmatrix} \mathbf{B}^{-1} \begin{bmatrix} \mathbf{C} & \mathbf{0} \end{bmatrix} = \tfrac{1}{2}\left(1 + \frac{\hbar^2\lambda^2\epsilon^2}{4m^2}\right) \begin{bmatrix} \mathbf{I}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$

where $\mathbf{I}_2$ represents the 2×2 identity matrix. Then the second factor in the expression for the determinant of $\mathbf{M}_{2N-2}$ can be calculated in the same way as the determinant of $\mathbf{M}_{2N-2}$ itself. Then the matrices $\mathbf{C}$ and $\mathbf{D}$ have the same expressions, but $\mathbf{B}$ becomes

$$\mathbf{B} = \left(2 - \tfrac{1}{2}\left(1 + \frac{\hbar^2\lambda^2\epsilon^2}{4m^2}\right)\right)\mathbf{I}_2$$

and we find the matrix $\mathbf{M}_{2N-6}$ in the bottom right block of the matrix. This pattern repeats until the matrix in the bottom right block is $2\mathbf{I}_2$.

We obtain the following recurrence relation. Let D_n denote the determinant of $\mathbf{M}_{2N}$, and then D_{n-1} the determinant of $\mathbf{M}_{2N-2}$, and so forth. Define $d_n^2 = D_n$, then

$$\begin{aligned} d_n &= d_{n-1}\left(2 - \frac{d_{n-2}}{d_{n-1}}\left(1 + \frac{\hbar^2\lambda^2\epsilon^2}{4m^2}\right)\right) \\ &= 2d_{n-1} - d_{n-2}\left(1 + \frac{\hbar^2\lambda^2\epsilon^2}{4m^2}\right) \end{aligned} \tag{4.9}$$

From this recurrence relation we can determine an explicit expression for d_n of the form

$$d_n = cr^n.$$

We find that

$$cr^n - 2cr^{n-1} + cr^{n-2}\left(1 + \frac{\hbar^2\lambda^2\epsilon^2}{4m^2}\right) = 0.$$

We can divide both sides by cr^{n-1} to get

$$r^2 - 2r + \left(1 + \frac{\hbar^2\lambda^2\epsilon^2}{4m^2}\right) = 0.$$

This characteristic equation has the two solutions

$$r_1 = \tfrac{1}{2}\left(2 + \sqrt{4 - 4\left(1 + \frac{\hbar^2\lambda^2\epsilon^2}{4m^2}\right)}\right) = 1 + i\frac{\hbar\lambda\epsilon}{2m},$$

$$r_2 = \tfrac{1}{2}\left(2 - \sqrt{4 - 4\left(1 + \frac{\hbar^2\lambda^2\epsilon^2}{4m^2}\right)}\right) = 1 - i\frac{\hbar\lambda\epsilon}{2m},$$

such that the general solution for d_n becomes

$$d_n = c_1 r_1^n + c_2 r_2^n = c_1\left(1 + i\frac{\hbar\lambda\epsilon}{2m}\right)^n + c_2\left(1 - i\frac{\hbar\lambda\epsilon}{2m}\right)^n \tag{4.10}$$

We can easily check that $d_0 = 1$ and $d_1 = 2$ (and $d_2 = 3 - \frac{\hbar^2\lambda^2\epsilon^2}{4m^2}$), so

