BSc Thesis Applied Mathematics

Structure of and algorithms for binary staircase matrices

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Preface

I would like to thank my supervisor Dr. Matthias Walter for providing guidance and feedback.

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Abstract

Staircase structures play an important role in many optimization problems involving linear programs. It has become apparent that systems with this characteristic structure can be solved in linear time, as opposed to standard linear programs which are usually less efficient. Recognizing a staircase structure can therefore be of great importance. This paper focuses specifically on binary matrices containing this characteristic structure. We say that a binary matrix is in staircase form if every 2-by-2 submatrix with 1s in the off-diagonal entries is the all-1s submatrix and the 1s in every row and column are consecutive. A binary matrix is staircase if its rows and columns can be permuted such that the resulting matrix is in staircase form. This paper investigates the structure of these staircase matrices from a graph-theoretic viewpoint and describes an improved linear-time algorithm that computes the staircase form of a given matrix or returns that the given matrix is not staircase.

1 Introduction

Staircase structures play an important role in many optimization problems involving linear programs. There has been a lot of research on solving staircase matrices and systems with staircase structure (see [1, 3, 4]). From this research, it has become apparent that systems with this characteristic structure can be solved in linear time, as opposed to standard linear programs which are usually less efficient. Exploiting the underlying structure of a given system and recognizing a staircase pattern can therefore be of great importance in solving these optimization problems more efficiently. The fastest algorithm known to detect a staircase structure and to compute its staircase form runs in polynomial time [2]. The goal of this thesis is to further investigate the structure of staircase matrices and to find a more efficient algorithm that detects staircase matrices.

Section 2 introduces the general notations and definitions that are used throughout this paper. We then describe two linear-time algorithms in Section 3. The first one is a known algorithm recognizes whether a given matrix is in staircase form [2]. The second one is an improved algorithm that decides whether a given matrix is staircase and, in the affirmative case, computes its staircase form. In Section 4, the minimal non-staircase matrices are determined. Finally, in Section 5, the conclusions are given and suggestions for further research are made.

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2 Notations and definitions

In this paper, the problem of finding a staircase matrix will be tackled from a graph-theoretic viewpoint. Instead of the binary matrix M one can analyze the bipartite graph $G(M) = (V_1 \cup V_2, E)$ that has a node for each row and for each column, where a row node r is connected to a column node c by an edge if and only if $M_{r,c} = 1$. For clarity, let us give an example.

Example 1. Let M be the binary matrix defined as

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

Then the graph $G(M) = (V_1 \cup V_2, E)$ in Figure 1 is its corresponding bipartite graph.

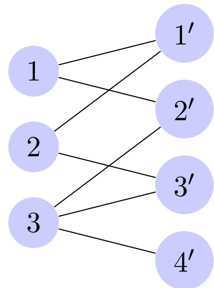


Figure 1: Bipartite graph $G(M)$ with vertex set $V_1 \cup V_2$ and edge set E such that each edge connects a vertex in V_1 to a vertex in V_2. We have that $V_1 = \{1, 2, 3\}$ is the set of nodes corresponding to the rows of M and $V_2 = \{1', 2', 3', 4'\}$ is the set of nodes corresponding to the columns of M. The edge set E resembles all the nonzero entries in the matrix M.

We denote the neighbourhood of a vertex $u \in V$ of a graph $G = (V, E)$ by $N_G(u) = \{v \in V \mid \{u, v\} \in E\}$. If it is clear which graph is referred to, we may simply write $N(u)$. A graph G is connected if and only if any vertex can be reached from any other vertex by a path in the graph. We say that a matrix M is connected if and only if $G(M)$ is connected. Furthermore, we say that the vertices $u, v \in V$ of a graph $G = (V, E)$ are identical if $N_G(u) = N_G(v)$ for $u \neq v$. Note that, if a binary matrix M contains no identical rows or columns, then $G(M)$ does not have any identical vertices.

The main focus of our research is the staircase structure of a matrix and its bipartite graph. It is defined as follows.

Definition 2.1 (Staircase form)**.** We say that a binary matrix M is in *staircase form* (SCF) if the following two conditions hold:
Condition 1: 1s in every row and column are consecutive:

$$\begin{pmatrix} 1 & \star & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \qquad \wedge \qquad \begin{pmatrix} 1 \\ \star \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \tag{SCF1}$$

Condition 2: Every 2-by-2 submatrix with 1s in the off-diagonal entries is the all-1s submatrix:

$$\begin{pmatrix} \star & 1 \\ 1 & \star \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{SCF2}$$

Similarly, we say that a bipartite graph is in staircase form if the two conditions illustrated in Figure 2 hold.

(A) SCF1 (B) SCF2

FIGURE 2: Illustration of the staircase-form conditions in a bipartite graph. If the solid edges are contained in the graph, then the dashed ones must be contained as well. For SCF1, the symmetric version also applies.

Example 2. The (7×6)-matrix M below is in staircase form.

$$M = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Definition 2.2 (SC)**.** A binary matrix M is *staircase* (SC) if its rows and columns can be permuted such that the resulting matrix is in staircase form. Similarly, we say that a bipartite graph G is staircase if the ordering of the nodes can be permuted such that the resulting graph is in staircase form.

We say that a matrix M is minimally non-staircase if M is not staircase, is connected and has no identical rows or columns, but removing one row or column would make it staircase. Similarly, a bipartite graph $G = (V_1 \cup V_2, E)$ is minimally non-staircase if G is connected, but not staircase and has no identical nodes, but removing one node would make it staircase.

The following definition introduces a matrix permutation that will be important when analyzing the structure of a staircase-form matrix.

Definition 2.3 (Reverse matrix)**.** Let M be an $m \times n$ matrix with ordered rows $(u_1, u_2, \dots u_m)$ and ordered columns $(v_1, v_2, \dots, v_n)$. The *reverse* of the matrix M is defined as the matrix M^* which has ordered rows $(u_m, u_{m-1}, \dots, u_1)$ and ordered columns $(v_n, v_{n-1}, \dots v_1)$. It can be computed as follows:

$$M^* = J_m M J_n,$$

where J_k is the $k \times k$ permutation matrix with 1s along the counter-diagonal and zeros everywhere else.

Remark. The reverse of a staircase-form matrix is also in staircase form.

3 Staircase-form matrices and graphs

In this section, we present two linear-time algorithms that decide whether a matrix is staircase and if it is in staircase form.

3.1 Recognizing staircase-form graphs

We first prove that it is possible in linear time to verify whether a given matrix and its corresponding bipartite graph are in staircase form. The recognition of staircase-form matrices was known before [2], but we state it here again because the result is necessary for the computation algorithm in Section 3.2. Let us first introduce the following lemma.

Lemma 3.1 (Recognition Algorithm; see [2])**.** Let M be an $m \times n$ matrix with ordered rows $(u_1, u_2, \dots u_m)$ and ordered columns $(v_1, v_2, \dots, v_n)$. We define $\lambda(x)$ as the number of zeros in row x before the first 1 and $\phi(x)$ is the number of zeros in row x after the last 1. In graph notation, this would be $\lambda(x) = |\{v_j \in V_2 \mid j < i \ \forall v_i \in N(x)\}|$ and $\phi(x) = |\{v_j \in V_2 \mid j > i \ \forall v_i \in N(x)\}|$. Then the following algorithm decides whether $G(M)$ (and hence M) is in staircase form.

Algorithm 1 SCF recognition on a bipartite graph

Input: A connected bipartite graph $G(M) = (V_1 \cup V_2, E)$ where $V_1 = (u_1, u_2, \dots, u_m)$ and $V_2 = (v_1, v_2, \dots, v_n)$.

Output: Returns whether $G(M)$ is in staircase form

if $n - \phi(u_1) - \lambda(u_1) > |N(u_1)|$ **then**
  return "$G(M)$ violates (SCF1)"
end if
for $i \in \{2, \dots, m\}$ **do**
  if $\lambda(u_i) < \lambda(u_{i-1})$ **or** $\phi(u_i) > \phi(u_{i-1})$ **then**
    return "$G(M)$ violates (SCF2)"
  end if
  if $n - \phi(u_i) - \lambda(u_i) > |N(u_i)|$ **then**
    return "$G(M)$ violates (SCF1)"
  end if
end for
return "$G(M)$ is SCF"

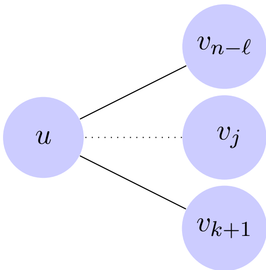
Remark. For an $m \times n$ SCF-matrix, we have that $|N(u)| = |n| - \lambda(u) - \phi(u)$ for all rows u.

Proof. In order to prove that the algorithm is correct, let us distinguish the following cases for $G(M)$:

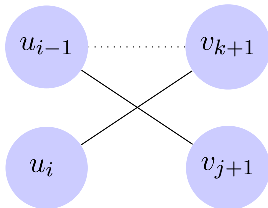
Case 1: $G(M)$ is in staircase form.
Let $u \in V_1$ and suppose first that there exists a vertex $u \in V_1$ such that $n - \lambda(u) - \phi(u) >$

$|N(u)|$. Let $k = \lambda(u)$, $\ell = \phi(u)$. Then by definition of $\lambda(u)$ and $\phi(u)$, the first vertex in the neighbourhood of u is v_{k+1} and the last vertex in the neighbourhood of u is $v_{n-\ell}$. If $n-k-\ell > |N(u)|$, then there exists a $j \in \{k+2, \ldots, n-\ell-1\}$ such that $\{u, v_j\} \notin E$. As shown in the subgraph in Figure 3a, this violates (SCF1). Hence, $n - \lambda(u) - \phi(u) \leq |N(u)|$ must hold for all $u \in V_1$ and so the checks in line 1 and 8 do not fail.

Now, suppose that there exists some $i \in \{2, ..., m\}$ such that $k = \lambda(u_i) < j = \lambda(u_{i-1})$. Then $v_{k+1} < v_{j+1}$. Moreover, we have that $\{u_i, v_{k+1}\}$, $\{u_{i-1}, v_{j+1}\} \in E$ and $\{u_{i-1}, v_k\} \notin E$ by definition of $\lambda(u_{i-1})$. However, as can be seen in the subgraph in Figure 3b, this is a violation of (SCF2). Therefore, $\lambda(u_i) \geq \lambda(u_{i-1})$ must hold for all $i \in \{2, ..., m\}$. Similarly, we can conclude that $\phi(u_i) \geq \phi(u_{i-1})$ holds for all $i \in \{2, ..., m\}$. Hence, Algorithm 1 works correctly if $G(M)$ is in staircase form.



(A) Violation of (SCF1)



(B) Violation of (SCF2)

FIGURE 3: SCF-violations in case 1

Case 2: $G(M)$ violates (SCF2).
In this case, we can find $i, k \in \{1, 2, ..., m\}$ and $j, \ell \in \{1, 2, ..., n\}$ with $i < k$ and $j < \ell$ such that vertices $u_i, u_k \in V_1$ and $v_j, v_\ell \in V_2$ and edges $\{u_i, v_\ell\}$, $\{u_k, v_j\} \in E$ such that either $\{u_i, v_j\} \notin E$ and/or $\{u_k, v_\ell\} \notin E$ holds. Without loss of generality, let us assume that $\{u_k, v_\ell\} \notin E$. If there exists a vertex $v_s \in N(u_k)$ such that $s > \ell$, then G violates (SCF1) as can be seen in Figure 4. Therefore, let us assume that $v_s \notin N(u_k)$ for all $s \in \{\ell, \ell+1, \ldots, n\}$. Hence, we have that $n - \ell < \phi(u_k) \leq n - j$. However, this immediately yields that $\phi(u_i) \leq n - \ell < \phi(u_k) \leq n - j$, i.e., $\phi(u_i) < \phi(u_k)$, and hence the check in line 5 fails. Similarly, we find that the check in line 5 fails if $\{u_i, v_j\} \notin E$ holds.