$$\begin{aligned}
d_0 &= c_1 + c_2 = 1 \Rightarrow c_2 = 1 - c_1\\
d_1 &= c_1\left(1 + i\frac{\hbar\lambda\epsilon}{2m}\right) + c_2\left(1 - i\frac{\hbar\lambda\epsilon}{2m}\right)\\
&= c_1\left(1 + i\frac{\hbar\lambda\epsilon}{2m}\right) + \left(1 - i\frac{\hbar\lambda\epsilon}{2m}\right) - c_1\left(1 - i\frac{\hbar\lambda\epsilon}{2m}\right)\\
&= c_1 i\frac{\hbar\lambda\epsilon}{m} + 1 - i\frac{\hbar\lambda\epsilon}{2m} = 2 \Rightarrow c_1 = -\frac{im}{\hbar\lambda\epsilon}\left(1 + i\frac{\hbar\lambda\epsilon}{2m}\right) = \tfrac{1}{2} - \frac{m}{\hbar\lambda\epsilon}i.\\
&\Rightarrow c_2 = 1 - \left(\tfrac{1}{2} - \frac{m}{\hbar\lambda\epsilon}i\right) = \tfrac{1}{2} + \frac{m}{\hbar\lambda\epsilon}i
\end{aligned}$$

Now we can use this to find the expression for the determinant of $\tilde{\mathbf{M}} = \frac{m}{2\hbar\epsilon}\mathbf{M}_{2N-2}$

$$\begin{aligned}
\det\tilde{\mathbf{M}} &= \left(\frac{m}{2\hbar\epsilon}\right)^{2N-2}\det\mathbf{M}_{2N-2} = \left(\frac{m}{2\hbar\epsilon}\right)^{2N-2} D_{N-1} = \left(\frac{m}{2\hbar\epsilon}\right)^{2N-2} d_{N-1}^2\\
&= \left(\frac{m}{2\hbar\epsilon}\right)^{2N-2}\left(\left(\tfrac{1}{2} - \frac{m}{\hbar\lambda\epsilon}i\right)\left(1 + i\frac{\hbar\lambda\epsilon}{2m}\right)^{N-1} + \left(\tfrac{1}{2} + \frac{m}{\hbar\lambda\epsilon}i\right)\left(1 - i\frac{\hbar\lambda\epsilon}{2m}\right)^{N-1}\right)^2.
\end{aligned} \tag{4.11}$$

Now recall the equation (4.8) and we can apply Theorem 3.1, since $\mathbf{M}$ is a real symmetric matrix with non-zero determinant. Filling in the expression that we found, we obtain

$$K_{\lambda,N} = A^{-2N}\left(\frac{i2\pi\hbar\epsilon}{m}\right)^{N-1}\left(\left(\tfrac{1}{2} - \frac{m}{\hbar\lambda\epsilon}i\right)\left(1 + i\frac{\hbar\lambda\epsilon}{2m}\right)^{N-1} + \left(\tfrac{1}{2} + \frac{m}{\hbar\lambda\epsilon}i\right)\left(1 - i\frac{\hbar\lambda\epsilon}{2m}\right)^{N-1}\right)^{-1}$$

(4.12)

Now that we have determined an expression for $K_{\lambda,N}$ as a function of N, we take the limit as N tends to infinity. We then substitute $\epsilon = T/N$:

$$K_{\lambda,N} = A^{-2N}\left(\frac{i2\pi\hbar T}{mN}\right)^{N-1}\left(\left(\tfrac{1}{2}-\frac{mN}{\hbar\lambda T}i\right)\left(1+i\frac{\hbar\lambda T}{2mN}\right)^{N-1}+\left(\tfrac{1}{2}+\frac{mN}{\hbar\lambda T}i\right)\left(1-i\frac{\hbar\lambda T}{2mN}\right)^{N-1}\right)^{-1}. \tag{4.13}$$

Using Euler's formula for complex numbers we can write this as

$$\begin{aligned} K_{\lambda,N} = &\left(\frac{i2\pi\hbar T}{mN}\right)^{N-1}\left(\sqrt{\tfrac{1}{4}+\frac{m^2N^2}{\hbar^2\lambda^2T^2}}\exp\left[-i\tan^{-1}\left(\frac{2mN}{\hbar\lambda T}\right)\right]\right.\\ &\times\left(\sqrt{1+\frac{\hbar^2\lambda^2T^2}{4m^2N^2}}\exp\left[i\tan^{-1}\left(\frac{\hbar\lambda T}{2mN}\right)\right]\right)^{N-1}\\ &+\sqrt{\tfrac{1}{4}+\frac{m^2N^2}{\hbar^2\lambda^2T^2}}\exp\left[i\tan^{-1}\left(\frac{2mN}{\hbar\lambda T}\right)\right]\\ &\left.\times\left(\sqrt{1+\frac{\hbar^2\lambda^2T^2}{4m^2N^2}}\exp\left[-i\tan^{-1}\left(\frac{\hbar\lambda T}{2mN}\right)\right]\right)^{N-1}\right)^{-1} \end{aligned} \tag{4.14}$$

We can use the proportionality constant $A^{-1} = \left(\frac{i2\pi\hbar T}{mN}\right)^{-\frac{1}{2}}$, see (2.10), before taking the limit. We also use that $\tan^{-1}(x) \approx \frac{\pi}{2}$ for very large x, $\tan^{-1}(x) \approx -\frac{\pi}{2}$ for very large negative values of x and $\tan^{-1}(x) \approx x$ for very small x.

$$\begin{aligned} K_\lambda = &\lim_{N\to\infty}\left(\frac{i2\pi\hbar T}{mN}\right)^{-N}\left(\frac{i2\pi\hbar T}{mN}\right)^{N-1}\left(\frac{mN}{\hbar\lambda T}\exp\left[-i\tfrac{\pi}{2}\right]\exp\left[i(N-1)\left(\frac{\hbar\lambda T}{2mN}\right)\right]\right.\\ &\left.+\frac{mN}{\hbar\lambda T}\exp\left[i\tfrac{\pi}{2}\right]\exp\left[-i(N-1)\left(\frac{\hbar\lambda T}{2mN}\right)\right]\right)^{-1}\\ = &\lim_{N\to\infty}\left(\frac{i2\pi\hbar T}{mN}\right)^{-1}\frac{\hbar\lambda Ti}{mN}\left(\exp\left[i\frac{\hbar\lambda T}{2m}\right]-\exp\left[-i\frac{\hbar\lambda T}{2m}\right]\right)^{-1}\\ = &\frac{\lambda}{4\pi i\sin\left(\frac{\hbar\lambda T}{2m}\right)} \end{aligned} \tag{4.15}$$

What remains in order to obtain the probability amplitude $K_{\mathcal{A}}$ is to apply a Fourier transform to K_λ with respect to lambda, since

$$K_{\mathcal{A}} = \int_{-\infty}^{\infty} e^{-i\lambda\mathcal{A}}K_\lambda d\lambda = \mathcal{F}_\lambda\left\{K_\lambda\right\}(\mathcal{A}). \tag{4.16}$$

Unfortunately, this integral does not converge for the propagator K_λ that we derived, because K_λ has a singularity whenever $\lambda = \frac{2\pi m}{\hbar T}n, \quad n \in \mathbb{Z}$. Therefore the Fourier transform does not exist. Perhaps the principles of quantum mechanics do not allow for such an area, see the discussion below in section 4.3, but we can take a step sideways to Brownian motion to see what result we would find in that semi-classical case.