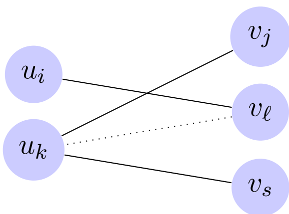


FIGURE 4: Violation of (SCF1) in case 2

Case 3: $G(M)$ violates (SCF1).
Suppose first that we can find vertices $u \in V_1$ and $v_i, v_j, v_s \in V_2$ such that (SCF1) is violated. Let $\lambda(u) = k$ and $\phi(u) = \ell$. Then, by definition of the neighbourhood, we must have that $|N(u)| \leq n - \ell - k - 1$, so $|N(u)| < n - \ell - k$, and hence the check in either line 1 (if $u = u_1$) or in line 8 will fail.

Suppose now that we can find vertices $u_i, u_j, u_s \in V_1$ and $v \in V_2$ such that (SCF1) is violated. Since $G(M)$ is connected, we have that $N(u_j) \neq \emptyset$. Hence, there exists some

$w \in V_2$ such that $\{u_j, w\} \in E$. If $w < v$, then the vertices u_i, u_j, w, v also violate (SCF2), as can be seen in Figure 5a. Otherwise, the vertices u_j, u_s, v, w violate (SCF2), as can be seen in Figure 5b. In both cases, we can apply the same arguments used in case 2.

(A) $w < v$ (B) $v < w$

FIGURE 5: Violation of (SCF2) in case 3

In all possible cases, Algorithm 1 correctly recognizes whether a graph is in staircase-form. This concludes the proof. $\square$

Theorem 3.2. It can be checked in time $\mathcal{O}(|V_1| + |V_2| + |E|)$ whether a matrix M and its bipartite graph $G(M) = ((V_1 \cup V_2), E)$ is in staircase form.

Proof. It is easy to see that the algorithm in Lemma 3.1 has a running time of $\mathcal{O}(|V_1| + |V_2| + |E|)$, since the computation of $\lambda(x)$, $\phi(x)$ and $N(x)$ can be done in this amount of time. This concludes our proof. $\square$

3.2 Computing staircase-form matrices

We next introduce an algorithm that computes the staircase form of a given bipartite graph if possible, and otherwise states that the graph is not staircase. This is equivalent to finding row and column permutations to turn a matrix into staircase form. This algorithm is an improvement of the computation algorithm in [2].

Before we state the algorithm, let us introduce important notation. We denote by d_i^r the length of the shortest path from node r to node i. We denote by B_i^r the set of nodes in the neighborhood of i that have distance $d_i^r - 1$, in notation: $B_i^r = \{u \in N(i) \mid d_u^r = d_i^r - 1\}$. Similarly, let us define A_i^r to be the set of nodes in the neighborhood of i that have distance $d_i^r + 1$, i.e. $A_i^r = \{u \in N(i) \mid d_u^r = d_i^r + 1\}$. Algorithm 2 then solves our problem.

Algorithm 2 Finding a staircase ordering of a bipartite graph

Input: A connected bipartite graph $G = (V_1 \cup V_2, E)$

Output: Lists L_1, L_2 of the vertices in V_1, V_2 representing a staircase ordering on G or the result that no staircase ordering exists

1: Let $r \in V_1$
2: Apply BFS to G starting in r and let $\Phi_v^r = (d_v^r, -|B_v^r|, |A_v^r|)\ \forall v \in V_1 \cup V_2$.
3: Let $s \in V_1 \cup V_2$ such that Φ_s^r is lexicographically largest $\forall v$: $\Phi_s^r \geq^{lex} \Phi_v^r\ \forall v$.
4: Apply BFS to G starting in s and let $\Phi_v^s = (d_v^s, -|B_v^s|, |A_v^s|)\ \forall v \in V_1 \cup V_2$.
5: Sort vertices $v \in V_1 \cup V_2$ in lexicographically increasing order by their Φ_v^s.
6: $L_1 = [s],\ L_2 = [\ ]$
7: **for** $v \in V_1 \cup V_2$ ordered as above **do**
8:     **if** d_v^s even **then**
9:         Append v to the list L_1
10:     **else**
11:         Append v to the list L_2
12:     **end if**
13: **end for**
14: **if** isSCF$(L_1 \cup L_2, E)$ **then**
15:     **if** $s \in V_1$ **then**
16:         **return** L_1, L_2
17:     **else**
18:         **return** L_2, L_1
19:     **end if**
20: **else**
21:     **return** "G is not SC"
22: **end if**

In order to find a candidate for the uppermost vertex in the SCF-graph, the algorithm first applies the Breadth-First Search method to an arbitrary vertex $r \in V_1$. Then, in line 3, the vertex with the lexicographically largest sequence Φ_v^r is chosen as the root. Next, line 5 constructs an ordering of the nodes. Finally, it is tested by Algorithm 1 whether the bipartite graph with the newly constructed ordering is in staircase form.

In order to prove the correctness of the algorithm, we will introduce the following lemmas.

Lemma 3.3. Let M be an $m \times n$ SCF-matrix with corresponding bipartite graph $G(M) = ((V_1 \cup V_2), E)$ and let $r \in V_1$. Then for rows $r < j < i$ exactly one of the following holds:

1. $d_j^r < d_i^r$
2. $d_j^r = d_i^r$ and $B_j^r \supset B_i^r$
3. $d_j^r = d_i^r$ and $B_j^r = B_i^r$ and $A_j^r \subseteq A_i^r$

Proof. We prove this lemma by contradiction. Let $r, j, i \in V_1$ with $r < j < i$.