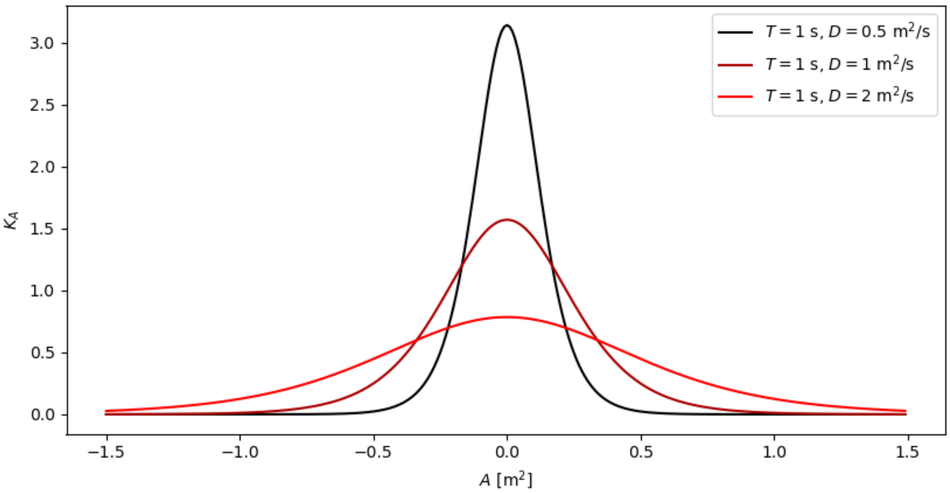

FIGURE 4: In this figure the probability distribution $K_{\mathcal{A},\text{Brownian}}$ is plotted as a function of the area $\mathcal{A}$ for different values of the diffusion coefficient D. The time T can be set to 1 s without loss of generality.

4.3 Brownian area

We can apply a Wick rotation to transition from quantum mechanics to Brownian motion. Consider the propagator K_λ that we found in the quantum mechanical case and let $T \to -i\tau$, then

$$K_\lambda = \frac{\lambda}{4\pi i \sin\left(\frac{\hbar\lambda T}{2m}\right)} \to \frac{\lambda}{4\pi \sinh\left(\frac{\hbar\lambda\tau}{2m}\right)} =: K_{\lambda,\text{Brownian}}.$$

The Fourier transform of this expression does exist and can be expressed as

$$K_{\mathcal{A},\text{Brownian}} = \mathcal{F}_\lambda \left\{K_{\lambda,\text{Brownian}}\right\}(\mathcal{A}) = \frac{m^2\pi}{2\tau^2\hbar^2} \cosh\left(\frac{m\pi}{\tau\hbar}\mathcal{A}\right)^{-2}. \tag{4.17}$$

To properly normalize this distribution, such that

$$\int_{-\infty}^{\infty} K_{\mathcal{A}} d\mathcal{A} = 1,$$

we multiply the expression by $\frac{\hbar\tau}{m}$. We can complete the transition to Brownian motion by making the correspondence

$$\frac{\hbar}{m} \to 2D,$$

where D represents the diffusion coefficient of the particle in the particular medium at a particular temperature. Then the probability distribution is given by

$$K_{\mathcal{A},\text{Brownian}} = \frac{\pi}{4\tau D} \cosh\left(\frac{\pi}{2\tau D}\mathcal{A}\right)^{-2}. \tag{4.18}$$

This distribution is graphically depicted in figure 4 for different values of the diffusion coefficient D, the time T is set to 1. We will discuss this result below.

4.4 Discussion of the results

In this chapter we have derived the probability distribution for the algebraic area enclosed by curves that start and end at the same point in space for Brownian motion. We have also discovered that our method does not yield a probability distribution in quantum mechanics. We will now discuss how to interpret the results and what their implications are.

The first interesting aspect is that the probability is nonzero for negative areas, while in geometry the area is always non-negative. This is a consequence of our definition of the area constraint. There is a subtle difference between the actual geometric area enclosed by a curve and the quantity that we have used, which is the algebraic area. The sign of the algebraic area depends on the direction in which the curve is traversed. Moreover, if the curve intersects itself, then the algebraic area is positive for one part of the enclosed area, while it is negative for the other. See figure 5 for examples of self-intersecting closed curves. This also results in more configurations that yield an area of zero, while geometrically only the situation in which every point of a curve has the same position as at least one other point of the curve yields an enclosed area equal to zero. This leads to the question: would we find a similar result if we only considered geometric area, rather than algebraic area? The calculation becomes more complicated, but this could be investigated in further research. The difference in area due to the orientations of the curves also explains the symmetry of the distribution. In the absence of a potential the contribution of a curve running in the clockwise direction is equal to that of the same curve running in the counterclockwise direction. This means that the probability of a certain positive area is the same as the probability of the negative area with the same absolute value.