Firstly, suppose that $d_j^r > d_i^r$. The proof of this case is based on the proof of Lemma 3.2 in [2]. Without loss of generality, let us assume that among all matrices M and rows $r < j < i$ for which $d_j^r > d_i^r$, we choose one with $(d_j^r + d_i^r)$ minimum. Let $P_{rj} = (r, \ldots, j)$

and $P_{ri} = (r, \ldots, i', x, i)$ be the shortest paths from r to j and from r to i, respectively. Note that, since $r, i \in V_1$ and the graph is bipartite, d_i^r is of even length greater than or equal to 2. We distinguish two cases:

Case $i' < j$:
If we have $j \in N(x)$, then P_{rj} is no longer the shortest path since we can construct a shorter one by simply choosing j instead of i as the last vertex in $P_{ri} = (r, \ldots, i', x, i)$. Hence, $j \notin N(x)$ must hold. However, Figure 6 illustrates that this violates (SCF1).

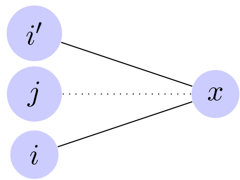


FIGURE 6: Contradiction for $i' < j$

Case $j < i'$:
In this case, we have $r < j < i' < i$ and $d_j^r > d_{i'}^r = d_i^r - 2$. Therefore, we have that $d_{i'}^r + d_j^r < d_i^r + d_j^r$. This contradicts the minimality of $d_i^r + d_j^r$.

Both cases lead to a contradiction, hence either statement 1 holds or both values must be equal.

Next, suppose that $d_j^r = d_i^r$ and $B_j^r \not\supset B_i^r$. We define N_k^r to be the set of nodes with shortest path (starting in r) of length k, i.e., $N_k^r = \{v \in V_1 \cup V_2 \mid d_v^r = k\}$. Let $d_j^r = d_i^r = k$, so that $j, i \in N_k^r$. Hence, we have $|N_k^r| \geq 2$. This can only happen if $k \geq 1$. Then $|N_{k-1}^r| \geq 1$. Since $B_j^r \not\supset B_i^r$, there exists a node $x \in B_i^r \backslash B_j^r$, i.e. $\{x, i\} \in E$ but $\{x, j\} \notin E$. We have $d_x^r = d_i^r - 1 = k - 1$, and so $x \in N_{k-1}^r$. Since $d_j^r = d_i^r$, there must be at least one other node $v \in N_{k-1}^r$ such that $\{v, j\} \in E$. Hence $|N_{k-1}^r| \geq 2$. This can only happen if $k - 1 \geq 1$, so $k \geq 2$. Thus, we have that $|N_{k-2}^r| \geq 1$. Now, let $w \in N_{k-2}^r$ such that $\{w, x\} \in E$. As illustrated in Figure 7a, the subgraph $N_{k-2}^r \cup N_{k-1}^r \cup N_k^r$ violates (SCF1). Hence, either the second case in the lemma must be true or $d_j^r = d_i^r$ and $B_j^r = B_i^r$.

Finally, suppose that $d_j^r = d_i^r$ and $B_j^r = B_i^r$ and $A_j^r \not\subseteq A_i^r$. Again, let $d_j^r = d_i^r = k$, so that $j, i \in N_k^r$ and $|N_k^r| \geq 2$. This can only happen if $k \geq 1$. Then $|N_{k-1}^r| \geq 1$. Let $w \in N_{k-1}^r$ such that $\{w, j\} \in E$ and $w \in B_j^r$. Since we have that $B_j^r = B_i^r$, we also have $w \in B_i^r$, so $\{w, i\} \in E$. Since $A_j^r \not\subseteq A_i^r$, there exists a node $x \in A_j^r$, but $x \notin A_i^r$ i.e., $\{j, x\} \in E$ but $\{i, x\} \notin E$. Then $d_x^r = d_j^r + 1 = k + 1$, so we have that $x \in N_{k+1}^r$ and $|N_{k+1}^r| \geq 1$. The subgraph $N_{k-1}^r \cup N_k^r \cup N_{k+1}^r$ in Figure 7b violates (SCF2). Hence we can conclude that $x \in A_i^r$ for all $x \in A_j^r$, thus $A_j^r \subseteq A_i^r$. This is a contradiction.

(A) Contradiction for $B_j^r \not\supset B_i^r$ (B) Contradiction for $B_j^r = B_i^r$, $A_j^r \not\subset A_i^r$

FIGURE 7: Subcases for $d_j^r = d_i^r$

We obtain a contradiction in all cases which concludes the proof. ☐

Remark. $N(i) = B_i^r \cup A_i^r$.

Remark. If $G(M)$ is in staircase form and we have that $d_i^r = d_j^r$, $B_i^r = B_j^r$ and $A_i^r = A_j^r$, then rows i and j are identical.

Lemma 3.4. Let M be an $m \times n$ SCF-matrix with corresponding bipartite graph $G(M) = ((V_1 \cup V_2), E)$ and let $r \in V_1$. The sequence $(d_i^r, -|B_i^r|, |A_i^r|)$ is lexicographically decreasing for rows $i = 1, \dots, r$ and increasing for rows $i = r, \dots, m$.

Proof. We prove this lemma by only considering the case $i \in \{r, \dots, m\}$. The proof for $i \in \{1, \dots, r\}$ will be similar, considering the fact that the reverse of an SCF-matrix is also in staircase form. From Lemma 3.3 we know that, for rows $r < j < i$, either $d_j^r < d_i^r$ or ($d_j^r = d_i^r$ and $B_j^r \supset B_i^r$) or ($d_j^r = d_i^r$ and $B_j^r = B_i^r$ and $A_j^r \subset A_i^r$). Hence, it can be easily seen that either $d_i^r > d_j^r$ or ($d_i^r = d_j^r$ and $|B_i^r| < |B_j^r|$) or ($d_i^r = d_j^r$ and $|B_i^r| = |B_j^r|$ and $|A_i^r| > |A_j^r|$). Hence, we conclude that the sequence $(d_i^r, -|B_i^r|, |A_i^r|)$ is lexicographically increasing for rows $i \in \{r, \dots, m\}$. ☐

Theorem 3.5. Algorithm 2 is correct.

Proof. It is clear that the algorithm will terminate in all cases. If G is not staircase, then in line 14, Algorithm 1 will show that the constructed ordering of the nodes fails. Otherwise, we can use Lemmas 3.3 and 3.4 to prove that constructed ordering of the nodes is correct. From Lemma 3.4, we can see that for any node r in an SCF-graph, the vertex v with lexicographically largest Φ_v^r must be the first or last row/column. Hence, the choice of the uppermost vertex in line 3 is correct. It can then be easily seen by Lemmas 3.3 and 3.4 that the constructed orderings of the nodes is correct. ☐

Theorem 3.6. Algorithm 2 runs in $\mathcal{O}(|V_1| + |V_2| + |E|)$ time.

Proof. The algorithm starts with Breadth-first search where it sorts the vertices in order of increasing distance. This has time-complexity $\mathcal{O}(|V_1| + |V_2| + |E|)$. Next, the algorithm sorts each set of nodes with equal distance by increasing $-|B_v^s|$. This can be done by counting sort. The time complexity of the counting sort algorithm is $\mathcal{O}(n + k)$, where n is the number of elements and k is the range of the elements. In our case, each set of nodes with equal distance $d_v^s = i$ has $|N_i|$ elements and the range of these elements is $\max_{v \in N_i} |B_v^s|$. Hence, the total time complexity for this step is bounded by the estimate

$$\sum_{i=1}^{|E|}(|N_i| + \max_{v \in N_i} |B_v^s|) \leq \sum_{i=1}^{|E|} |N_i| + \sum_{i=1}^{|E|} \max_{v \in N_i} |B_v^s| \leq |V_1| + |V_2| + |E|.$$

After that, the algorithm sorts each set of nodes with equal distance and equal $|B_v^s|$ by increasing $|A_v^s|$. This can be done in the same way via counting sort and the time complexity for this step is bounded by the estimate

$$\sum_{i=1}^{|E|}(|N_i| + \max_{v \in N_i} |A_v^s|) \leq \sum_{i=1}^{|E|} |N_i| + \sum_{i=1}^{|E|} \max_{v \in N_i} |A_v^s| \leq |V_1| + |V_2| + |E|.$$

Finally, the new ordering is checked. As stated in Theorem 3.2, this can be done in $\mathcal{O}(|V_1| + |V_2| + |E|)$ time. Combining all steps yields a total time complexity of $\mathcal{O}(4(|V_1| + |V_2| + E|))$. Since we only look at the most crucial term, we can write this as $\mathcal{O}(|V_1| + |V_2| + |E|)$. This completes the proof. ☐

4 Minimal non-staircase matrices

In this section, we consider the uniqueness of staircase-form matrices and use this result to determine the minimal non-staircase matrices.

4.1 Uniqueness of staircase-form matrices

In order to analyse the structure of non-staircase matrices, we first state an important theorem about the staircase form of a staircase matrix. This result and its proof are based on the uniqueness theorem in [2].

Theorem 4.1 (Uniqueness; see [2])**.** Let M be a binary $m \times n$ matrix that is connected and has no identical rows or columns. Then the staircase form of M is unique up to reversal.

Proof. We prove this theorem by contradiction. Suppose that there exist two different staircase forms M' and M'' of the matrix M such that M' and M'' are not each others reverses, i.e., $(M')^* \neq M''$ and $(M'')^* \neq M'$. Let us look at the bipartite graphs $G(M')$ and $G(M'')$. Since M is connected, $G(M)$ must be connected as well. Then we have that $G(M')$ and $G(M'')$ are also connected because turning a graph into staircase form is just a reordering of the nodes and no edges are removed nor added. Since we do not have any identical rows or columns, we have that $N(u) \neq N(v)$ for all vertices u, v. Because of these two properties, we can find vertices $u, v \in V_1$ and $x, y \in V_2$ such that $\{u, x\}$, $\{v, y\}$, $\{v, x\} \in E$ and $\{u, y\} \notin E$, see Figure 8.

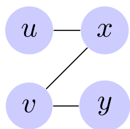
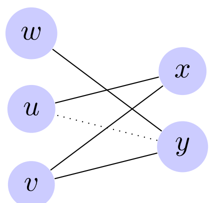


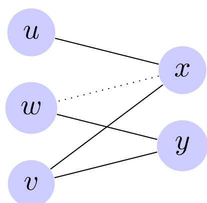
FIGURE 8: Subgraph $H = ((u, v) \cup (x, y), E) \subseteq G(M)$.

The subgraph H in Figure 8 shows the ordering $u < v$ and $x < y$. If we would reverse one of these orderings, then the subgraph would violate (SCF2). Hence, the ordering $u < v$ implies the ordering $x < y$. We can now assume w.l.o.g. that there exist vertices $u, v \in V_1$ such that $u < v$ for both $G(M')$ and $G(M'')$.

Since $G(M') \neq G(M'')$, let us assume w.l.o.g. that there exists a node $w \in V_1$ with $\{w, y\} \in E$ such that $u < v < w$ holds in $G(M')$ but not in $G(M'')$. Then we must have that either $w < u < v$ or $u < w < v$ holds in $G(M'')$. In Figure 9, we see the subgraphs of these orderings.



(A) $w < u < v$.



(B) $u < w < v$.

FIGURE 9: Possible subgraphs $H(M'') = ((u, v, w) \cup (x, y), E) \subseteq G(M'')$.