In classical physics, the only closed curve that a free particle can have is a single point, when the particle is stationary. The area enclosed by this classical curve is simply zero, and it can be seen in the distribution that areas which are close to zero have the largest contribution. Furthermore, when the diffusion coefficient D, or analogously the fraction $\frac{\hbar}{m}$, is taken to zero we obtain the δ-distribution, which can also be seen in figure 4. This indicates that, as expected, in the classical limit, the only possible value of the area enclosed by the trajectory of a free particle is zero. The same holds for the quantum mechanical propagator K_λ, if we consider the limit $\frac{\hbar}{m} \to 0$. Then

$$\frac{\lambda}{4\pi i \sin\left(\frac{\hbar\lambda T}{2m}\right)} \approx \frac{\lambda}{4\pi i \frac{\hbar\lambda T}{2m}} = \frac{m}{2\pi i \hbar T}.$$

Here we applied the first order Taylor approximation of the sine-function. For small x

$$\sin(x) = x + O\left(x^3\right). \tag{4.19}$$

This expression does have a Fourier transform, which is

$$K_{\mathcal{A}} = \mathcal{F}_\lambda\left\{\frac{m}{2\pi i \hbar T}\right\}(\mathcal{A}) = \frac{m}{i\hbar T}\delta(\mathcal{A}). \tag{4.20}$$

To obtain a normalized probability amplitude, this should be multiplied by $\frac{\hbar T}{m}$, which is analogous to the normalization factor applied to the Brownian distribution described in section 4.2. Equation (4.19) can also be applied to the result for small λ. If we take the limit as $\lambda \to 0$, we regain the free particle propagator 3.15 for $x_0 = x_N$:

$$\lim_{\lambda \to 0} \frac{\lambda}{4\pi i \sin\left(\frac{\hbar\lambda T}{2m}\right)} = \frac{\lambda}{4\pi i \frac{\hbar\lambda T}{2m}} = \frac{m}{2\pi i \hbar T} \tag{4.21}$$

This corresponds to our expectations, since the constrained action (4.4) reduces to the free particle action (3.17), if $\lambda = 0$.

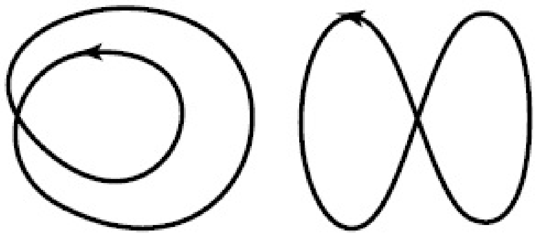

FIGURE 5: Examples of self intersecting curves, of which the algebraic area differs from the geometric area. The right curve actually has an enclosed algebraic area of zero.

With the Brownian description of area we can imagine a generalization for the area of geometric shapes, such as a triangle or a square. The results from this chapter can be applied, in the case of a triangle, to three curves connecting three points. The propagators of these three curves can then be simply multiplied with one another. However, first those probability amplitudes should be calculated for curves which are not closed. The computations are then slightly more complicated, since there is an additional exponential factor that one needs to take the path integral over, and subsequently the Fourier transform. Once one can describe the area of a two dimensional shape, I expect that the description of the surface area of a three dimensional shape can also be derived. One could consider the application of similar methods to describe the volume of three dimensional objects, as well. Another interesting application of this theory is the area of curved planes in three-dimensional space. This may be achieved using coordinate transformations from the flat areas, considered in this chapter, to a curved space. In summary, with this formulation of Brownian area there is plenty of further research that can be done to expand the concept of Brownian geometry. Perhaps, other methods, or other geometric concepts, can yield feasible quantum descriptions, but we have not been able to show this in this chapter regarding the area.