As one can see in Figure 9a, in order to not violate (SCF1), we must have that $\{u, y\} \in E$. However, this is a contradiction to our hypothesis. Hence, the ordering $w < u < v$

in $G(M'')$ is not possible. Now, if we look at Figure 9b, we see that, in order to not violate both (SCF1) and (SCF2), we must have that $\{w, x\} \in E$. But then we have that $N_H(v) = N_H(w)$. Since we know that $N_G(v) \neq N_G(w)$, there exists some $z \in V_2$ such that $z \in N_G(v)$, $z \notin N_G(w)$ or $z \in N_G(w)$, $z \notin N_G(v)$. Without loss of generality, let us assume that $z \in N_G(w)$, $z \notin N_G(v)$. Then we must have that either $z < x < y$ or $x < y < z$ holds. We can assume, without loss of generality that $x < y < z$ holds. Figure 10 illustrates the subgraphs $L(M') \subseteq G(M')$ and $L(M'') \subseteq G(M'')$ with vertex set $\{u, v, w, x, y, z\}$.

(A) $L(M')$ with $u < v < w$.

(B) $L(M'')$ with $u < w < v$.

FIGURE 10: Possible subgraphs of $G(M')$ and $G(M'')$ with vertex set $\{u, v, w, x, y, z\}$.

In Figure 10a, we see that $L(M')$ does not violate the SCF conditions. However, in Figure 10b, we must have that $\{v, z\} \in E$ or else there would be an (SCF2)-violation. But this is a contradiction to our hypothesis. From this we can conclude that, for all $w \in V_1$ with $u < v < w$ in $G(M')$, we must have that $u < v < w$ holds for $G(M'')$. The same holds for any $w \in V_1$ with $w < u < v$ or $u < w < v$ in $G(M')$. This is a contradiction.

$\square$

4.2 Forbidden sub-matrices

We will now determine the minimal non-staircase matrices and characterize staircase matrices in terms of these forbidden sub-matrices. Let us first introduce the following definitions and lemmas.

Definition 4.1. A W_k-matrix is a binary $k \times k$ matrix that has two 1's in each row and column and zeroes everywhere else such that $G(W_k)$ is connected. After permutation of rows and columns, a W_k-matrix will look like this:

$$W_k = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 1 \\ 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & 0 & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & 1 & 0 \\ 0 & \dots & 0 & 0 & 1 & 1 \end{pmatrix}$$

Lemma 4.2. For $k \geq 3$, a W_k-matrix is not staircase.

Proof. For a W_k-matrix ($k \geq 3$), we have that $G(W_k)$ is a cycle graph with $2k$ nodes. Let $(v_1, v_2, ..., v_{2k})$ be the ordered nodes of $G(W_k)$. Let us run Algorithm 2. Without loss of generality, assume that v_k is the starting node in the algorithm. Then the BFS Algorithm will result in the graph illustrated in Figure 11.

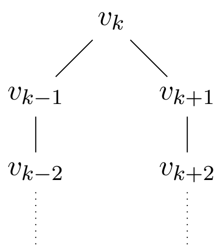


FIGURE 11: Start of the BFS Algorithm for $G(W_k)$. Here, the vertical positioning of the nodes is based on the distance from v_k. We have that $N_0^{v_k} = \{v_k\}$, $N_1^{v_k} = \{v_{k-1}, v_{k+1}\}$ and $N_2^{v_k} = \{v_{k-2}, v_{k+2}\}$.

We can see that $d_{v_{k-i}}^{v_k} = i$ and $d_{v_{k+i}}^{v_k} = i$ for all $i \in \{0, \dots, k-1\}$. Also Figure 11 illustrates that, for each $v \in N_i^{v_k}$, we have that $|B_v^{v_k}|$ and $|A_v^{v_k}|$ are equal, and hence, the re-ordering does not matter. Looking at the subgraph $N_0^{v_k} \cup N_1^{v_k} \cup N_2^{v_k}$ illustrated in Figure 12, we can see can see that $G(W_k)$ violates both (SCF1) and (SCF2). This completes the proof.

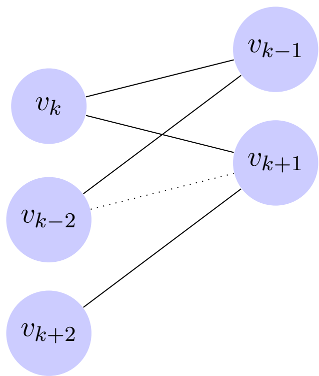


FIGURE 12: Subgraph of a $G(W_k)$ graph

$\square$

Definition 4.2. We define Q to be the 3×4-matrix:

$$Q = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Lemma 4.3. *Q is not staircase.*

Proof. Let $V_1 = (u_1, u_2, u_3, u_4)$ and $V_2 = (v_1, v_2, v_3)$ be the sets of ordered rows and columns Q, respectively. It is easy to see that Q is not in staircase form, since $\phi(u_4) > \phi(u_3)$ and hence by Algorithm 1, Q violates (SCF2). In order to check whether Q is staircase, let us run Algorithm 2 with input $G(Q) = (V_1 \cup V_2, E)$. Without loss of generality, we take $r = u_1$. The BFS algorithm will then result in the graph in Figure 13.

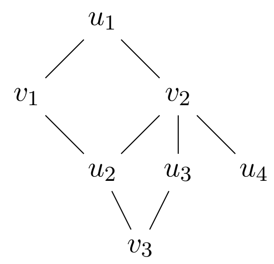


FIGURE 13: BFS Algorithm for $G(Q)$ starting in u_1. The vertical positioning of the nodes is based on the distance from u_1.

From Figure 13, it is clear to see that $d_{v_3}^{u_1} > d_v^{u_1}$ for all $v \in V_1 \cup V_2$. Hence, $\Phi_{v_3}^{u_1}$ is lexicographically largest and so we run the BFS algorithm again with starting node v_3. This results in the graph in Figure 14.

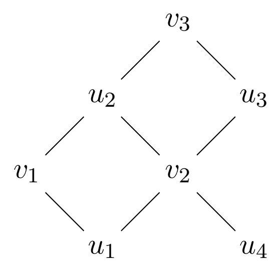


FIGURE 14: BFS Algorithm for $G(Q)$ starting in v_3. The vertical positioning of the nodes is based on the distance from v_3.