5 The path to a Brownian generalization of trigonometry

In the previous chapter we have found that classical area has a Brownian generalization. The question is whether areas suffice to provide a Brownian generalization of classical geometry. At first sight, this seems unlikely, given that lengths and angles provide the elementary quantities in classical geometry, while areas are a derived concept. As such, if the lengths of the sides of a triangle are known, a single unique triangle can be constructed. With a combination of the lengths and angles the same can be achieved. Then its area can also be calculated. In ordinary trigonometry there are theorems, such as the law of cosines, relating the properties of a triangle. Therefore, lengths and angles are the only quantities required to describe trigonometry. Remarkably, it turns out that this is also possible with areas as the fundamental quantity, rather than lengths, and introducing a new quantity, which we call spread. This is the basis of rational trigonometry.

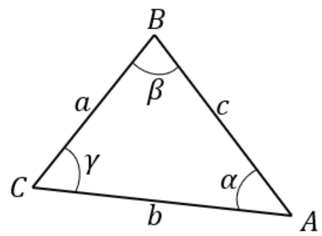

FIGURE 6: A triangle ABC in the plane. The sides are labeled with a, b and c. The angles are labeled with α, β and γ.

5.1 Classical rational trigonometry

Consider an arbitrary triangle ABC in the plane, see figure 6. In ordinary trigonometry this triangle has a unique combination of side lengths a, b, c and angles α, β, γ, defining the triangle. These quantities have theorems relating them to each other, as well as to other properties of the triangle. The concept of rational trigonometry replaces these quantities with the concepts of *quadrance* and *spread* as the fundamental quantities of trigonometry. [6]

A quadrance is defined as the squared length of the side of a triangle, but this can be interpreted as the area of the square adjacent to that side of the triangle, see figure 7. We define the quadrances Q_a, Q_b, Q_c respective to the sides a, b, c. Furthermore, we define the spreads s_a, s_b, s_c respective to the angles α, β, γ as the square of the sine of an angle

$$\begin{aligned} Q_a &:= a^2 \quad s_a := \sin(\alpha)^2 \\ Q_b &:= b^2 \quad s_b := \sin(\beta)^2 \\ Q_c &:= c^2 \quad s_c := \sin(\gamma)^2. \end{aligned} \tag{5.1}$$

With these new concepts trigonometry can be reformulated. The spreads can be expressed in terms of the quadrances. In ordinary trigonometry the sine of the angles can be expressed in terms of the side lengths using the law of cosines

$$\cos(\gamma) = \frac{a^2 + b^2 - c^2}{2ab}. \tag{5.2}$$

Squaring both sides we find

$$\cos(\gamma)^2 = \frac{a^4 + b^4 + c^4 + 2a^2b^2 - 2a^2c^2 - 2b^2c^2}{4a^2b^2}. \tag{5.3}$$

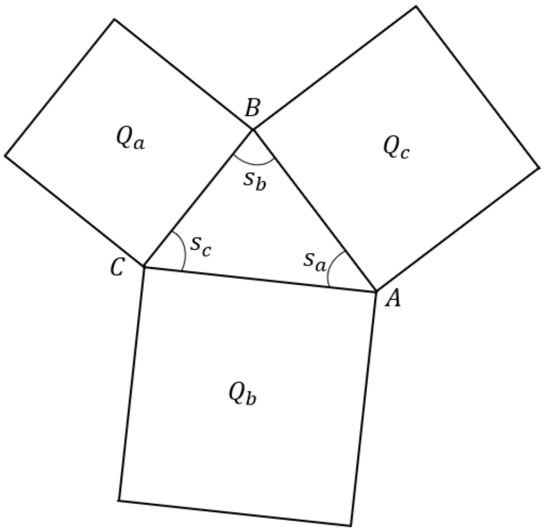

FIGURE 7: Depiction of the quadrances $(Q_a, Q_b, Q_c)$of the triangle in figure 6. A quadrance can be interpreted as the area of the square adjacent to that side of the triangle. The spreads s_a, s_b, s_c have replaced the concept of angles.