After sorting the nodes $v \in V_1 \cup V_2$ by lexicographically increasing $\Phi_v^{v_3}$, the new ordering will be: $(v_3, u_3, u_2, v_2, v_1, u_1, u_4)$. So, we have that $L_1 = [v_3\ v_2\ v_1]$ and $L_2 = [u_3\ u_2\ u_1\ u_4]$. Figure 15 shows that this new ordering of vertices in $G(Q)$ is not SCF.

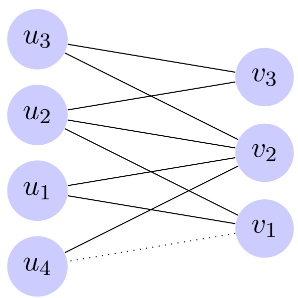


FIGURE 15: The graph $G(Q)$ with vertex set $(L_2 \cup L_1)$ and edge set E. The dotted line from node u_4 to node v_1 shows the (SCF2)-violation.

Hence, we can conclude that the matrix Q is not staircase. $\square$

Remark. Any permutation of the Q-matrix is not staircase.

We can now state the theorem about an important property of non-staircase matrices.

Theorem 4.4. Let M be an $m \times n$ binary matrix that is not staircase. Then it must either contain a W_k-submatrix ($k \geq 3$) or a Q-matrix in any permuted form.

Proof. Without loss of generality, suppose that the matrix M is minimally non-staircase and that removing row 1 or column n would make it staircase. It follows from minimality that every row and column of M is nonzero. We define $M/\{1\}$ to be the $(m-1)\times n$ matrix after removing row 1 and we define $M\backslash\{n\}$ to be the $m \times (n-1)$ matrix after removing column n. Also, let $M/\{1\}\backslash\{n\}$ be the $(m-1)\times(n-1)$ matrix after removing both row 1 and column n. Then we have that $M/\{1\}\backslash\{n\}$ is also staircase.

Now, without loss of generality, let us assume that matrix $M\backslash\{n\}$ is in staircase form. Then we must have that $M/\{1\}\backslash\{n\}$ is in staircase form as well. Since $M/\{1\}$ is staircase, there exists a permutation of $M/\{1\}$ such that the resulting matrix $\bar{M}/\{1\}$ is in staircase form. But then also $\bar{M}/\{1\}\backslash\{n\}$ is in staircase form. From Theorem 4.1, we know that the staircase form of a matrix is unique up to reversal. Hence, we must have that $\bar{M}/\{1\}\backslash\{n\} = M/\{1\}\backslash\{n\}$. This is can only be if the permutation on $M/\{1\}$ is the insertion of column n at some position k, i.e., the relative order of the columns $1,\ldots,n-1$ is the same as that of $M\backslash\{n\}$. Let us define M' to be this permuted matrix M. Then we have that $M'/\{1\}$ is in staircase form and hence $M'/\{1\}\backslash\{n\}$ is in staircase form. Since the only difference between matrix M' and M is the position of column n with respect to the relative order of the columns $1,\ldots n-1$, we have that $M'\backslash\{n\}$ is also in staircase form. Hence, no insertion is necessary and we can assume that $M' = M$.

Let us now look at the structure of M. Since M is not staircase and hence also not in staircase form, there must be an SCF-violation. This SCF-violation must be a sub-matrix of size 1×3, 3×1, or 2×2 (see Figure 2). Also, it must involve the top-right entry $M_{1,n}$ since it would otherwise also induce a violation of $M/\{1\}$ or $M\backslash\{n\}$. Each of these sub-matrices has a 1-entry in its top-right. Hence we can conclude that $M_{1,n} = 1$. Let us now distinguish the possible (SCF)-violations;

Case 1: (SCF2)-violation. We can have three different kinds of (SCF2)-violations, see Figure 16.

$$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad \begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix} \qquad \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}$$

(A) (B) (C)

FIGURE 16: Different kinds of (SCF2)-violations.

Note that the two kinds of (SCF2)-violations in Figure 16b and 16c are symmetric. Hence, we will only proof type b. The proof of the violation in Figure 16c will be similar.

Let us first consider the (SCF2)-violation in Figure 16a. If we have that $M_{m,n} = 0$, then we must have that $M_{i,n} = 0$ for all rows $i \in \{2,\ldots m\}$ because otherwise row m would be a zero-row, which contradicts our minimality assumption. But then M is staircase because we can just insert column n in front of the first column, and the permuted matrix is in staircase form. Hence, we have that $M_{m,n} = 1$. Similarly, we must have that $M_{1,1} = 1$.

Suppose that $M_{m,1} = 1$. Then the first column is the all-1s column and row m is the all-1s row because otherwise, $M/\{1\}$ or $M\backslash\{n\}$ violates (SCF1) in row m or column 1, respectively. Note that, in this case, we must have that $M_{i,n} = 0$ for all $i \in \{2,\ldots m-1\}$, because otherwise there would be identical rows. Similarly, we must have that $M_{1,j} = 0$ for all $j \in \{2,\ldots n-1\}$. Figure 17 illustrates what matrix M would look like given our current knowledge of rows $1, m$ and columns $1, n$. We can see $M/\{1\}\backslash\{n\}$ must be either

$$\begin{array}{cccc|c} 1 & 0 & \dots & 0 & 1 \\ \hline 1 & & & & 0 \\ \vdots & & & & \vdots \\ 1 & & & & 0 \\ 1 & 1 & \dots & 1 & 1 \end{array}$$

FIGURE 17: Illustration of the matrix M when there exists an (SCF2)-violation of the type in Figure 16a and $M_{m,1} = 1$. Here, only the first and last rows and columns are illustrated.

a lower-triangular matrix or the all-1s matrix. If it is the all-1s matrix, then we have must have that $m = 3$, or there would be identical rows. Then it is easy to see that M can be permuted into staircase form. Hence, we must have that $M/\{1\}\backslash\{n\}$ is a lower-triangular matrix. Then we can find rows $i, k \in \{2, \dots m-1\}$ with $i < k$ and a column $j \in \{2, \dots n-1\}$ such that $M_{i,j} = 0$ and $M_{k,j} = 1$. Let us look at the submatrix $H \subseteq M$ in Figure 18.

$$\begin{array}{c} \\ \color{red}{1} \\ \color{red}{i} \\ \color{red}{k} \\ \color{red}{m} \end{array} \begin{array}{c} \begin{array}{ccc} \color{blue}{1} & \color{blue}{j} & \color{blue}{n} \end{array} \\ \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \end{array}$$

FIGURE 18: Submatrix $H \subseteq M$ with rows $1, i, k, m$ and columns $1, j, n$.

From Figure 18, it can be easily checked that H can be permuted into the Q-matrix. Hence, M contains a permuted Q-matrix.

Suppose now that $M_{m,1} = 0$. Since M is connected, it is easy to see that M contains a W_k-submatrix.

Next, suppose there exists an (SCF2)-violation of the type in Figure 16b. Then there exists a row $k \in \{2, \dots, m\}$ and a column $\ell \in \{2, \dots, n-1\}$ such that $M_{1,\ell} = 0$, $M_{k,\ell} = 1$, and $M_{k,n} = 1$. By the same arguments as before, we must have that $M_{1,1} = 1$ and $M_{m,n} = 1$. Since $M_{1,j} = 0$ and $M_{k,j} = 1$ for all $j \in \{\ell+1, n-1\}$, We can assume w.l.o.g. that $\ell = n-1$ and that k is the second nonzero entry of column n.

Suppose that $M_{m,1} = 1$. Then again, we have that $M_{i,1} = 1$ for all rows $i \in \{2, m-1\}$ and $M_{m,j} = 1$ for all columns $j \in \{2, \dots n-1\}$. Then we must have that $k = m$ because otherwise there will exist identical rows. Then the matrix M would again look like the matrix in Figure 17 and we can use the same arguments as before to conclude that M contains a permuted Q-matrix.

Supose now that $M_{m,1} = 0$. If $k \neq 2$, then it is easy to see that M contains a W_k-matrix. On the other hand, if $k = 2$, then column n consists of only nonzero entries. We must then have that $M/\{1\}$ is an upper-triangular matrix. Note that, if $m = 3$, we can do some permutations and M would be in staircase form. Hence, we have that $m > 3$. If $M_{1,2} = 0$ and $M_{2,1} = 0$, then again, we can do some row and columns permutations so that M is in staircase form. Hence, we must have that $M_{1,2} = 1$ and/or $M_{2,1} = 1$. Then we can find rows $i, s \in \{2, \dots, m-1\}$ with $i < s$ and columns $j, t \in \{1, \dots n-1\}$ with $j < t$ that induce the submatrix in Figure 19. It is easy to see that this submatrix is a permutation of Q.

$$\begin{array}{c|c} & \begin{array}{ccc} j & t & n \end{array} \\ \begin{array}{c} 1 \\ i \\ s \\ m \end{array} & \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

FIGURE 19: Submatrix $H \subseteq M$ with rows $1, i, s, m$ and columns j, t, n.

In short, if there exists an (SCF2)-violation of any of the three types in Figure 16, then M must contain a W_k-submatrix or a permuted Q-matrix.

Case 2: (SCF1)-violation. We can have two types of (SCF1)-violations, see Figure 20.

$$\begin{pmatrix} 1 & 0 & 1 \end{pmatrix}$$

(A)

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

(B)

FIGURE 20: Different kinds of (SCF1)-violations.

The two kinds of (SCF1)-violations in Figure 20 are symmetric, and hence the proof of these will be similar. Therefore, we will only give the proof of the (SCF1)-violation in Figure 20a.

Since the violation must involve the top-right entry $M_{1,n}$ of matrix M, we must have that the violation in Figure 20a happens in the first row. Hence, we have that $M_{1,n-1} = 0$ and $M_{1,\ell} = 1$ for some column $\ell \in \{1, \dots n-2\}$. Since column $n-1$ may not be a zero column due to our minimality assumption, there exists some row $k \in \{2, \dots m\}$ such that $M_{k,n-1} = 1$. If we then look at the submatrix of rows $1, k$ and columns $n-1, n$ in Figure 21, we can see that there is also an (SCF2)-violation. Specifically, if $M_{k,n} = 0$, there is an (SCF2)-violation of the type in Figure 16a. Otherwise, if $M_{k,n} = 1$, then there is an (SCF2)-violation of the type in Figure 16b. We have already proven our statement for these types of SCF-violations. Hence, we can use the same arguments to verify that M must contain a W_k-submatrix or a Q-matrix.

$$\begin{array}{c|c} & \begin{array}{cc} n-1 & n \end{array} \\ \begin{array}{c} 1 \\ k \end{array} & \begin{pmatrix} 0 & 1 \\ 1 & \star \end{pmatrix} \end{array}$$

FIGURE 21: Submatrix $H \subset M$ with rows $1, k$ and columns $n-1, n$. The $\star$ in the bottom-right entry indicates that we do not know the value of this entry yet.

In all possible cases, M must contain a W_k-submatrix or a Q-matrix. This concludes the proof. $\square$

5 Conclusions

We have found a linear-time algorithm (see Algorithm 2) that detects whether a given matrix is staircase and find its staircase form (if applicable). Since the fastest known algorithm was a polynomial-time algorithm (see [2]), we can conclude that we have successfully found a more efficient algorithm. Furthermore, we have found that every non-staircase matrix contains either a W_k-submatrix or a Q-submatrix.

For further research it could be interesting to extend Algorithm 2 as to finding the forbidden sub-matrices W_k and Q if the input is not staircase. Also, since staircase structures can be viewed as special kinds of block-tridiagonal structures, it might be interesting to investigate if Algorithm 2 can be modified so that it will be applicable to matrices and systems that have other kinds of block structures.

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