We employ the identity

$$1 - \cos(\gamma)^2 = \sin(\gamma)^2 = s_c. \tag{5.4}$$

Then we can express s_c after inserting the definitions of quadrance

$$\begin{aligned} s_c &= 1 - \frac{Q_a^2 + Q_b^2 + Q_c^2 + 2Q_aQ_b - 2Q_aQ_c - 2Q_bQ_c}{4Q_aQ_b} \\ &= 1 - \frac{(Q_a + Q_b - Q_c)^2}{4Q_aQ_b}. \end{aligned} \tag{5.5}$$

Analogously, we can express s_a and s_b in terms of the quadrances. Therefore, the quadrances are sufficient for the description of rational trigonometry. Moreover, the ordinary trigonometric theorems can be translated to rational geometry. The advantage of this alternative description is that, if the points defining the triangle are on an integer lattice, all quantities, including the quadrance and spread, will be rational numbers, avoiding the need for irrational numbers. In regular trigonometry, the appearance of square roots in the calculation of distances impedes this. For example, Pythagoras' Theorem becomes simply the addition of quadrances of a triangle, while in ordinary trigonometry one needs to take the square root to find the side length.

We now have established that area, in the form of quadrances, is a quantity which alone is sufficient to formulate trigonometry. Then it seems that the concept of Brownian area, which we have described in this work, could be used to formulate a Brownian trigonometry. With this a Brownian generalization of classical geometry appears to be in reach.

5.2 Brownian generalization

In section 4, we almost, but not quite, calculated the probability distributions one needs to determine the fluctuation of the area of a square around its classical value, due to the restricted time that was available for the research to be conducted on this thesis. However,

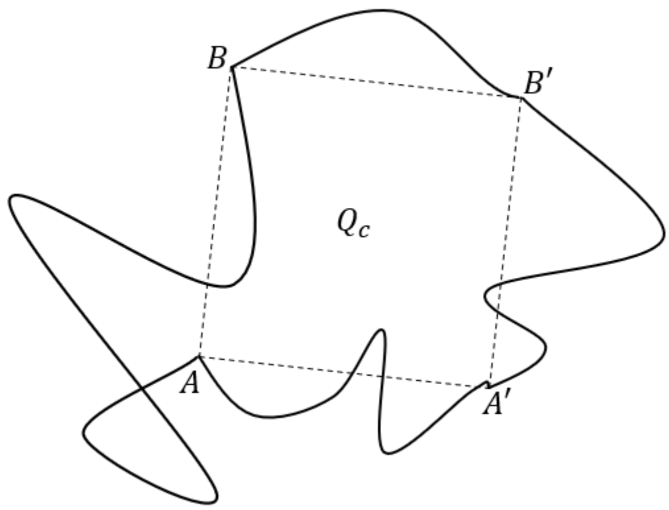

FIGURE 8: A possible combination of curves connecting the four vertices of the square corresponding to the quadrance Q_c. The classical edges of the square are represented by the dotted line.

it seems perfectly feasible to redo the calculation for this slightly more general case. And then we can formulate a generalized distribution for the quadrances defined in this chapter for rational trigonometry. With this we can formulate Brownian trigonometry in the same way as we did for classical trigonometry in section 5.1. For this generalization we can define the spread according to the relation in (5.5) with the operations of addition and multiplication of scalars replaced with those for probability distributions.

Consider a piece-wise smooth closed curve γ: $[0,1] \to \mathbb{R}$ consisting of the curves

$$\begin{aligned} &\gamma_{AB} : [0, t_1] \to \mathbb{R} \quad \text{with } \gamma_{AB}(0) = A,\ \gamma_{AB}(t_1) = B \\ &\gamma_{BB'} : [t_1, t_2] \to \mathbb{R} \quad \text{with } \gamma_{BB'}(t_1) = B,\ \gamma_{BB'}(t_2) = B' \\ &\gamma_{B'A'} : [t_2, t_3] \to \mathbb{R} \quad \text{with } \gamma_{B'A'}(t_2) = B',\ \gamma_{B'A'}(t_3) = A' \\ &\gamma_{A'A} : [t_3, 1] \to \mathbb{R} \quad \text{with } \gamma_{A'A}(t_3) = A',\ \gamma_{A'A}(1) = A. \end{aligned} \tag{5.6}$$

Here $0 < t_1 < t_2 < t_3 < 1$ and the points A, B, A', B' form a square with side lengths equal to the side c of the triangle ABC, see figure 8. Recall from section 3.2 that the enclosed area of γ can be expressed by the area functional (3.27). The distribution of the Brownian area of this square can then be found by path integrating

$$K_c(Q_c) = \int [d\gamma] \delta(S_{\text{area}} - Q_c) \exp\left[-S[\gamma]\right] \tag{5.7}$$

with

$$\begin{aligned} S[\gamma] &= \int_0^1 \frac{1}{4D} \dot{\gamma}(t)^2 dt \\ &= \frac{1}{4D} \left(\int_0^{t_1} \dot{\gamma}_{AB}(t)^2 dt + \int_{t_1}^{t_2} \dot{\gamma}_{BB'}(t)^2 dt + \int_{t_2}^{t_3} \dot{\gamma}_{B'A'}(t)^2 dt + \int_{t_3}^{1} \dot{\gamma}_{A'A}(t)^2 dt \right). \end{aligned} \tag{5.8}$$

The same can be done for the quadrances Q_a and Q_b. We expect that this calculation will yield a symmetric distribution around the classical area of the square $Q_c = (B - A)^2$, which is expected to have the largest contribution, similar to the result in section 4.3. The Brownian generalization of trigonometry will then be built up from the probability distributions $K_a(Q_a), K_b(Q_b)$ and $K_c(Q_c)$ of the respective quadrances.

Brownian probability distributions of the spreads can then be derived from the distributions for the quadrances using the relation in (5.5). The operations of addition and multiplication then correspond to the addition and multiplication of probability distributions. Addition of the distributions f_X and f_Y of the respective independent random variables X and Y is defined as

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx. \tag{5.9}$$

Multiplication of (independent) probability distributions is according to

$$f_{X\times Y}(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z/x) \frac{1}{|x|} dx. \tag{5.10}$$

Using these operations distributions for the spreads can be obtained and we have a foundation for a Brownian generalization of trigonometry.

6 Conclusions

We have demonstrated that the geometric concept of area can not be generalized using quantum mechanical path integrals. Path integration employing the algebraic area constraint (4.6) yields no result. However, with the Brownian path integral we found a distribution for the area enclosed by particle trajectories that start at a point and end up at the point again after a finite amount of time. This probability distribution is described by (4.18) and plotted in figure 4. The Brownian result, as well as the limit case of quantum mechanics converge to a δ-distribution centered at zero, equivalent to the classical result.

Rational trigonometry has been shown to be a theory for which a generalized Brownian geometry can be developed. By extending the result from section 4 to the area of squares, Brownian distributions for quadrances can be calculated. From these distributions one can then derive other geometric concepts, such as spread, or trigonometric theorems.

These results indicate that a quantum or Brownian description using path integrals of geometry is a realistic goal, although this work also reveals some difficulties that might arise when attempting to evaluate these path integrals. In the case of the area we found that this concept could not be defined in quantum mechanics. For quantum mechanics one needs to use other quantities than area in order to generalize geometric concepts. In Brownian motion, however, the result is well-defined, but is only valid for the algebraic area. It could be interesting to consider the geometric area instead or to generalize rational geometry as to include oriented areas, which can become negative.

Obviously, not all of classical geometry can be described using only rational trigonometry. The concept of circles for example has not been considered in this research. Furthermore, one could look at higher dimension, as we only looked at two-dimensional geometry. In conclusion, plenty of further research is still required, but this project has been an insightful first step.

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